Classification of the behaviour of embedded curves on smooth orientable surfaces under Curve Shortening Flow

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Abstract

Building upon Grayson's theorem, this paper answers the question: *on a smooth orientable surface, which embedded curves converge to a geodesic and which shrink to a point*? The paper, first, asks this question on surfaces with the metric induced from an embedding in Euclidean space. We obtain a classification for, beyond the well known example of the round sphere, the rotational torus, surfaces with negative Gaussian curvature everywhere, and surfaces that satisfy the condition of Fiala-Huber's isoperimetric inequality. Looking at surfaces with a Riemannian metric in which they have constant Gaussian curvature, justified by Uniformisation theorem, the paper obtains a classification for embedded curves under the flow. We also obtain a classification of the embedded curves that converge to the systoles of a flat torus and hyperbolic surfaces.

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Analysis of the behaviour of curves under Curve Shortening Flow is a field that is not traditionally linked to topological, geometrical ideas such as the Teichmüller space. The paper connects arguments of Geometry, Topology, and Analysis to answer questions where the answer to them are not explicitly written elsewhere. Independent work done in the paper are written below in chronological order with the essential results being highlighted.

- Section 2.3; Calculations in the Gauss-Bonnet argument and Classification of the embedded curves on a torus under Curve Shortening Flow
- **Example in Section 2.3: a geodesic of a surface homeomorphic to a torus that bounds a disc**
- Classification in Section 2.5.2: the $\int_\Omega \kappa^+ dA < 2\pi$ case[1]
- Classification in Section 3.1: the negative curvature case
- Section 3.2; Theorem 12 and 13: Classification of the embedded curves that converge to the systoles of a flat torus under Curve Shortening Flow
- **Section 3.3.2; Theorem 14: Classification of the embedded curves that converge to the systoles of hyperbolic surfaces under Curve Shortening Flow**

[1]This condition is enforced by Fiala-Huber in [1]. Curve Shortening Flow on surfaces in 2.5.2 and 3.1, albeit being simple arguments, is not explicitly dealt elsewhere.

1 Introduction

We introduce some concepts to motivate the classification problem of the behaviour of embedded curves under Curve Shortening Flow.

Definition 1.1 *Let M and N be smooth manifolds. A smooth function $f : M \to N$ is an* ***embedding*** *if M is diffeomorphic to its image f(M) in N.* ***Embedded curves*** *on a smooth manifold M are curves $\gamma : S^1 \to M$, S^1 being a circle, where the map from S^1 to M is an embedding. A curve $\gamma : S^1 \to M$ is* ***immersed*** *if it is a smooth map from S^1 to M that has non-zero derivatives everywhere.*

From the definition, we can note that all embeddings are immersions. Yet, not all immersions are embeddings. The figure ∞ would be an example of an immersion that is not an embedding. It's not injective; hence, can't be a diffeomorphism.

Definition 1.2 *For a vector v starting at point p of a manifold M, there exists a unique geodesic $\gamma_v(t)$ where $\gamma_v(0) = p$ with its tangent vector at p being v. Let's define an exponential map*[2] $\exp_p(v) = \gamma_v(1)$. *A* ***Complete*** *Riemannian manifold is manifold M in which the exponential map at p $\exp_p : S \subset T_pM \to M$ is defined on the entire tangent space T_pM at any p of M. Hence, there can't be punctures or boundaries.*

We will now define what it means to apply the Curve Shortening Flow to a closed immersed curve on a smooth complete orientable 2-manifold M. The Completeness of the manifold is enforced to avoid issues where the flow hits the boundary of the manifold. The orientability is necessary to define the normal vector of M; normal direction and tangent direction of curve γ.

For a point p on an orientable surface, there are two candidates of choosing the normal vector of the surface M. Let's denote them as $\mathbf{N}$ and $-\mathbf{N}$. Choose an orientation on S^1 for $\gamma : S^1 \to M$, where $\gamma(s) = p$ for a $s \in S^1$, to define the normal direction and tangent direction of γ at p. The unit vector in the normal direction of γ is denoted as $\mathbf{n}$. This allows us to define the Geodesic curvature κ_g to apply the Curve Shortening Flow. Note that reversing the signs of $\mathbf{N}$ would reverse those of $\mathbf{n}$ and κ_g. Thus, it would not change the value $\kappa_g \cdot \mathbf{n}$. This motivates the following definition.

[2] An exponential map is a map from the subset of a tangent space T_pM at p of M to M. Definition 1.2 is a summary of key concepts in [2].

Definition 1.3 *For a 2-dimensional Smooth Complete Orientable Manifold, denote the initial closed immersed curve as $\gamma_0(u) : S^1 \to M$ and let $T > 0$. We say that a family of curves $\gamma(u,t) : S^1 \times [0,T) \to M$* ***evolves under Curve Shortening Flow*** *if it satisfies the following initial value problem.*

$$\begin{aligned} \frac{\partial \gamma}{\partial t} &= \kappa_g \cdot \boldsymbol{n} \\ \gamma(u,0) &= \gamma_0(u) \end{aligned} \tag{1}$$

By definition, we can note that for geodesics, κ_g would be 0 and hence under Curve Shortening Flow, geodesics are stationary.

As M is an orientable manifold, we can define a vector starting from a point p on a curve $\partial\Omega$ to be *pointing inwards* when parallel to $\mathbf{n}$. The region enclosed by the curve can be denoted as Ω. The equation for $\frac{dA}{dt}$, of area A enclosed by an embedded curve, is given below [3].

$$\frac{dA}{dt} = -\int_{\partial\Omega} \kappa_g ds \tag{2}$$

It's also worth noting that the arc-length L of an embedded curve is monotone under Curve Shortening Flow [3].

$$\frac{dL}{dt} = -\int_{\partial\Omega} \kappa_g^2 ds \tag{3}$$

Definition 1.4 *An* ***Isotopy*** *is a Homotopy H: M $\times$ [0,1] $\to$ N from one embedding of M in N to another such that it is an embedding for all t $\in$ [0,1]. If two embedded curves γ_1 and γ_2 are* ***isotopic*** *to each other, there exists a homotopy H: S^1 $\times$ [0,1] $\to$ M with H(0) $= \gamma_1$ and H(1) $= \gamma_2$ such that H(t) for all t $\in$ [0,1] is an embedded curve.*

All results we have for homotopy classes, we can have for isotopy classes when dealing with embedded curves evolved under Curve Shortening Flow because of the following result.

Theorem 1 *Embedded curves remain embedded under Curve Shortening Flow.*

For the full proof of this theorem, we refer the reader to [3] and [4]. Here, we would sketch the proof for the case of $\mathbb{R}^2$.

- As S^1 is compact, it suffices to prove that $\gamma : S^1 \to M$ is injective at an arbitrary time t to show that it's an embedding[3]. Let's define the extrinsic distance function $d : S^1 \times S^1 \times [0, t_0] \to \mathbb{R}$ for γ as

$$d(x, y, t) = |\gamma(y, t) - \gamma(x, t)|$$

- Note that d is zero on the diagonal subset $\{(x, x) : x \in S^1\}$. The curvature is bounded by compactness. Say the upper bound is K. We will obtain a lower bound of d on a neighbourhood of this diagonal subset in terms of K. (Theorem 2)

- We apply the maximum principle [5] to d on $\{(x, y, t) \in S^1 \times S^1 \times [0, t_0] : l(x, y, t) \geq \frac{\pi}{K}\}$, where $l(x, y, t)$ is the arc-length between x and y at time t, to yield a positive lower bound for d for all $y \neq x$ and t.

Theorem 2 *For an immersed curve $\gamma : S^1 \to \mathbb{R}^2$ with $\kappa_g(x) \leq K$ for all $x \in S^1$,*

$$d(x, y) \geq \frac{2}{K} \cdot \sin(\frac{K \cdot l(x, y)}{2})$$

for all (x,y) such that arc-length $l(x, y) \leq \frac{\pi}{K}$.

For the proof of Theorem 2, we refer the reader to [3]. Theorem 1 holds for embeddings on orientable surfaces as well. The proof is done by the same idea as the $\mathbb{R}^2$ case. Yet, the extrinsic distance function needs to be bounded in a more sophisticated way, which was presented by Gage in [4].

We now state Gage,Hamilton, and Grayson's theorem for the $\mathbb{R}^2$ case.

Theorem 3 *Let the initial curve $\gamma_0 : S^1 \to \mathbb{R}^2$ be embedded. Then the Curve Shortening Flow has a solution up to some maximal time T. The curve γ_t is smooth for all $t \in [0, T)$ and shrinks to a round point [4] as $t \to T$.*

We sketch a proof of the theorem done by Ben Andrews. The key of the proof is obtaining a curvature bound of the embedding that holds as long as the length is positive. For a rigorous statement of the curvature bound, we refer the reader to [3].

[3] An embedding is a proper injective immersion. Let γ, a closed immersed curve, be injective. As S^1 is compact, its image $\gamma(S^1)$ is compact. As M is Hausdorff by definition, $\gamma(S^1)$ is closed. For a compact subset C of $\gamma(S^1)$, it is closed in M and hence its pre-image is also closed. A closed subset of a compact set is compact. Hence, γ is proper.

[4] The embedding converges to a circle as it shrinks to a point.

- The existence of solutions for the Curve Shortening Flow for all time $t \in [0, T)$ follows from the curvature bound described above. In the process, we obtain a bound for all the spatial derivatives of κ_g as well.
- We first prove that (Theorem 2.12 in [3])

$$\lim_{t \to T} \sup \quad \sup_{S^1 \times t} \{|\kappa_g(s,t)| : s \in S^1\} = \infty \tag{4}$$

  Note that L is monotone (3). By observing that κ_g^2 remains bounded if we have a non-zero lower bound for L, we get a contradiction. Hence, L tends to zero. Embedded curves shrink to a point as $t \to T$.
- We obtain estimates of the total arc-length $L(t)$ using the curvature bound to obtain that the isoperimetric ratio: $\frac{L^2}{4\pi A}$, A being the enclosed area, converges to 1 as $t \to T$. The isoperimetric ratio is 1 for a circle.

Theorem 4 ***Grayson's theorem*** *Let (M,g) be a smooth, orientable, and complete surface that is* ***convex at infinity*** [5]. *For the initial embedded curve $\gamma_0 : S^1 \to M$, there exists a solution of the Curve Shortening Flow such that it exists for all $t \in [0, T)$ for $T \in \mathbb{R}^+ \cup \infty$ and the embedding either shrinks to a point in finite time or converges to a geodesic, as $t \to \infty$.*

For the infinite T case, we first obtain the result that the total length of the curve converges to a certain value: $L_\infty > 0$ as $t \to \infty$ and then prove that κ_g converges to 0 in the C^∞ norm. For the proof, we refer the reader to [6].

The paper would be exclusively dealing with smooth, orientable, closed, and connected manifolds[6]. From section 2 and onwards, we would use the terminology *surfaces* to represent closed and connected 2-manifolds. This terminology is not conventional, so we emphasize this to the reader. Grayson's theorem, albeit being a powerful theorem, does not help us determine whether an embedding would converge to a geodesic or shrink to a point. Classification of the embedded curves that converge to a geodesic and the ones that shrink to a point would be a natural goal. Among the surfaces in which we

[5] M is *convex at infinity* when the intersection of all convex subsets containing an arbitrary compact subset of M is compact.

[6] Grayson, in [6], gives an example of a torus with complete hyperbolic structure that is not convex at infinity where Grayson's theorem does not hold. Closed manifolds are complete and convex at infinity.

would be able to do so, we would try to obtain a classification of the embedded curves that converge to the shortest closed geodesics. We introduce theorems that help us answer these questions.

1.1 Theorems in Topology and Geometry

Theorem 5 *Euler Characteristics $\chi(\Omega)$ are homotopy invariants [7]*

Hence, Euler Characteristics are isotopy invariants.

Theorem 6 ***Uniformisation theorem for Riemann Surfaces*** *Every Riemann surface (X,J), X being the topological space and J the holomorphic atlas, has a Riemannian metric g compatible with its conformal structure in which X has constant Gaussian Curvature of -1 or 0 or 1 [8].*

The Uniformisation theorem for Riemann surfaces also hold for smooth closed orientable surfaces. The following theorem justifies this.

Theorem 7 *The Riemannian metric of every smooth closed Riemannian surface induces a complex structure [9]*

Let's apply the Gauss-Bonnet theorem to Theorem 6. From Theorem 7 we can deduce the following.

Theorem 8 *Let M be a smooth closed orientable surface.*

- *If $\chi(M) > 0$, there exists a Riemannian metric such that M has constant Gaussian Curvature 1.*
- *If $\chi(M) = 0$, there exists a Riemannian metric such that M has constant Gaussian Curvature 0.*
- *If $\chi(M) < 0$, there exists a Riemannian metric such that M has constant Gaussian Curvature -1.*

Finally for hyperbolic surfaces[7], we have the following theorem that is used to obtain our classification of embedded curves that converge to the systoles of hyperbolic surfaces.

Theorem 9 *Let M be a hyperbolic surface. If Γ is a closed curve in M that is not homotopic into a neighbourhood of a puncture, then Γ is homotopic to a unique closed geodesic [10].*

[7] Hyperbolic surfaces are surfaces with constant negative curvature.

1.2 Teichmüller space

Definition 1.5 *Fix a smooth closed orientable surface S of genus $g \geq 2$. We define a* ***marked hyperbolic surface*** *(X, f) to be a hyperbolic surface X together with a diffeomorphism $f : S \to X$. Such a surface X exists by the Uniformisation theorem. Two marked hyperbolic surfaces, (X, f) and (Y, g), are equivalent if $gf^{-1} : X \to Y$ is homotopic to an isometry. A* ***Teichmüller space*** *of S, denoted as $\tau(S)$, is defined as the set of equivalence classes $[(X, f)]$ of marked hyperbolic surfaces*[8]*.*

Note that we don't have to consider the neighbourhood of a puncture case in Theorem 9 with the compactness condition enforced in Definition 1.5. We will define the following terms: pants decomposition, length parameters, and twist parameters.

Definition 1.6 *A pair of pants is a compact surface of genus 0 with three boundary-components. A* ***pants decomposition*** *P of S is a collection of disjoint simple closed curves in S such that cutting along the curves split S into a disjoint union of pairs of pants.*

Definition 1.7 *Define S likewise to Theorem 1.5. X would be a hyperbolic surface. The dimension of its Teichmüller space is 6g-6 [10]. Take 3g-3 simple closed curves $\gamma_1, \ldots, \gamma_{3g-3}$ to be its pants decomposition P and 3g-3 curves $\beta_1, \ldots, \beta_{3g-3}$ that cross two curves in the same pair of pants pairwise. For $\chi = [(X, f)] \in \tau(S)$ and a curve $\gamma_i \in P$ of S, the* ***length parameter*** *of χ: $l_i(\chi)$ is the length $l(\gamma_i)$ of the unique geodesic Γ_i, by Theorem 9, homotopic to $f(\gamma_i)$.*

For γ_i and γ_j of the same pair of pants in P, there would exist a β_k intersecting them. For β_k and β_l crossing γ_i, $f(\beta_k)$ and $f(\beta_l)$ are homotopic to some b_k and b_l which meet with Γ_i at two points i_1 and i_2. By giving an orientation to Γ_i, we can denote the signed distance along Γ_i from i_1 to i_2 as $t_L - t_R$. The ***twist parameters*** *$\theta_i(\chi)$ of $\chi \in \tau(S)$ are defined as $2\pi \frac{t_L - t_R}{l(\gamma_i)}$.*

Definition 1.8 *For S with genus $g \geq 2$, the* ***Fenchel-Nielsen coordinates*** *of $\chi \in \tau(S)$ are defined as the numbers $(l_1, \theta_1, l_2, \theta_2, \ldots, l_{3g-3}, \theta_{3g-3})$*

[8]Theorem 8 allows us to rephrase this for S being a torus and X being a flat torus. The equivalence between Definition 1.5 and the set of isotopy classes of hyperbolic metrics on S for $\chi(S) < 0$ is shown in [10].

2 Classification of embedded curves under Curve Shortening Flow on surfaces defined with a Euclidean Metric

After establishing Grayson's theorem of embedded curves on smooth orientable surfaces, a natural question arises: how would you classify the embedded curves that converge to a geodesic and the ones that shrink to a point? In this paper, we would tackle this question case by case for the surface the curves are embedded in. Section 2 would exclusively deal with surfaces with the metric induced from their embedding in Euclidean space. We will use the term *Euclidean metrics* to represent these metrics.

2.1 Classification of embedded curves on a round sphere

On a round sphere, by Jordan curve theorem, an embedded curve on the surface splits the sphere into two faces. Both faces are homeomorphic to discs. The Gauss-Bonnet formula[9] reads as the following.

$$\int_{\Omega} \kappa dA + \int_{\partial\Omega} \kappa_g ds \;=\; 2\pi\chi(\Omega)$$

where on top of the variables in equation (2) and (3), κ is the Gaussian Curvature and $\chi(\Omega)$ is the Euler characteristic of Ω. Here, the region enclosed by a curve on the sphere would be homeomorphic to a disc; thus, $\chi(\Omega)$ is 1. Note that as derived in equation (2),

$$\frac{dA}{dt} = -\int_{\partial\Omega} \kappa_g ds$$

Plugging (2) into Gauss-Bonnet, we obtain

$$\frac{dA}{dt} = \int_{\Omega} \kappa dA - 2\pi \tag{5}$$

Note that κ is 1 for a unit sphere, and; thus, the equation reads as

$$\frac{dA}{dt} = A(t) - 2\pi$$

[9]Remind ourselves that compactness of Ω is required to apply the theorem. The compactness condition in the term *surfaces* allows this.

Solving this differential equation, we obtain that $A(t)$ is $ce^t + 2\pi$ where the constant c is forced to be $A(0) - 2\pi$ by the initial conditions. Thus, if $A(0) < 2\pi$, the curve shrinks to a point when $t = \log(2\pi) - \log(2\pi - A(0))$.

When the enclosed area defined, likewise to the introduction, under the conventional choice of normal vector $\mathbf{N}$ is larger than 2π, $\frac{dA}{dt}$ obtains a positive value for all t. This insists that the area we would like to observe the outcome of is actually the complement. We can resolve this issue by taking $-\mathbf{N}$ to reverse the normal direction $\mathbf{n}$ such that the enclosed area would be defined as the area smaller than 2π. Thus, the curve would shrink to a point. For $A(0) = 2\pi$, $A(t)$ is constantly 2π; thus, the curve would exist for an infinite time. Grayson's theorem implies that such curves would converge to a geodesic.

2.2 Applying the Gauss-Bonnet Argument for Curve Shortening Flow

In Theorem 5, we have introduced that the Euler Characteristic is an isotopy invariant. Thus, if the Euler characteristic of the two regions (or the whole surface if the curve is not separating[10]) are not 1, we could conclude that this curve would not shrink to a point, which has Euler Characteristic 1. We denote these cases as when *embedded curves do not bound discs*. By Grayson, such embedded curves would converge to geodesics. In this section, we cautiously define enclosed areas to apply the Gauss-Bonnet Argument for Curve Shortening Flow.

Prior to the construction, let's remind ourselves that orientable surfaces that are not homeomorphic to spheres can not have an embedded curve separating them into two (regions homeomorphic to) discs. Thus, when dealing with surfaces that are not homeomorphic to spheres, which would be the case for the rest of the section, there would be only one surface that would be homeomorphic to a disc if existent.

The isotopy argument above implies that cases when the curve bounds

[10]It is conventional to define curves to be separating when the curve splits the surface into two components.

regions with Euler Characteristic 1 would be the only ones where the curve can shrink to a point. Thus, it is quite natural to set the enclosed area as the area bounded by the disc. We would use the term: *an embedded curve that bounds a disc* regularly to describe this case in the paper. We formalise this similarly to the $A(0) > 2\pi$ case for the round sphere.

For the case when the enclosed area defined with $\mathbf{N}$[11] is the region homeomorphic to a disc, we are done. Say, however, that the enclosed area defined like above is the complement of the region homeomorphic to a disc. We can take $-\mathbf{N}$ so that $\mathbf{n}$ of $\partial\Omega$ would now be pointing towards the interior of the region homeomorphic to a disc. We saw that this would not change the value of $\kappa_g \cdot \mathbf{n}$. From now on, we will think of doing this process automatically when necessary and just define the enclosed region to be the region homeomorphic to a disc.

Before we introduce some ideas using the Gauss-Bonnet formula, we would like to note that the enclosed area defined as above might not necessary be the region shrinking to a point or converging to a geodesic. It could be the complement.

Following the same procedure as the sphere case, we can insert equation

$$\frac{dA}{dt} = -\int_{\partial\Omega} \kappa_g ds$$

into the Gauss-Bonnet formula to obtain

$$\frac{dA}{dt} = \int_{\Omega} \kappa dA - 2\pi$$

For the surfaces where the right hand side of the equality is smaller than 0 for all embedded curves, we can conclude that such a surface does not have a geodesic bounding a region homeomorphic to a disc (Geodesics are stationary). Thus, by Grayson, all curves that bound a region homeomorphic to a disc would shrink to a point.

Say that $\frac{dA}{dt}$ is larger or equal to 0 initially, looking at a surface not homeomorphic to a sphere. If $\frac{dA}{dt}$ remain to be of non-negative value for all

[11] We remind the reader of the construction in page 4 of the Introduction.

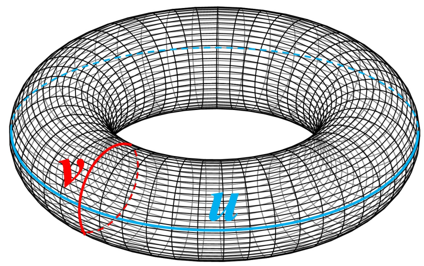

Figure 1: Torus

time t, this would imply that the complement of the enclosed area would be the one shrinking to a point or converging to a geodesic. As this region does not have Euler Characteristic 1, by the isotopy argument, this would indicate that the embedded curve would converge to a geodesic. Yet, this information is not given to us necessarily when given an initial curve[12]. As far as we know, although the initial curve increases in area, $\frac{dA}{dt}$ might be negative as the curve evolves under Curve Shortening Flow.

2.3 Classification of embedded curves on a rotational torus

In this section, the surface of interest would be a rotational torus defined under a Euclidean Metric. We would parameterise the torus using the following terms: c would be the major radius; a the minor radius; and u,v angles defined as in Figure 1. We would be only looking at the conventional ring torus, where c is larger than a. Let v be from $-\pi$ to π and u from 0 to 2π. The parameterisation reads as

$$x = (c + acosv)cosu$$

$$y = (c + acosv)sinu$$

$$z = asinv$$

Let's first apply the isotopy argument to the torus case. For embedded curves that do not bound regions homeomorphic to discs, as the Euler Characteristic of the bounded region is not 1, they can't shrink to a point. By

[12]unless we know that the area is proportional to the integral of the Gaussian curvature over the area such as the sphere case

Grayson, this implies that they converge to a geodesic. For embedded curves that bound a region homeomorphic to a disc, we can first attempt to create an analogous argument to the sphere case.

Calculating the first fundamental form and second fundamental form of the torus, we obtain the Gaussian curvature and $|\partial\gamma/\partial u \times \partial\gamma/\partial v|$, where the curve is represented as γ. They are precisely $cos(v)/a(c + acos(v))$ and $a(c + acos(v))$, resulting the integral $\int_\Omega \kappa dA$ to read as

$$\int\int_\Gamma cos(v)/a(c + acos(v)) \cdot a(c + acos(v))dudv$$

where Γ is the enclosed area Ω re-parameterised in terms of u, v. Returning to the form

$$\frac{dA}{dt} = \int_\Omega \kappa dA - 2\pi\chi(\Omega),$$

the equation would read as

$$\frac{dA}{dt} = \int\int_\Gamma cos(v)dudv - 2\pi \tag{6}$$

since the Euler Characteristic of the region enclosed by the curve is 1. Note that

$$\int\int_\Gamma cos(v)dudv < \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} cos(v)dudv = 4\pi$$

This inequality holds as the right hand side is when the integral is maximum. The strictness follows from the fact that a region with the integral equal to the right hand side exists uniquely on the surface of a torus and is not bounded by an embedding. We can justify this. The region of u varying from 0 to 2π and v from $-\pi/2$ to $\pi/2$ in Figure 1 is not bounded by an embedding. The complement of this region has negative Gaussian curvature at every point and every point in the interior of the region has positive Gaussian curvature. Hence, the integral of the Gaussian curvature of this region is maximum and unique.

An area enclosed by an embedding where the integral of the Gaussian curvature is larger than 2π would increase initially. Thus, an analytic argument using the Gauss-Bonnet formula would not give a complete classification for embedded curves under Curve Shortening Flow on a rotational torus. Yet,

Table 1: Geodesic characterisation, drawn likewise to Barret O'Neill [11]

$\|h\|$	**Geodesics**
0	Meridians
$0 < \|h\| < c-a$	Alternately cross both equators ("unbounded" geodesics)
$c-a$	The inner equator, and geodesics asymptotic to it
$c-a < \|h\| < c+a$	Cross outer equator but not inner equator ("bound" geodesics)
$c+a$	The outer equator
$\|h\| > c+a$	There are no real solutions to the geodesic equation

for the case of a rotational torus, the geodesics are characterised. The idea used to characterise geodesics is using Clairaut's Parameterisation with the Geodesic equation. Geodesics of $X(u, v)$ depend on one parameter: the geodesic slant, which is $(c+acos(v))sin(\phi)$, where ϕ is the angle between the geodesic and X_u [11]. Clairaut's relation implies that the geodesic slant is constant along a geodesic. The characterisation of the geodesics of the torus is given in table 1.

Note that we are looking at non self-intersecting geodesics only, as we are looking at embedded curves and non self-intersecting curves can not self intersect (Theorem 1), evolving under Curve Shortening Flow. We can see that the geodesics can not bound a disc, as none of the geodesics are separating. For a visualisation of the geodesics, we would refer the reader to [11].

As geodesics of the rotational torus do not bound a disc, by an isotopy argument, we can conclude that embedded curves on a torus that bound a disc can not converge to a geodesic. By Grayson, this would imply that such curves shrink to a point.

Theorem 10 *If an embedded curve on a rotational torus bounds a disc, it would shrink to a point and otherwise, it would converge to a geodesic.*

A question arises from the observation made above: would the topological property that a geodesic does not bound a disc hold for a surface homeomor-

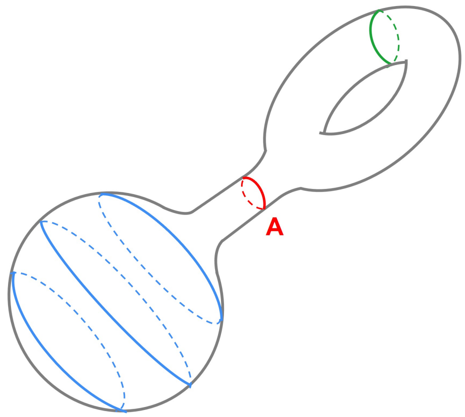

Figure 2: Geodesic bounding a disc of a surface homeomorphic to a torus

phic to a rotational torus i.e, a general torus? If we could show that this is the case, this could be a suitable approach to classify embedded curves on more general surfaces in the later sections. However, this is not the case. The red curve A of Figure 2, constructed so that the derivative of the tangent vector at points of A are all parallel to the normal vectors of the surface[13], bounds a disc.

Let's bring attention to the fact that two topologically equivalent surfaces did not agree on a topological property of the geodesics: whether there exists a geodesic that bounds a disc. This implies that in regards of determining whether a geodesic bounds a disc, using a topological argument of the surface would not hold. [**Remark 2.1**]

2.4 Embedded curves on a g-genus torus

In this section, the object of interest would be the g-genus torus M_g obtained by gluing g tori when each torus is parameterised likewise to the section before. The smoothness would imply that we would have to *smoothen out*, alter the Gaussian curvature of points on the torus, such that the gluing boundary would be smooth everywhere. By Gauss-Bonnet, $\int_{M_g} \kappa dA = 2\pi\chi(M_g)$, we know that the integral of the curvature over the surface is of negative value for a g-genus torus. Thus, we can note that in the smoothen-

[13]This is a geodesic [11].

ing out process, we would normally introduce negative Gaussian Curvature. How we define this smoothening out procedure would alter the geodesics of the g-genus torus.

Let's say for now that we specify a well-defined smoothening out procedure and; hence, get a g-genus torus with parameterisation Γ. Would there be cases where we get a parameterisation in terms of angles u, v? If so, likewise to the torus case, we would get a explicit expression for the Gaussian Curvature and perhaps be able to characterise the geodesics of the g-genus torus. Yet, such a parameterisation does not exist. Let's prove this rigorously.

Theorem 7 states that Riemannian metrics of smooth orientable surfaces induce complex structures. This allows us to interpret the g-genus torus with the *Euclidean metric* as a Riemann surface with its induced complex structure. We can view parameterisations in terms of u, v as maps defined on the flat torus, viewed as the quotient space: $\mathbb{R}^2/(2\pi\mathbb{Z})^2$. The Riemann-Hurwitz Formula reads as

$$\chi_X = \chi_Y \cdot d - \sum_{i=1}^{n}(e_i - 1)$$

[14] for a meromorphic map $f : X \to Y$ in which this case, X is the torus. This implies that $f(T)$, the image of the torus under the map f, can not have an Euler Characteristic of negative value.

Hence, the Gauss-Bonnet argument or geometric arguments using an explicit form of the Gaussian Curvature κ would not be applicable to the g-genus torus case. Yet, an isotopy argument would still stand. Thus, for all embedded curves that do not bound a region homeomorphic to a disc, the curves would converge to a geodesic under the Curve Shortening Flow. For embedded curves that bound a disc, for now, the observations above imply that when defined under a Euclidean Metric, there would not exist direct applications of the methods used for the round sphere or the rotational torus. This motivates viewing the g-genus torus in a different Riemannian metric so that we can get an explicit parameterisation for κ.

[14] χ is the Euler Characteristic, d the degree of the meromorphic map, and e_i the ramification index. n indicates the number of ramification points.

2.5 Surfaces under a Euclidean Metric with full classifications

Likewise to the g-genus torus case, the isotopy argument using the fact that the Euler Characteristic is a homotopy invariant can be applied to a general surface. When an embedding does not bound a region homeomorphic to a disc, the area would not be able to shrink to a point which has Euler Characteristic 1. By Grayson's theorem, this implies that such curves converge to a geodesic of the surface.

In this section, the paper would illustrate some surfaces on which we can make interesting observations for when $\chi(\Omega)$ is 1.

2.5.1 Surfaces with negative Gaussian curvature everywhere

Albeit not existing as a surface embedded in R^3, surfaces with negative Gaussian curvature on all of its points can be embedded in higher dimensions [12]. We can obtain a classification for a family of surfaces that include surfaces with negative Gaussian curvature everywhere.

2.5.2 Surfaces that satisfy $\int_\Omega \kappa^+ dA < 2\pi$

One classification of surfaces that we could classify would be the ones that we can use the isoperimetric inequality on given by Fiala-Huber [1].

$$L^2 \geq 2A(2\pi - \int_\Omega \kappa dA)$$

In [1], the condition for the inequality is given as

$$\int_\Omega \kappa^+ dA < 2\pi$$

where κ^+ is defined as the Gaussian curvature κ when it is positive and 0 when negative. As we are integrating over compact sets, for the area A of an arbitrary enclosed region homeomorphic to a disc, there would exist a real number ϵ such that $\frac{dA}{dt} < -\epsilon$. Geodesics are stationary i.e., $\frac{dA}{dt} = 0$. Hence, this implies that a geodesic bounding a disc would not exist. (This argument was introduced in section 2.2.) Thus, by Grayson, embedded curves bounding a disc would shrink to a point.

3 Classification of embedded curves that converge to Systoles under Curve Shortening Flow

The Uniformisation theorem states that there are Riemannian metrics under which a surface M has constant Gaussian Curvature. Dealt in section 3.1, having constant curvature would give us a complete classification of the outcome of embedded curves on orientable surfaces defined under these Riemannian metrics. Yet, even in this abstract setting, the behaviour of the embedded curves such as which specific limit they would converge to if converging to a geodesic remains not answered.

Systoles of a surface are the shortest closed geodesics of the surface. Extending from the complete classification of outcomes (whether the embedding would shrink to a point or converge to a geodesic) of embedded curves in section 3.1, we attempt to classify curves that would converge to these geometrically significant objects. The isotopy class of embedded curves on the surfaces would be the subject of interest.

3.1 Classification of embedded curves on surfaces under a Riemannian metric in which they have constant Gaussian curvature

Applying the Uniformisation theorem, we obtain a classification for embedded curves under the Curve Shortening Flow on every oriented surface when defined under a Riemannian metric in which it has constant Gaussian curvature. Note that this is different from classifying the behaviour of embedded curves under the Curve Shortening Flow when given an arbitrary surface and an arbitrary metric.

We have established that the Uniformisation theorem can be extended to smooth orientable surfaces in Theorem 8. Deducing from Theorem 8 and section 2.2, in Figure 3, we give a complete classification.

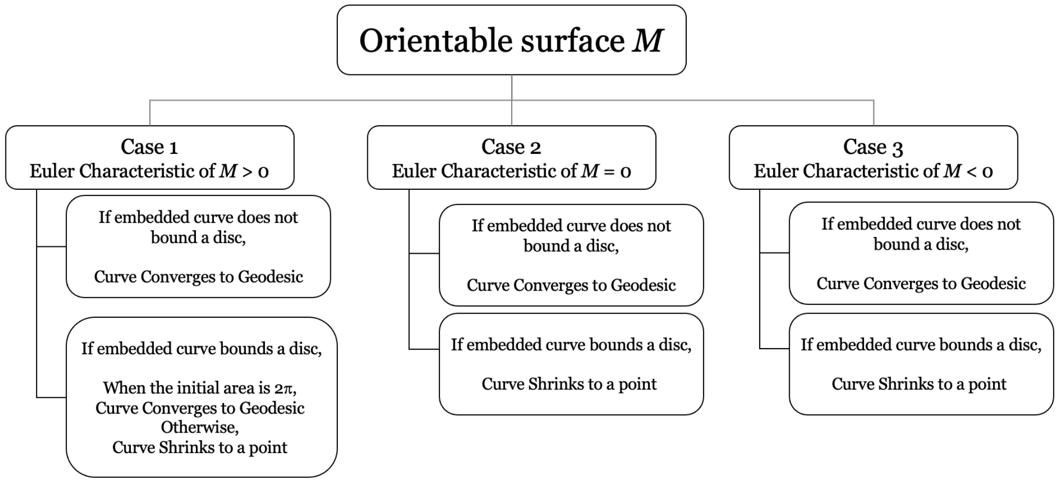

Figure 3: Classification of the behaviour of embedded curves on surfaces with constant curvature; Uniformisation Theorem

Here are some brief justifications for the result given above.

- For a surface M with positive Euler characteristic, by Uniformisation theorem, it can be given a Riemannian metric where its Gaussian Curvature is 1. We can complete the classification of embedded curves on such surfaces using an analogous argument of the Gauss-Bonnet argument on the round sphere for embedded curves bounding a disc. The criteria for the initial area $A(0) = 2\pi$ is applicable as well, as the Gauss-Bonnet formula for surface M: $\int_M \kappa dA = 2\pi\chi(M)$ suggests that the area of surface M with constant curvature 1 is 4π.

- For a surface M with zero Euler Characteristic, by Uniformisation, it can be defined with a Riemannian metric where its Gaussian curvature is 0. Thus, we can complete the classification by using the Gauss-Bonnet argument. $\frac{dA}{dt}$ would be -2π for all embedded curves bounding a disc; hence, shrink to a point.

- For a surface M with negative Euler Characteristic, by Uniformisation, there exists a Riemannian metric where its Gaussian Curvature is -1. The classification follows from the result we've got from section 2.5 for surfaces with negative Gaussian curvature everywhere.

3.2 Classification of embedded curves that converge to the Systoles of a flat torus

By Uniformisation theorem, we can note that there exists Riemannian Metrics where the torus is a flat torus. The planar model of a Torus suffices to demonstrate this: $\mathbb{R}^2/\mathbb{Z}^2$. We can further note that the Teichmüller space characterising such Riemannian metrics is the upper half plane with the group action of $\mathrm{PSL}(2,\mathbb{Z})$: the projective special linear group of degree 2 over the integers. We can briefly justify this. Each parallelogram can be rotated or translated such that one of its vertices is the origin and one of the edges containing the origin lies on the x-axis. We can correspond the antipodal point of the origin on the parallelogram to a point on the upper half plane. It is natural to think of the rotations and translations done on these tori as the action of $\mathrm{PSL}(2,\mathbb{Z})$. Thus, the result holds. For a rigorous proof and description of this result, we would refer the reader to [13]. We would try to understand how the parameters in the Teichmüller space correspond to the geometry of the flat torus and discuss their geodesics and systoles.

3.2.1 Geodesics and Systoles of a flat torus

The geodesics of a plane are lines. As we are looking for closed geodesics on a flat torus, the union of line segments we are interested in should start from a point and end at the point it's identified with on the boundary. Looking at the fundamental group of the torus would give the following description of the closed geodesics.

Theorem 11 *The fundamental group of a torus is $\mathbb{Z}\times\mathbb{Z}$. Each isotopy class of the closed embedded geodesics of a flat torus, which would be a union of line segments [15] starting from a point on the boundary and ending at its identified point on the boundary, would be represented with* ***winding numbers*** *(x,y) where x would be amount of times the geodesic wraps around the flat torus horizontally, parallel to the x-axis, and y vertically, parallel to the other pair of sides of the parallelogram[16]. Diagonal lines would have (1,1) as their winding number. Horizontal lines would have (1,0) and vertical lines (0,1). Geodesics with different winding numbers would not be isotopic to each other [7]. Figure 4 illustrates the closed geodesics.*

[15] not intersecting with each other apart from their end points

[16] With the parallelogram rotated or translated such that one vertex is on the origin and one side on the x-axis.

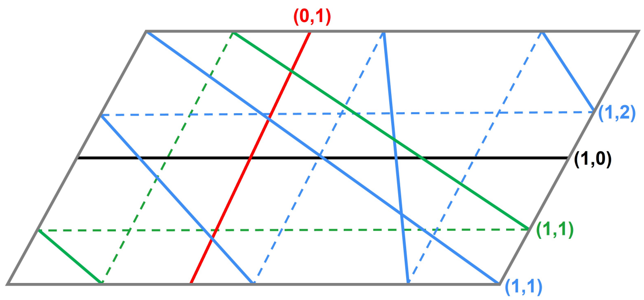

Figure 4: Closed geodesics and its winding numbers

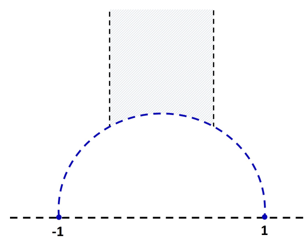

Figure 5: Fundamental domain of the Teichmüllerspace of a torus; drawn likewise to [13]

When determining which one of the closed geodesics are systoles, we would consider the variable: L^2/A, with L being the length of the curve and A being the area of the flat torus, instead of just the length L. The infinite case motivates this. When the side lengths of the parallelogram are infinite, it would not be possible to compare the lengths of the closed geodesics. By Loewner's inequality [14], we have an upper bound for our variable: $2/\sqrt{3}$. For the angle being close to 0, this variable would also be helpful as well.

Note that with the variable L^2/A, a torus with length l_1 on its side on the x-axis and l_2 on its other side with angle θ would have the exact geodesics and systoles as its scalar scaled version of length 1 on the x-axis, length $\tau = \frac{l_2}{l_1}$ on its other side and angle θ. We can think of the parameters of the Teichmüllerspace as angle θ and length τ. On this setting, let's remind ourselves what the fundamental domain of the action of $(\mathrm{PSL2}, \mathbb{Z})$ on the upper half plane looks like (Figure 5).

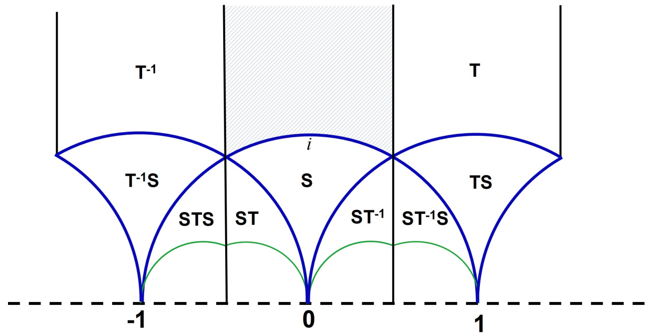

Figure 6: Drawn likewise to Jean P.Serre [13]

Theorem 12 *Calculation of the length of the line segments gives us the following classification of systoles when $\tau = 1$.*

- *For $\pi/3 < \theta < 2\pi/3$, the systoles are the horizontal lines and the vertical lines*
- *For $\theta = \pi/3$ $\theta = 2\pi/3$, the systoles are the horizontal lines, vertical lines, and (1,1) lines parallel to the shorter diagonal.*
- *For $\theta < \pi/3$ or $\theta > 2\pi/3$, the systoles are (1,1) lines parallel to the shorter diagonal.*

It is shown in [13] that for the fundamental domain of the action of $\mathrm{PSL}(2,\mathbb{Z})$ on H, on the points $\tau = 1$; $\theta = \pi/3, 2\pi/3$, the group does not act freely[17]. We can observe that this is where we have systoles with winding numbers: (1,0), (0,1), (1,1). For θ smaller than $\pi/3$ or larger than $2\pi/3$, which are outside the fundamental domain of the Teichmüller space, we can map this back to the fundamental domain by using Figure 6, where S and T are the rotation, translation matrices forming the basis of $\mathrm{PSL}(2,\mathbb{Z})$.

3.2.2 Classification of embedded curves that converge to the Systoles of a flat torus

From Section 3.1, as embedded curves in the interior of the flat torus would bound a disc, we can deduce that embedded curves that start and end

[17] A group acts freely if its stabilisers are trivial.

at points that are identified to each other on the boundary are the only ones that converge to closed geodesics.

Let's first introduce the concept of Strip-distance for embedded curves that have winding number (1,0) with the horizontal lines of the flat torus. Strip-distance for embedded curves with winding number (0,1) or (1,1) would be done analogously. Note that such curves are not isotopic to each other; hence, the notion of Strip-distance is well defined.

Definition 3.1 ***Strip-distance*** *of A from B* ***along x*** *would be defined as the sum of signed distance, as x changes its value, say from 0 to 1 in the setting from Section 3.2.1, between two embedded curves A and B where it is given a positive sign when for a certain x, A is above B and negative sign when for x, B is above A.*

We would claim using this definition that two embedded curves evolved under Curve Shortening Flow would have Strip-distance 0. A simple argument using the definition of Curve Shortening Flow and the Fundamental theorem of Calculus suffices to prove this. Note that we can interpret embedded curves on a flat torus as graphs. Say f and g. As the curves are closed, we know that these graphs are periodic. Let's integrate $f - g$ for x from 0 to 1. This would be exactly $F(1) - F(0)$, by Fundamental theorem of Calculus, for some integrand F. Yet, as these two curves are related to each other by Curve Shortening Flow, this F would be the flow between the two embedded curves. As $x = 1$ is identified to $x = 0$, and hence the same point, $F(1) - F(0)$ would be 0. Hence, the Strip-distance would be zero.

By Grayson, embedded curves not bounding discs converge to geodesics. They don't converge to geodesics that are not isotopic to them; hence, the horizontal lines are the candidates. Among these, the ones with non zero Strip-distance can not be the geodesics the embedded curves converge to from the argument above. Thus, they would have to converge to the geodesic with Strip-distance 0. We can duplicate both directions with Strip-distances defined for vertical lines and for lines with winding number (1,1).

Theorem 13 *Embedded curves on the flat torus converge to systoles under Curve Shortening Flow if and only if they do not bound discs and have Strip-distance 0 with the systoles they converge to.*

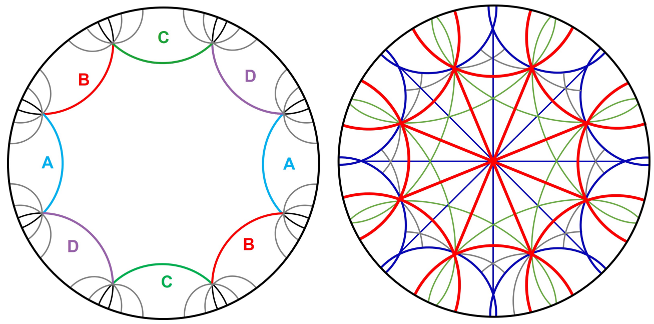

Figure 7: Hyperbolic double torus in Poincaré disc model

3.3 Classification of embedded curves that converge to the Systoles of hyperbolic surfaces

For the flat torus, we could describe it as a parallelogram with side identifications. A hyperbolic surface could be thought of as in a similar way using the Poincaré Disc Model. One way to construct a hyperbolic double torus would be an arc-length regular octagon with side identifications on the Poincaré disc. We illustrate this in Figure 7.

Geodesics of the Poincaré disc are arcs orthogonal to the boundary circle or diameters of the circle. The planar models should be *hyperbolic tilings*, i.e., the angle between the two incident arc-lengths forming the octagon should have angle 45 degrees. The second picture in Figure 7 helps us calculate the arc lengths of the hyperbolic octagon. The arc lengths of the hyperbolic octagon are the second longest length of a triangle of angles $\pi/2$,$\pi/3$, $\pi/8$. Using the $\cosh(c) = \cot(\alpha)\cot(\beta)$ relation, we get that the arc lengths of the octagon are $\cosh^{-1}(\frac{2\cos(\pi/8)}{\sqrt{3}}) = 2\cosh^{-1}(1+\sqrt{2})$.

3.3.1 The Systoles of a double torus and its Teichmüller space of marked hyperbolic surfaces

Figure 7 in the earlier section has a specific name: the Bolza surface. We will state and roughly derive the length spectrum of the Bolza surface that indicates that the regular arcs of the octagon are the systoles of this surface. For further details, we will refer to the works of Aurich [15]. Let's think about the symmetry group for the Bolza surface. Rotations in the Poincaré disc model would not be different from the canonical sense. Yet, the translation would need a new definition, as the lengths are defined differently in this model[18]. Let's denote the subgroup of the symmetry group of the Bolza surface that contains only hyperbolic translations as G. The generators of G can be derived to be in the following form [15], where k represents the kth vertex, counting from 0 to 7.

$$g_k = \begin{pmatrix} 1+\sqrt{2} & (2+\sqrt{2})(\sqrt{\sqrt{2}-1})(\exp \frac{ik\pi}{4}) \\ (2+\sqrt{2})(\sqrt{\sqrt{2}-1})(\exp \frac{-ik\pi}{4}) & 1+\sqrt{2} \end{pmatrix}$$

For all closed geodesics, we can get the expression of its corresponding hyperbolic translation as products of these generators and its inverse. We can express this as the following[19].

$$g = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}$$

Aurich, looking at closed geodesics as periodic orbits, motivates that the length of the geodesic would be two times the real part of a in the matrix above. This gives us the explicit expression of the length spectrum of the closed geodesics of the Bolza surface. They are written as $m + n\sqrt{2}$ for natural numbers m, n where n goes from 1 to all the natural numbers whilst m satisfies to be the odd number that minimises $|m - n\sqrt{2}|$ [16]. This gives us that the regular arcs are indeed the systoles and gives an explicit expression for the other lengths of closed geodesics.

A pants decomposition of the Bolza surface is given in Figure 8. Given a Fenchel–Nielsen coordinate of the Bolza surface, we can note the following about the embedded curves that converge to the systoles using equation (3). One of the most commonly used coordinates for the surface is $(l_1, t_1; l_2, t_2; l_3, t_3) = (2\cosh^{-1}(1+\sqrt{2}), 0; 2\cosh^{-1}(1+\sqrt{2}), 0; 2\cosh^{-1}(3+2\sqrt{2}), 1/2)$ [15]. We note that there are no closed geodesics of length[20] between l_1 and l_3. Using the fact that Curve Shortening Flow is length decreasing for non-geodesic curves, we can describe a family of embedded

[18]The length between two points p and q are $\ln(\frac{|aq||pb|}{|ap||qb|})$ where points a and b are defined as the two points the geodesic passing through p and q meet with the boundary circle.

[19]Here, * denotes complex conjugates. Justification can be done by noting that the symmetry group would be the pseudo unitary group: $SU(1,1)/\pm 1$ [15].

[20]The length spectrum of the Bolza surface justifies that l_3 is the second shortest length of closed geodesic.

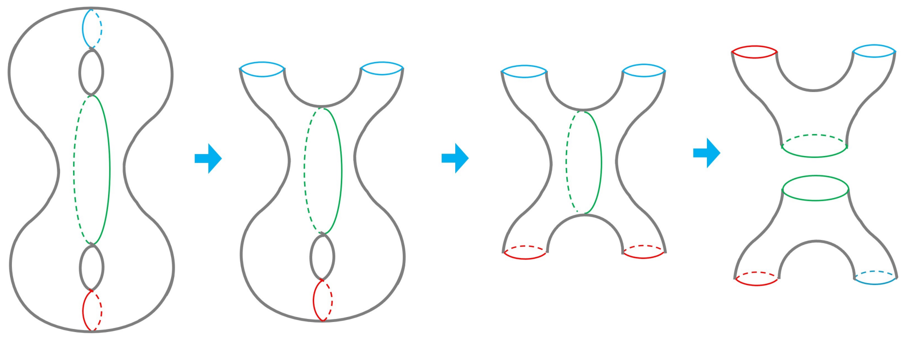

Figure 8: Pants decomposition of hyperbolic double torus, drawn likewise to [10]

curves that converge to the systoles as embedded curves that do not bound discs and have length shorter than l_3. Yet, this would not be a full classification which we will find out in the next section.

Given a hyperbolic surface, we can identify its systolic length by describing its group of hyperbolic translations as products of generators and getting its length spectrum likewise to the example of the Bolza surface. We refer the reader to [16] for calculations on higher genus surfaces as well such as the Klein Quartic.

3.3.2 Classification of the embedded curves that converge to the Systoles of hyperbolic surfaces

By Theorem 9, a geodesic of a hyperbolic surface would be the unique one in its isotopy class of embedded curves evolved under Curve Shortening Flow. Note that if the systole does not bound a disc, then the elements of its isotopy class would not be able to shrink to a point. Hence, by Grayson, they would converge to a geodesic in the isotopy class. We have that such a geodesic is uniquely the systole in this case; hence, the embedded curves of the isotopy class of the systole all converge to the systole under Curve Shortening Flow. By the isotopy argument, we can conclude that embedded curves in other isotopy classes can not converge to the systole; hence, this is a complete classification.

We can conclude that in fact, geodesics of hyperbolic surfaces can't bound discs using section 3.1. We have the result by using the Gauss-Bonnet argument, that all embedded curves that bound a disc would shrink to a point. Hence, if there exists such a geodesic, the geodesic would shrink to a point under Curve Shortening Flow. Geodesics are stationary; hence, a contradiction.

Theorem 14 *Embedded curves of hyperbolic surfaces converge to a systole if and only if they are in the isotopy class of the systole.*

4 Conclusion

In this paper, we obtained classifications of the behaviour of embedded curves under Curve Shortening Flow whilst varying the surface it was defined on. We restricted the *surfaces* of interest to smooth closed connected orientable 2-manifolds.

We first scrutinised surfaces with the metric induced from their embedding in Euclidean space. Combining the results given by Barrett O'Neill on the geodesics of a rotational torus [11] and Theorem 5 [7], we obtained the dichotomy: **if an embedded curve on a rotational torus bounds a disc, it would shrink to a point and otherwise, it would converge to a geodesic**. For the g-genus torus, using the Riemann-Hurwitz formula, we justified why a likewise geometric argument would not work. For general surfaces, embedded curves bounding discs were the curves that we could not determine the outcome of. We could, however, adapt the Gauss-Bonnet argument to obtain a classification for surfaces that satisfy $\int_\Omega \kappa^+ dA < 2\pi$. This held significance, as this is the condition for the isoperimetric inequality given by Fiala-Huber [1].

We, next, looked at the orientable surfaces under the Riemannian metrics in which they have constant Gaussian curvature. From Uniformisation theorem and the results above, we could naturally obtain a complete classification for orientable surfaces under such metrics (Figure 3, Section 3.1). Defining the notion of Strip-distance and using Theorem 11, we obtained that **embedded curves on the flat torus converge to systoles under Curve Shortening Flow if and only if they do not bound discs and have Strip-distance 0 with the systoles they converge to**. The Teichmüller space of marked hyperbolic surfaces up to a homotopy gave us the foundation to formulate arguments when dealing with hyperbolic surfaces (such as the Fenchel-Nielson coordinates). A geometric result about hyperbolic surfaces: Theorem 9 [10] combined with Theorem 5 [7] and Grayson's theorem gave us the desired classification: **embedded curves of hyperbolic surfaces converge to a systole if and only if they are in the isotopy class of the systole**. That the arc-length L of embedded curves is monotone under Curve Shortening Flow [3] combined with results from Aurich [15], gave a description of a subset of the isotopy class of the systoles: embedded curves not bounding discs with L smaller than the second shortest length of closed geodesics would be in the isotopy class of the systoles.

The isotopy class of embedded curves on the flat torus was not enough to determine which geodesic the embedding would converge to, as there were many different geodesics of the same isotopy class. When not dealing with hyperbolic surfaces, we do not have the uniqueness of geodesics in each isotopy class. For surfaces with positive constant curvature, due to this, it would be a completely different type of question to the flat torus or the hyperbolic case. Geodesics of round spheres, for instance, are not only in the same isotopy class but also have Strip-distance 0 to each other. We would need to introduce more sophisticated arguments to deal with the positive curvature case. Finding an argument that would help one determine whether a surface would contain geodesics that bound discs would be helpful in completing the classification for the surfaces we failed to do so. The example in section 2.3 demonstrates that an argument using the topological properties of the surface would not be effective. This steers the direction of further research on the classification of embedded curves under Curve Shortening Flow.

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