Geometry advanced class
Construction of non-locally symmetric Einstein manifolds with negative curvature, Part II

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1 Reminders from Part I

In the last lecture of this advanced class, we study the analytical aspects of the construction of non-locally symmetric negatively curved compact Einstein manifolds [1, 4]. In the previous lecture, we explained how to construct a family of hyperbolic manifolds (M_k, g_k) and branched cyclic l-fold covers $\pi_{k,l} : X_{k,l} \to M_k$ ramified along a null-homologous codimension 2 totally geodesic submanifold $\Sigma_k \subset M_k$ ($[\Sigma_k] = 0 \in H_{n-2}(M_k)$, where the dimension $n \geq 4$. Recall moreover that we had the following properties (Prop. 2.2 in the previous lecture):

(i) The injectivity radius $R_k \to \infty$.

(ii) Σ_k has a most two connected components.

(iii) The normal injectivity radius of $\Sigma_k \subset M_k$, $R_k^\nu \geq R_k/2$.

(iv) $\frac{\operatorname{diam}(\Sigma_k)}{R_k^\nu} \to 0$.

(v) $\operatorname{vol}(\Sigma_k) \leq C \exp((n-3)\operatorname{diam}(\Sigma_k))$.

These are particular examples of *Gromov-Thurston manifolds* [3], and they cannot admit any locally symmetric Einstein metrics (or any homogeneous Einstein metrics if $n = 4$). Nevertheless, we saw that $X_{k,l}$ can be endowed with a negatively curved metric $g_{k,l}$ which is almost Einstein, and we had the following estimates (Prop. 4.1 and the following remark in the previous lecture):

(i) $\|\operatorname{Ric}(g_{k,l} + (n-1)g_{k,l}\|_{C^m} \leq C(m,n,l)U_k^{-(n-1)}$ for any $m \in \mathbb{N}$.

(ii) $\sec(g_{k,l}) \leq c(m,n,l) < 0$.

(iii) For $U < U_{\max}$, $\operatorname{vol}(\{\frac{1}{2}U < u < U\}) \leq C(m,n,l)\operatorname{vol}(\Sigma_k)U_k^{n-1}$.

(iv) $\|\operatorname{Ric}(g_{k,l} + (n-1)g_{k,l}\|_{L^2} \to 0$ as $k \to \infty$ (for fixed l).

Here U_k is a gluing parameter satisfying $U_k \leq U_{k,\max} = \cosh(R_k^\nu/2)$ and $u = \cosh(d(\Sigma_k, \cdot))$.

The goal of this lecture is to explain how to deform $g_{k,l}$ to a nearby Einstein metric (at least for k, l large enough). This makes use of the following version of the Banach fixed-point theorem:

Theorem 1.1. *Let $(A_1, \|\cdot\|_1), (A_2, \|\cdot\|_2)$ be Banach spaces and $f : B_\delta \subset A_1 \to A_2$ be a continuous function defined on the ball of radius δ centred at 0 which can be written as*

$$f(u) = f(0) + L(u) + F(u)$$

where $L : A_1 \to A_2$ *is a bounded linear map which has a bounded inverse and* $F : B_\delta \to A_2$ *is a continuous map such that*

$$\|F(u_1) - F(u_2)\|_2 \leq C\|u_1 - u_2\|_1(\|u_1\|_1 + \|u_2\|_1)$$

for some constant $C > 0$*. Let* Q *be the operator norm of* L^{-1} *and assume that the following inequalities hold:*

$$Q|f(0)| < \delta/2, \quad CQ\delta \leq 1/4.$$

Then there is a unique $u \in B_\delta$ *such that* $f(u) = 0$*, and moreover*

$$\|u\|_1 \leq 2Q\|f(0)\|_2.$$

An obvious issue in our situation is that the Einstein equation is invariant under diffeomorphisms, and as such there is no hope for the linearised deformation operator to be elliptic (it always has an infinite-dimensional kernel). Therefore, our first task will be to rewrite the equation in order to fix the diffeomorphism invariance. We will then spend the remainder of the lecture explaining how to find estimates on the inverse of the linearised operator in order to run the fixed-point theorem, following the method of Hamendstädt–Jäckel.

2 Rewriting the Einstein equation

In this section we let $(X, \overline{g})$ be a compact Riemannian manifold. Before writing the linearisation of the Einstein equation, we need to introduce some notations. For any vector feld v on X, we will denote by $\mathrm{Ric}_{\overline{g}}(v)$ the vector field dual to the 1-form $\mathrm{Ric}(\overline{g})(v, \cdot)$ with respect to $\overline{g}$. If $h \in C^\infty(S^2T^*M)$ we define

$$\mathrm{Ric}_{\overline{g}}(h)(v, w) = h(\mathrm{Ric}_{\overline{g}}(v), w) + h(v, \mathrm{Ric}_{\overline{g}}(w)) - 2\,\mathrm{tr}_{\overline{g}}\, h(R(\cdot, v)w, \cdot). \tag{2.1}$$

Then $\mathrm{Ric}_{\overline{g}}(h)$ is also a symmetric $(0, 2)$-tensor. Then we can define the *Lichnerowicz Laplacian* by

$$\Delta_L = \nabla^*\nabla h + \mathrm{Ric}_{\overline{g}}(h) \tag{2.2}$$

where ∇^* is the adjoint operator of ∇, implicitly defined with respect to $\overline{g}$. Note that ∇_L is an *elliptic* operator.

If T is a $(0, k)$-tensor, we may also define its divergence $\delta_{\overline{g}}T$ by

$$\delta T = -\,\mathrm{tr}_{\overline{g}}(\nabla T) \tag{2.3}$$

where we take the trace with respect to the first two indices by convention. Then it can be shown that if $g_t = \overline{g} + th$ is a family of metrics, the variation of the Ricci tensor is given by the following formula (see for instance Peter Topping's notes on the Ricci flow):

$$\left.\frac{\partial\,\mathrm{Ric}(g_t)}{\partial t}\right|_{t=0} = \frac{1}{2}\Delta_L h - \frac{1}{2}\mathcal{L}_{\beta_{\overline{g}}(h)^\sharp}\overline{g} \tag{2.4}$$

where $\mathcal{L}$ is the Lie derivative, $\sharp$ the musical isomorphism mapping 1-forms to their dual vector field with respect to $\overline{g}$, and $\beta_{\overline{g}}(h)$ is the Bianchi operator:

$$\beta_{\overline{g}}(h) = \delta h + \frac{1}{2} d\,\mathrm{tr}_{\overline{g}}(h). \tag{2.5}$$

In order to fix the diffeomorphism invariance and compensate the second term in order to make the linearisation of the Einstein equation elliptic, the idea is to introduce the operator

$$\Phi_{\overline{g}}(g) = \mathrm{Ric}(g) + (n-1)g + \frac{1}{2}\mathcal{L}_{\beta_{\overline{g}}(g)^\sharp} g \tag{2.6}$$

for another Riemannian metric g. Then the linearisation of $\Phi_{\overline{g}}$ at $g = \overline{g}$ is to be

$$(D\Phi_{\overline{g}})_{\overline{g}}(h) = \frac{1}{2}\Delta_L h + (n-1)h \tag{2.7}$$

for any symmetric $(0,2)$-tensor h. This uses the identity $\beta_{\overline{g}}(\overline{g}) = 0$ which is clear. Another useful identity is $\beta_{\overline{g}}(\mathrm{Ric}(\overline{g})) = 0$ which follows by tracing the Bianchi identity. Perhaps for this reason, $\beta_{\overline{g}}$ is called the Bianchi operator, and a metric satisfying $\beta_{\overline{g}}(g) = 0$ is said to be *in Bianchi gauge with respect to* $\overline{g}$.

This looks like an ad hoc way of making the equations elliptic, but the point is that the equation $\Phi_{\overline{g}}(g) = 0$ implies that g is a Einstein metric at least when $\mathrm{Ric}(g) < 0$ (see [1, Lem. 4.2], with somewhat different notations):

Lemma 2.1. *Let g be a Riemannian metric with $\mathrm{Ric}(g) < 0$ on X. Then $\Phi_{\overline{g}}(g) = 0$ if and only if $\beta_{\overline{g}}(g) = 0$ and $\mathrm{Ric}(g) = -(n-1)g$.*

Proof. The idea is to apply the Bianchi operator β_g of g to the equation $\Phi_{\overline{g}}(g) = 0$, which yields $\beta_g(\mathcal{L}_{\beta_{\overline{g}}(g)^\sharp} g) = 0$. Now a fastidious computation yields

$$\beta_g(\mathcal{L}_{\beta_{\overline{g}}(g)^\sharp} g) = \frac{1}{2}\nabla^{*_g}\nabla^g \beta_{\overline{g}}(g) - \frac{1}{2}\mathrm{Ric}(g)(\beta_{\overline{g}}(g)^\sharp, \cdot) \tag{2.8}$$

and since $\mathrm{Ric}(g) < 0$ and X is compact, then $\mathrm{Ric}(g) < -\lambda g$ for some $\lambda > 0$ and the result follows by integration by parts:

$$0 = \int_X \langle \beta_g(\mathcal{L}_{\beta_{\overline{g}}(g)^\sharp} g), \beta_{\overline{g}}(g)\rangle \, \mathrm{dvol}_g \geq \int_M |\nabla^g \beta_{\overline{g}}(g)|_g^2 \, \mathrm{dvol}_g + \lambda \int_M |\beta_{\overline{g}}(g)|_g^2 \, \mathrm{dvol}_g \tag{2.9}$$

which implies $\beta_{\overline{g}}(g)^\sharp = 0$. ☐

In order to construct Einstein metrics on the manifolds $X_{k,l}$ using the fixed-point theorem, the most important step is to prove that the linearised deformation operator $L_{k,l} = \frac{1}{2}\Delta_L + (n-1)$ is invertible on $(0,2)$-tensors, with uniform estimates on the operator norm of its inverse. In the next two sections we explain the argument of Hamendstädt–Jäckel who found a very slick way to do so in any dimension $n \geq 4$.

3 The deformation argument

The invertibility of $L_{k,l}$ is a consequence of the following proposition [1, Prop. 4.3]:

Proposition 3.1. *For any $l \geq 2$, there exists a constant $\lambda = \lambda(n,l) > 0$ such that for any sufficiently large k we have*

$$\int_{X_{k,l}} \langle L_{k,l} h, h\rangle_{g_{k,l}} \, \mathrm{dvol}_{g_{k,l}} \geq \lambda \int_{X_{k,l}} |h|^2_{g_{k,l}} \, \mathrm{dvol}_{g_{k,l}}$$

for all $h \in C^\infty(S^2 T^* X_{k,l})$.

The proof proceeds by integration by parts using (2.2), which yields

$$\int_{X_{k,l}} \langle L_{k,l} h, h\rangle_{g_{k,l}} \, \mathrm{dvol}_{g_{k,l}} \geq (n-1) \int_{X_{k,l}} |h|^2_{g_{k,l}} \, \mathrm{dvol}_{g_{k,l}} + \frac{1}{2} \int_{X_{k,l}} \langle \mathrm{Ric}_{g_{k,l}}(h), h\rangle_{g_{k,l}} \, \mathrm{dvol}_{g_{k,l}}. \tag{3.1}$$

Then the idea is to use the fact that $\mathrm{Ric}(g_{k,l}) \to -(n-1)$ and the uniform upper bound $\sec(g_{k,l}) \leq -c$ in order to estimate the right-hand side, given the explicit expression (2.1) of $\mathrm{Ric}_{g_{k,l}}(h)$. The proof of Fine–Premoselli is essentially an adaptation of an argument of Koiso

[6], who proves that for any compact Einstein manifold (M^n, g) such that $\mathrm{Ric}(g) = -(n-1)$ and $\sec(g) \leq -K < 0$, the deformation operator L_g satisfies

$$\int_M \langle L_g h, h\rangle_g \,\mathrm{dvol}_g \geq \frac{n-2}{2} K \int_M |h|_g^2 \,\mathrm{dvol}_g \tag{3.2}$$

which in particular implies that (M, g) is *rigid*.

In order to find uniform bounds on the inverse of $L_{k,l}$, the key result of Hamenstädt–Jäckel is the following local C^0-estimate [4, Lem. 2.2]:

Lemma 3.2. *For all $n \in \mathbb{N}$, $\Lambda \geq 0$ and $i_0 > 0$ there exist constants $\rho, C > 0$ with the following property. Let (X, g) be a Riemannian n-manifold with $|\sec(g)| \leq \Lambda$ and $\mathrm{inj}(X, g) \geq i_0$. Let $f \in C^0(S^2T^*X)$ and $h \in C^2(T^*X)$ satisfying*

$$\frac{1}{2}\Delta_L h + (n-1)h = f.$$

Then for all $x \in X$ it holds

$$|h|(x) \leq C(\|h\|_{L^2(B_\rho(x))} + \|f\|_{C^0(B_\rho(x))}).$$

We will outline the proof of these local estimates in the next section. Before this, we explain how this is enough to run the deformation argument in order to perturb $g_{k,l}$ to an Einstein metric $\widetilde{g}_{k,l}$ in the remainder of this section.

The idea is to work with the Banach spaces $C^{0,\alpha}(S^2T^*X_{k,l})$ and $C^{2,\alpha}(T^*X_{k,l})$, equipped with hybrid norms

$$\|h\|_0 = \max\{\|h\|_{C^{0,\alpha}(g_{k,l})}, \|h\|_{L^2(g_{k,l})}\}$$
$$\|h\|_2 = \max\{\|h\|_{C^{2,\alpha}(g_{k,l})}, \|h\|_{L^2_2(g_{k,l})}\}$$

where the L^2_2 of h norm is defined as the L^2-norm of $(h, \nabla h, \nabla^2 h)$. The norms $\|\cdot\|_i$ are equivalent to the $C^{i,\alpha}$-norms and hence $C^{i,\alpha}(S^2T^*X_{k,l})$ is still a Banach space for the norm $\|\cdot\|_i$. The point of introducing these norms is that with the previous lemma one can show that the operator norm of the inverse of $L_{k,l}$ is uniformly bounded (for l fixed and k large enough) [4, Prop. 4.2]:

Proposition 3.3. *There exists a constant $C = C(\alpha, n, l)$ such that*

$$L_{k,l} : (C^{2,\alpha}(S^2T^*X_{k,l}), \|\cdot\|_2) \to (C^{0,\alpha}(S^2T^*X_{k,l}), \|\cdot\|_0)$$

is invertible, and the operator norms $\|L_{k,l}\|, \|L_{k,l}^{-1}\| \leq C$.

Proof. The basic observation is that for a fixed value of l, all the spaces $X_{k,l}$ are locally the same, and therefore the operator norm of $L_{k,l}$ is uniformly bounded? Moreover, classical elliptic regularity gives uniform a priori estimates

$$\|h\|_{C^{2,\alpha}} \leq C(\|L_{k,l}h\|_{C^{0,\alpha}} + \|h\|_{C^0}) \tag{3.3}$$
$$\|h\|_{L^2_2} \leq C(\|L_{k,l}h\|_{L^2} + \|h\|_{L^2}). \tag{3.4}$$

Using Proposition 3.1, one can improve the second bound to

$$\|h\|_{L^2_2} \leq C\|L_{k,l}h\|_{L^2}. \tag{3.5}$$

It just remains to improve the first a priori estimate using the C^0-bound of Lemma 4, which implies

$$\|h\|_{C^0} \leq C(\|h\|_{L^2} + \|L_{k,l}h\|_{C^0}) \leq C(\|L_{k,l}h\|_{L^2} + \|L_{k,l}h\|_{C^{0,\alpha}} \tag{3.6}$$

whence we deduce $\|h\|_2 \leq C\|L_{k,l}h\|_0$ uniformly in k for fixed n, l, α. ☐

Once we have this results, then for k large enough the fixed-point theorem (applied to the function $\Phi_{g_{k,l}}$) implies that $g_{k,l}$ can be deformed to a nearby Einstein metric $\tilde{g}_{k,l}$ with $\|g_{k,l} - \tilde{g}_{k,l}\|_2 \leq C\|\operatorname{Ric}(g_{k,l}) + (n-1)g_{k,l}\|_0 \to 0$. This is where it is important to have the L^2-estimate. This use the fact that the nonlinear part of $\Phi_{g_{k,l}}$ satisfies quadratic bounds uniform in k.

Remark 3.4. Since $\|g_{k,l} - \tilde{g}_{k,l}\|_{C^2} \to 0$ and the sectional curvature of $g_{k,l}$ is *nonconstant* and bounded above away from zero, then the same holds for the sectional curvature of $\tilde{g}_{k,l}$; in particular the metric cannot be locally homogeneous.

4 The local estimates

In this section we explain where the local C^0-estimates of Proposition come from. In fact, the estimates are not specific to the metrics $g_{k,l}$, they only depend on a lower bound on the injectivity radius and a double-sided bound on the sectional curvatures. Hence to prove the inequality, me let (M, g) be a compact manifold with injectivity radius $\operatorname{inj}(M, g) \geq i_0 > 0$ and $\sec(g) \leq \Lambda$. Then if $\alpha \in (0, 1)$, a result of Jost–Karcher [5] implies that there exists $\rho > 0$, depending only on α, i_0, Λ and the dimension n, such that at every $p \in M$ there there are harmonic coordinates $(x_1, \ldots, x_n)$ on the ball $B(p, 2\rho)$ such that in these coordinates $\|g_{ij} - \delta_{ij}\|_{C^{1,\alpha}} \leq C$.

The local estimates will come from (a version of) the DeGiorgi–Nash–Moser estimates, which can be stated as follows. Let us consider a uniformly elliptic linear partial differential operator L acting on functions on $B_{2R} \subset \mathbb{R}^n$ with bounded coefficients, and let f be a bounded function and $u \in C^2(B_{2R})$[1] satisfying $Lu \geq f$. Then the following interior estimates hold:

$$\sup_{B_R} u \leq C(\|u^+\|_{L^2(B_{2R})} + \|f\|_{C^0(B_{2R})}) \tag{4.1}$$

for some constant C depending only on R, the bounds on the coefficients of L and the ellipticity of the principal symbol [2, Th. 8.17]. Here u^+ denotes the nonnegative part of u.

Of course in our case we are considering sections of a vector bundle and not functions so the estimates do not apply directly. Instead, the key idea is to show that the norm $|h|$ satisfies a differential inequation (at least if h does not vanish, which we can assume to be the case by a density argument since generic sections of S^2T^*M do not intersect the zero section). Let us first look at $|h|^2$: the rough Laplacian $\Delta = \nabla^*\nabla$ satisfies $\frac{1}{2}\Delta(|h|^2) = \langle \Delta h, h\rangle - |\nabla h|^2$. Now we have the equation $\frac{1}{2}\Delta h = f - \frac{1}{2}\operatorname{Ric}_g(h) - (n-1)h$ where $\operatorname{Ric}_g(h)$ may be bounded only in terms of the sectional curvature of g. Therefore using the Cauchy–Schwarz inequality we obtain an inequality of the form

$$-\Delta(|h|^2) \geq -|f||h| + C|h|^2 + |\nabla h|^2. \tag{4.2}$$

In order to get rid of the annoying term $|\nabla h|^2$, one can use the formula for the Laplacian acting on the square of a function:

$$\frac{1}{2}\Delta(|h|^2) = |h|(\Delta|h|) - (\nabla|h|)^2. \tag{4.3}$$

Moreover $(\nabla|h|)^2 \leq |\nabla h|^2)$, and putting everything together one obtains

$$-\frac{1}{2}\Delta|h| + C|h| \geq -|f| \tag{4.4}$$

and the DeGiorgi–Nash–Moser estimate can be applied to this equation.

[1] One could also considerably relax these regularity assumptions: e.g. one could take $u \in L^2_1$ and $g \in L^q(B_{2\rho})$ for some $q > 2n$, but this will not be needed.

Remark 4.1. The original proof of Fine–Premoselli relied on the use of weighted Hölder norms, and a key part of the argument was to prove that the operator norm of the inverse of the linearised deformation operator did not grow too quickly. Ultimately this relies on close examination of the Green's functions on the model spaces, and the authors could only prove sufficiently good estimates in dimension $n = 4$. The main insight of Hamendstädt–Premoselli was to introduce the mixed Sobolev/Hölder norms where the DeGiorgi–Nash–Moser estimates imply uniform bounds on the operator norm of the inverse of the deformation operator. On the geometric side, this required a refinement of the geometric construction in order to prove the L^2-smallness of $\mathrm{Ric}(g_{k,l}) + (n-1)g_{k,l}$.

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