Index of the Lichnerowicz Laplacian for Böhm metrics on products of spheres

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Research project carried out as part of the M1 Hadamard Master's program at ENS Paris-Saclay

April - July 2026

Acknowledgements

First and foremost, I would like to thank Professor Jason D. Lotay for his supervision, his kindness, and the many interesting discussions we had throughout these three months. He suggested a subject that was interesting, intellectually stimulating and at the interface of many fields that I especially appreciate, such as analysis, spectral theory, and general relativity.

I would also like to thank Qiu Shi Wang for answering my questions and for explaining his code for computing Böhm metrics and allowing me to use it.

I am also very grateful to Professors Matthias Wink and Christoph Böhm who agreed to answer my questions about their papers. They also provided invaluable assistance in developing the proof presented in Section 3.2.

Finally, I would like to thank all the PhD students, Post-docs and researchers at the Mathematical Institute for welcoming me so warmly during this project.

Résumé

La théorie de la relativité générale, établie par Albert Einstein au début du XXe siècle, stipule notamment que la géométrie de notre espace-temps de dimension 4 est décrite par un système d'équations, appelées "équations d'Einstein".

Les métriques d'Einstein sont une généralisation naturelle des solutions des équations d'Einstein en toute dimension. Elles vérifient la relation

$$\mathrm{Ric} = \Lambda g$$

pour une constante réelle Λ.

Les métriques de Böhm, découvertes en 1998 par C. Böhm [Böh98], sont les premiers exemples non triviaux de telles métriques sur des espaces de petite dimension, comme S^{k+l+1} ou $S^{k+1} \times S^l$ avec $k, l \geqslant 2$ et $k + l \leqslant 8$. Sur chacun de ces espaces, il existe une suite de métriques d'Einstein (g_i), de cohomogénéité un et non isométriques.

Cependant, bientôt trente ans après leur découverte, leurs propriétés géométriques sont toujours méconnues, notamment leur stabilité. Plus précisément, si l'on perturbe une métrique de Böhm, aura-t-on tendance à revenir vers elle (stabilité), ou à s'en éloigner (instabilité) ? Il se peut très bien que cela dépende de la direction dans laquelle on perturbe notre métrique, et dans ce cas, le nombre de directions instables permet de mieux comprendre l'ensemble des métriques d'un de ces espaces au voisinage d'une métrique de Böhm.

Pour étudier ce problème, un outil important intervient : le laplacien de Lichnerowicz. Il apparaît naturellement quand on linearise le tenseur de Ricci. Le nombre de directions instables est lié au nombre de valeurs propres négatives de ce laplacien. Si g_i désigne une métrique de Böhm, nous notons ν_i le nombre de valeurs propres négatives de cet opérateur associé à g_i, appelé *indice* du laplacien de Lichnerowicz.

Une autre notion de stabilité peut être reliée à l'étude spectrale du laplacien de Lichnerowicz. En physique théorique, il est possible d'associer un espace-temps généralisé à une métrique d'Einstein. Le trou noir obtenu, appelé espace-temps généralisé de Schwarzschild-Tangherlini, est dit *instable* si, quand on le perturbe un peu, il s'effondre sur lui-même ou se dilate à l'infini. Gibbons et Hartnoll ont trouvé un critère reliant cette instabilité à la plus petite valeur propre du laplacien de Lichnerowicz [GH02].

En 2003, Gibbons, Hartnoll et Pope ont prouvé que sur $S^3 \times S^2$, $S^3 \times S^3$ et S^5, toutes les métriques de Böhm admettent au moins une direction instable (donc $\nu_i \geqslant 1$ pour tout i) [GHP03]. Ils ont conjecturé que ce résultat demeure valable en dimension supérieure. Dans ce rapport, nous apportons une réponse partielle à cette conjecture, dans le cas des produits de sphères $S^{k+1} \times S^l$. Plus précisément, nous montrons le résultat suivant (voir la partie 6 pour un énoncé plus précis) :

Théorème A. *Si (g_i) désigne une suite de métriques de Böhm sur $S^{k+1} \times S^l$ avec $k, l \geqslant 2$, $k + l \leqslant 8$, alors*

$$\nu_i \underset{i \to +\infty}{\longrightarrow} +\infty.$$

Par conséquent, sur les produits de sphères, les métriques de Böhm sont instables (pour les deux définitions), sauf éventuellement un nombre fini d'entre elles. De plus, notre théorème affirme que, plus on progresse dans la suite, plus il existe de directions instables.

Dans ce rapport, après quelques brefs rappels de géométrie riemannienne, nous calculons le laplacien de Lichnerowicz associé aux métriques de Böhm. À notre connaissance, c'est la première fois qu'une expression explicite de cet opérateur est donnée. Outre ce calcul, nous démontrons deux résultats. En premier lieu, l'équation aux valeurs propres du laplacien prenant la forme d'une équation différentielle ordinaire, nous étudions en détail cette équation, avant de réduire le problème d'existence des valeurs propres à celui des zéros d'une fonction scalaire. Ce résultat ouvre ainsi la voie à une méthode numérique pour approcher les valeurs propres de ce laplacien. Par ailleurs, il est connu que la suite des métriques de Böhm converge vers une métrique *cône* en un certain sens. L'étude approfondie du laplacien de Lichnerowicz associée à cette métrique couplée à des arguments de théorie spectrale et d'analyse asymptotique permettent d'aboutir au Théorème A.

Abstract

The theory of general relativity, established at the beginning of the 20^{th} century, states that the geometry of our 4-dimensional spacetime is described by a system of equations: Einstein's equations.

Einstein metrics naturally generalize the solutions to these equations to higher dimensions. They satisfy the relation

$$\mathrm{Ric} = \Lambda g$$

for some real constant Λ.

Böhm metrics, discovered in 1998 by C. Böhm [Böh98], are the first non-trivial examples of such metrics on low-dimensional spaces, such as S^{k+l+1} or $S^{k+1} \times S^l$, with $k, l \geqslant 2$ and $k + l \leqslant 8$. On each of these spaces, there exists a sequence of cohomogeneity one and non-isometric Einstein metrics (g_i).

However, almost thirty years after their discovery, the geometric properties of these metrics remain poorly understood, in particular their stability. More precisely, if we perturb a Böhm metric, does it tend to move back toward it (stability) or move away from it (instability)? Of course, the stability may depend on the direction in which we perturb the metric. In this case, the number of unstable directions allows us to better understand the set of Riemannian metrics on one of these spaces near a Böhm metric.

To study this problem, a key tool is involved: the Lichnerowicz Laplacian. It appears naturally when one linearizes the Ricci tensor. The number of unstable directions is related to the number of negative eigenvalues of this Laplacian. If g_i denotes a Böhm metric, we denote by ν_i the number of negative eigenvalues of this operator associated with g_i, called the *index* of the Lichnerowicz Laplacian.

Another notion of stability is linked to the spectral study of the Lichnerowicz Laplacian. In theoretical physics, it is possible to associate a generalized spacetime with an Einstein metric, called the generalized Schwarzschild-Tangherlini spacetime. It is said to be *unstable* if, when it is perturbed, it collapses or expands indefinitely. Gibbons and Hartnoll established a criterion that connects this instability to the smallest eigenvalue of the Lichnerowicz Laplacian [GH02].

In 2003, Gibbons, Hartnoll and Pope proved that, on $S^3 \times S^2$, $S^3 \times S^3$ and S^5, all the Böhm metrics admit at least one unstable direction (more specifically that $\nu_i \geqslant 1$ for all i) [GHP03]. They also conjectured that this result remains true in higher dimensions. In this report, we give a partial answer to this conjecture, in the case of the products of spheres $S^{k+1} \times S^l$. More precisely, we prove the following result (see Section 6 for a more precise statement):

Theorem A. *If (g_i) denotes a sequence of Böhm metrics on $S^{k+1} \times S^l$, with $k, l \geqslant 2$, $k + l \leqslant 8$, then*

$$\nu_i \underset{i \to +\infty}{\longrightarrow} +\infty.$$

Consequently, on products of spheres, Böhm metrics are unstable (in both senses), except possibly for a finite number of them. Moreover, this theorem states that the further along the sequence one goes, the greater the number of unstable directions.

In this report, after a brief review of Riemannian geometry, we compute the Lichnerowicz Laplacian for Böhm metrics. To the best of our knowledge, this is the first time that an explicit expression for this operator is given. In addition to this calculation, we prove two results. First, since the eigenvalue equation is an ordinary differential equation, we study this equation in detail, before reducing the eigenvalue existence problem to the study of the zeros of a scalar function. This result leads to a numerical method to approximate the eigenvalues of this Laplacian. Moreover, it is well-known that the sequence of Böhm metrics converges to a *cone* metric, in some sense. A thorough study of the Lichnerowicz Laplacian associated with this metric, combined with ideas from spectral theory and asymptotic analysis, leads to Theorem A.

MSC 2020: 53C25 (primary), 47A75, 83E15, 58J50 (secondary).

Keywords: Einstein Metric, Lichnerowicz Laplacian, Böhm metrics, stability of Einstein metrics, cohomogeneity one, eigenvalue problems, Schwarzschild-Tangherlini spacetimes.

Contents

1 Introduction

The first non-trivial Einstein metrics on spheres were discovered in the 1970s, by Jensen on S^{4m+3} [Jen73] and by Bourguignon and Karcher on S^{15} [BK78]. Concerning low-dimensional spaces, such as spheres and products of spheres with dimensions between 5 and 9, Böhm's work in the late 1990s provided the first such construction [Böh98].

When we study Einstein metrics, one of the natural questions concerns their stability. Indeed, Einstein metrics on a manifold M are the critical points of the Einstein-Hilbert action [Hil15]:

$$\begin{aligned} S : \mathcal{M}_V &\longrightarrow \mathbb{R} \\ g &\longmapsto \int_M \mathrm{scal}_g \, dV_g, \end{aligned}$$

where $\mathcal{M}_V$ denotes the set of Riemannian metrics on M of fixed volume $V > 0$, and scal_g the scalar curvature. Thus, studying the stability of these metrics means studying the nature of the critical points of S. The nature of these critical points depends on the direction in which we perturb the metric. This leads to the notion of *unstable directions*, that is, directions along which S increases. The behavior of S in some directions, such as those given by conformal deformations (see [Bes87], chapter 4), is already well understood. More precisely, these directions are unstable and infinitely many. Moreover, S is invariant under the action of diffeomorphisms. Hence, the relevant directions are those orthogonal to the conformal directions and diffeomorphisms: transverse-traceless tensors (see Section 2.2). This restriction is necessary if one wants to study the number of unstable directions: this number is infinite when we look at all the directions but finite when we focus on transverse-traceless perturbations.

Furthermore, there is another notion of stability, more related to general relativity: the stability of generalized black holes associated with these metrics [GH02].

Regardless of the notion of stability, a tool allows us to study whether an Einstein metric is stable: the Lichnerowicz Laplacian. Originally introduced in the 1960s to study perturbations of the solutions of Einstein's equations in general relativity, it appears naturally when one linearizes the Ricci tensor. This is the reason why it is related to the first notion of stability. Regarding the second one, a criterion established in 2002 by Gibbons and Hartnoll [GH02] links the stability of generalized Schwarzschild-Tangherlini black holes to the smallest eigenvalue of the Lichnerowicz Laplacian.

Regarding Böhm metrics, the only known results about their stability were obtained in 2003 on $S^3 \times S^2$, $S^3 \times S^3$, and S^5 [GHP03]. That paper focuses on the second notion of stability and provides little information about how unstable Böhm metrics are, in the sense of the first definition.

In this report, we are interested in the number of unstable directions of Böhm metrics. More precisely, let (g_i) be a sequence of Böhm metrics on a product of spheres $S^{k+1} \times S^l$ ($k, l \geqslant 2$, $k + l \leqslant 8$) constructed in [Böh98]. We aim to answer the following question: how does the number of unstable directions evolve as we move along the sequence? Theorem A gives an answer to this question (see Section 6 for a precise statement). It also provides a partial answer to Gibbons, Hartnoll and Pope's conjecture [GHP03]: for all the products $S^{k+1} \times S^l$ that admit Böhm metrics, generalized Schwarzschild-Tangherlini black holes are unstable, except possibly for finitely many of them.

To prove Theorem A, we proceed as follows. We know that a sequence of Böhm metrics converges to the cone metric (see [Böh98], Theorem 3.7). Thus, we study the Lichnerowicz Laplacian associated with this metric. Kröncke proved that this operator is unbounded below in exactly these dimensions (see Corollary 1.6 [Krö21]). We need something stronger: explicit unstable directions, which allow us to pass to the limit in the quadratic form and then apply the min-max theorem.

In the first two parts of this report, we fix the notation and conventions we use. We also recall some notions of Riemannian geometry, and define the important objects and notions, such as the Lichnerowicz Laplacian, the different notions of stability and Böhm metrics. In Section 3.2, we prove a new result about the convergence of Böhm metrics, which will be useful in the proof of Theorem A.

Then we obtain an explicit expression for the Lichnerowicz Laplacian of Böhm metrics restricted to a particular class of tensors: G-invariant tensors. To the best of our knowledge, this is the first time that an explicit formula for the Lichnerowicz Laplacian is given for these metrics.

In the next part, we study the eigenvalue problem of the Lichnerowicz Laplacian using tools from the theory of ordinary differential equations. The main result of this part is the reduction of this problem to the study of the zeros of a scalar function. This allows us to compute numerical approximations of the eigenvalues of the Lichnerowicz Laplacian (Section 5.4).

Finally, Section 6 is devoted to the proof of Theorem A, following the method detailed above.

In the appendix, the reader will find the details of the computation of the Lichnerowicz Laplacian (Appendix A), the details of the proof of Theorem 5.2 (Appendix B) and the Python code used in the numerical simulations of Section 5.4 (Appendix C).

The proof of Theorem A presented in Section 6, and a generalization to the case of spheres which is not included in this report, constitute the basis of a research paper currently in preparation.

2 General settings

In this first part, we recall some elementary notions of Riemannian geometry. Since many definitions depend on conventions and may vary from one reference to another, this section provides a good opportunity to fix these conventions.

In this section, we consider a compact and orientable smooth Riemannian manifold (M,g) such that $\dim M \geqslant 4$. We denote by ∇ the Levi-Civita connection on M. We also denote by $\mathscr{X}(M)$ the set of smooth vector fields on M.

2.1 Einstein manifolds

The goal of this section is to define the notion of Einstein manifolds. To do so, we first define the Riemann and Ricci tensors.

Definition 2.1 (Riemann curvature tensor). *The Riemann curvature tensor is the map $R : \mathscr{X}(M)\times\mathscr{X}(M)\times\mathscr{X}(M) \to \mathscr{X}(M)$ defined for all $X,Y,Z \in \mathscr{X}(M)$ by*

$$R(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z.$$

We choose to use the same convention as in [Bes87], which is the opposite convention to the one adopted in most textbooks of Riemannian geometry.

In fact, the Riemann tensor gives "too much information" about the curvature of the manifold M. This is why we define the Ricci tensor.

Definition 2.2 (Ricci curvature tensor). *The Ricci curvature tensor is the map $\mathrm{Ric} : \mathscr{X}(M)\times\mathscr{X}(M) \to \mathcal{C}^\infty(M)$ defined for all $X,Y \in \mathscr{X}(M)$ by*

$$\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R(X,Z)Y).$$

We also denote by Ric *the map* $\mathrm{Ric} : \mathscr{X}(M) \to \mathscr{X}(M)$ *defined by*

$$\forall X,Y \in \mathscr{X}(M),\ \ \mathrm{Ric}(X,Y) = g(\mathrm{Ric}(X),Y).$$

Now that the Ricci tensor has been defined, we can introduce the notion of Einstein manifolds.

Definition 2.3 (Einstein manifold). *A Riemannian manifold (M,g) is called an Einstein manifold if* $\mathrm{Ric} = \Lambda g$, *where Λ is a real constant.*

In this case, g is called an Einstein metric.

Remark: Since Ric can also be viewed as a map from $\mathscr{X}(M)$ to $\mathscr{X}(M)$, the Einstein condition becomes $\mathrm{Ric} = \Lambda$.

Remark: Einstein manifolds are generalizations of surfaces of constant curvature to higher dimensions. However, since there exist several notions of curvature, choosing the Ricci tensor could seem arbitrary. Since the Riemann tensor gives complete information about the curvature, the condition "Riemann tensor=constant" (more precisely $R_{ijkl} = \kappa(g_{ik}g_{jl} - g_{il}g_{jk})$) is too restrictive, since such manifolds are completely classified (see [Lee18], Theorem 12.4). On the other hand, prescribing the scalar curvature to be constant is too weak, because it is always possible to find infinitely many metrics with constant scalar curvature on a compact manifold [LP87]. Thus, the condition $\mathrm{Ric} = cst$ is the right compromise.

Remark: We can always rescale the metric by a positive constant so that Λ takes any prescribed value with the same sign.

We have an interesting characterization of Einstein metrics, due to Hilbert [Hil15].

Proposition 2.4. *Let $\mathcal{M}_V$ be the set of Riemannian metrics on M of fixed volume $V > 0$. Define the Einstein-Hilbert action by*

$$\begin{aligned} S : \mathcal{M}_V &\longrightarrow \mathbb{R} \\ g &\longmapsto \int_M \mathrm{scal}_g \, dV_g, \end{aligned}$$

where $\mathrm{scal}_g = \mathrm{tr}_g(\mathrm{Ric})$.
Einstein metrics are the critical points of S.

We finally define another curvature tensor, which appears in the expression of the Lichnerowicz Laplacian.

Definition 2.5. *Let* T *be a* $(0,2)$*-tensor on* M *and* (e_A) *an orthonormal basis of the tangent space at a point. We define:*

$$\mathring{R}(T)(X,Y) = \sum_A T(R(X,e_A)Y, e_A).$$

See [Bes87] 1.131b for more details.

2.2 G-invariant symmetric transverse-traceless tensors

In this report, we study the Lichnerowicz Laplacian. It can be defined for all symmetric tensors, but when studying perturbations of an Einstein metric, it is quite natural to study symmetric traceless and transverse tensors, as mentioned in the introduction. In the next section, we will see another reason why transverse-traceless tensors are the right tensors to study.

In this section, we define transverse-traceless tensors. We also define another class of tensors: G-invariant tensors.

Since we only work on $(0,2)$-tensors, we only study this case, even though the definitions can be generalized.

G denotes a Lie group that acts on the manifold M, $\mathscr{T}_2{}^0(M)$ denotes the set of symmetric $(0,2)$-tensors on M.

We first define "transverse-traceless" tensors (called TT tensors).

Definition 2.6 (Traceless tensor)**.** *Let* $T \in \mathscr{T}_2{}^0(M)$. T *is said to be traceless (or trace-free) if*

$$\mathrm{tr}_g(T) = 0.$$

To define transverse tensors, we need to define an operator called *divergence* on tensors.

Definition 2.7 (Divergence)**.** *Let* $T \in \mathscr{T}_2{}^0(M)$ *and* (e_A) *be an orthonormal basis of the tangent space at a given point. The divergence of* T *is defined for all* $X \in \mathscr{X}(M)$ *by*

$$(\delta T)(X) = -\sum_A (\nabla_{e_A} T)(e_A, X).$$

We can now give the definition of a transverse tensor.

Definition 2.8 (Transverse tensor)**.** *Let* $T \in \mathscr{T}_2{}^0(M)$. T *is said to be transverse if*

$$\delta T = 0.$$

Definition 2.9 (TT tensor)**.** $T \in \mathscr{T}_2{}^0(M)$ *is said to be TT if it is traceless and transverse. The set of TT tensors is denoted by* $\mathscr{T}_{TT}(M) \subset \mathscr{T}_2{}^0(M)$.

The space $\mathscr{T}_{TT}(M)$ is difficult to describe explicitly. When studying the Lichnerowicz Laplacian, we restrict our attention to a subspace of $\mathscr{T}_{TT}(M)$: the set of G-invariant TT tensors. This restriction is natural since Böhm metrics are G-invariant, as we shall see in Section 3.1.

Definition 2.10 (G-invariant tensor)**.** *Let* $T \in \mathscr{T}_2{}^0(M)$. T *is said to be* G*-invariant if for all* $x \in M$, $a \in G$ *and* $X, Y \in \mathscr{X}(M)$*:*

$$T_x(X,Y) = T_{a\cdot x}(da_x(X), da_x(Y)),$$

where da_x *denotes the differential of the action at* x.

For instance, the round metric on the sphere S^d (*i.e.* the metric induced by the Euclidian inner product on $\mathbb{R}^{d+1}$) is $SO(d+1)$-invariant.

We denote by $\mathscr{T}^G_{TT}(M) \subset \mathscr{T}_{TT}(M)$ the subspace of G-invariant TT tensors.

2.3 The Lichnerowicz Laplacian

This Laplacian was introduced by A. Lichnerowicz in 1961 in [Lic61]. It appears naturally when one linearizes Einstein's equations and studies gravitational waves. It is also related to the notion of stability of Einstein metrics.

It belongs to the family of Laplace-type operators admitting a Weitzenböck decomposition (see [Bes87], chapter 1 for more details).

Definition 2.11 (Lichnerowicz Laplacian)**.** *The Lichnerowicz Laplacian is the linear operator* $\Delta_L : \mathscr{T}_2{}^0(M) \to \mathscr{T}_2{}^0(M)$ *defined by*

$$\forall T \in \mathscr{T}_2{}^0(M), \;\; \Delta_L T = \overline{\Delta} T + \mathfrak{R} T,$$

where $\overline{\Delta} = \nabla^* \nabla$ *is the Bochner Laplacian, and*

$$\forall T \in \mathscr{T}_2{}^0(M), \;\; \forall X, Y \in \mathscr{X}(M), \;\; \mathfrak{R}(T)(X,Y) = T(\mathrm{Ric}(X), Y) + T(X, \mathrm{Ric}(Y)) - 2\mathring{R}(T)(X,Y).$$

We have the following property (see [Bes87] 1.174d).

Theorem 2.12. *Let* $T \in \mathscr{T}_2{}^0(M)$ *be a symmetric tensor. Then*

$$d\,\mathrm{Ric}_g(T) = \frac{1}{2}\Delta_L(T) - \delta^*\delta(T) - \frac{1}{2}D_g d(\mathrm{tr}_g(T)),$$

where δ^* *is defined for all 1-forms* ω *by*

$$\delta^*\omega(X,Y) = \frac{1}{2}((\nabla_X \omega)(Y) + (\nabla_Y \omega)(X)),$$

and $D_g d(u)(X,Y) = X(Y(u)) - (\nabla_X Y)u$ *is the Hessian of a smooth function* u*.*

This theorem motivates the choice of TT tensors. Indeed, if $T \in \mathscr{T}_{TT}(M)$, we get:

$$d\,\mathrm{Ric}_g(T) = \frac{1}{2}\Delta_L(T).$$

Finally, if we define Δ_L on $\mathscr{T}_{TT}(M)$, we have the standard properties: Δ_L is self-adjoint, elliptic and bounded below (see [Krö16] for more details).

From now on, we assume that Δ_L is defined on $\mathscr{T}_{TT}(M)$.

2.4 Stability of Einstein metrics

There are several notions of stability for Einstein metrics. We present here two of them, both related to the Lichnerowicz Laplacian.

Geometric stability. This is the most classical notion of stability. It is related to Proposition 2.4. See [Krö21] Definition 1.2 for more details.

Definition 2.13 (Stable metric)**.** *Let* g *be an Einstein metric.* g *is called stable if* $S''_{g\,|\mathscr{T}_{TT}(M)}$ *is non-positive.*

By Theorem 2.12, this definition is related to the Lichnerowicz Laplacian.

Proposition 2.14 (A stability criterion)**.** *Let* g *be an Einstein metric and* Λ *the Einstein constant associated with* g*. Then* g *is stable if and only if*

$$\Delta_L \geqslant 2\Lambda.$$

Proof. It is an immediate consequence of the following formula (see [Krö16]):

$$\left.\frac{d^2}{dt^2}\right|_{t=0} S(g+th) = -\frac{1}{2}\int_M \langle h, \Delta_L h - 2\Lambda h \rangle dV_g.$$

$\square$

Remark: Since Δ_L is elliptic and self-adjoint on a smooth compact manifold, its spectrum is discrete and bounded below with finite multiplicities. Hence the number of unstable directions is finite.

Physical stability. This notion is related to theoretical physics. Given an Einstein metric g, we can associate a generalized spacetime with this metric and study the stability of this spacetime (see [GH02]). Gibbons and Hartnoll established the following instability criterion [GH02; GHP03].

Theorem 2.15. *A generalized Schwarzschild-Tangherlini spacetime associated with an Einstein metric g is unstable if the smallest eigenvalue $\mu_{\min}$ of Δ_L satisfies:*

$$\mu_{\min} < \mu_* := 4 - \frac{(5-d)^2}{4},$$

where d is the dimension of the Einstein manifold.

Although these notions arise in different contexts, they are both governed by the spectrum of the Lichnerowicz Laplacian.

In our case, we study Böhm metrics on products of spheres. The dimension of the manifold satisfies $d \in [5,9]$, thus $\mu_* \geqslant 0$. Moreover, for these metrics, $\Lambda > 0$. Therefore, if we show that the Lichnerowicz Laplacian admits negative eigenvalues, we can conclude that Böhm metrics are geometrically and physically unstable. This is the aim of this report.

3 Böhm Metrics on products of spheres

3.1 Background on Böhm metrics

In this part, we briefly review Böhm's construction of Einstein metrics from [Böh98].

Let $k, l \geqslant 2$, such that $k + l \leqslant 8$.

We consider the product of spheres $M = S^{k+1} \times S^l$.

We denote by G the group $SO(k+1) \times SO(l+1)$. G acts on M with cohomogeneity one, *i.e.* $\dim(M/G) = 1$. Since M is compact, we can write: $M/G = [0, \Upsilon]$ for some $\Upsilon > 0$.

The canonical projection $\pi : M \to M/G$ can be seen as a height function (see Figure 1).

There are two kinds of orbits:

- *Principal orbits* (in blue in Figure 1). All orbits but two are principal. They are diffeomorphic to the product $S^k \times S^l$. The principal orbits are hypersurfaces of M.
- *Singular orbits* S_+ (for $t = 0$) and S_- (for $t = \Upsilon$) (in red in Figure 1). The dimension of a singular orbit is strictly smaller than the dimension of the principal orbits. In our case $S_+ = S_- = S^l$.

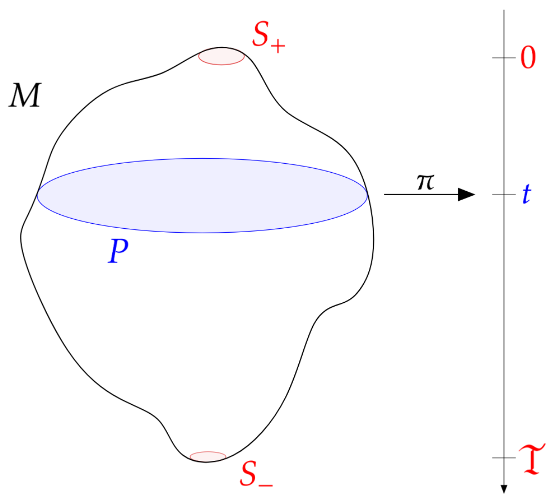

Figure 1: Principal and singular orbits

Böhm looked for Einstein metrics which, on the principal orbits, take the form:

$$g = dt^2 + f(t)^2 g^k + h(t)^2 g^l, \tag{1}$$

where g^k and g^l denote the round metrics on S^k and S^l, and f, h two smooth functions. The Einstein condition $\mathrm{Ric} = \Lambda g$ for these metrics is as follows:

$$\Lambda = -\left(k\frac{f''}{f} + l\frac{h''}{h}\right), \tag{2}$$

$$\Lambda = -\frac{f''}{f} + (k-1)\frac{1-f'^2}{f^2} - l\frac{f'h'}{fh}, \tag{3}$$

$$\Lambda = -\frac{h''}{h} + (l-1)\frac{1-h'^2}{h^2} - k\frac{f'h'}{fh}. \tag{4}$$

See [GHP03] for further details.

Remark: We adopt Böhm's normalization for the Einstein constant, *i.e.* $\Lambda = k + l$.

Böhm imposes the following boundary conditions for f and h

$$f(0) = 0,\ f'(0) = 1,\ h(0) = \bar{h} > 0, h'(0) = 0, \quad f(\Upsilon) = 0,\ f'(\Upsilon) = -1,\ h(\Upsilon) = \bar{h}, h'(\Upsilon) = 0. \tag{5}$$

A metric of the form (1), with f and h solutions of (2), (3), (4), with initial conditions (5) is denoted by $g(\bar{h})$.

Remark: Such a metric is *symmetric*, *i.e.* it satisfies

$$f(\Upsilon - t) = f(t), \qquad h(\Upsilon - t) = h(t).$$

This means that the metric is even with respect to the middle of the manifold.

Notation (Mean curvature)**.** We denote by H the mean curvature. More precisely,

$$H = k\frac{f'}{f} + l\frac{h'}{h}. \tag{6}$$

Now that Böhm metrics have been defined, it is natural to ask whether there exist metrics of this form on M. The following theorem, proved in 1998, answers this question.

Theorem 3.1 (Theorem 3.4 [Böh98])**.** *Let $k, l \geqslant 2$ such that $k + l \leqslant 8$. $S^{k+1} \times S^l$ carries infinitely many non-isometric Einstein metrics with positive scalar curvature. On all of these manifolds the full isometry group acts with cohomogeneity one.*

More precisely, Böhm proved that there exists a decreasing sequence $(\bar{h}_i)$, tending to 0, of initial conditions such that for all $i \in \mathbb{N}$, $g_i := g(\bar{h}_i)$ is an Einstein metric on M for the Einstein constant $\Lambda = k + l$.

Remark: For all $i \in \mathbb{N}$, we denote by $f_i, h_i : [0, \Upsilon_i] \to \mathbb{R}$ the functions associated with g_i. More precisely, (1) becomes:

$$g_i = dt^2 + f_i(t)^2 g^k + h_i(t)^2 g^l.$$

The behavior of the sequence (Υ_i) is known (see [Böh98], Section 6).

Proposition 3.2. *For all $i \in \mathbb{N}$, $\Upsilon_i \leqslant \pi$ and*

$$\Upsilon_i \underset{i\to+\infty}{\longrightarrow} \pi.$$

Böhm also studied the behavior of the sequence (g_i) as $i \to +\infty$. He proved the following theorem.

Theorem 3.3 (Theorem 3.7 [Böh98])**.**

$$g_i \underset{i\to\infty}{\longrightarrow} g_c$$

in the Gromov-Hausdorff distance where $g_c = dt^2 + f_c(t)^2 g^k + h_c(t)^2 g^l$ (the cone metric) is defined by :

$$f_c(t) = \sqrt{\frac{k-1}{\Lambda - 1}}\sin(t), \qquad h_c(t) = \sqrt{\frac{l-1}{\Lambda - 1}}\sin(t).$$

Remark: In this case, $\Upsilon = \pi$.

Remark: The cone metric is not smooth at the singular orbits. Indeed, it is singular because $f_c(t) \to 0$ and $h_c(t) \to 0$ when $t \to 0$ and $t \to \pi$.

This convergence is quite weak because the sequence (g_i) cannot converge near the singular orbits (due to the boundary conditions).

However, if we remain sufficiently far from the singular orbits, we can prove a better convergence. This is the goal of the next section.

3.2 Convergence of the Böhm functions away from the singular orbits

Our first result is a stronger convergence of Böhm metrics than the one established in Theorem 3.3 when we are far from the singular orbits.

Theorem 3.4. *Let (g_i) be a sequence of Böhm metrics on $S^{k+1} \times S^l$. For every compact subset K of $(0,\pi)$, and $n \in \mathbb{N}$,*

$$\left\| f_i^{(n)} - f_c^{(n)} \right\|_{\infty,K} \underset{i\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad \left\| h_i^{(n)} - h_c^{(n)} \right\|_{\infty,K} \underset{i\to+\infty}{\longrightarrow} 0.$$

Remark: Since $\Upsilon_i \to \pi$, restrictions of f_i and h_i to K are well defined for i sufficiently large.

To the best of our knowledge, this refinement does not appear in the literature, although it is presumably known to experts.

The goal of this section is to prove Theorem 3.4. More specifically, we prove the following result.

Theorem 3.5. *Let K be a compact subset of $(0,\pi)$. Let $Y_i = (f_i, f_i', h_i, h_i')$ and $Y_c = (f_c, f_c', h_c, h_c')$. Then*

$$\|Y_c - Y_i\|_{\infty,K} \underset{i\to+\infty}{\longrightarrow} 0.$$

The following lemma shows that it is sufficient to prove Theorem 3.5.

Lemma 3.6. *Theorem 3.5 implies Theorem 3.4.*

Proof. Let

$$X(f,f',h,h') = \left(f', -\Lambda f + (k-1)\frac{1-f'^2}{f} - l\frac{f'h'}{h}, h', -\Lambda h + (l-1)\frac{1-h'^2}{h} - k\frac{f'h'}{f}\right),$$

defined on $\Omega = \{(f,f',h,h') \in \mathbb{R}^4 \mid f > 0 \text{ and } h > 0\}$. The Einstein equations are just

$$Y' = X(Y). \tag{7}$$

Let $\delta = d(Y_c(K), \Omega^c) > 0$, and let $\tilde{K} = \{Y \in \Omega \mid d(Y, Y_c(K)) \leqslant \delta/2\}$.

Since X is $\mathcal{C}^1$ on $\tilde{K}$, it is L-Lipschitz for some $L > 0$. Moreover, we have $Y_c' = X(Y_c)$ and $Y_i' = X(Y_i)$. Thus, for all $t \in K$,

$$|Y_c'(t) - Y_i'(t)| \leqslant L|Y_c(t) - Y_i(t)| \leqslant L\|Y_c - Y_i\|_{\infty,K}.$$

Hence, we also have the uniform convergence of the second derivatives of f_i and h_i. Since X is smooth, an induction leads to Theorem 3.4. Thus, it is sufficient to prove Theorem 3.5.

□

We first record a lemma that will be needed below.

Lemma 3.7. *Let $X : \Omega \to \mathbb{R}^4$ be a $\mathcal{C}^\infty$ function defined on an open subset Ω of $\mathbb{R}^4$. Let $s_- < 0 < s_+$ and $Z_c : [s_-, s_+] \to \Omega$ be a solution of $Z' = X(Z)$.*

We define $K = Z_c([s_-, s_+])$ which is a compact subset of Ω. Let $\eta_0 > 0$ be such that $T := \{x \in \Omega \mid d(x,K) \leqslant \eta_0\}$ is a compact subset of Ω. Since X is locally Lipschitz, it admits a Lipschitz constant L on T.

Finally, let $P \in \mathbb{R}^4$ such that $|P - Z_c(0)| < \eta_0 e^{-LS}$, where $S = \max(-s_-, s_+)$.

Then, the maximal solution Z_P of

$$\begin{cases} Z' = X(Z) \\ Z(0) = P \end{cases} \qquad \textit{is defined on } [s_-, s_+]$$

and

$$\sup_{[s_-,s_+]} |Z_P - Z_c| \leqslant |P - Z_c(0)| e^{LS}.$$

Proof. Adapted from [HSD13], Section 17.4. □

We can now prove Theorem 3.5.

Proof. First, we notice that a solution Y to (7) is injective. Indeed, if there exist s, s' such that $Y(s) = Y(s')$, then $H(s) = H(s')$, where

$$H := k\frac{f'}{f} + l\frac{h'}{h} \quad \text{(the mean curvature)}.$$

However, (2) yields

$$H' \leqslant -\Lambda,$$

which implies $s = s'$.

By Theorem 5.7 and Section 7 in [Böh98], there exists P_i on the trajectory of Y_i, such that $P_i \to Y_c(\pi/4)$. Since Y_i is injective, there exists a unique $\tau_i \in (0, \Upsilon_i)$ such that $P_i = Y_i(\tau_i)$.

Now, we want to prove that $\tau_i \to \frac{\pi}{4}$.

Let $\varepsilon \in (0, \pi/8)$, and $J_\varepsilon = \left[\varepsilon - \frac{\pi}{4}, \frac{3\pi}{4} - \varepsilon\right]$. We set $Z_c(t) = Y_c\left(t + \frac{\pi}{4}\right) = (z_c^1(t), z_c^2(t), z_c^3(t), z_c^4(t))$.

$K_\varepsilon := Z_c(J_\varepsilon)$ is a compact subset of Ω.

For all $t \in J_\varepsilon$, we have

$$z_c^1(t) \geqslant \underbrace{\sqrt{\frac{k-1}{\Lambda-1}}}_{=:a_1} \sin(\varepsilon) \quad \text{and} \quad z_c^3(t) \geqslant \underbrace{\sqrt{\frac{l-1}{\Lambda-1}}}_{=:b_1} \sin(\varepsilon).$$

Hence, if we set $\eta_\varepsilon = \frac{1}{2}\min(a_1, b_1)\sin(\varepsilon)$, $T_\varepsilon = \{x \in \mathbb{R}^4 \mid d(x, K_\varepsilon) \leqslant \eta_\varepsilon\}$ is a compact subset of Ω. We denote by $L(\varepsilon)$ the Lipschitz constant of X on T_ε. Let $S(\varepsilon) = \frac{3\pi}{4} - \varepsilon$.

Let Z_i be the solution to

$$\begin{cases} Z' = X(Z) \\ Z(0) = P_i \end{cases}.$$

By the Picard-Lindelöf theorem, Z_i is defined on a maximal interval (α_i, β_i). Moreover, since the equation is autonomous, by uniqueness, we have $Z_i = Y_i(\tau_i + \cdot)$. Since Y_i is defined on $(-\tau_i, \Upsilon_i - \tau_i)$, we have:

$$(-\tau_i, \Upsilon_i - \tau_i) \subset (\alpha_i, \beta_i).$$

We show that $\alpha_i = -\tau_i$. We assume that $\alpha_i < -\tau_i$. Then, $Z_i(-\tau_i)$ is well defined and lies in Ω. However, $Z_i(-\tau_i) = Y_i(0) = (0, 1, \bar{h}_i, 0) \notin \Omega$.

We show similarly that $\beta_i = \Upsilon_i - \tau_i$.

By definition, $P_i \to Y_c(\pi/4)$, so there exists $i_\varepsilon \in \mathbb{N}$ such that,

$$\forall i \geqslant i_\varepsilon, \;\; |P_i - Z_c(0)| < \eta_\varepsilon e^{-L(\varepsilon)S(\varepsilon)}.$$

By Lemma 3.7, for $i \geqslant i_\varepsilon$, we have that Z_i is defined on J_ε. This implies:

$$J_\varepsilon \subset (-\tau_i, \Upsilon_i - \tau_i).$$

In other words, for all $i \geqslant i_\varepsilon$:

$$\begin{cases} \tau_i - \frac{\pi}{4} > -\varepsilon \\ \tau_i - \frac{\pi}{4} < \varepsilon + \Upsilon_i - \pi \end{cases}.$$

Finally, by Proposition 3.2, $\Upsilon_i \leqslant \pi$, so

$$\forall i \geqslant i_\varepsilon, \ \left|\tau_i - \frac{\pi}{4}\right| < \varepsilon.$$

So, $\tau_i \to \frac{\pi}{4}$.

We can now prove the uniform convergence. Let $K = [\delta, \pi - \delta]$ be a compact subset of $(0, \pi)$.

Let $J = J_{\delta/4}$, and $T = T_{\delta/4}$.

By Lemma 3.7,

$$\sup_{J_{\delta/4}} |Z_i - Z_c| \leqslant |P_i - Z_c(0)| e^{L(\delta/4)S(\delta/4)} =: \eta_i.$$

By the definition of P_i, $\eta_i \to 0$.

Then, there exists $i_\delta \in \mathbb{N}$ such that, for all $i \geqslant i_\delta$, $\left|\tau_i - \frac{\pi}{4}\right| < \frac{3\delta}{4}$.

For all $t \in K$ and $i \geqslant i_\delta$,

$$t - \tau_i \geqslant \delta - \tau_i \geqslant \delta - \frac{\pi}{4} - \frac{3\delta}{4} = \frac{\delta - \pi}{4} \quad \text{and} \quad t - \tau_i \leqslant \pi - \delta - \tau_i \leqslant \frac{3\pi - \delta}{4}.$$

Thus, $t - \tau_i \in J$.

Finally, let $C = \sup_J |Z_c'|$. For all $t \in K$, and $i \geqslant i_\delta$

$$\begin{aligned}
|Y_i(t) - Y_c(t)| &= |Z_i(t - \tau_i) - Z_c(t - \pi/4)| \\
&\leqslant |Z_i(t - \tau_i) - Z_c(t - \tau_i)| + |Z_c(t - \tau_i) - Z_c(t - \pi/4)| \\
&\leqslant \eta_i + C|\tau_i - \pi/4|.
\end{aligned}$$

Since $\eta_i + C|\tau_i - \pi/4| \to 0$, the theorem is proved.

□

4 Computation of the Lichnerowicz Laplacian for Böhm metrics

Throughout this section, we use the notation introduced in the previous one. We saw in Section 2.3 that it is natural to define the Lichnerowicz Laplacian on TT tensors. However, it is sufficient to study the restriction of Δ_L to $\mathscr{T}_{TT}^G(M)$, in order to get a lower bound on the number of negative eigenvalues. In the first subsection, we characterize this space in our particular case $M = S^{k+1} \times S^l$ and $G = SO(k+1) \times SO(l+1)$.

4.1 G-invariant TT tensors on $S^{k+1} \times S^l$

Before computing the Lichnerowicz Laplacian on $M = S^{k+1} \times S^l$, equipped with the Böhm metrics, we first characterize the space $\mathscr{T}_{TT}^G(M)$.

We first prove the following elementary lemma.

Lemma 4.1. *Let T be an $SO(n+1)$-invariant symmetric 2-tensor on S^n ($n \geqslant 1$). Then, there exists a constant c such that:*

$$T = cg^n.$$

Proof. We denote by N the north pole of S^n. Let $B = T_N$. Since $T_N S^n \simeq \mathbb{R}^n$, $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bilinear form.

Let $G' = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \ A \in SO(n) \right\} \simeq SO(n)$.

For all $A' \in G'$, $A'N = N$ hence, for all $X, Y \in \mathbb{R}^n$, $B(AX, AY) = B(X, Y)$.

The action of $SO(n)$ on S^{n-1} is transitive. Therefore for all $u, v \in \mathbb{R}^n$ such that $\|u\| = \|v\| = 1$, $B(u, u) = B(v, v) =: c \in \mathbb{R}$.

So for all $u \in \mathbb{R}^n$, $B(u, u) = c\|u\|^2$.

Moreover, for $u, v \in \mathbb{R}^n$, using the symmetry of B,

$$c\|u\|^2 + c\|v\|^2 + 2c\langle u, v\rangle = c\|u + v\|^2 = B(u + v, u + v) = c\|u\|^2 + c\|v\|^2 + 2B(u, v).$$

Hence, $B(u,v) = c\langle u,v\rangle = cg^n(u,v)$.

Finally, let $p \in S^n$. Since the action of $SO(n+1)$ is transitive, there exists $a \in SO(n+1)$ such that $ap = N$. Therefore, for all X, Y,

$$T_p(X,Y) = T_{ap}(aX,aY) = B(aX,aY) = cg^n(X,Y).$$

□

Using this result, we obtain a characterization of G-invariant symmetric tensors on M.

Theorem 4.2. *Let T be a G-invariant symmetric 2-tensor on M. Then, there exist three smooth functions a, ϕ, ψ such that:*

$$T = a(t)dt^2 + \phi(t)f(t)^2 g^k + \psi(t)h(t)^2 g^l.$$

We define a basis of the tangent space as follows:

$$e_0 = \partial_t,$$

$$\forall i \in [\![1,k]\!],\ e_i = \frac{1}{f}E_i^k \quad \text{(where } (E_i^k) \text{ is a local geodesic orthonormal frame on } S^k\text{)},$$

$$\forall \alpha \in [\![k+1,d-1]\!],\ e_\alpha = \frac{1}{h}E_{\alpha-k}^l \quad \text{(where } (E_{\alpha-k}^l) \text{ is a local geodesic orthonormal frame on } S^l\text{)}.$$

To make things clearer, we denote by $i, j\ldots$ an element of $[\![1,k]\!]$, by $\alpha, \beta\ldots$ an element of $[\![k+1,d-1]\!]$.

To prove Theorem 4.2, we need the following lemma.

Lemma 4.3. *Let T be a G-invariant symmetric 2-tensor on M. Then, there exist three tensors T_0, T^k, T^l such that, in the basis (e_0, e_i, e_α),*

$$T = \begin{pmatrix} T_0 & 0 & 0 \\ 0 & T^k & 0 \\ 0 & 0 & T^l \end{pmatrix}.$$

Proof. Let $x \in M$. Let $t = \pi(x)$. For every tangent vector X, we write $X = (X_0, X^k, X^l)$ with $X_0 \in \mathbb{R}$, $X^j \in \mathbb{R}^j$.

We assume that x is in a principal orbit $P \simeq S^k \times S^l$. We can write $x = (x^k, x^l)$.

Let $B : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}$ defined by

$$B(X^k, Y^l) = T_{(x^k,x^l)}((0,X^k,0),(0,0,Y^l)).$$

The stabilizer of x is denoted by G_x. G_x is isomorphic to $SO(k) \times SO(l)$. Hence,

$$B(AX^k, Y^l) = B(X^k, CY^l) = B(X^k, Y^l),$$

with $(A, C) \in SO(k) \times SO(l)$.

Let $Y^l \in \mathbb{R}^l$. We define $\varphi \in (\mathbb{R}^k)^*$ by $\varphi(X^k) = B(X^k, Y^l)$.

For all $A \in SO(k)$, $\varphi(AX^k) = \varphi(X^k)$. Therefore, there exists $K \in \mathbb{R}$ such that: for all $X^k \in \mathbb{R}^k$ such that $\|X^k\| = 1$, $\varphi(X^k) = K$.

But, $K = \varphi(X_k) = \varphi(-X_k) = -K$. Thus $K = 0$ and $\varphi = 0$. Hence, $B = 0$.

Similarly, we define $B : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ defined by

$$B(X_0, Y^k) = T_{(x^k,x^l)}((X_0,0,0),(0,Y^k,0)).$$

With the same argument, $B = 0$, and the lemma follows. □

We can now prove the theorem.

Proof. Let $T_{x^k}^k(X^k, Y^k) = T_{x^k,0}((0,X^k,0),(0,Y^k,0))$.

By Lemma 4.1, on each principal orbit, there exists a constant c such that $T_{x^k}^k = cg^k$. More precisely, for all t, there exists a constant, denoted by $b(t)$ such that $T_{x^k}^k = b(t)g^k$. We then set $\phi(t) = b(t)/f(t)^2$. The smoothness of ϕ at the singular orbits follows from Lemmas 1.1 and 1.2 from [EW00].

Similarly, $T_{x^l}^l = \psi(t)h(t)^2 g^l$.

Finally, T_0 is a 2-tensor on $\mathbb{R}$. It can be written $a(t)dt^2$. □

Moreover, the transverse-traceless property adds some conditions.

Proposition 4.4. *Let $T = a(t)dt^2 + \phi(t)f(t)^2 g^k + \psi(t)h(t)^2 g^l \in \mathscr{T}^G_{TT}(M)$. Then*

$$\begin{aligned} a(t) + k\phi(t) + l\psi(t) &= 0 && \textit{(traceless)}, \\ a'(t) + H(t)a(t) - k\frac{f'}{f}\phi(t) - l\frac{h'}{h}\psi(t) &= 0 && \textit{(transverse)}, \end{aligned}$$

where $H(t) = k\dfrac{f'(t)}{f(t)} + l\dfrac{h'(t)}{h(t)}$ (the mean curvature).

Proof. The two identities follow from direct calculations, using the definition of the trace and δ and the expression of the connection (see Proposition A.3). □

4.2 Expression of the Lichnerowicz Laplacian

Let $g = g(\bar{h})$ be a Böhm metric on $S^{k+1} \times S^l$. We now compute the Lichnerowicz Laplacian associated with this metric.

Theorem 4.5 (Explicit expression for the Lichnerowicz Laplacian)**.** *In the basis (e_0, e_i, e_α), the Lichnerowicz Laplacian of a G-invariant symmetric 2-tensor $T = a(t)dt^2 + \phi(t)f(t)^2 g^k + \psi(t)h(t)^2 g^l$ is diagonal, with*

$$\begin{aligned} (\Delta_L T)_{00} &= -a''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)a'(t) + 2\left(\Lambda + k\frac{f'^2}{f^2} + l\frac{h'^2}{h^2}\right)a(t) + 2k\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)\phi(t) + 2l\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right)\psi(t), \\ (\Delta_L)T_{ii} &= -\phi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\phi'(t) + \left(2\Lambda - 2\frac{k-1}{f^2} + 2k\frac{f'^2}{f^2}\right)\phi(t) + 2\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)a(t) + 2\frac{f'h'}{fh}l\psi(t), \\ (\Delta_L T)_{\alpha\alpha} &= -\psi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\psi'(t) + \left(2\Lambda - 2\frac{l-1}{h^2} + 2l\frac{h'^2}{h^2}\right)\psi(t) + 2\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right)a(t) + 2k\frac{f'h'}{fh}\phi(t). \end{aligned}$$

Using the Einstein equations, these expressions can be rewritten as

$$\begin{aligned} (\Delta_L T)_{00} &= -a''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)a'(t) + 2k\left(\frac{f'^2}{f^2} - \frac{f''}{f}\right)(a(t) - \phi(t)) + 2l\left(\frac{h'^2}{h^2} - \frac{h''}{h}\right)(a(t) - \psi(t)), \\ (\Delta_L T)_{ii} &= -\phi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\phi'(t) - 2\left(\frac{f'^2}{f^2} - \frac{f''}{f}\right)(a(t) - \phi(t)) + 2\frac{f'h'}{fh}l(\psi(t) - \phi(t)), \\ (\Delta_L T)_{\alpha\alpha} &= -\psi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\psi'(t) - 2\left(\frac{h'^2}{h^2} - \frac{h''}{h}\right)(a(t) - \psi(t)) + 2\frac{f'h'}{fh}k(\phi(t) - \psi(t)). \end{aligned}$$

Proof. There are two ways to obtain this result. The first consists in using Definition 2.11, whereas the second relies on Theorem 2.12.

These calculations are quite long and are presented in Appendix A. □

For convenience, we set:

$$A_f = \left(\frac{f'}{f}\right)' = \frac{f''}{f} - \frac{f'^2}{f^2}, \qquad A_h = \left(\frac{h'}{h}\right)' = \frac{h''}{h} - \frac{h'^2}{h^2}.$$

5 Study of the eigenvalue problem

In the previous section, we obtained an explicit expression of the Lichnerowicz Laplacian for G-invariant tensors.

In this section, we consider a Böhm metric

$$g = dt^2 + f(t)^2 g^k + h(t)^2 g^l,$$

and our goal is to prove the existence of some negative eigenvalue for the Lichnerowicz Laplacian. To this end, we study the eigenvalue equation:

$$\Delta_L T = \mu T, \tag{8}$$

with $\mu < 0$.

The main result of this part is the following theorem.

Theorem 5.1. *There exists a C^∞ function $F : \mathbb{R} \to \mathbb{R}$ such that $\mu \in \mathbb{R} \setminus \{\Lambda + 1\}$ is an eigenvalue of Δ_L (i.e. there exists $T \in \mathscr{T}^G_{TT}(M)$ a non-zero solution of (8)) if and only if $F(\mu) = 0$.*

Remark: The value $\Lambda + 1$ seems to be an artifact due to the formula obtained for the function F (see Section 5.3.

This function F is hard to study theoretically, but it can be useful to compute numerical approximations of the negative eigenvalues.

Remark: Since we are only focusing on G-invariant TT tensors, we may miss some eigenvalues.

In the first subsection, we prove that we can construct a solution to $\Delta_L T = \mu T$ in a neighborhood of the two singular orbits. The second one focuses on the extension of these tensors to the whole manifold outside a neighborhood of the other singular orbit. The final subsection is devoted to the proof of Theorem 5.1.

5.1 Existence and uniqueness in a neighborhood of the singular orbits

In this section, we prove that, in a neighborhood of $S_\pm$, there exists a solution to the equation (8).

The main goal of this part is to prove the following theorem.

Theorem 5.2. *Let $\mu \in \mathbb{R}$. There exists t_B such that, for all $a_0 \in \mathbb{R}$, there exists a unique $T = a(t)dt^2 + \phi(t)f(t)^2 g^k + \psi(t)h(t)^2 g^l \in \mathscr{T}^G_{TT}(M)$ ($t \in [0, t_B]$), analytic, such that*

$$\begin{cases} \Delta_L T = \mu T \\ a(0) = \phi(0) = a_0 \\ \psi(0) = -\frac{k+1}{l} a_0 \\ a'(0) = \phi'(0) = \psi'(0) = 0 \end{cases}.$$

The proof is quite long, and the details are presented in Appendix B. Since the system is singular at $t = 0$, the overall strategy is to study analytic solutions and find a relation between the coefficients of the formal power series. However, this recursion involves a matrix with eigenvalues in $\mathbb{N}$, hence classical theorems such as Theorem 5.5 in [Was18] cannot be applied. To prove our result, we have to introduce a degree of freedom to prove the existence of G-invariant eigentensors in a neighborhood of a singular orbit. These solutions are not unique, but imposing the TT condition restores uniqueness.

5.2 Extension of solutions

In the previous section, we showed that if we consider equation (8), there exist $\Upsilon > t_B^+ > 0$ and a unique solution T_+ defined on $[0, t_B^+]$ (with suitable initial conditions).

$[0, t_B^+]$ can be seen as a neighborhood of the singular orbit S_+ (green in Figure 2).

The same construction can be carried out near S_- (by applying Theorem 5.2 after replacing t with $\Upsilon - t$). This yields another tensor T_- defined in a small neighborhood $[t_B^-, \Upsilon]$ of S_-.

The main objective is to find, for some $\mu < 0$, a global solution defined on $[0, \Upsilon]$, and we seek to match the two tensors T_- and T_+. But to answer this question, we need to prove that there exists a $t_* \in [0, \Upsilon]$ such that T_+ and T_- are both defined at t_*.

More precisely, we prove the following theorem (we state it only for tensors defined in a neighborhood of S_+, but it can also be applied near S_- by symmetry).

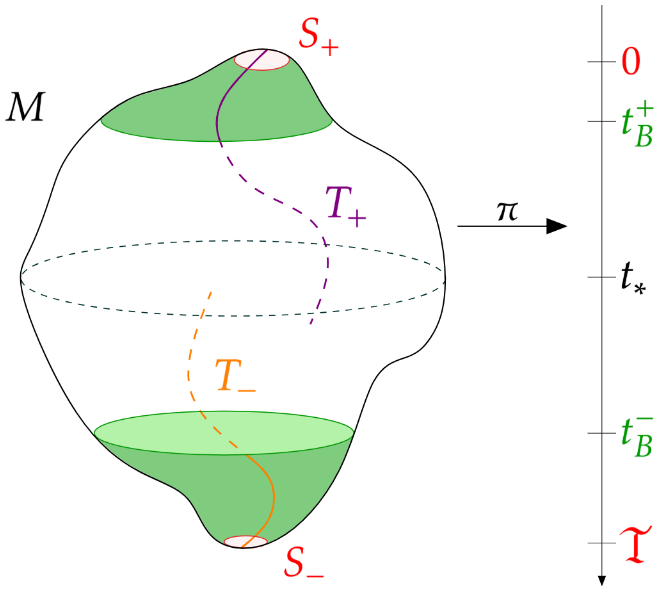

Figure 2: Extension of solutions on the sphere

Theorem 5.3. *Let $T : [0, t_B] \to \mathscr{T}^G_{TT}(M)$ be a solution of (8) as in Theorem 5.2. For all $t_f < \Upsilon$, there exists a unique $\tilde{T} : [0, t_f] \to \mathscr{T}^G_{TT}(M)$ solution of 5.2 such that*

$$\tilde{T}_{|[0,t_B]} = T.$$

Proof. Let $t_1 < t_B$. With the notation of the previous section, equation (8) can be written

$$Y'(t) = C(t)Y(t), \tag{9}$$

with $Y = (a, ta', \phi, t\phi', \psi, t\psi')^T$ and $C : (0, \Upsilon) \to \mathcal{M}_6(\mathbb{R})$ (see Appendix B for the exact expression of C).
Theorem 5.2 provides the existence of $Y_1 : [0, t_B] \to \mathbb{R}^6$ solution of (9) which is analytic at 0. This solution is the unique vector field that defines a TT tensor (see Appendix B for the detailed argument).
Let $U = Y_1(t_1)$.
Since $f(t) > 0$ and $h(t) > 0$ for all $t \in (0, \Upsilon)$, $C(t)$ is analytic for all $t \in [t_1, t_f]$ (for all $t_f < \Upsilon$).
Therefore, the Cauchy problem:

$$\begin{cases} Y'(t) = C(t)Y(t) & (\forall t \in [t_1, t_f]) \\ Y(t_1) = U \end{cases}, \tag{10}$$

has a unique analytic solution $Y_2 : [t_1, t_f] \to \mathbb{R}^6$.
Moreover, Y_1 is a solution of (10) on $[t_1, t_B]$. Hence, for all $t \in [t_1, t_B]$, $Y_1(t) = Y_2(t)$.
We define $Y : [0, t_f] \to \mathbb{R}^6$ by:

$$\forall t \in [0, t_1], \ \ Y(t) := Y_1(t), \qquad \forall t \in [t_1, t_f], \ \ Y(t) := Y_2(t).$$

Thus, Y is an analytic solution of (9) defined for all $t \in [0, t_f]$.
Finally, the uniqueness of Y follows from the uniqueness of Theorem 5.2 and the uniqueness part of the Picard-Lindelöf theorem.

□

5.3 Reduction of the matching problem

In the previous sections, we showed that we can construct an eigentensor $T_+ : [0, \Upsilon - \varepsilon] \to \mathscr{T}^G_{TT}(S^{k+1} \times S^l)$, for any $\varepsilon > 0$, and another eigentensor $T_- : [\varepsilon, \Upsilon] \to \mathscr{T}^G_{TT}(S^{k+1} \times S^l)$.

For some $\mu < 0$, we want to prove that there exists $T : [0, \Upsilon] \to \mathscr{T}^G_{TT}(S^{k+1} \times S^l)$ defined on the entire manifold, which is a solution of (8). Our problem is now a matching problem. Can the two local eigentensors $T_+ = a_+ dt^2 + \phi_+ f^2 g^k + \psi_+ h^2 g^l$ and $T_- = a_- dt^2 + \phi_- f^2 g^k + \psi_- h^2 g^l$ coincide and define a global eigentensor T?

Due to the Picard-Lindelöf theorem and Theorem 5.3, it is sufficient to prove that there exists a point $t_* \in (0, \Upsilon)$ such that $T_+(t_*) = T_-(t_*)$ and $T'_+(t_*) = T'_-(t_*)$.

The goal of this part is to prove Theorem 5.1.

To prove this theorem, we exploit the existence of the maximum-volume orbit, defined in [Böh98], section 4.

In $M = S^{k+1} \times S^l$, Böhm metrics are symmetric, which means that the functions f and h are invariant under the reflection $t \mapsto \Upsilon - t$.

We set $t_* = \Upsilon/2$. By [Böh98], Section 6, t_* is the maximum volume orbit.

Since f and h are even at t_*, we get the following relations:

$$H(t_*) = 0, \qquad f'(t_*) = 0, \qquad h'(t_*) = 0.$$

Before proving the theorem, we need the following lemma.

Lemma 5.4. *Let $T_\mu = a(t)dt^2 + \phi(t)f(t)^2 g^k + \psi(t)h(t)^2 g^l$ be an eigentensor corresponding to an eigenvalue $\mu \in \mathbb{R}$. Then T_μ is even or odd with respect to $t_* = \Upsilon/2$.*

Proof. Let $\sigma : t \mapsto \Upsilon - t$. σ is an isometry of M since f and h are even with respect to t_*. Since dt^2, g^k and g^l are invariant, the pullback

$$\sigma^* : T \mapsto (t \mapsto T(\Upsilon - t))$$

commutes with Δ_L.
Hence

$$\Delta_L \sigma^* T_\mu = \mu \sigma^* T_\mu.$$

Theorem 5.2 yields the existence of $c \in \mathbb{R}^*$ such that

$$\sigma^* T_\mu = c T_\mu,$$

since the space of G-invariant TT tensors that are solutions of (8) is one-dimensional.
Thus, since $(\sigma^*)^2 = Id$,

$$T_\mu = (\sigma^*)^2 T_\mu = c^2 T_\mu,$$

therefore $c = \pm 1$.

□

We now have everything needed to prove the theorem.

Proof. If T_+ and T_- define an eigentensor T, because of the lemma, it is even or odd with respect to t_*.
Hence $T(t_*) = 0$ (odd) or $T'(t_*) = 0$ (even).

Even case. We just need to find the values of μ such that

$$T'_+(t_*) = 0,$$

i.e. $a'_+(t_*) = \phi'_+(t_*) = \psi'_+(t_*) = 0$, because we can define $T_-(t) = T_+(\Upsilon - t)$, and the matching is immediate.
Since $T_\pm \in \mathscr{T}^G_{TT}(M)$, we have, for all $t \in [0, \Upsilon]$:

$$\begin{aligned} a_\pm(t) + k\phi_\pm(t) + l\psi_\pm(t) &= 0 && \text{(traceless)}, \\ a'_\pm(t) + H(t)a_\pm(t) - k\frac{f'}{f}\phi_\pm(t) - l\frac{h'}{h}\psi_\pm(t) &= 0 && \text{(transverse)}. \end{aligned}$$

Thus, $a'_\pm = -k\phi'_\pm - l\psi'_\pm$
In, $t = t_*$, we get $a'_\pm(t_*) = 0$, thus:

$$\phi'_\pm(t_*) = -\frac{l}{k}\psi'_\pm(t_*).$$

Therefore $T_+'(t_*) = 0$ if and only if $\phi_+'(t_*) = 0$ or $\psi_+'(t_*) = 0$.

We can set $F_e(\mu) = \psi_+'(t_*)$.

Thus, it is enough to study T_+. Since the dependence on the parameter μ now plays a central role, we will denote the tensor $T_+(t) = a_+(t)dt^2 + \phi_+(t)f(t)^2 g^k + \psi_+(t)h(t)^2 g^l$ by $T_\mu = a_\mu(t)dt^2 + \phi_\mu(t)f(t)^2 g^k + \psi_\mu(t)h(t)^2 g^l$.

Hence, $F_e(\mu) = \psi_\mu'(t_*)$.

Finally, smoothness of F_e follows from Theorem 2.11 in [Tes12].

Odd case. Similarly, we just need to find the values of μ such that $T_+(t_*) = 0$, because we can define $T_-(t) = -T_+(\Upsilon - t)$. In this case, $T(t_*) = 0$ if and only if $\phi_+(t_*) = \psi_+(t_*) = 0$.

If we differentiate the transverse condition and evaluate it at t_*, we get:

$$a_+''(t_*) + \underbrace{\left(k\frac{f''(t_*)}{f(t_*)} + l\frac{h''(t_*)}{h(t_*)}\right)}_{-\Lambda} a_+(t_*) - k\frac{f''(t_*)}{f(t_*)}\phi_+ - l\frac{h''(t_*)}{h(t_*)}\psi_+ = 0.$$

Then, we use the first equation from (8):

$$-a_+''(t_*) + (2\Lambda - \mu)a_+(t_*) + 2k\frac{f''(t_*)}{f(t_*)}\phi_+(t_*) + 2l\frac{h''(t_*)}{h(t_*)}\psi_+(t_*) = 0.$$

Thus, using the traceless condition:

$$k\left(\frac{f''(t_*)}{f(t_*)} - \Lambda + \mu\right)\phi_+(t_*) + l\left(\frac{h''(t_*)}{h(t_*)} - \Lambda + \mu\right)\psi_+(t_*) = 0.$$

Here, the coefficients of $\phi_+(t_*)$ and $\psi_+(t_*)$ may vanish, therefore considering only $\psi_+(t_*)$ to check if μ is an eigenvalue is not sufficient.

Let

$$F_o(\mu) = k\left(\frac{h''(t_*)}{h(t_*)} - \Lambda + \mu\right)\phi_+(t_*) - l\left(\frac{f''(t_*)}{f(t_*)} - \Lambda + \mu\right)\psi_+(t_*).$$

Thus, $F_o(\mu) = 0$ if and only if

$$\begin{cases} k\left(\frac{f''(t_*)}{f(t_*)} - \Lambda + \mu\right)\phi_+(t_*) + l\left(\frac{h''(t_*)}{h(t_*)} - \Lambda + \mu\right)\psi_+(t_*) = 0, \\ k\left(\frac{h''(t_*)}{h(t_*)} - \Lambda + \mu\right)\phi_+(t_*) - l\left(\frac{f''(t_*)}{f(t_*)} - \Lambda + \mu\right)\psi_+(t_*) = 0. \end{cases}$$

The determinant of the system is

$$\Delta = -kl\left[\left(\frac{f''(t_*)}{f(t_*)} - \Lambda + \mu\right)^2 + \left(\frac{h''(t_*)}{h(t_*)} - \Lambda + \mu\right)^2\right].$$

Hence, if $\Delta = 0$, then

$$-\Lambda - \Lambda^2 + \Lambda\mu = 0,$$

because of (2), thus $\mu = \Lambda + 1$.

The smoothness of F_o follows from the same argument.

Conclusion. Finally, we can set

$$F(\mu) = F_o(\mu)F_e(\mu).$$

We notice that F_o and F_e cannot both be equal to 0 for the same μ, because it would imply that $T = 0$.

$\square$

5.4 Numerical results

Theorem 5.1 has limited theoretical use because the function F is not explicit. Hence, it is quite difficult to study its properties. However, computing this function numerically is relatively easy, and we can get an estimate of the eigenvalues of the Lichnerowicz Laplacian. The Python code is given in Appendix C.

We plot F_o and F_e, and not $F = F_o \times F_e$, because if a zero of F_o and a zero of F_e are very close, the program could miss some eigenvalues (see Figure 3).

To compute F, instead of solving the previous system with six equations, we eliminate a, a' and ψ' using the TT condition. This makes the computation faster and improves numerical accuracy.

To compute the functions f and h, we solve equations (2) - (4). The only non-trivial step is determining the initial condition $\bar{h}$. To compute it, we use Wang's code [Wan26].

In Figure 4, we plot the graphs of F for some Böhm metrics on $S^3 \times S^2$ and $S^4 \times S^2$ (on the left). The zeros of F_e (*i.e.* the "even" eigenvalues of Δ_L) are represented by a vertical red line, and the "odd" eigenvalues are represented in green.

On the right, we plot the eigentensor associated with the smallest eigenvalue found numerically.

In the case of the standard product metric, we observe that 0 is an eigenvalue for a constant eigentensor. This can also be verified analytically.

Moreover, three observations can be made.

- The behavior of F_e and F_o strongly depends on the metric. However, the influence of the dimension is less clear, and the behavior of F_e and F_o appears to be quite similar.
- For all the Böhm metrics considered in our numerical experiments, the Lichnerowicz Laplacian appears to admit negative modes, which is consistent with the conjecture stated in [GHP03].
- Between two "even" eigenvalues $\mu_e^1 < \mu_e^2$ (in red), there appears to be an "odd" eigenvalue μ_o (in green) which lies very close to μ_e^1 (see Figure 3). For instance, the gap $\delta = |\mu_e^1 - \mu_o|$ can be of order 10^{-1} (see Figure 3 for g_1) or even 10^{-3} (see Figure 3 for g_2).

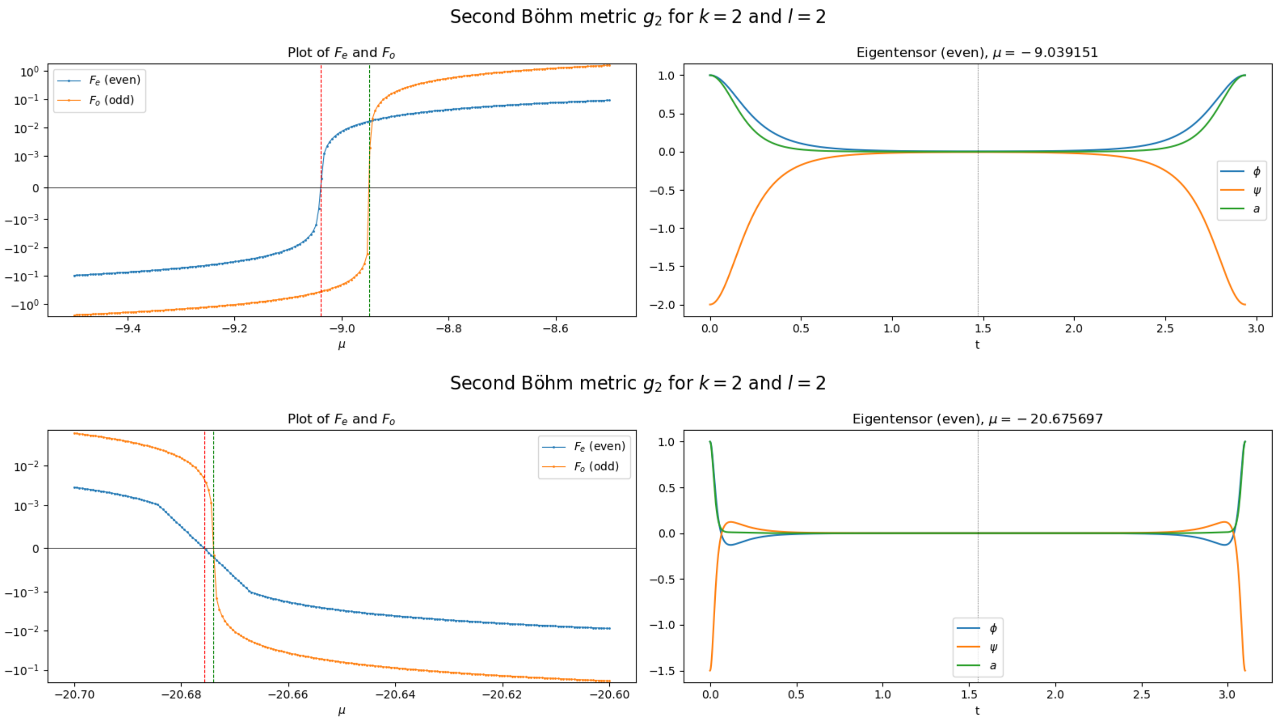

Figure 3: Even and odd eigenvalues can be close (F_e seems to be non-smooth because of the use of a `symlog` scale)

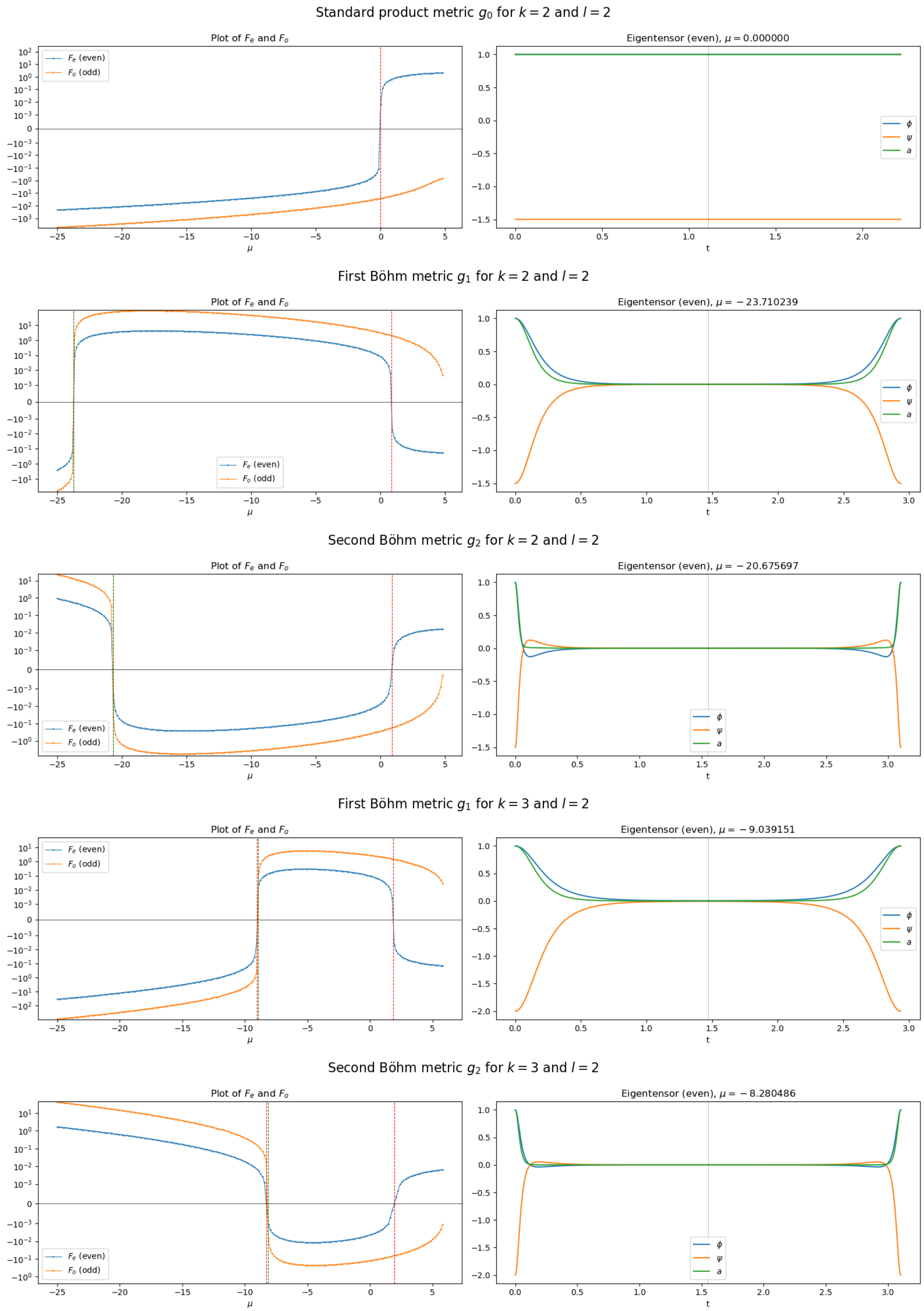

Figure 4: Numerical results

6 Index of the Lichnerowicz Laplacian for Böhm metrics

In this section, we prove Theorem A. We recall the statement of the theorem.

Theorem A. *Let (g_i) be a sequence of Böhm metrics on the product $S^{k+1} \times S^l$ $(k, l \geqslant 2,\ k + l < 9)$. We denote by Δ_L^i the Lichnerowicz Laplacian associated with g_i. If ν_i denotes the number of negative eigenvalues of Δ_L^i, we have:*

$$\nu_i \xrightarrow[i \to +\infty]{} +\infty.$$

Remark: ν_i is called the *index* of Δ_L^i.

Remark: We recall that the index (which is directly related to the number of unstable directions) is finite according to Section 2.4.

The idea of the proof is as follows. Since we know that $g_i \to g_c$, we first study the Lichnerowicz Laplacian Δ_L^c associated with the cone metric g_c. In some sense, we prove that it admits an infinite number of negative eigenvalues (see Theorem 6.1). We then show that this property also holds for Böhm metrics by means of a continuity argument for the quadratic form associated with the Lichnerowicz Laplacian. Finally, we conclude the proof using the min-max theorem.

6.1 Cone metric

We are now working with the cone metric $g_c = dt^2 + f_c^2 g^k + h_c^2 g^l$, with

$$f_c(t) = \sqrt{\frac{k-1}{\Lambda - 1}} \sin(t), \qquad h_c(t) = \sqrt{\frac{l-1}{\Lambda - 1}} \sin(t).$$

We define a TT tensor as follows:

$$T = a(t) dt^2 + \phi(t) f_c(t)^2 g^k + \psi(t) h_c(t)^2 g^l,$$

with:

- $a(t) = 0$,
- $\psi(t) = -\frac{k}{l} \phi(t)$.

We first check that T is TT:

- $a(t) + k\phi(t) + l\psi(t) = k\phi(t) + l\psi(t) = 0$,
- $a'(t) + H(t)a(t) - k\frac{f_c'}{f_c}\phi(t) - l\frac{h_c'}{h_c}\psi(t) = -\cot(t)(k\phi(t) + l\psi(t)) = 0.$

So T is TT.

Now, we want to compute $\langle \Delta_L^c T, T \rangle$. By definition,

$$\langle \Delta_L^c T, T \rangle = \int_0^\pi g_c(t)(\Delta_L^c T, T) V(t) dt = \int_0^\pi V(t)(k(\Delta_L^c T)_{ii}\phi(t) + l(\Delta_L^c T)_{\alpha\alpha}\psi(t)) dt,$$

where $V(t) = f_c(t)^k h_c(t)^k$ We recall that $A_f = (f'/f)'$ and $A_h = (h'/h)'$. For all $t \in (0, \pi)$:

$$\begin{aligned}
\Delta_L^c T_{ii} &= -\phi''(t) - H(t)\phi'(t) - 2A_{f_c}\phi(t) + 2\frac{f_c' h_c'}{f_c h_c} l(\psi(t) - \phi(t)) \\
&= -\phi'' - H\phi' - 2A_{f_c}\phi - 2\frac{f_c' h_c'}{f_c h_c}(k + l)\phi, \\
\Delta_L^c T_{\alpha\alpha} &= -\psi''(t) - H(t)\psi'(t) - 2A_h\psi(t) + 2\frac{f_c' h_c'}{f_c h_c} k(\phi(t) - \psi(t)) \\
&= -\psi'' - H\psi' - 2A_{h_c}\psi - 2\frac{f_c' h_c'}{f_c h_c}(k + l)\psi.
\end{aligned}$$

We have $f_c'/f_c = h_c'/h_c = \cot$. Hence,

$$A_{f_c} = A_{h_c} = \cot' = -\frac{1}{\sin^2}.$$

Let $W(t) = \frac{2}{\sin(t)^2} - 2(k+l)\cot^2(t)$. We have

$$\Delta_L^c T_{ii} = -\phi'' - H\phi' + W\phi \quad \text{and} \quad \Delta_L^c T_{\alpha\alpha} = -\psi'' - H\psi' + W\psi.$$

Thus,

$$\begin{aligned}
\langle \Delta_L^c T, T\rangle &= \int_0^\pi V(t)(k(-\phi'' - H\phi' + W\phi)\phi(t) + l(-\psi'' - H\psi' + W\psi)\psi(t))dt \\
&= -\int_0^\pi V(t)(k\phi''\phi + l\psi''\psi)dt - \int_0^\pi V(t)H(t)(k\phi'\phi + l\psi'\psi)dt + \int_0^\pi V(t)W(t)(k\phi^2 + l\psi^2)dt \\
&= \left(k + \frac{k^2}{l}\right)\left[-\int_0^\pi V(t)\phi''\phi dt - \int_0^\pi V(t)H(t)\phi'\phi dt + \int_0^\pi V(t)W(t)\phi^2 dt\right] \\
&= \left(k + \frac{k^2}{l}\right)\left[\int_0^\pi V(t)(\phi'^2 + W(t)\phi^2)dt\right] \\
&= C\int_0^\pi \sin^\Lambda(t)\phi'^2 + 2\sin^{\Lambda-2}(t)(1 - \Lambda\cos^2(t))\phi^2 dt \\
&= C\int_0^\pi \sin^\Lambda(t)\phi'^2 - 2(\Lambda-1)\sin^{\Lambda-2}(t)\phi^2 + 2\Lambda\sin^\Lambda\phi^2 dt.
\end{aligned}$$

We define

$$Q(\phi) = \langle \Delta_L^c T, T\rangle / C = \int_0^\pi \sin^\Lambda(t)\phi'^2 - 2(\Lambda-1)\sin^{\Lambda-2}(t)\phi^2 + 2\Lambda\sin^\Lambda\phi^2 dt.$$

We now prove the following theorem.

Theorem 6.1. *For all $N \in \mathbb{N}$, there exists an N-dimensional vector space E_N, such that, for all $\phi \in E_N \setminus \{0\}$, $Q(\phi) < 0$.*

Proof. We want to make the integral negative. The term $2(\Lambda-1)\sin^{\Lambda-1}$ must be dominant. A natural idea is to look for a function ϕ whose support is contained in $(0,\delta)$, with $\delta > 0$. If δ is small enough, we can use Taylor expansion $\sin^\Lambda(t) = t^\Lambda + O(t^{\Lambda+2})$.

Let η be such that $\mathrm{supp}(\eta) \subset (0,1)$.

Let $\phi_\delta(t) = \eta\left(\frac{t}{\delta}\right)$ be a function that satisfies $\mathrm{supp}(\phi_\delta) \subset (0,\delta)$. Later, we will construct such a function ϕ_δ (and some η) which satisfy $Q(\phi_\delta) < 0$.

Now, we prove the following lemma.

Lemma 6.2. *Let*

$$\tilde{Q}(\phi) = \int_0^\pi t^\Lambda\phi'^2 - 2(\Lambda-1)t^{\Lambda-2}\phi^2 dt.$$

If δ is small enough, and $\tilde{Q}(\phi_\delta) < 0$, then $Q(\phi_\delta) < 0$.

Proof. We use the following Taylor expansions:

$$\sin^\Lambda(t) = t^\Lambda + \varepsilon_1(t), \qquad \sin^{\Lambda-2}(t) = t^{\Lambda-2} + \varepsilon_2(t),$$

with $\varepsilon_1(t) = O(t^{\Lambda+2})$ and $\varepsilon_2(t) = O(t^\Lambda)$.

$$\begin{aligned}
Q(\phi_\delta) &= \int_0^\delta (t^\Lambda + \varepsilon_1(t))\phi_\delta'^2 - 2(\Lambda-1)(t^{\Lambda-2} + \varepsilon_2(t))\phi_\delta^2 + 2\Lambda(t^\Lambda + \varepsilon_1(t))\phi_\delta^2 dt \\
&= \hat{Q}(\phi_\delta) + R(\delta),
\end{aligned}$$

where

$$\begin{aligned}
\hat{Q}(\phi_\delta) &= \int_0^\delta t^\Lambda\phi_\delta'^2 - 2(\Lambda-1)t^{\Lambda-2}\phi_\delta^2 + 2\Lambda t^\Lambda\phi_\delta^2 dt = I_1 - I_2 + I_3, \\
R(\delta) &= \int_0^\delta \frac{1}{\delta^2}\varepsilon_1(t)\eta'^2\left(\frac{t}{\delta}\right) - 2(\Lambda-1)\varepsilon_2(t)\eta^2\left(\frac{t}{\delta}\right) + 2\Lambda\varepsilon_1(t)\eta^2\left(\frac{t}{\delta}\right)dt.
\end{aligned}$$

Hence,

$$\int_0^\delta |\varepsilon_1(t)|dt = O(\delta^{\Lambda+3}), \qquad \int_0^\delta |\varepsilon_2(t)|dt = O(\delta^{\Lambda+1}).$$

Thus,

$$\begin{aligned}|R(\delta)| &\leqslant \frac{\|\eta'\|_\infty^2}{\delta^2}\int_0^\delta |\varepsilon_1(t)|dt + 2(\Lambda-1)\|\eta\|_\infty^2 \int_0^\delta |\varepsilon_2(t)|dt + 2\Lambda\|\eta\|_\infty^2 \int_0^\delta |\varepsilon_1(t)|dt \\ &= O(\delta^{\Lambda+1}).\end{aligned}$$

We now focus on the leading-order terms.

$$\hat{Q}(\phi_\delta) = I_1^\delta - I_2^\delta + I_3^\delta.$$

We have the following:

$$|I_1^\delta| \leqslant \frac{\|\eta'\|_\infty^2}{\delta^2}\int_0^\delta t^\Lambda dt = O(\delta^{\Lambda-1}), \qquad |I_2^\delta| \leqslant 2(\Lambda-1)\|\eta\|_\infty^2 \int_0^\delta t^{\Lambda-2}dt = O(\delta^{\Lambda-1}),$$

$$|I_3^\delta| \leqslant 2\Lambda\|\eta\|_\infty^2 \int_0^\delta t^\Lambda dt = O(\delta^{\Lambda+1}).$$

Therefore, only I_1 and I_2 play a role when δ becomes small, hence

$$Q(\phi_\delta) = \tilde{Q}(\phi_\delta) + O(\delta^{\Lambda+1})$$

Moreover,

$$\tilde{Q}(\phi_\delta) = \delta^{\Lambda-1}\int_0^1 \left(\frac{t}{\delta}\right)^\Lambda \eta'\left(\frac{t}{\delta}\right)^2 - 2(\Lambda-1)\left(\frac{t}{\delta}\right)^{\Lambda-2}\eta\left(\frac{t}{\delta}\right)^2 \frac{dt}{\delta} = \delta^{\Lambda-1}\tilde{Q}(\eta)$$

Hence,

$$\frac{Q(\phi_\delta)}{\delta^{\Lambda-1}} = \tilde{Q}(\eta) + O(\delta^2),$$

and since $\tilde{Q}(\phi_\delta)$ and $\tilde{Q}(\eta)$ have the same sign, this concludes the proof of the lemma. □

Now, we make the following substitution:

$$\phi_\delta(t) = t^{-\beta} v_\delta(\ln(t)),$$

with $\beta = \frac{\Lambda-1}{2}$.

$$\phi_\delta'(t) = -\beta t^{-\beta-1} v_\delta(\ln(t)) + t^{-\beta-1} v_\delta'(\ln(t)) = t^{-\beta-1}(v_\delta'(\ln(t)) - \beta v_\delta(\ln(t))).$$

Therefore,

$$\begin{aligned}\int_0^\delta t^\Lambda \phi_\delta'(t)^2 dt &= \int_0^\delta t^{\Lambda-2\beta-2}(v_\delta'(\ln(t)) - \beta v_\delta(\ln(t)))^2 dt \\ &= \int_0^\delta (v_\delta'(\ln(t)) - \beta v_\delta(\ln(t)))^2 \frac{dt}{t} \\ &\underset{s=\ln t}{=} \int_{-\infty}^{\ln(\delta)} (v_\delta'(s) - \beta v_\delta(s))^2 ds \\ &= \int_{-\infty}^{\ln(\delta)} v_\delta'(s)^2 + \beta^2 v_\delta(s)^2 ds - 2\beta\int_{-\infty}^{\ln(\delta)} v_\delta'(s) v_\delta(s) ds.\end{aligned}$$

But

$$2\int_{-\infty}^{\ln(\delta)} v_\delta'(s) v_\delta(s) ds = [v_\delta(s)^2]_{-\infty}^{\ln(\delta)}$$

and $v(s) = \phi_\delta(e^s)e^{\beta s}$. Since $\phi_\delta(\delta) = 0$ and $\exp(\beta s) \to 0$ when $s \to -\infty$, the integral vanishes.

Thus,

$$\int_0^{\delta} t^{\Lambda}\phi'(t)^2 dt = \int_{-\infty}^{\ln(\delta)} v_\delta'(s)^2 + \beta^2 v_\delta(s)^2 ds,$$

$$\int_0^{\delta} t^{\Lambda-2}\phi(t)^2 dt = \int_0^{\delta} t^{\Lambda-2\beta-2} v_\delta(\ln(t))^2 dt = \int_0^{\delta} v_\delta(\ln(t))^2 \frac{dt}{t} = \int_{-\infty}^{\ln(\delta)} v_\delta(s)^2 ds.$$

Hence,

$$\tilde{Q}(\phi_\delta) = \int_{-\infty}^{\ln(\delta)} v_\delta'(s)^2 + \beta^2 v_\delta(s)^2 ds - 2(\Lambda-1)\int_{-\infty}^{\ln(\delta)} v_\delta(s)^2 ds.$$

We want to find ϕ_δ such that

$$\tilde{Q}(\phi_\delta) < 0.$$

Thus, $\tilde{Q}(\phi_\delta) < 0$ is equivalent to

$$\int_{-\infty}^{\ln(\delta)} v_\delta'(s)^2 ds < [2(\Lambda-1)-\beta^2]\int_{-\infty}^{\ln(\delta)} v_\delta(s)^2 ds.$$

Thus, (we can integrate to $+\infty$ because $v(s)=0$ for $s>\ln(\delta)$):

$$S(v_\delta) := \frac{\displaystyle\int_{-\infty}^{+\infty} v_\delta'(s)^2 ds}{\displaystyle\int_{-\infty}^{+\infty} v_\delta(s)^2 ds} < 2(\Lambda-1)-\beta^2 =: \alpha^2.$$

Note that $\alpha^2 = 2(\Lambda-1)-\beta^2 = \frac{(9-\Lambda)(\Lambda-1)}{4} > 0$ since $\Lambda\in[4,8]$.

S measures the frequency of the function v_δ. This naturally suggests considering v_δ in the following form:

$$v_\delta(s) = \delta^{\beta}\sin\left(\frac{\pi(s-s_0)}{L}\right)\mathbb{1}_{[s_0,s_0+L]}.$$

v_δ is not $\mathcal{C}^\infty$, but is H^1.

$$S(v_\delta) = \frac{\displaystyle\int_{s_0}^{s_0+L}\frac{\pi^2}{L^2}\cos^2\left(\frac{\pi(s-s_0)}{L}\right)ds}{\displaystyle\int_{s_0}^{s_0+L}\sin^2\left(\frac{\pi(s-s_0)}{L}\right)ds} = \frac{\pi^2}{L^2}.$$

We want $S(v_\delta) < \alpha^2$, therefore we have to choose $L > \frac{\pi}{\alpha}$.

We fix such a L. It is important that L does not depend on δ. However, it is also important to have $\mathrm{supp}(v_\delta)\subset(-\infty,\ln(\delta)]$. We can, for example, consider $s_0 = \ln(\delta)-L$.

We just need to verify that ϕ_δ can be written $\phi_\delta(t) = \eta\left(\frac{t}{\delta}\right)$.

$$\begin{aligned}\phi_\delta(t) &= t^{-\beta}v_\delta(\ln(t))\\ &= -\left(\frac{t}{\delta}\right)^{-\beta}\sin\left(\frac{\pi}{L}\ln\left(\frac{t}{\delta}\right)\right)\mathbb{1}_{[\ln(\delta)-L,\ln(\delta)]}(\ln(t))\\ &= -\left(\frac{t}{\delta}\right)^{-\beta}\sin\left(\frac{\pi}{L}\ln\left(\frac{t}{\delta}\right)\right)\mathbb{1}_{[\exp(-L),1]}\left(\frac{t}{\delta}\right).\end{aligned}$$

Thus, by the lemma, ϕ_δ satisfies $Q(\phi_\delta)<0$ for δ small enough.

Let $N\in\mathbb{N}^*$. For all $1\leqslant n\leqslant N$, we set:

$$\tilde{\phi}_n(t) = \left(\frac{t}{\delta}\right)^{-\beta}\sin\left(\frac{\pi}{L}\ln\left(\frac{t}{\delta}\right)\right)\mathbb{1}_{[\exp(-2nL),\exp(-(2n-1)L)]}\left(\frac{t}{\delta}\right)$$

(we just replace s_0 by $s_n = \ln(\delta)-2nL$).

Therefore, for all $1 \leqslant n \leqslant N$, $Q(\tilde{\phi}_n) < 0$.

Let $B(\phi, \psi) = \langle \Delta_L^c T_\phi, T_\psi \rangle$ For all $(i, j) \in [\![1, N]\!]^2$:

$$B(\tilde{\phi}_i, \tilde{\phi}_j) = \left(k + \frac{k^2}{l}\right)\left[-\int_0^\pi V(t)\tilde{\phi}_i''\tilde{\phi}_j dt - \int_0^\pi V(t)H(t)\tilde{\phi}_i'\tilde{\phi}_j dt + \int_0^\pi V(t)W(t)\tilde{\phi}_i\tilde{\phi}_j dt\right].$$

Since the supports of $\tilde{\phi}_i$ and $\tilde{\phi}_j$ are disjoint for each $i \neq j$, $B(\tilde{\phi}_i, \tilde{\phi}_j) = 0$.

We define $\tilde{E}_N = \mathrm{Span}(\tilde{\phi}_1, \cdots, \tilde{\phi}_N)$.

Let $\phi = \sum_{i=1}^{N} \lambda_i \tilde{\phi}_i \in \tilde{E}_N \setminus \{0\}$.

$$Q(\tilde{\phi}) = \sum_{i=1}^{N} \lambda_i^2 Q(\tilde{\phi}_i) < 0.$$

Finally, later in the proof, we will need to differentiate these functions twice. Since $\tilde{\phi}$ is only H^1, we replace it by a smooth regularization of this function. Moreover, for the cone metric, $\Upsilon = \pi$, but it will no longer be the case for Böhm metric, thus we will need to change this parameter π (we will replace it by Υ_i for a Böhm metric g_i). These two remarks lead to the introduction of the following function.

We define $\tilde{\rho}_n : (0, \pi) \times (0, 2\pi) \to \mathbb{R}$ by:

$$\tilde{\rho}_n(t, \omega) = \sin\left(\frac{\omega}{L} \ln\left(\frac{t}{\delta}\right)\right) \mathbb{1}_{[\exp(-2nL\pi/\omega), \exp(-(2n-1)L\pi/\omega)]}\left(\frac{t}{\delta}\right).$$

By density of $\mathcal{C}^\infty$, we can consider a regularization ρ_n of $\tilde{\rho}_n$.

Note that we can assume that the bounds of $\mathrm{supp}(\rho_n(\cdot, \omega))$ are continuous with respect to ω (for instance, it is the case if ρ_n is defined by a convolution).

We can set

$$\phi_n(t) = \left(\frac{t}{\delta}\right)^{-\beta} \rho_n(t, \pi).$$

It satisfies $Q(\phi_n) < 0$ and we can assume that the supports of ϕ_n remain disjoint, because the distances between the supports of $\tilde{\phi}_n$ are positive. More precisely, we assume that

$$\mathrm{supp}(\rho_n(\cdot, \omega)) \subset \left[\exp\left(-\left(2n + \frac{1}{2}\right)L\frac{\pi}{\omega}\right), \exp\left(-\left(2n - \frac{3}{2}\right)L\frac{\pi}{\omega}\right)\right].$$

Let $\psi_n(t) = \phi_n(\pi - t)$.

Finally, we set

$$\gamma_n(t) = \phi_n(t) + \psi_n(t).$$

We introduce γ_n so that the associated tensor is symmetric with respect to $\pi/2$.

For δ sufficiently small, $\mathrm{supp}(\phi_n) \cap \mathrm{supp}(\psi_n) = \emptyset$, therefore $Q(\gamma_n) < 0$.

Hence, $E_N = \mathrm{Span}(\gamma_1, \cdots, \gamma_N)$ satisfies the required property, since the supports of (γ_n) are disjoint.

This concludes the proof of the theorem.

□

Remark: For $1 \leqslant n \leqslant N$, let $T_c^n(t) = \gamma_n(t) f_c(t)^2 g^k - (k/l)\gamma_n(t) h_c(t)^2 g^l$ (the G-invariant TT tensor associated with γ_n). By a slight abuse of notation, E_N will also denote $\mathrm{Span}(T_c^1, \cdots, T_c^N)$.

The previous theorem proved that for all $T \in E_N \setminus \{0\}$, $Q(T) < 0$.

6.2 Böhm metrics close to the cone metric

Let $N \in \mathbb{N}$ and $1 \leqslant n \leqslant N$.

We consider a sequence of Böhm metrics $(g_i) = (g(\bar{h}_i))$ with $\bar{h}_i \to 0$.

In this section, we construct a G-invariant tensor T_i^n, TT for g_i such that $T_i^n \to T_c^n$.

Let Υ_i be the height of the second singular orbit (the first is at $t = 0$). We recall that Böhm proved that $\Upsilon_i \to \pi$ ([Böh98], Section 6).

We fix δ and L as in the previous section.

We define

$$\phi_i^n(t) = \left(\frac{t}{\delta}\right)^{-\beta} \rho_n(t, \Upsilon_i)$$

and $\psi_i^n(t) = \phi_i^n(\Upsilon_i - t)$.

We define $\gamma_i^n = \phi_i^n + \psi_i^n$. We notice that γ_i^n is even with respect to $\Upsilon_i/2$ (the maximum volume orbit).

Unlike in the cone case, we cannot simply define $T_i^n = \gamma_i^n f_i^2 g^k - (k/l)\gamma_i^n h_i^2 g^l$, because this tensor is not TT for the metric g_i. We need to introduce a correction term, inspired by equation (62) in [GHP03].

Let $\tilde{R}_i = \frac{k}{l}\left(\frac{h_i'}{h_i} - \frac{f_i'}{f_i}\right)$, $W_i(t) = f_i(t)^k h_i(t)^{l+1}$, and

$$\chi_i^n(t) = \frac{1}{W_i(t)} \int_0^t W_i(s)\tilde{R}_i(s)\gamma_i^n(s)ds.$$

We set the tensor:

$$T_i^n(t) = -l\chi_i^n(t)dt^2 + \gamma_i^n(t)f_i(t)^2 g^k + \left(\chi_i^n(t) - \frac{k}{l}\gamma_i^n(t)\right)h_i(t)^2 g^l,$$

Lemma 6.3. *T_i^n is a G-invariant TT tensor (for the metric g_i), defined on $[0, \Upsilon_i]$, with compact support $K_i^n \subset (0, \Upsilon_i)$. Moreover, there exists K_N a compact subset of $(0, \pi)$ such that*

$$\bigcup_{\substack{i \in \mathbb{N} \\ 1 \leqslant n \leqslant N}} K_i^n \subset K_N.$$

Proof. We show that T_i^n is well defined on the entire manifold. $\gamma_i^n(t)$ is well defined for all $t \in [0, \Upsilon_i]$. Therefore, it remains to verify that χ_i^n is also well defined. Since $W_i(t) \to 0$ when $t \to 0$ or Υ_i, we need to verify that χ_i^n does not diverge at the singular orbits.

Let

$$J_i(\gamma) := \int_0^{\Upsilon_i} f_i(t)^k h_i(t)^{l+1} \left(\frac{h_i'(t)}{h_i(t)} - \frac{f_i'(t)}{f_i(t)}\right)\gamma(t)dt.$$

Let $D_i(t) = f_i(t)^k h_i(t)^{l+1}\left(\frac{h_i'(t)}{h_i(t)} - \frac{f_i'(t)}{f_i(t)}\right)$. We notice that D_i is odd with respect to $\Upsilon_i/2$. Thus, since γ is even with respect to $\Upsilon_i/2$, we have $J_i(\gamma) = 0$.

We show that χ_i^n does not diverge when $t \to 0$.

$$\chi_i^n(t) = \frac{1}{W_i(t)} \int_0^t W_i(s)\tilde{R}_i(s)\gamma_i^n(s)ds =: \frac{N_i(t)}{W_i(t)}.$$

Since $N_i(t) \to 0$ and $W_i(t) \to 0$ when $t \to 0$,

$$\lim_{t\to 0^+} \chi_i^n(t) = \lim_{t\to 0^+} \frac{N_i'(t)}{W_i'(t)} = \lim_{t\to 0^+} \frac{\tilde{R}_i(t)\gamma_i^n(t)}{G_i(t)} = \lim_{t\to 0^+} \frac{-(k/l)\frac{1}{t}\gamma_i^n(t)}{k/t} = -\frac{1}{l}\gamma_i^n(0) = 0$$

where $G_i = k\frac{f_i'}{f_i} + (l+1)\frac{h_i'}{h_i}$. W_i satisfies $W_i' = G_i W_i$.

Since $J_i(\gamma_i^n) = 0$, we have $N_i(t) \to 0$ and $W_i(t) \to 0$ when $t \to \Upsilon_i$. Thus, the same argument applies at $t = \Upsilon_i$.

We now prove that the support is compact and establish the existence of K_N.

We notice that

$$\operatorname{supp}(\gamma_i^n) \subset \left[\delta \exp\left(-\left(2n + \frac{1}{2}\right)L\frac{\pi}{\Upsilon_i}\right), \delta \exp\left(-\left(2n - \frac{3}{2}\right)L\frac{\pi}{\Upsilon_i}\right)\right] \cup \left[\Upsilon_i - \delta \exp\left(-\left(2n + \frac{1}{2}\right)L\frac{\pi}{\Upsilon_i}\right), \Upsilon_i - \delta \exp\left(-\left(2n - \frac{3}{2}\right)L\frac{\pi}{\Upsilon_i}\right)\right]$$

and

$$\operatorname{supp}(\gamma_n) \subset \left[\delta \exp\left(-\left(2n+\frac{1}{2}\right)L\right), \delta \exp\left(-\left(2n-\frac{3}{2}\right)L\right)\right] \cup \left[\pi - \delta \exp\left(-\left(2n+\frac{1}{2}\right)L\right), \pi - \delta \exp\left(-\left(2n-\frac{3}{2}\right)L\right)\right].$$

Hence, for all $t \leqslant \delta \exp\left(-\left(2n+\frac{1}{2}\right)L\frac{\pi}{\Upsilon_i}\right)$,

$$\chi_i^n(t) = 0.$$

Moreover, for all $t \in \left[\Upsilon_i - \delta \exp\left(-\left(2n+\frac{1}{2}\right)L\frac{\pi}{\Upsilon_i}\right), \Upsilon_i\right)$,

$$\chi_i^n(t) = \frac{1}{W_i(t)} J_i(\gamma_i^n) - \frac{1}{W_i(t)} \int_t^{\Upsilon_i} W_i(s)\tilde{R}_i(s)\gamma_i^n(s)ds = 0 - 0.$$

Let $C_i^n = [\inf(\operatorname{supp}(\gamma_i^n)), \sup(\operatorname{supp}(\gamma_i^n))]$.
Thus, $\operatorname{supp}(\chi_i^n) \subset C_i^n$.
This yields $K_i^n \subset C_i^n$.
Finally, for each n, the lower bound of C_i^n is bounded below by some $\delta_n > 0$ (because it is positive and converges to $\delta \exp(-(2n+1/2)L) > 0$ by Proposition 3.2). The same argument can be applied for the upper bound. Hence, we get $\eta_n > 0$ such that

$$\bigcup_{i \in \mathbb{N}} K_i^n \subset [\eta_n, \pi - \eta_n].$$

Thus we may define

$$K_N = \bigcup_{n=1}^{N} [\eta_n, \pi - \eta_n],$$

which is also a compact subset of $(0, \pi)$.
Now, we need to prove that T_i^n is transverse (the traceless property and the G-invariance are immediate).

$$\begin{aligned}
\delta T_i^n &= l\chi_i^{n\prime} + lH_i\chi_i^n + k\frac{f_i'}{f_i}\gamma_i^n + l\frac{h_i'}{h_i}\chi_i^n - k\frac{h_i'}{h_i}\gamma_i^n \\
&= l(\chi_i^{n\prime} + G_i(t)\chi_i^n) - l\tilde{R}_i\gamma_i^n \\
&= 0 \quad \text{(by definition of } \chi_i^n).
\end{aligned}$$

□

Our tensor is well defined. Thus, we can study the convergence.

Lemma 6.4. *Let K be a compact subset of $(0, \pi)$. For all $1 \leqslant n \leqslant N$,*

$$\|T_i^n - T_c^n\|_{\infty,K} \to 0.$$

Proof. It suffices to show

$$\gamma_i^n \to \gamma_n \quad \text{and} \quad \chi_i^n \to 0 \quad \text{uniformly on } K.$$

For all $t \in K$,

$$|\phi_i^n(t) - \phi_n(t)| \leqslant \left(\frac{t}{\delta}\right)^{-\beta} |\rho_n(t, \Upsilon_i) - \rho_n(t, \pi)|.$$

Since ρ_n is $\mathcal{C}^1$ on a compact set, there exists $C > 0$ such that, for all $t \in K$,

$$|\rho_n(t, \Upsilon_i) - \rho_n(t, \pi)| \leqslant C|\Upsilon_i - \pi|.$$

Hence,

$$|\phi_i^n(t) - \phi_n(t)| \leqslant C'|\pi - \Upsilon_i|,$$

where $C' = C\sup_K (t/\delta)^{-\beta}$. The right-hand side does not depend on t and tends to 0 when $i \to +\infty$. Therefore,

$$\|\phi_i^n - \phi_n\|_{\infty,K} \to 0.$$

This implies the uniform convergence of γ_i^n.

We now consider χ_i^n. We recall that $\tilde{R}_i = \frac{k}{l}\left(\frac{h_i'}{h_i} - \frac{f_i'}{f_i}\right)$, $W_i(t) = f_i(t)^k h_i(t)^{l+1}$, and

$$\chi_i^n(t) = \frac{1}{W_i(t)} \int_0^t W_i(s)\tilde{R}_i(s)\gamma_i^n(s)ds.$$

On K, the numerator converges uniformly to 0, due to Theorem 3.4, and the denominator converges uniformly to W_c, which is strictly positive on K. Hence,

$$\|\chi_i^n\|_{\infty,K} \to 0.$$

□

6.3 Negative modes for Böhm metrics close to the cone metric

We can now prove Theorem A.

Proof. Let $N \in \mathbb{N}$. We show that there exists $i(N) \in \mathbb{N}$, such that $\nu_i \geqslant N$ for all $i \geqslant i(N)$.

Let $Q_i(T) = \langle \Delta_L^i T, T\rangle$.

It suffices to construct an N-dimensional vector space E_i^N such that, for sufficiently large i, and for all $T_i \in E_i^N \setminus \{0\}$, $Q_i(T_i) < 0$.

We set $E_i^N = \mathrm{Span}(T_i^1, \cdots, T_i^N)$.

Since the supports of (T_i^n) are no longer disjoint, we introduce the Gram matrix associated with the bilinear form $B_i : T, T' \mapsto \langle \Delta_L^i T, T'\rangle$:

$$\mathfrak{G}_i = (\langle \Delta_L^i T_i^p, T_i^q\rangle)_{1\leqslant p,q\leqslant N}.$$

Likewise, let

$$\mathfrak{G}_c = \mathrm{diag}(Q_c(T_c^1), \cdots, Q_c(T_c^N)).$$

We first prove that $\mathfrak{G}_i \to \mathfrak{G}_c$.

If $B_c(T,T') := \langle \Delta_L^c T, T'\rangle$, it is enough to prove that

$$\forall 1 \leqslant n,m \leqslant N, \qquad B_i(T_i^n, T_i^m) \to B_c(T_c^n, T_c^m).$$

Let $1 \leqslant n,m \leqslant N$. Let K_N be as in Lemma 6.3.

$$\begin{aligned} B_i(T_i^n, T_i^m) &= \int_0^{\Upsilon_i} \left[(\Delta_L^i T_i^n)_{00}(-l\chi_i^m) + k(\Delta_L^i T_i^n)_{jj}\gamma_i^m + l(\Delta_L^i T_i^n)_{\alpha\alpha}\left(\chi_i^m - \frac{k}{l}\gamma_i^m\right)\right] V_i(t)dt \\ &= \int_{K_N} \left[(\Delta_L^i T_i^n)_{00}(-l\chi_i^m) + k(\Delta_L^i T_i^n)_{jj}\gamma_i^m + l(\Delta_L^i T_i^n)_{\alpha\alpha}\left(\chi_i^m - \frac{k}{l}\gamma_i^m\right)\right] V_i(t)dt. \end{aligned}$$

Define

$$\Psi_i^{n,m} = \left[(\Delta_L^i T_i^n)_{00}(-l\chi_i^m) + k(\Delta_L^i T_i^n)_{jj}\gamma_i^m + l(\Delta_L^i T_i^n)_{\alpha\alpha}\left(\chi_i^m - \frac{k}{l}\gamma_i^m\right)\right] V_i(t)$$

and

$$\Psi_c^{n,m} = \left[k(\Delta_L^c T_c^n)_{jj}\gamma_m + l(\Delta_L^c T_c^n)_{\alpha\alpha}\left(-\frac{k}{l}\gamma_m\right)\right] V_c(t).$$

Lemma 6.4, Theorem 3.4 and the expression of the Lichnerowicz Laplacian imply that

$$\|\Psi_i^{n,m} - \Psi_c^{n,m}\|_{\infty,K_N} \to 0,$$

because, on K_N, f_i, h_i, f_c, h_c do not vanish.

Consequently,

$$|B_i(T_i^n,T_i^m)-B_c(T_c^n,T_c^m)|\leqslant \int_{K_N}|\Psi_i^{n,m}-\Psi_c^{n,m}|\leqslant |K_N|\cdot\|\Psi_i^{n,m}-\Psi_c^{n,m}\|_{\infty,K_N}\to 0,$$

This implies

$$\mathfrak{G}_i\to\mathfrak{G}_c.$$

Theorem 6.1 yields that $\mathfrak{G}_c$ is negative definite. Thus, there exists $i(N)\in\mathbb{N}$, such that

$$\forall i\geqslant i(N),\qquad \mathfrak{G}_i<0.$$

In particular, for all $i\geqslant i(N)$, (T_i^n) are independent, and E_i^N is N-dimensional.

Hence, for all $i\geqslant i(N)$ and $T\in E_i^N\setminus\{0\}$ $Q_i(T)<0$.

Δ_L^i is bounded below and has a compact resolvent (because it is elliptic on a compact manifold), therefore,

$$\mathrm{Sp}(\Delta_L^i)=\{\mu_1^i\leqslant\mu_2^i\cdots\}.$$

Since Δ_L^i is self-adjoint, the min-max theorem yields:

$$\mu_n^i=\inf_{\substack{V\subset\mathcal{T}_{TT}((M,g_i))\\ \dim V=n}}\ \max_{T\in V\setminus\{0\}}\frac{Q_i(T)}{\|T\|^2}.$$

Hence, for all $i\geqslant i(N)$, $\mu_N^i<0$.

Therefore,

$$\forall i\geqslant i(N),\qquad \nu_i\geqslant N.$$

□

7 Conclusion

7.1 Academic conclusion

In this work, we have studied the stability of a particular class of Einstein metrics, the Böhm metrics. We have gained a better understanding of the geometry of these metrics, *i.e.* the behavior of the space of Riemannian metrics near Böhm metrics.

More precisely, we understood the behavior of the index of the Lichnerowicz Laplacian associated with Böhm metrics as they approach a singular metric: the cone metric. Since Böhm introduced these metrics in 1998, this is, to the best of our knowledge, the first result describing the asymptotic behavior of this sequence of indices.

In addition to the geometric interest, this work also has applications in theoretical physics. For instance, we were able to partially confirm a conjecture by Gibbons, Hartnoll and Pope [GHP03]: all the generalized Schwarzschild-Tangherlini spacetimes associated with Böhm metrics on products of spheres are unstable, except possibly finitely many of them.

Despite this progress, many questions remain open, such as giving a final answer to the conjecture of Gibbons, Hartnoll and Pope. The result of Section 5 could be useful for a numerical study of the first terms of the Böhm sequence.

Moreover, we now know that the sequence of these indices (ν_i) tends to infinity, but studying the divergence rate of this sequence would be interesting to better understand the behavior of these metrics as they approach the cone metric.

Finally, since there are also Böhm sequences of metrics on spheres, extending Theorem A to the spheres could be a very interesting step forward. We are still working and aim to extend the theorem.

The proof of Theorem A, together with this extension to the case of spheres, is the subject of a research paper currently in preparation.

7.2 Personal conclusion

This research project was very rewarding in many ways. First, from a mathematical point of view, it was insightful, because differential geometry is at the interface of many fields of mathematics, such as analysis, spectral theory, ordinary differential equations... It was particularly enriching to connect the different concepts I learned in courses, to discover new theories and their applications.

Furthermore, since I have always been interested in physics, realizing that my results may have applications, however modest, in modern physics was very satisfying.

During this project, I also faced difficulties in my research work, such as the complexity of the computation of the Lichnerowicz Laplacian, or the limited theoretical use of Theorem 5.1. These difficulties or failures were frustrating, but very instructive. They taught me to look at the problem from a different point of view, and to use some results in ways I had not anticipated, as when Theorem 5.1 turned out to be useful numerically rather than theoretically.

Finally, discovering the research world in an environment as stimulating as Oxford, talking with researchers, Post-docs and PhD students about their research problems and attending seminars reinforced my desire to be part of this amazing world.

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A Calculation of the Lichnerowicz Laplacian

A.1 Direct calculation

A.1.1 Setting

We work on the manifold $M = S^{k+1} \times S^l$ with $k, l \geqslant 2$ and $k + l \leqslant 8$.

To compute the Lichnerowicz Laplacian, we first determine the Levi-Civita connection. To calculate it, we use the following formula.

Theorem A.1 (Koszul formula)**.**

$$2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]).$$

We recall the definition of the basis.

$$e_0 = \partial_t,$$

$$\forall i \in [\![1,k]\!],\ e_i = \frac{1}{f} E_i^k \quad \text{(where } (E_i^k) \text{ is a local geodesic orthonormal frame on } S^k\text{)},$$

$$\forall \alpha \in [\![k+1,d-1]\!],\ e_\alpha = \frac{1}{h} E_{\alpha-k}^l \quad \text{(where } (E_{\alpha-k}^l) \text{ is a local geodesic orthonormal frame on } S^l\text{)}.$$

To make the notation clearer, we denote by $i, j...$ an element of $[\![1,k]\!]$, by $\alpha, \beta...$ an element of $[\![k+1,d-1]\!]$ and by $A, B...$ an element of $[\![0,d-1]\!]$.

Since this basis is orthogonal, when X, Y, Z are different vectors of this basis, the first three terms in Koszul formula vanish.

A.1.2 Lie Brackets and connection

We only need the connection when X is a vector of the basis. To compute it, we first look at the Lie brackets between the vectors of the basis.

Proposition A.2.

$$[e_0, e_i] = -\frac{f'}{f} e_i, \quad [e_0, e_\alpha] = -\frac{h'}{h} e_\alpha, \quad [e_i, e_j] = \frac{1}{f^2}[E_i^k, E_j^k], \quad [e_\alpha, e_\beta] = \frac{1}{h^2}[E_{\alpha-k}^l, E_{\beta-k}^l], \quad [e_i, e_\alpha] = 0.$$

Proof. We use the following formula:

$$[X, \phi Y] = X(\phi) Y + \phi [X, Y].$$

Thus,

$$[e_0, e_i] = [\partial_t, \frac{1}{f} E_i^k] = \partial_t(\frac{1}{f}) E_i^k + \frac{1}{f} 0 = -\frac{f'}{f} e_i.$$

Similarly,

$$[e_0, e_\alpha] = -\frac{h'}{h} e_\alpha.$$

Since f only depends on t (the first coordinate), $E_i^k(1/f) = 0$ and

$$[e_i, e_j] = \frac{1}{f^2}[E_i^k, E_j^k].$$

Similarly,

$$[e_\alpha, e_\beta] = \frac{1}{h^2}[E_{\alpha-k}^l, E_{\beta-k}^l].$$

And we also get

$$[e_i, e_\alpha] = 0.$$

□

With these relations, we can compute the connection :

Proposition A.3.

$$\begin{aligned}
&\nabla_{e_0} e_A = 0, && && \\
&\nabla_{e_p} e_i = -\frac{f'}{f}\delta_{ip} e_0 + \nabla^k_{e_p} e_i, && \nabla_{e_p} e_0 = \frac{f'}{f} e_p, && \nabla_{e_p} e_\alpha = 0, \\
&\nabla_{e_\pi} e_i = 0, && \nabla_{e_\pi} e_0 = \frac{h'}{h} e_\pi, && \nabla_{e_\pi} e_\alpha = -\frac{h'}{h}\delta_{\pi\alpha} e_0 + \nabla^l_{e_\pi} e_\alpha.
\end{aligned}$$

where ∇^n denotes the Levi-Civita connection on (S^n, g^n).

Proof. We only give the details for ∇_{e_p}. The other cases are similar.
First, $\nabla_{e_p} e_i$:

$$2g(\nabla_{e_p} e_i, e_j) = -g(e_p, [e_i, e_j]) + g(e_i, [e_j, e_p]) + g(e_j, [e_p, e_i]) = 2g(\nabla^k_{e_p} e_i, e_j),$$

where ∇^k denotes the Levi-Civita connection on S^k with the round metric.

$$2g(\nabla_{e_p} e_i, e_\alpha) = -g(e_p, [e_i, e_\alpha]) + g(e_i, [e_\alpha, e_p]) + g(e_\alpha, [e_p, e_i]) = 0.$$

$$2g(\nabla_{e_p} e_i, e_0) = -g(e_p, [e_i, e_0]) + g(e_i, [e_0, e_p]) + g(e_0, [e_p, e_i]) = -g(e_p, \frac{f'}{f} e_i) + g(e_i, -\frac{f'}{f} e_p) = -2\frac{f'}{f}\delta_{ip}$$

Thus,

$$\nabla_{e_p} e_i = -\frac{f'}{f}\delta_{ip} e_0 + \nabla^k_{e_p} e_i.$$

Then, $\nabla_{e_p} e_0$:

$$2g(\nabla_{e_p} e_0, e_j) = -g(e_p, [e_0, e_j]) + g(e_0, [e_j, e_p]) + g(e_j, [e_p, e_0]) = -g(e_p, -\frac{f'}{f} e_j) + 0 + g(e_j, \frac{f'}{f} e_p) = 2\frac{f'}{f}\delta_{jp},$$

$$2g(\nabla_{e_p} e_0, e_0) = -g(e_p, [e_0, e_0]) + g(e_0, [e_0, e_p]) + g(e_0, [e_p, e_0]) = 0,$$

$$2g(\nabla_{e_p} e_0, e_\alpha) = -g(e_p, [e_0, e_\alpha]) + g(e_0, [e_\alpha, e_p]) + g(e_\alpha, [e_p, e_0]) = -g(e_p, -\frac{h'}{h} e_\alpha) + 0 + g(e_\alpha, \frac{f'}{f} e_p) = 0.$$

because $\alpha \neq p$.
Hence,

$$\nabla_{e_p} e_0 = \frac{f'}{f} e_p.$$

Finally, $\nabla_{e_p} e_\alpha$:

$$2g(\nabla_{e_p} e_\alpha, e_j) = -g(e_p, [e_\alpha, e_j]) + g(e_\alpha, [e_j, e_p]) + g(e_j, [e_p, e_\alpha]) = 0,$$

$$2g(\nabla_{e_p} e_\alpha, e_\beta) = -g(e_p, [e_\alpha, e_\beta]) + g(e_\alpha, [e_\beta, e_p]) + g(e_\beta, [e_p, e_\alpha]) = 0,$$

$$2g(\nabla_{e_p} e_\alpha, e_0) = -g(e_p, [e_\alpha, e_0]) + g(e_\alpha, [e_0, e_p]) + g(e_0, [e_p, e_\alpha]) = 0.$$

$$\nabla_{e_p} e_\alpha = 0.$$

□

A.1.3 Bochner Laplacian for G-invariant TT tensors

We only compute the Laplacian on G-invariant TT tensors.
G-invariant TT tensors can be written

$$T = a(t)dt^2 + \phi(t)f(t)^2 g^k + \psi(t)h(t)^2 g^l.$$

Writing $X = X_0\partial_t + X^k + X^l$, $Y = Y_0\partial_t + Y^k + Y^l$, we have,

$$T(X,Y) = a(t)X_0Y_0 + b(t)g^k(X^k,Y^k) + c(t)g^l(X^l,Y^l).$$

We can calculate $\overline{\Delta}$ with the following formula:

$$\overline{\Delta}T = -\sum_A (\nabla_{e_A}\nabla_{e_A}T - \nabla_{\nabla_{e_A}e_A}T),$$

(see [Bes87] 1.135 and 1.55).
We first need to calculate $\nabla_{e_A}T$.

Proposition A.4. *We have the following formulas:*

$$\begin{aligned}
&(\nabla_{e_i}T)(\partial_t,\partial_t) = 0, \qquad (\nabla_{e_i}T)(\partial_t,e_j) = \frac{f'}{f}(a(t)-\phi(t))\delta_{ij}, \qquad (\nabla_{e_i}T)(\partial_t,e_\alpha) = 0,\\
&(\nabla_{e_i}T)(e_j,e_p) = 0, \qquad (\nabla_{e_i}T)(e_j,e_\alpha) = 0, \qquad (\nabla_{e_i}T)(e_\alpha,e_\beta) = 0.
\end{aligned}$$

$$\begin{aligned}
&(\nabla_{e_\alpha}T)(\partial_t,\partial_t) = 0, \qquad (\nabla_{e_\alpha}T)(\partial_t,e_\beta) = \frac{h'}{h}(a(t)-\psi(t))\delta_{\alpha\beta}, \qquad (\nabla_{e_\alpha}T)(\partial_t,e_i) = 0,\\
&(\nabla_{e_\alpha}T)(e_i,e_j) = 0, \qquad (\nabla_{e_\alpha}T)(e_i,e_\beta) = 0, \qquad (\nabla_{e_\alpha}T)(e_\beta,e_\gamma) = 0.
\end{aligned}$$

$$\begin{aligned}
&(\nabla_{\partial_t}T)(\partial_t,\partial_t) = a'(t), \qquad (\nabla_{\partial_t}T)(\partial_t,e_i) = 0, \qquad (\nabla_{\partial_t}T)(\partial_t,e_\alpha) = 0,\\
&(\nabla_{\partial_t}T)(e_i,e_j) = \phi'(t)\delta_{ij}, \qquad (\nabla_{\partial_t}T)(e_i,e_\alpha) = 0, \qquad (\nabla_{\partial_t}T)(e_\alpha,e_\beta) = \psi'(t)\delta_{\alpha\beta}.
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
T(e_i,e_j) &= \phi(t)\delta_{ij},\\
T(e_\alpha,e_\beta) &= \psi(t)\delta_{\alpha\beta},\\
T(\partial_t,\partial_t) &= a(t).
\end{aligned}$$

By the Leibniz rule :

$$(\nabla_{e_i}T)(X,Y) = e_i(T(X,Y)) - T(\nabla_{e_i}X,Y) - T(X,\nabla_{e_i}Y).$$

Therefore,

$$\begin{aligned}
(\nabla_{e_i}T)(\partial_t,\partial_t) =& e_i(T(\partial_t,\partial_t)) - T(\nabla_{e_i}\partial_t,\partial_t) - T(\partial_t,\nabla_{e_i}\partial_t)\\
=& e_i(a(t)) - T\left(\frac{f'}{f}e_i,\partial_t\right) - T\left(\partial_t,\frac{f'}{f}e_i\right)\\
=& 0.
\end{aligned}$$

$$
\begin{aligned}
(\nabla_{e_i} T)(\partial_t, e_j) &= e_i(T(\partial_t, e_j)) - T(\nabla_{e_i}\partial_t, e_j) - T(\partial_t, \nabla_{e_i} e_j) \\
&= e_i(0) - T\left(\frac{f'}{f} e_i, e_j\right) - T\left(\partial_t, -\frac{f'}{f}\delta_{ij}\partial_t + \nabla^k_{e_i} e_j\right) \\
&= -\frac{f'}{f} b(t)\frac{1}{f^2}\delta_{ij} + a(t)\frac{f'}{f}\delta_{ij} \\
&= \frac{f'}{f}\left(a(t) - \frac{b(t)}{f(t)^2}\right)\delta_{ij} \\
&= \frac{f'}{f}\left(a(t) - \phi(t)\right)\delta_{ij}.
\end{aligned}
$$

$$
\begin{aligned}
(\nabla_{e_i} T)(\partial_t, e_\alpha) &= e_i(T(\partial_t, e_\alpha)) - T(\nabla_{e_i}\partial_t, e_\alpha) - T(\partial_t, \nabla_{e_i} e_\alpha) \\
&= e_i(0) - T\left(\frac{f'}{f} e_i, e_\alpha\right) - T\left(\partial_t, 0\right) \\
&= 0.
\end{aligned}
$$

The other calculations are very similar.

$\square$

Replacing T by $\nabla_{e_i} T$ in the previous proposition yields

Proposition A.5.

$$
\begin{aligned}
&(\nabla_{e_i}\nabla_{e_i} T)(\partial_t, \partial_t) = -2\frac{f'^2}{f^2}(a(t) - \phi(t)), \qquad (\nabla_{e_i}\nabla_{e_i} T)(\partial_t, e_j) = 0, \qquad (\nabla_{e_i}\nabla_{e_i} T)(\partial_t, e_\alpha) = 0, \\
&(\nabla_{e_i}\nabla_{e_i} T)(e_j, e_p) = 2\frac{f'^2}{f^2}(a(t) - \phi(t))\delta_{ij}\delta_{ip}, \qquad (\nabla_{e_i}\nabla_{e_i} T)(e_j, e_\alpha) = 0, \qquad (\nabla_{e_i}\nabla_{e_i} T)(e_\alpha, e_\beta) = 0.
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla_{e_\alpha}\nabla_{e_\alpha} T)(\partial_t, \partial_t) = -2\frac{h'^2}{h^2}(a(t) - \psi(t)), \qquad (\nabla_{e_\alpha}\nabla_{e_\alpha} T)(\partial_t, e_i) = 0, \qquad (\nabla_{e_\alpha}\nabla_{e_\alpha} T)(\partial_t, e_\beta) = 0. \\
&(\nabla_{e_\alpha}\nabla_{e_\alpha} T)(e_i, e_j) = 0, \qquad (\nabla_{e_\alpha}\nabla_{e_\alpha} T)(e_i, e_\beta) = 0, \qquad (\nabla_{e_\alpha}\nabla_{e_\alpha} T)(e_\beta, e_\gamma) = 2\frac{h'^2}{h^2}(a(t) - \psi(t))\delta_{\alpha\beta}\delta_{\alpha\gamma}.
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla_{\partial_t}\nabla_{\partial_t} T)(\partial_t, \partial_t) = a''(t), \qquad (\nabla_{\partial_t}\nabla_{\partial_t} T)(\partial_t, e_i) = 0, \qquad (\nabla_{\partial_t}\nabla_{\partial_t} T)(\partial_t, e_\alpha) = 0, \\
&(\nabla_{\partial_t}\nabla_{\partial_t} T)(e_i, e_j) = \phi''(t)\delta_{ij}, \qquad (\nabla_{\partial_t}\nabla_{\partial_t} T)(e_i, e_\alpha) = 0, \qquad (\nabla_{\partial_t}\nabla_{\partial_t} T)(e_\alpha, e_\beta) = \psi''(t)\delta_{\alpha\beta}.
\end{aligned}
$$

Similar calculations lead to

Proposition A.6.

$$
\begin{aligned}
&\nabla_{\nabla_{e_i} e_i} T(\partial_t, \partial_t) = -\frac{f'}{f} a'(t), \qquad \nabla_{\nabla_{e_i} e_i} T(\partial_t, e_j) = 0, \qquad \nabla_{\nabla_{e_i} e_i} T(\partial_t, e_\alpha) = 0, \\
&\nabla_{\nabla_{e_i} e_i} T(e_j, e_p) = -\frac{f'}{f}\phi'(t)\delta_{jp}, \qquad \nabla_{\nabla_{e_i} e_i} T(e_j, e_\alpha) = 0, \qquad \nabla_{\nabla_{e_i} e_i} T(e_\alpha, e_\beta) = -\frac{f'}{f}\psi'(t)\delta_{\alpha\beta}.
\end{aligned}
$$

$$
\begin{aligned}
&\nabla_{\nabla_{e_\alpha} e_\alpha} T(\partial_t, \partial_t) = -\frac{h'}{h} a'(t), \qquad \nabla_{\nabla_{e_\alpha} e_\alpha} T(\partial_t, e_j) = 0, \qquad \nabla_{\nabla_{e_\alpha} e_\alpha} T(\partial_t, e_\beta) = 0, \\
&\nabla_{\nabla_{e_\alpha} e_\alpha} T(e_j, e_p) = -\frac{h'}{h}\phi'(t)\delta_{jp}, \qquad \nabla_{\nabla_{e_\alpha} e_\alpha} T(e_i, e_\beta) = 0, \qquad \nabla_{\nabla_{e_\alpha} e_\alpha} T(e_\beta, e_\gamma) = -\frac{h'}{h}\psi'(t)\delta_{\beta\gamma}.
\end{aligned}
$$

$$
\nabla_{\nabla_{\partial_t}\partial_t} T = 0.
$$

Finally, we can compute $\overline{\Delta}$:

Proposition A.7 (Expression of $\overline{\Delta}$)**.**

$$\overline{\Delta}T = -T'' - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)T' + \Gamma(T),$$

where (in the basis (e_A)):

$$\Gamma(T) = \begin{pmatrix} -2k\frac{f'^2}{f^2}\phi - 2l\frac{h'^2}{h^2}\psi + 2\left(k\frac{f'^2}{f^2} + l\frac{h'^2}{h^2}\right)a(t) & 0 & 0 \\ 0 & -2\frac{f'^2}{f^2}(a(t)-\phi(t))I_k & 0 \\ 0 & 0 & -2\frac{h'^2}{h^2}(a(t)-\psi(t))I_l \end{pmatrix}.$$

Proof. It follows from Propositions A.5 and A.6.

We detail the calculation of $(\overline{\Delta}T)(e_j, e_p)$:

$$\begin{aligned} -(\overline{\Delta}T)(e_j,e_p) &= (\nabla_{\partial_t}\nabla_{\partial_t}T - \nabla_{\nabla_{\partial_t}\partial_t}T)(e_j,e_p) + \sum_{i=1}^{k}(\nabla_{e_i}\nabla_{e_i}T - \nabla_{\nabla_{e_i}e_i}T)(e_j,e_p) + \sum_{\alpha=k+1}^{d-1}(\nabla_{e_\alpha}\nabla_{e_\alpha}T - \nabla_{\nabla_{e_\alpha}e_\alpha}T)(e_j,e_p) \\ &= \phi''(t)\delta_{jk} + \sum_{i=1}^{k}\left(2\frac{f'^2}{f^2}(a(t)-\phi(t))\delta_{ij}\delta_{ik} + \frac{f'}{f}\phi'(t)\delta_{jk}\right) + \sum_{\alpha=k+1}^{d-1}\frac{h'}{h}\phi'(t)\delta_{jk} \\ &= \phi''(t)\delta_{jk} + 2\frac{f'^2}{f^2}(a(t)-\phi(t))\delta_{jk} + \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\phi'(t)\delta_{jk}. \end{aligned}$$

□

A.1.4 Calculation of $\mathfrak{K}$

Now, we want to compute $\mathfrak{K}$.

$$\mathfrak{K}(T)(X,Y) = T(\operatorname{Ric}X, Y) + T(X, \operatorname{Ric}Y) - 2\mathring{R}(T)(X,Y) = 2\Lambda T(X,Y) - 2\mathring{R}(T)(X,Y).$$

We recall that R, Ric and $\mathring{R}$ were defined in Section 2.1.

We first compute the Riemann curvature tensor. This follows directly from its definition and Propositions A.2 and A.3.

Proposition A.8.

$$R(e_i,e_j)e_p = \frac{f'^2-1}{f^2}(\delta_{jp}e_i - \delta_{ip}e_j), \qquad R(e_i,e_j)e_\alpha = 0, \qquad R(e_i,e_j)\partial_t = 0,$$

$$R(e_\alpha,e_\beta)e_\gamma = \frac{h'^2-1}{h^2}(\delta_{\beta\gamma}e_\alpha - \delta_{\alpha\gamma}e_\beta), \qquad R(e_\alpha,e_\beta)e_i = 0, \qquad R(e_\alpha,e_\beta)\partial_t = 0,$$

$$R(e_i,e_\alpha)e_j = -\frac{f'h'}{fh}\delta_{ij}e_\alpha, \qquad R(e_i,e_\alpha)e_\beta = \frac{f'h'}{fh}\delta_{\alpha\beta}e_i, \qquad R(e_i,e_\alpha)\partial_t = 0,$$

$$R(\partial_t,\partial_t) = 0,$$

$$R(e_i,\partial_t)e_j = -\frac{f''}{f}\delta_{ij}\partial_t, \qquad R(e_i,\partial_t)\partial_t = \frac{f''}{f}e_i, \qquad R(e_i,\partial_t)e_\alpha = 0,$$

$$R(e_\alpha,\partial_t)e_i = 0, \qquad R(e_\alpha,\partial_t)\partial_t = \frac{h''}{h}e_\alpha, \qquad R(e_\alpha,\partial_t)e_\beta = -\frac{h''}{h}\delta_{\alpha\beta}\partial_t.$$

We now compute $\mathring{R}$.

Proposition A.9. *In the basis* (e_A)

$$\mathring{R}(T)=\begin{pmatrix}0&0&0\\0&-\frac{f''}{f}I_k&0\\0&0&-\frac{h''}{h}I_l\end{pmatrix}a(t)+\begin{pmatrix}-k\frac{f''}{f}&0&0\\0&(k-1)\frac{1-f'^2}{f^2}I_k&0\\0&0&-\frac{f'h'}{fh}kI_l\end{pmatrix}\phi(t)+\begin{pmatrix}-l\frac{h''}{h}&0&0\\0&-\frac{f'h'}{fh}lI_k&0\\0&0&(l-1)\frac{1-h'^2}{h^2}I_l\end{pmatrix}\psi(t).$$

Proof. We detail the calculation of $\mathring{R}(T)(e_i,e_j)$:

$$\begin{aligned}
\mathring{R}(T)(e_i,e_j)&=T(R(e_i,\partial_t)e_j,\partial_t)+\sum_{p=1}^{k}T(R(e_i,e_p)e_j,e_p)+\sum_{\alpha=k+1}^{d-1}T(R(e_i,e_\alpha)e_j,e_\alpha)\\
&=T\left(-\frac{f''}{f}\delta_{ij}\partial_t,\partial_t\right)+\sum_{p=1}^{k}T\left(\frac{f'^2-1}{f^2}(\delta_{pj}e_i-\delta_{ij}e_p),e_p\right)+\sum_{\alpha=k+1}^{d-1}T\left(-\frac{f'h'}{fh}\delta_{ij}e_\alpha,e_\alpha\right)\\
&=-\frac{f''}{f}\delta_{ij}a(t)+\frac{f'^2-1}{f^2}\sum_{p=1}^{k}(\delta_{pj}\phi(t)\delta_{ip}-\delta_{ij}\phi(t))-\frac{f'h'}{fh}l\psi(t)\delta_{ij}\\
&=-\frac{f''}{f}\delta_{ij}a(t)+\frac{f'^2-1}{f^2}(\delta_{ij}\phi(t)-\delta_{ij}k\phi(t))-\frac{f'h'}{fh}l\psi(t)\delta_{ij}\\
&=-\left(\frac{f''}{f}a(t)+(k-1)\phi(t)\frac{f'^2-1}{f^2}+\frac{f'h'}{fh}l\psi(t)\right)\delta_{ij}.
\end{aligned}$$

□

Finally, the expression of $\mathfrak{R}$ follows from Propositions A.8 and A.9.

Proposition A.10. *In the basis* (e_A)*:*

$$\begin{aligned}\mathfrak{R}(T)=&\begin{pmatrix}2\Lambda&0&0\\0&2\frac{f''}{f}I_k&0\\0&0&2\frac{h''}{h}I_l\end{pmatrix}a(t)+\begin{pmatrix}2k\frac{f''}{f}&0&0\\0&\left[2\Lambda-2(k-1)\frac{1-f'^2}{f^2}\right]I_k&0\\0&0&2\frac{f'h'}{fh}kI_l\end{pmatrix}\phi(t)\\
&+\begin{pmatrix}2l\frac{h''}{h}&0&0\\0&2\frac{f'h'}{fh}lI_k&0\\0&0&\left[2\Lambda-2(l-1)\frac{1-h'^2}{h^2}\right]I_l\end{pmatrix}\psi(t).\end{aligned}$$

A.1.5 Formula for the Lichnerowicz Laplacian

Thus, we can finally calculate the Lichnerowicz Laplacian, and prove Theorem 4.5:

$$\begin{aligned}
\Delta_L T_{00}&=-a''(t)-\left(k\frac{f'}{f}+l\frac{h'}{h}\right)a'(t)+2\left(k\frac{f'^2}{f^2}+l\frac{h'^2}{h^2}\right)a(t)-2k\frac{f'^2}{f^2}\phi(t)-2l\frac{h'^2}{h^2}\psi(t)+2\Lambda a(t)+2k\frac{f''}{f}\phi(t)+2l\frac{h''}{h}\psi(t)\\
&=-a''(t)-\left(k\frac{f'}{f}+l\frac{h'}{h}\right)a'(t)+2\left(\Lambda+k\frac{f'^2}{f^2}+l\frac{h'^2}{h^2}\right)a(t)+2k\left(\frac{f''}{f}-\frac{f'^2}{f^2}\right)\phi(t)+2l\left(\frac{h''}{h}-\frac{h'^2}{h^2}\right)\psi(t),
\end{aligned}$$

$$\begin{aligned}
\Delta_L T_{ii}&=-\phi''(t)-2\frac{f'^2}{f^2}(a(t)-\phi(t))-\left(k\frac{f'}{f}+l\frac{h'}{h}\right)\phi'(t)+2\frac{f''}{f}a(t)+\left(2\Lambda-2(k-1)\frac{1-f'^2}{f^2}\right)\phi(t)+2\frac{f'h'}{fh}l\psi(t)\\
&=-\phi''(t)-\left(k\frac{f'}{f}+l\frac{h'}{h}\right)\phi'(t)+\left(2\Lambda-2(k-1)\frac{1-f'^2}{f^2}+2\frac{f'^2}{f^2}\right)\phi(t)+2\left(\frac{f''}{f}-\frac{f'^2}{f^2}\right)a(t)+2\frac{f'h'}{fh}l\psi(t)\\
&=-\phi''(t)-\left(k\frac{f'}{f}+l\frac{h'}{h}\right)\phi'(t)+\left(2\Lambda-2\frac{k-1}{f^2}+2k\frac{f'^2}{f^2}\right)\phi(t)+2\left(\frac{f''}{f}-\frac{f'^2}{f^2}\right)a(t)+2\frac{f'h'}{fh}l\psi(t),
\end{aligned}$$

$$\begin{aligned}\Delta_L T_{\alpha\alpha} &= -\psi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\psi'(t) - 2\frac{h'^2}{h^2}(a(t)-\psi(t)) + 2\frac{h''}{h}a(t) + 2\frac{f'h'}{fh}k\phi(t) + \left(2\Lambda - 2(l-1)\frac{1-h'^2}{h^2}\right)\psi(t)\\ &= -\psi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\psi'(t) + \left(2\Lambda - 2(l-1)\frac{1-h'^2}{h^2} + 2\frac{h'^2}{h^2}\right)\psi(t) + 2\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right)a(t) + 2k\frac{f'h'}{fh}\phi(t)\\ &= -\psi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\psi'(t) + \left(2\Lambda - 2\frac{l-1}{h^2} + 2l\frac{h'^2}{h^2}\right)\psi(t) + 2\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right)a(t) + 2k\frac{f'h'}{fh}\phi(t).\end{aligned}$$

Since g is Einstein, the previous expressions reduce, by (2), (3) and (4), to

$$\begin{aligned}\Delta_L T_{00} &= -a''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)a'(t) + 2\left(\Lambda + k\frac{f'^2}{f^2} + l\frac{h'^2}{h^2}\right)a(t) + 2k\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)\phi(t) + 2l\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right)\psi(t)\\ &= -a''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)a'(t) + 2k\left(\frac{f'^2}{f^2} - \frac{f''}{f}\right)(a(t)-\phi(t)) + 2l\left(\frac{h'^2}{h^2} - \frac{h''}{h}\right)(a(t)-\psi(t)),\end{aligned}$$

$$\begin{aligned}\Delta_L T_{ii} &= -\phi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\phi'(t) + \left(2\Lambda - 2(k-1)\frac{1-f'^2}{f^2} + 2\frac{f'^2}{f^2}\right)\phi(t) + 2\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)a(t) + 2\frac{f'h'}{fh}l\psi(t)\\ &= -\phi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\phi'(t) - 2\left(\frac{f'^2}{f^2} - \frac{f''}{f}\right)(a(t)-\phi(t)) + 2\frac{f'h'}{fh}l(\psi(t)-\phi(t)),\end{aligned}$$

$$\Delta_L T_{\alpha\alpha} = -\psi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\psi'(t) - 2\left(\frac{h'^2}{h^2} - \frac{h''}{h}\right)(a(t)-\psi(t)) + 2\frac{f'h'}{fh}k(\phi(t)-\psi(t)).$$

A.2 Calculation using the linearization of Ricci

The previous calculations were quite long, so finding the same expression by another method provides an independent verification of the result.
Theorem 2.12 yields

$$\frac{\mathrm{Ric}(g+\varepsilon T) - \mathrm{Ric}\, g}{\varepsilon} \to \frac{1}{2}\Delta_L T - \delta^*\delta T - \frac{1}{2}D_g d(\mathrm{tr} T).$$

Thus, it remains to compute the Ricci tensor.

A.2.1 Calculation of the derivative of Ricci

We need to compute Ric which is defined by

$$\mathrm{Ric}(g)(X,Y) = \mathrm{tr}(Z \mapsto R(X,Z)Y).$$

Thus,

$$\mathrm{Ric}(g)(X,Y) = \sum_{A=0}^{d-1} g(R(X,e_A)Y, e_A).$$

Proposition A.11. *Let $T = a(t)dt^2 + \phi(t)f(t)^2 g^k + \psi(t)h(t)^2 g^l$ be a G-invariant tensor. Then:*

$$\begin{aligned}d\,\mathrm{Ric}_g(T)_{00} &= -\frac{1}{2}(k\phi'' + l\psi'') - \left(k\frac{f'}{f}\phi' + l\frac{h'}{h}\psi'\right) + \frac{1}{2}Ha',\\ d\,\mathrm{Ric}_g(T)_{jj} &= -\frac{1}{2}\phi'' - H\phi' - (k-1)\frac{\phi-a}{f^2} + \frac{lh'}{2h}\phi' - \frac{lf'}{2f}\psi' + \frac{a'f'}{2f} + \Lambda(\phi-a),\\ d\,\mathrm{Ric}_g(T)_{\beta\beta} &= -\frac{1}{2}\psi'' - H\psi' - (l-1)\frac{\psi-a}{h^2} + \frac{kf'}{2f}\psi' - \frac{kh'}{2h}\phi' + \frac{a'h'}{2h} + \Lambda(\psi-a).\end{aligned}$$

Proof. To prove this result, we introduce a perturbed metric ${}^{\varepsilon}g$, defined for all $\varepsilon > 0$ by

$$
{}^{\varepsilon}g := g + \varepsilon T = \begin{pmatrix} 1+\varepsilon a(t) & 0 & 0 \\ 0 & (\varepsilon\phi(t)+1)f(t)^2 g^k & 0 \\ 0 & 0 & (\varepsilon\psi(t)+1)h(t)^2 g^l \end{pmatrix}.
$$

We have:

$$
{}^{\varepsilon}g = (1+\varepsilon a(t)) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\varepsilon\phi(t)+1}{1+\varepsilon a(t)} f(t)^2 g^k & 0 \\ 0 & 0 & \frac{\varepsilon\psi(t)+1}{1+\varepsilon a(t)} h(t)^2 g^l \end{pmatrix}.
$$

We define

$$
f_\varepsilon(t) = \sqrt{\frac{1+\varepsilon\phi(t)}{1+\varepsilon a(t)}} f(t) \quad \text{and} \quad h_\varepsilon(t) = \sqrt{\frac{1+\varepsilon\psi(t)}{1+\varepsilon a(t)}} h(t).
$$

Thus,

$$
{}^{\varepsilon}g = (1+\varepsilon a(t))\tilde{g}.
$$

$\tilde{g}$ is very close to the metric of the previous section (we just have to replace f and h by f_ε and h_ε). Therefore, we already know the Riemann tensor, because of Proposition A.8.
We have (we do not write $o(\varepsilon)$) :

$$
\begin{aligned}
f_\varepsilon(t) &= \left(1 + \frac{\varepsilon}{2}(\phi(t) - a(t))\right) f(t), \\
f_\varepsilon'(t) &= f'(t) + \frac{\varepsilon}{2} f'(t)(\phi(t) - a(t)) + \frac{\varepsilon}{2} f(t)(\phi'(t) - a'(t)), \\
f_\varepsilon''(t) &= f''(t) + \frac{\varepsilon}{2} f''(t)(\phi(t) - a(t)) + \varepsilon f'(t)(\phi'(t) - a'(t)) + \frac{\varepsilon}{2} f(t)(\phi''(t) - a''(t)).
\end{aligned}
$$

Moreover, we already know that (see [GHP03] (31)):

$$
\begin{aligned}
\mathrm{Ric}_{00} &= -\left(k\frac{f''}{f} + l\frac{h''}{h}\right), \\
\mathrm{Ric}_{jj} &= -\frac{f''}{f} + (k-1)\frac{1-f'^2}{f^2} - l\frac{f'h'}{fh}, \\
\mathrm{Ric}_{\alpha\alpha} &= -\frac{h''}{h} + (l-1)\frac{1-h'^2}{h^2} - k\frac{f'h'}{fh}.
\end{aligned}
$$

We also have:

$$
\frac{f_\varepsilon'}{f_\varepsilon} = \frac{f'}{f} + \frac{\varepsilon}{2}(\phi' - a'),
$$

$$
\frac{f_\varepsilon''}{f_\varepsilon} = \frac{f''}{f} + \frac{\varepsilon}{2}\left(\phi'' - a'' + 2\frac{f'}{f}(\phi' - a')\right),
$$

$$
\frac{f_\varepsilon' h_\varepsilon'}{f_\varepsilon h_\varepsilon} = \frac{f'h'}{fh} + \frac{\varepsilon}{2}\frac{h'}{h}(\phi' - a') + \frac{\varepsilon}{2}\frac{f'}{f}(\psi' - a'),
$$

$$
\frac{1 - f_\varepsilon'^2}{f_\varepsilon^2} = \frac{1 - f'^2}{f^2} + \varepsilon\left(-\frac{\phi - a}{f^2} - \frac{f'}{f}(\phi' - a')\right).
$$

We denote by $\widetilde{\mathrm{Ric}}$ the Ricci tensor of $\tilde{g}$.

$$\begin{aligned}
\widetilde{\mathrm{Ric}}_{00} &= \Lambda + \frac{\varepsilon}{2}(d-1)a'' - \frac{\varepsilon}{2}(k\phi'' + l\psi'') - \varepsilon\left(k\frac{f'}{f}\phi' + l\frac{h'}{h}\psi'\right) + \varepsilon a'\left(k\frac{f'}{f} + l\frac{h'}{h}\right),\\
\widetilde{\mathrm{Ric}}_{jj} &= \Lambda + \varepsilon\left(\frac{1}{2}a'' - \frac{1}{2}\phi'' - \frac{f'}{f}(\phi'-a') - (k-1)\frac{\phi-a}{f^2} - (k-1)\frac{f'}{f}(\phi'-a') - \frac{lh'}{2h}(\phi'-a') - \frac{lf'}{2f}(\psi'-a') + \Lambda(\phi-a)\right),\\
\widetilde{\mathrm{Ric}}_{\alpha\alpha} &= \Lambda + \varepsilon\left(\frac{1}{2}a'' - \frac{1}{2}\psi'' - \frac{h'}{h}(\psi'-a') - (l-1)\frac{\psi-a}{h^2} - (l-1)\frac{h'}{h}(\psi'-a') - \frac{kf'}{2f}(\psi'-a') - \frac{kh'}{2h}(\phi'-a') + \Lambda(\psi-a)\right).
\end{aligned}$$

Finally, to compute $\mathrm{Ric}({}^{\varepsilon}g)$ we use the formula for a conformal change of metrics (see [Bes87] 1.159d):

Lemma A.12. *Let (M,g) be a Riemannian manifold of dimension n. Let u be a smooth function on M and $\hat{g} = e^{2u}g$. Then*

$$(\mathrm{Ric}_{\hat{g}})_{ij} = (\mathrm{Ric}_g)_{ij} - (n-2)\nabla_i\nabla_j u + (n-2)\nabla_i u\nabla_j u + \left(\Delta u - (n-2)|\nabla u|^2\right)g_{ij},$$

where

$$\nabla_i\nabla_j u = [D_g du]_{ij},$$

and $\Delta u = -\mathrm{tr}(D_g du)$.

In our setting, $u(t) = \frac{1}{2}\ln(1+\varepsilon a(t))$. Thus,

$$u'(t) = \frac{1}{2}\frac{\varepsilon a'(t)}{1+\varepsilon a(t)} = \frac{1}{2}\varepsilon a'(t),$$

$$u''(t) = \frac{1}{2}\varepsilon a''(t).$$

$$\begin{aligned}
Ddu_{00} &= u'' = \frac{\varepsilon}{2}a'',\\
Ddu_{ij} &= (-\nabla_{e_i}e_j)u = \frac{f'}{f}\delta_{ij}\partial_t u = \frac{f'}{f}u'\delta_{ij} = \frac{f'}{f}\frac{\varepsilon}{2}a'\delta_{ij},\\
Ddu_{\alpha\beta} &= (-\nabla_{e_\alpha}e_\beta)u = \frac{h'}{h}\delta_{\alpha\beta}\partial_t u = \frac{h'}{h}u'\delta_{\alpha\beta} = \frac{h'}{h}\frac{\varepsilon}{2}a'\delta_{\alpha\beta},
\end{aligned}$$

$$\Delta u = -\frac{\varepsilon}{2}(a'' + Ha').$$

Therefore, at the first order in ε,

$$\begin{aligned}
\mathrm{Ric}({}^{\varepsilon}g)_{00} &= \widetilde{\mathrm{Ric}}_{00} - (d-1)\frac{\varepsilon}{2}a'' - \frac{\varepsilon}{2}Ha',\\
\mathrm{Ric}({}^{\varepsilon}g)_{jj} &= \widetilde{\mathrm{Ric}}_{jj} - \varepsilon\frac{d-2}{2}a'\frac{f'}{f} - \frac{\varepsilon}{2}a'' - \varepsilon\frac{a'}{2}H,\\
\mathrm{Ric}({}^{\varepsilon}g)_{\alpha\alpha} &= \widetilde{\mathrm{Ric}}_{\alpha\alpha} - \varepsilon\frac{d-2}{2}a'\frac{h'}{h} - \frac{\varepsilon}{2}a'' - \varepsilon\frac{a'}{2}H.
\end{aligned}$$

Rearranging the previous expressions yields

$$\begin{aligned}
\mathrm{Ric}(g)'(T)_{00} &= -\frac{1}{2}(k\phi'' + l\psi'') - \left(k\frac{f'}{f}\phi' + l\frac{h'}{h}\psi'\right) + \frac{1}{2}Ha',\\
\mathrm{Ric}(g)'(T)_{jj} &= -\frac{1}{2}\phi'' - H\phi' - (k-1)\frac{\phi-a}{f^2} + \frac{lh'}{2h}\phi' - \frac{lf'}{2f}\psi' + \frac{a'f'}{2f} + \Lambda(\phi-a),\\
\mathrm{Ric}(g)'(T)_{\beta\beta} &= -\frac{1}{2}\psi'' - H\psi' - (l-1)\frac{\psi-a}{h^2} + \frac{kh'}{2h}\psi' - \frac{kh'}{2h}\phi' + \frac{a'h'}{2h} + \Lambda(\psi-a).
\end{aligned}$$

□

A.2.2 Calculation of $\delta^*\delta$ and $D_g d(\mathrm{tr}_g(T))$

To recover the expression of Δ_L, we also need to compute $\delta^*\delta(T)$ and $D_g d(\mathrm{tr}_g(T))$.

Proposition A.13. *If $T = a(t)dt^2 + \phi(t)f(t)^2 g^k + \psi(t)h(t)^2 g^l$, then, in the basis (e_A)*

$$\begin{aligned}
(\delta^*\delta T)_{00} &= -a''(t) - k\frac{f''f-(f'^2)}{f^2}(a-\phi) - k\frac{f'}{f}(a'-\phi') - l\frac{h''h-(h'^2)}{h^2}(a-\psi) - l\frac{h'}{h}(a'-\psi'),\\
(\delta^*\delta T)_{jj} &= -\frac{f'}{f}a' - k\frac{f'^2}{f^2}(a-\phi) - l\frac{f'h'}{fh}(a-\psi),\\
(\delta^*\delta T)_{\beta\beta} &= -\frac{h'}{h}a' - l\frac{h'^2}{h^2}(a-\psi) - k\frac{f'h'}{fh}(a-\phi).
\end{aligned}$$

And the other terms vanish.

Proof. We have the following formula (see [Bes87] 1.60):

$$\delta^*\delta T(X,Y) = \frac{1}{2}((\nabla_X \delta T)(Y) + (\nabla_Y \delta T)(X)),$$

But:

$$\begin{aligned}
(\nabla_X \delta T)(Y) &= \nabla_X((\delta T)(Y)) - (\delta T)(\nabla_X Y)\\
&= -\sum_{i=1}^{n}[\nabla_X(\nabla_{e_i}T(e_i,Y)) - (\nabla_{e_i}T)(e_i,\nabla_X Y)].
\end{aligned}$$

Now, we can compute it easily, since we already know the connection.
For instance, we detail the calculation of $(\delta^*\delta T)_{00}$.

$$\begin{aligned}
(\nabla_{\partial_t}\delta T)(\partial_t) &= -\sum_{A=0}^{d-1}[\nabla_{\partial_t}(\nabla_{e_A}T(e_A,\partial_t)) - (\nabla_{e_A}T)(e_A,\nabla_{\partial_t}\partial_t)]\\
&= -\nabla_{\partial_t}(\nabla_{\partial_t}T(\partial_t,\partial_t)) - \sum_{i=1}^{k}\nabla_{\partial_t}(\nabla_{e_i}T(e_i,\partial_t)) - \sum_{\alpha=k+1}^{d-1}\nabla_{\partial_t}(\nabla_{e_\alpha}T(e_\alpha,\partial_t))\\
&= -\nabla_{\partial_t}(a'(t)) - \sum_{i=1}^{k}\nabla_{\partial_t}\left(\frac{f'}{f}(a-\phi)\right) - \sum_{\alpha=k+1}^{d-1}\nabla_{\partial_t}\left(\frac{h'}{h}(a-\psi)\right)\\
&= -a''(t) - k\frac{f''f-(f'^2)}{f^2}(a-\phi) - k\frac{f'}{f}(a'-\phi') - l\frac{h''h-(h'^2)}{h^2}(a-\psi) - l\frac{h'}{h}(a'-\psi').
\end{aligned}$$

Thus,

$$(\delta^*\delta)_{00} = -a''(t) - k\frac{f''f-(f'^2)}{f^2}(a-\phi) - k\frac{f'}{f}(a'-\phi') - l\frac{h''h-(h'^2)}{h^2}(a-\psi) - l\frac{h'}{h}(a'-\psi').$$

□

Now, we calculate $D_g d(\mathrm{tr}_g(T))$.
We recall that (see [Bes87] 1.54),

$$D_g df(X,Y) = X(Y(f)) - (\nabla_X Y)f.$$

Here $f = u = a + k\phi + l\psi$. This yields the following proposition.

Proposition A.14.

$$\begin{aligned}
Ddu_{00} &= u'',\\
Ddu_{ij} &= \frac{f'}{f}u'\delta_{ij},\\
Ddu_{\alpha\beta} &= \frac{h'}{h}u'\delta_{\alpha\beta}.
\end{aligned}$$

and the other terms vanish.

A.2.3 Expression of Δ_L

Now, we can recover the expression of the Lichnerowicz Laplacian, using Propositions A.11, A.13 and A.14.

$$\Delta_L T = 2\operatorname{Ric}'_g(T) + 2\delta^*\delta + D_g du.$$

Thus,

$$\begin{aligned}
\Delta_L T_{00} &= -(k\phi'' + l\psi'') - 2\left(k\frac{f'}{f}\phi' + l\frac{h'}{h}\psi'\right) + Ha' - 2a''(t) \\
&\quad - 2k\frac{f''f - (f'^2)}{f^2}(a-\phi) - 2k\frac{f'}{f}(a'-\phi') - 2l\frac{h''h - (h'^2)}{h^2}(a-\psi) - 2l\frac{h'}{h}(a'-\psi') + a'' + (k\phi'' + l\psi'') \\
&= -a'' - Ha' - \left(2k\frac{f''}{f} + 2l\frac{h''}{h}\right)a + 2\left(k\frac{f'^2}{f^2} + l\frac{h'^2}{h^2}\right)a + 2k\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)\phi + 2l\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right)\psi.
\end{aligned}$$

This is the same expression as before, since $\Lambda = -k\dfrac{f''}{f} - l\dfrac{h''}{h}$

$$\Delta_L T_{jj} = -\phi'' - H\phi' + 2l\frac{f'h'}{fh}\psi + \phi\left(2k\frac{(f')^2 - 1}{f^2} + \frac{2}{f^2} + 2\Lambda\right) + a\left(2k\frac{1 - f'^2}{f^2} - \frac{2}{f^2} - 2l\frac{f'h'}{fh} - 2\Lambda\right),$$

which is also the same expression using the fact that the metric is Einstein. The last expression also follows by symmetry.

B Proof of Theorem 5.2

We first recall the statement of the theorem:

Theorem B.1. *Let $\mu \in \mathbb{R}$. There exists t_B such that, for all $a_0 \in \mathbb{R}$, there exists a unique analytic tensor $T = a(t)dt^2 + \phi(t)f(t)^2g^k + \psi(t)h(t)^2g^l \in \mathscr{T}^G_{TT}(M)$ ($t \in [0, t_B]$), such that*

$$\begin{cases} \Delta_L T = \mu T \\ a(0) = \phi(0) = a_0 \\ \psi(0) = -\frac{k+1}{l}a_0 \\ a'(0) = \phi'(0) = \psi'(0) = 0 \end{cases}.$$

To prove this theorem, we start by proving the following result, only for G-invariant tensors.

Theorem B.2. *Let $\mu \in \mathbb{R}$. There exists $t_B > 0$ such that, for all $a_0, \psi_0 \in \mathbb{R}$, there exists an analytic G-invariant symmetric $(0,2)$-tensor $T = a(t)dt^2 + \phi(t)f(t)^2g^k + \psi(t)h(t)^2g^l$ ($t \in [0, t_B]$) such that*

$$\begin{cases} \Delta_L T = \mu T \\ a(0) = \phi(0) = a_0 \\ \psi(0) = \psi_0 \\ a'(0) = \phi'(0) = \psi'(0) = 0 \end{cases}.$$

However, the solution is not unique.

Proof. We recall that the equation can be written as

$$\begin{cases}
-a''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)a'(t) + 2\left(\Lambda + k\frac{f'^2}{f^2} + l\frac{h'^2}{h^2}\right)a(t) + 2k\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)\phi(t) + 2l\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right)\psi(t) = \mu a(t) \\[2ex]
-\phi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\phi'(t) + \left(2\Lambda - 2\frac{k-1}{f^2} + 2k\frac{f'^2}{f^2}\right)\phi(t) + 2\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)a(t) + 2\frac{f'h'}{fh}l\psi(t) = \mu\phi(t) \\[2ex]
-\psi''(t) - \left(k\frac{f'}{f} + l\frac{h'}{h}\right)\psi'(t) + \left(2\Lambda - 2\frac{l-1}{h^2} + 2l\frac{h'^2}{h^2}\right)\psi(t) + 2\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right)a(t) + 2k\frac{f'h'}{fh}\phi(t) = \mu\psi(t)
\end{cases},$$

for $T = a(t)dt^2 + \phi(t)f(t)^2 g^k + \psi(t)h(t)^2 g^l$.
We set:

$$Y = (a, ta', \phi, t\phi', \psi, t\psi')^T = (y_1, \cdots, y_6)^T$$

We have:

$$\begin{aligned}
y_1' &= \frac{1}{t} y_2, \\
y_2' &= ta'' + a' \\
&= \frac{y_2}{t} - Hy_2 + t\left(2\Lambda + 2l\frac{h'^2}{h^2}\right) y_1 + \frac{2kf'^2}{t}\frac{t^2}{f^2} y_1 + 2kt\frac{f''}{f} y_3 - \frac{2kf'^2}{t}\frac{t^2}{f^2} y_3 + 2lt\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right) y_5 - t\mu y_1, \\
y_3' &= \frac{1}{t} y_4, \\
y_4' &= \phi' + t\phi'' \\
&= \frac{y_4}{t} - Hy_4 + 2t\Lambda y_3 + \frac{2kf'^2}{t}\frac{t^2}{f^2} y_3 - \frac{2(k-1)}{t}\frac{t^2}{f^2} y_3 + 2t\frac{f''}{f} y_1 - 2t\frac{f'^2}{f^2} y_1 + 2tl\frac{f'h'}{fh} y_5 - t\mu y_3, \\
y_5' &= \frac{1}{t} y_6, \\
y_6' &= \frac{y_6}{t} - Hy_6 + 2t\Lambda y_5 + \frac{2lh'^2}{t}\frac{t^2}{h^2} y_5 - \frac{2(l-1)}{t}\frac{t^2}{h^2} y_5 + 2t\frac{h''}{h} y_1 - 2t\frac{h'^2}{h^2} y_1 + 2tk\frac{f'h'}{fh} y_3 - t\mu y_5.
\end{aligned}$$

Hence, this system can be written as $Y' = \frac{1}{t}A(t)Y$ with A analytic at 0

$$A(0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2k & 1-k & -2k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 2 & 1-k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1-k \end{pmatrix},$$

since $f(0) = 0$, $f'(0) = 1$, $h(0) = \bar{h} > 0$ and $h'(0) = 0$.
Let

$$J = \begin{pmatrix} 2 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & -k-1 & & \\ & & & & 1-k & \\ & & & & & 1-k \end{pmatrix},$$

$$P = \frac{1}{2}\begin{pmatrix} -k & 0 & 2 & \frac{2k}{k+1} & 0 & \frac{2}{1-k} \\ -2k & 0 & 0 & -2k & 0 & 2 \\ 1 & 0 & 2 & -\frac{2}{k+1} & 0 & \frac{2}{1-k} \\ 2 & 0 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 0 & \frac{2}{1-k} & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix},$$

We have:

$$A(0) = PJP^{-1}.$$

We rewrite the differential equation as

$$\begin{aligned}
tY' &= A(t)Y(t) \\
&= A_0 Y(t) + (A(t) - A_0)Y(t).
\end{aligned}$$

Let $M(t) = \frac{A(t)-A_0}{t}$. Since A is analytic at 0, M is analytic as well. Thus:

$$tY' = A_0 Y(t) + tM(t)Y(t).$$

Let $V = P^{-1}Y$ and $B(t) = P^{-1}M(t)P$. We have:

$$tV'(t) = JV(t) + tB(t)V(t). \tag{11}$$

Since $B(t)$ is analytic at 0, there exists $t_B > 0$ such that

$$\forall t \in [0, t_B), \;\; B(t) = \sum_{p=0}^{+\infty} B_p t^p.$$

We are looking for analytic solutions. We write

$$V(t) = \sum_{n=0}^{+\infty} V_n t^n.$$

Thus, (11) yields

$$\sum_{n=1}^{+\infty} nV_n t^n = JV_0 + \sum_{n=1}^{+\infty} t^n \left[JV_n + \sum_{p=0}^{n-1} B_{n-1-p} V_p \right].$$

Comparing the coefficients of equal powers of t, we obtain

$$\begin{cases} JV_0 = 0 \\ (nI - J)V_n = \displaystyle\sum_{p=0}^{n-1} B_{n-1-p} V_p \end{cases} . \tag{12}$$

Hence, we must take $V_0 \in \ker(J) = \{(0, \alpha, \beta, 0, 0, 0)^T, \;\; \alpha, \beta \in \mathbb{R}\}$. We now fix $\alpha, \beta \in \mathbb{R}$ and $V_0 = (0, \alpha, \beta, 0, 0, 0)^T$?
Then, we can define V_1 by

$$V_1 = (I - J)^{-1} B_0 V_0$$

since 1 is not an eigenvalue of J.
One easily checks that $A'(0) = 0$ (since $f''(0) = 0$ due to (2)). Hence $V_1 = 0$.
However, to define V_2, the situation is more delicate since 2 is an eigenvalue of J. Let

$$R = B_1 V_0 + B_0 V_1 = B_1 V_0 = \frac{1}{2} P^{-1} A''(0) P V_0.$$

We need to show that there exists a solution to the equation:

$$(2I - J)X = R.$$

Therefore, we need to show that $R \in \operatorname{im}(2I - J)$.

$$2I - J = \begin{pmatrix} 0 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & k+3 & & \\ & & & & k+1 & \\ & & & & & k+1 \end{pmatrix}.$$

Hence, $\operatorname{im}(2I - J) = \{0\} \times \mathbb{R}^5$. Therefore, we need to show that $R_1 = 0$.
Let $\xi = 4\Lambda - 2\mu$.
We have $A''(0) =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \xi + \frac{8k}{3} f'''(0) & -\frac{2k}{3} f'''(0) - 2l \frac{h''(0)}{\bar{h}} & \frac{4k}{3} f'''(0) & 0 & 4l \frac{h''(0)}{\bar{h}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{3} f'''(0) & 0 & \xi + \left(4k - \frac{4}{3}\right) f'''(0) & -\frac{2k}{3} f'''(0) - 2l \frac{h''(0)}{\bar{h}} & 4l \frac{h''(0)}{\bar{h}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 \frac{h''(0)}{\bar{h}} & 0 & 4k \frac{h''(0)}{\bar{h}} & 0 & \xi - \frac{4(l-1)}{\bar{h}^2} & -\frac{2k}{3} f'''(0) - 2l \frac{h''(0)}{\bar{h}} \end{pmatrix}$$

We just need to know the first line of B_1. A direct computation shows that

$$L_1(B_1) = (*, 0, 0, *, 0, 0).$$

In particular, for all $\alpha, \beta \in \mathbb{R}$, we obtain:

$$R_1 = 0.$$

Thus, $R \in \operatorname{im}(2I - J)$, and we can choose $V_2 \in \mathbb{R}^6$ such that

$$(2I - J)V_2 = R.$$

Then, since $(nI - J)$ is invertible for all $n \geqslant 3$, we recursively define the sequence (V_n) using (12). We note that the solution is not unique, due to the choice of V_2.

Finally, Theorem 5.3 in [Was18] proves that the power series $\sum V_n t^n$ converges for all $t < t_B$.

We now examine the initial conditions. Since $Y = PV$,

$$Y(0) = PV_0 = \frac{1}{2}\begin{pmatrix} -k & 0 & 2 & \frac{2k}{k+1} & 0 & \frac{2}{1-k} \\ -2k & 0 & 0 & -2k & 0 & 2 \\ 1 & 0 & 2 & -\frac{2}{k+1} & 0 & \frac{2}{1-k} \\ 2 & 0 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 0 & \frac{2}{1-k} & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \alpha \\ \beta \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta \\ 0 \\ \beta \\ 0 \\ \alpha \\ 0 \end{pmatrix}.$$

Moreover, $Y'(0) = 0$ thus, $a'(0) = \phi'(0) = \psi'(0) = 0$.

Consequently, $a(0) = \phi(0) = \beta \in \mathbb{R}$, $a'(0) = 0$, $\phi'(0) = 0$, $\psi(0) = \alpha \in \mathbb{R}$ and $\psi'(0) = 0$.

□

We want to study traceless and transverse tensors. These assumptions add two additional conditions:

$$a + k\phi + l\psi = 0,$$
$$a' + Ha - k\frac{f'}{f}\phi - l\frac{h'}{h}\psi = 0.$$

Let $F = a + k\phi + l\psi$ and $D = a' + Ha - k\frac{f'}{f}\phi - l\frac{h'}{h}\psi$.

The following lemma will be needed.

Lemma B.3. *F and D satisfy*

$$F'' + HF' + \mu F = 0,$$
$$D'' + HD' + (H' + \mu)D = 0.$$

Proof. Both identities follow directly from $\Delta_L T = \mu T$ and the Einstein equations. □

Now, we are looking for some conditions on a_0 and ψ_0 to have $T \in \mathscr{T}_{TT}^G(M)$.

Proof of Theorem 5.2. We denote by a_i, f_i, and h_i the i-th derivative of a, f, h evaluated at 0.

With the previous lemma, the two functions F and D must satisfy

$$F(0) = F'(0) = D(0) = D'(0) = 0.$$

We have:

$$F(0) = (k+1)a_0 + l\psi_0,$$
$$F'(0) = a_1 + k\phi_1 + l\psi_1,$$
$$D(0) = a_1 + k(a_1 - \phi_1).$$

However, $D'(0)$ also depends on $a''(0)$, and we cannot replace it using (8).

Indeed,

$$D'(0) = \frac{k+2}{2} a_2 - \frac{k}{2} \phi_2 + l \frac{h_2}{\bar{h}} (a_0 - \psi_0).$$

We also have, with the equation (8) (this expression is well defined since $a_1 = \phi_1 = 0$):

$$a_2 + k\phi_2 = -\mu a_0 - 2l \frac{h_2}{\bar{h}} (a_0 - \psi_0).$$

Combining these two identities yields

$$\frac{k+3}{2} a_2 = -\frac{\mu}{2} a_0 - 2 \frac{l h_2}{\bar{h}} (a_0 - \psi_0).$$

Finally, using the traceless condition $F(0) = 0$, we can replace ψ_0.

$$\frac{k+3}{2} a_2 = -\frac{\mu}{2} a_0 - 2d \frac{h_2}{\bar{h}} a_0. \tag{13}$$

Thus, we obtain the following conditions

$$\begin{cases} \phi_0 = a_0 \\ \psi_0 = -\frac{k+1}{l} a_0 \end{cases}.$$

And, T is transverse if and only if (13) is satisfied. The idea is to exploit the analyticity of the solution, and also the fact that 2 is an eigenvalue of J. We must find an eigenvector to obtain the desired value for a_2.

The solution Y can be written:

$$Y(t) = \sum_{n=0}^{+\infty} Y_n t^n.$$

$[Y_2]_1 = \frac{a''(0)}{2}$.

We also have

$$Y_2 = PV_2 \qquad (2I - J)V_2 = R.$$

We compute R:

$$2R = P^{-1} A''(0) Y_0 = \begin{pmatrix} 0 \\ \dfrac{Q}{k-1} \\ \dfrac{S}{k-1} \\ 0 \\ Q \\ S \end{pmatrix},$$

with

$$S := \beta(\xi + 4kf'''(0)) + 4\alpha l \frac{h''(0)}{\bar{h}} = -2a_0\mu - 4\frac{h''(0)}{\bar{h}}(\Lambda + 1)a_0$$

(the last equality is a consequence of (2) in $t = 0$) an

$$Q := 4\beta(1+k)\frac{h''(0)}{\bar{h}} + \alpha\left(\xi - \frac{4(l-1)}{\bar{h}^2}\right),$$

where $\beta = a_0$ and $\alpha = \psi_0 = -\frac{k+1}{l} a_0$.

Therefore,

$$\begin{pmatrix} 0 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & k+3 & & \\ & & & & k+1 & \\ & & & & & k+1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ \frac{Q}{k-1} \\ \frac{S}{k-1} \\ 0 \\ Q \\ S \end{pmatrix}.$$

Moreover,

$$Y_2 = PV_2 = \frac{1}{2} \begin{pmatrix} -k & 0 & 2 & \frac{2k}{k+1} & 0 & \frac{2}{1-k} \\ -2k & 0 & 0 & -2k & 0 & 2 \\ 1 & 0 & 2 & -\frac{2}{k+1} & 0 & \frac{2}{1-k} \\ 2 & 0 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 0 & \frac{2}{1-k} & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}.$$

Hence,

$$\begin{aligned} [Y_2]_1 &= \frac{1}{2}\left(-kv_1 + 2v_3 + \frac{2k}{k+1}v_4 + \frac{2}{1-k}v_6\right) \\ &= \frac{1}{2}\left(-kv_1 + \frac{1}{2}\frac{S}{k-1} + \frac{1}{1-k^2}S\right). \end{aligned}$$

Thus, $T \in \mathscr{T}^G_{TT}(M)$ if and only if

$$-\frac{\mu}{k+3}a_0 - 4d\frac{h_2}{\bar{h}(k+3)}a_0 = -kv_1 + \frac{1}{2}\frac{S}{k-1} + \frac{1}{1-k^2}S.$$

Therefore, if we set:

$$v_1 = \frac{1}{k}\left(\frac{\mu}{k+3}a_0 + 4d\frac{h_2}{\bar{h}(k+3)}a_0 + \frac{1}{2}\frac{S}{k-1} + \frac{1}{1-k^2}S\right) = \frac{2a_0}{k(k+1)(k+3)}\left(-\mu + d(k-1)\frac{h_2}{\bar{h}}\right),$$

$T \in \mathscr{T}^G_{TT}(M)$, and it is the only choice.

This concludes the proof of Theorem 5.2. □

C Python code

Since the coefficients diverge at $t = 0$, we compute an approximation of the solution from `T_init` > 0, and then we can solve the equation using `solve_ivp`. We first solve the Einstein equations. The values of $\bar{h}$ are determined using Wang's code [Wan26] (some of the values are in the code), and then we can solve $\Delta_L T = \mu T$.

We do not solve the system with 6 equations, because using the traceless condition, we need only track ϕ and ψ. Moreover, the transverse condition lets us eliminate ψ', hence we just solve an equation of the following form:

$$Y' = S(Y),$$

with $Y = (\phi, t\phi', \psi)^T$.

```python
import numpy as np
from scipy.integrate import solve_ivp


import matplotlib.pyplot as plt

plt.close("all")

# ========== Parameters ==========
```

```
k,l=2,2
d=k+l+1
Lam=k+l

a0=1.0
psi0=-(k+1)/l*a0
alpha,beta=psi0,a0

Omega=42.4756543567637 # \bar h = \sqrt{l-1}/Omega

# Some values of Omega (obtained using Q.S. Wang's code)

# ==== k=2, l=2 ====
#standard product: sqrt(Lambda)
#product 1: 8.48950409449637
#product 2: 42.4756543567637
#product 3: 271.836709312228

# ==== k=3, l=2 ====
#standard product: sqrt(Lambda)
#product 1: 7.914353185403161
#product 2: 37.19215141230961

# ==== k=2, l=3 ====
#standard product: sqrt(Lambda)
#product 1: 9.776871812879108
#product 2: 46.82379512616899

hbar=np.sqrt(l-1)/Omega

T_geom=1e-5 # initial time for solving Einstein equations
T_init=1e-4 # initial time for the eigentensor
T_max=2.5
Rtol_G,Atol_G=1e-13, 1e-16 # tolerance parameters for the integration of the Einstein equations
Rtol_T,Atol_T=1e-11, 1e-13 # tolerance for the integration of the eigenvalue equation

mu_min,mu_max,N_scan=-25,2,200 # Min/max value of the eigenvalue mu and number of points

# ========== Solving Einstein equations ==========

def EinF(f,fp,h,hp):
    return f*(-Lam+(k-1)*(1-fp**2)/f**2-l*fp*hp/(f*h))

def EinH(f,fp,h,hp):
    return h*(-Lam+(l-1)*(1-hp**2)/h**2-k*fp*hp/(f*h))

def taylor_coeffs(hini):
    h2=(-Lam*hini+(l-1)/hini)/(k+1)
    f3=-Lam/k-(l/k)*(h2/hini)
    return h2,f3

def systemEE(t,Y):
    f,fp,h,hp=Y
    return [fp, EinF(f,fp,h,hp),hp, EinH(f,fp,h,hp)]

def solve_EE(hini):
    h2, f3_ = taylor_coeffs(hini)
    t = T_geom
    Y0 = [t + f3_ * t**3 / 6, 1 + f3_ * t**2 / 2,
          hini + h2 * t**2 / 2, h2 * t]
    return solve_ivp(systemEE,
```

```
                    (t, T_max), Y0, method="DOP853",
                    rtol=Rtol_G, atol=Atol_G, dense_output=True)

def dicho(fun, ta, tb, tol=1e-12):
    fa,fb=fun(ta),fun(tb)
    if fa*fb>0:
        return None
    while tb-ta>tol:
        tm=0.5*(ta+tb)
        fm=fun(tm)
        if fm==0.0:
            return tm
        if fa*fm<0:
            tb,fb=tm,fm
        else:
            ta,fa=tm,fm
    return 0.5*(ta+tb)

def first_zero(fun, ta, tb, n=5000,tol=1e-12):
    ts = np.linspace(ta, tb, n)
    v  = fun(ts)
    i = 0
    while i<n-1 and v[i]*v[i+1]>0 :
        i+=1
    if i>=n-1:
        return None
    return dicho(fun,ts[i],ts[i+1],tol)

def find_tstar(sol):
    return first_zero(lambda t: sol.sol(t)[1], 1e-3, T_max)

sol_geom=solve_EE(hbar)

tstar=find_tstar(sol_geom)

f_ts,_,h_ts,_=sol_geom.sol(tstar)
F2=-Lam+(k-1)/f_ts**2    # f''/f(t*)
H2=-Lam+(l-1)/h_ts**2    # h''/h(t*)

# ========== Some coefficients ==========

h2z, f3 = taylor_coeffs(hbar)
c = d * h2z / hbar

def order2_coeffs(mu):
    xi=4*Lam-2*mu
    S=beta*(xi+4*k*f3)+4*alpha*l*h2z/hbar
    Q=4*beta*(1+k)*h2z/hbar+alpha*(xi-4*(l-1)/hbar**2)
    v1=2*a0*(-mu+(k-1)*c)/(k*(k+1)*(k+3))
    return S,Q,v1+S/(2*(k+1)),Q/(2*(k+1))
# ========== Solving the eigenvalue equation ==========

def t_psip(t,w1,w2,w3):    #Returns t*psi'(t), obtained from the TT conditions
    f,fp,h,hp=sol_geom.sol(t)
    tf,th=fp/f,hp/h
    tH=t*(k*tf+l*th)
    return -(k/l)*(tH+t*tf)*w1-(k/l)*w2-(tH+t*th)*w3

def integrate(mu):
    _, _, phi2, psi2 = order2_coeffs(mu)
    W0 = np.array([a0+0.5*phi2*T_init**2,phi2*T_init**2,psi0+0.5*psi2*T_init**2])
    def syst(t,W):
```

```
        w1,w2,w3=W
        f,fp,h,hp=sol_geom.sol(t)
        fpp=EinF(f,fp,h,hp)
        tf,th=fp/f,hp/h
        Af=fpp/f-tf**2
        tH=t*(k*tf+l*th)
        B1=(t**2)*(2*Lam-2*(k-1)/f**2+2*k*tf**2-2*k*Af)
        B2=(t**2)*(2*l*tf*th-2*l*Af)
        tpsip=t_psip(t,w1,w2,w3)
        return (w2/t,((B1-mu*t**2)*w1+(1-tH)*w2+B2*w3)/t,tpsip/t)
    return solve_ivp(syst,(T_init, tstar),W0,method="DOP853",rtol=Rtol_T,atol=Atol_T,dense_output=True)

def eval_t_star(mu):
    Y0,Y1,Y2 = integrate(mu).y[:, -1]
    return Y0,Y1/tstar,Y2,t_psip(tstar,Y0,Y1,Y2)/tstar

def F_e(mu):
    return eval_t_star(mu)[3]

def F_o(mu):
    phi,_,psi,_=eval_t_star(mu)
    return k*(H2-Lam+mu)*phi-l*(F2-Lam+mu)*psi

# ========== Root finding ==========

mus=np.linspace(mu_min,mu_max,N_scan)
even_vals=np.array([F_e(m) for m in mus])
odd_vals=np.array([F_o(m) for m in mus])

def search_roots(fun, vals):
    ind=[]
    for i in range(len(vals)-1):
        if vals[i]*vals[i+1]<0:
            ind.append(i)
    list_roots=[]
    for i in ind:
        list_roots.append(dicho(fun,mus[i],mus[i+1]))
    return list_roots

roots_even=search_roots(F_e,even_vals)
roots_odd=search_roots(F_o,odd_vals)

# ========== Plots ==========

def eigentensor(mu,n=1500):
    sol=integrate(mu)
    tt=np.linspace(T_init,tstar,n)
    Y=sol.sol(tt)
    phi,psi=Y[0],Y[2]
    a_=-k*phi-l*psi
    return tt,phi,psi,a_

def plot_full(mu, parity, ax):
    tt,phi,psi,a_=eigentensor(mu)
    if parity=="even":
        s=1
    else:
        s=-1
    tt_full=np.concatenate([tt,2*tstar-tt[::-1]]) # mirror the grid
    phi_full=np.concatenate([phi,s*phi[::-1]])   # We extend the eigentensor to the entire manifold
    psi_full=np.concatenate([psi, s*psi[::-1]])
    a_full=np.concatenate([a_,  s*a_[::-1]])
```

```
    ax.plot(tt_full,phi_full,label=r"$\phi$")
    ax.plot(tt_full,psi_full,label=r"$\psi$")
    ax.plot(tt_full,a_full,label=r"$a$")
    ax.axvline(tstar,color="k",lw=0.5,ls=":")
    ax.legend();ax.set_xlabel("t")

fig, axes = plt.subplots(1,2,figsize=(16, 4.5))

axes[0].plot(mus,even_vals,".-",lw=0.8,ms=2,label=r"$F_e$ (even)")
axes[0].plot(mus,odd_vals,".-",lw=0.8,ms=2,label=r"$F_o$ (odd)")
axes[0].axhline(0,color="k",lw=0.5)
for r in roots_even:
    axes[0].axvline(r,color="r",ls="--",lw=0.8)
for r in roots_odd:
    axes[0].axvline(r,color="g",ls="--",lw=0.8)
axes[0].set_yscale("symlog", linthresh=1e-3)
axes[0].set_xlabel(r"$\mu$"); axes[0].legend()
axes[0].set_title("Plot of $F_e$ and $F_o$")

mu_show, parity = min([(r,'even') for r in roots_even]+[(r,'odd') for r in roots_odd])

plot_full(mu_show, parity, axes[1])
axes[1].set_title(rf"Eigentensor ({parity}), $\mu = {mu_show:.6f}$")

fig.suptitle(r"Second Böhm metric $g_2$ for $k=2$ and $l=2$", fontsize=16)
fig.tight_layout()
plt.show()
```