Hermitian and G_2-structures with large symmetry groups

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To dad.

Abstract

In the context of Hermitian geometry, the Hull–Strominger system is a system of non-linear PDEs on heterotic string theory, over a six-dimensional manifold endowed with an SU(3)-structure. Its seven-dimensional analogue, the heterotic G_2 system, is a system for both geometric fields and gauge fields over a manifold with a G_2-structure. In this thesis, we study manifolds with geometric structures compatible with the Hull–Strominger system and the heterotic G_2 system in the cohomogeneity one setting. In the former case, we develop a case-by-case analysis to provide a non-existence result for balanced non-Kähler SU(3)-structures which are invariant under a cohomogeneity one action on a simply connected six-manifold. In the latter case, we study two different $\mathrm{SU}(2)^2$-invariant cohomogeneity one manifolds, one non-compact $M = \mathbb{R}^4 \times S^3$, and one compact $M = S^4 \times S^3$. For $\mathbb{R}^4 \times S^3$, we prove the existence of a family of coclosed (but not necessarily torsion-free) G_2-structures which is given by three smooth functions satisfying certain boundary conditions around the singular orbit and a non-zero parameter. Moreover, any coclosed G_2-structure constructed from a half-flat SU(3)-structure is in this family. For $S^4 \times S^3$, we prove that there are no $\mathrm{SU}(2)^2$-invariant coclosed G_2-structures constructed from half-flat SU(3)-structures. Then, we study the existence of $\mathrm{SU}(2)^2$-invariant G_2-instantons on $\mathbb{R}^4 \times S^3$ manifold with the coclosed G_2-structures found. We find two 1-parameter families of smooth $\mathrm{SU}(2)^3$-invariant G_2-instantons with gauge group SU(2) on $\mathbb{R}^4 \times S^3$ and study its "bubbling" behaviour. We also provide existence results for locally defined $\mathrm{SU}(2)^2$-invariant G_2-instantons.

Statement of Originality

The content of this thesis is based on my own work unless stated otherwise. In particular:

Chapter 2 of the thesis is based on [AS22], written in collaboration with Francesca Salvatore.

1. **On the existence of balanced metrics on six-manifolds of cohomogeneity one**

   I. Alonso, F. Salvatore

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Chapter 3 of the thesis is based on [Alo22].

2. **Coclosed G_2-structures on $\mathbf{SU}(2)^2$-invariant cohomogeneity one manifolds**

   I. Alonso

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Chapter 4 of the thesis is based on [Alo23].

3. **New examples of G_2-instantons on $\mathbb{R}^4 \times S^3$**

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Contents

Introduction

In 1955, Berger presented a complete classification of the groups which can possibly occur as the holonomy groups of simply connected, irreducible, non-symmetric Riemannian manifolds [Ber55]. If the holonomy group of a manifold is contained in either of $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(n)\mathrm{Sp}(1)$, G_2 or $\mathrm{Spin}(7)$, we say that the manifold has *special holonomy.* Two of these groups have gathered special interest from the mathematical community, the special unitary group of dimension n, $\mathrm{SU}(n)$; and the Lie group G_2, which (together with the group $\mathrm{Spin}(7)$) is one of the so-called *exceptional holonomy groups.* A manifold with special holonomy group H admits a special type of geometric structure, known as an H-structure, which has vanishing torsion. The H-structures which present some torsion expand the concept of special holonomy, and naturally appear in physical theories such as heterotic string theory and M-theory.

In the context of Hermitian geometry, an $\mathrm{SU}(3)$-structure on a six-dimensional manifold consists of the data of a Riemannian metric, an orthogonal almost complex structure and a nowhere-vanishing holomorphic $(3,0)$-form satisfying a normalisation condition. They can be uniquely determined in terms of a 2-form and a 3-form which satisfy certain properties. An $\mathrm{SU}(3)$-structure may give rise to different types of Hermitian metrics; such as Kähler metrics and balanced metrics, depending on whether the aforementioned 2-form, known as *Hermitian form* or *fundamental form*, is closed, or its second power is closed. At the present time, we do not have a general method to determine whether a manifold admits a balanced structure.

Seven-dimensional manifolds with a torsion-free G_2-structure are known as G_2-manifolds.

They exhibit noteworthy properties: they have holonomy group contained in G_2 and are Ricci-flat. The construction of examples of G_2-manifolds turned out to be a difficult problem, with the first complete examples being found on non-compact manifolds by Bryant and Salamon in 1989 [BS89], and on compact manifolds by Joyce in 1996 [Joy96]. A G_2-structure is a 3-form which is pointwise linearly identified with a certain 3-form on $\mathbb{R}^7$. There are different types of G_2-structures with non-vanishing torsion, which have also attracted substantial interest: closed, coclosed and nearly-parallel. Coclosed G_2-structures are particularly relevant in the context of gauge theory as they provide the setting to construct G_2-instantons. Coclosed G_2-structures exist on any oriented spin seven-manifold [CN15].

On manifolds with a G_2-structure, we consider a special type of connections, G_2-instantons, whose study lies at the intersection of special holonomy and gauge theory. The study of higher dimensional gauge theories (i.e. gauge theories on manifolds of six, seven and eight dimensions) is motivated by three- and four-dimensional gauge theories. In particular, G_2-instantons, anti-self-dual instantons in four-manifolds and flat connections on three-manifolds are absolute minimizers of a Chern–Simons type energy functional. Following Donaldson–Thomas' suggestion [DT98] we can think about using G_2-instantons to define invariants for compact G_2-manifolds, inspired by Donaldson's work on anti-self-dual connections on four-manifolds. Some details of this idea were later worked out by Donaldson and Segal [DS11]. However, the naïve count of G_2-instantons on a compact G_2-manifold cannot produce a deformation-invariant, and the construction of such an invariant is an important open problem.

These structures are not only of great interest for the mathematics community, but also for mathematical physicists working in string theory. In 1986, Hull and Strominger independently introduced a system of partial differential equations in the context of heterotic string theory [Hul86,Str86], known as the *Hull–Strominger system*. The Hull–Strominger system involves a manifold with an SU(3)-structure, a pair of Hermitian metrics and two gauge fields. Since then, there has been plenty of work on the Hull–Strominger system, both from mathematicians and physicists, but the difficulty to understand one of their equations, the *Bianchi identity*, has made it difficult to find solutions. Its seven-dimensional analogue, known as the *heterotic G_2 system*, is a system for both geometric fields and gauge fields over a manifold with a G_2-structure. In particular, it requires the G_2-structure to be coclosed (after possibly a conformal transformation) and for the existence of two G_2-instantons whose curvatures are related through the *heterotic Bianchi identity*. At the present time, solutions to the Hull–Strominger system have been found under restrictive hypotheses in [AGF12, FY15, Fei16, FHP21, FTY09, FY08, GF18b, Gra11, OUV17]. For the heterotic G_2 system, exact and approximate solutions have been found in [FIUV11, FIUV15, CGFT16, CGFT20, LE21, dlOG21], but with the exception of the last two references, these solutions correspond to Lorentzian internal spaces of zero scalar curvature.

In this thesis, we study special differential-geometric structures in six and seven dimensions. In particular: balanced SU(3)-structures, coclosed G_2-structures and G_2-instantons. Our methodology involves using the theory of homogeneous spaces and Lie groups to implement a symmetry reduction. These methods lead to considering a specific kind of manifolds, called *cohomogeneity one manifolds*, whose metrics can be described using functions of one parameter. A cohomogeneity one manifold is a Riemannian manifold with an action by isometries of a compact Lie group having a generic orbit of codimension one. Cohomogeneity one techniques have played a significant role in the construction of examples of structures in special holonomy (see [BS89, BGGG01, Bog13, FHN21b, Cla14, LO18, Oli14, MNT22, ST23]). Moreover, until [FHN21a] and except for [KN10], the known examples of complete non-compact G_2–manifolds were of cohomogeneity one. Therefore, one can hope that it will be similarly fruitful to utilise this tool in our settings. It is also the natural next step after the use of homogeneous spaces, also known as cohomogeneity zero, to construct examples.

We provide a summary of the main results of this thesis. A more detailed layout is provided at the beginning of each chapter.

Summary of the results

We divide our results into three parts, corresponding respectively to Chapters 2, 3 and 4 of this thesis.

Balanced metrics on six-manifolds of cohomogeneity one

In Chapter 2 of this thesis we study the existence of balanced SU(3)-structures over cohomogeneity one manifolds with six dimensions. We first give a local result.

Theorem A. [AS22] Let M be a six-dimensional simply connected cohomogeneity one manifold under the almost effective action of a connected Lie group G, and let K be the principal isotropy group. Then, the principal part M^{princ} admits a G-invariant balanced non-Kähler SU(3)-structure (g, J, Ψ) if and only if $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathbb{R})$ and $\mathbb{R}$ is diagonally embedded in $\mathfrak{g}$ or M is compact and $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus 2\mathbb{R}, \{0\})$.

The idea of the proof is to first write a classification of the possible pairs $(\mathfrak{g}, \mathfrak{k})$, and then study which of these pairs admit a G-invariant balanced non-Kähler SU(3)-structures on the principal part of the manifold, by solving systems of ODEs and algebraic equations.

Then, we use the Eschenburg–Wang method to show that when $(\mathfrak{g}, \mathfrak{k}) \neq (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \Delta\mathbb{R})$ neither of these structures can smoothly extend to the singular orbits. This gives the main result of this chapter, which is a non-existence theorem.

Theorem B. [AS22] Let M be a six-dimensional simply connected cohomogeneity one manifold under the almost effective action of a connected Lie group G, and let K be the principal isotropy group. Assume $(\mathfrak{g}, \mathfrak{k}) \neq (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \Delta\mathbb{R})$. Then M admits no G-invariant balanced non-Kähler SU(3)-structures.

This result gives us a better understanding of manifolds with balanced structures, the interaction of the conditions of being balanced and being of cohomogeneity one, and how the cohomogeneity one technique can be used to study the existence of new examples of solutions to the Hull–Strominger system.

$\mathrm{SU}(2)^2$-invariant coclosed G_2-structures

In Chapter 3 of this thesis we study another type of special geometric structures: coclosed G_2-structures. We study two different $\mathrm{SU}(2)^2$-invariant cohomogeneity one manifolds, one non-compact $M = \mathbb{R}^4 \times S^3$ and one compact $M = S^4 \times S^3$, and look for coclosed $\mathrm{SU}(2)^2$-invariant G_2-structures constructed from half-flat SU(3)-structures, that are not necessarily torsion-free. We first prove the following theorem (see Chapter 3 for a more detailed statement).

Theorem C. [Alo22] On the cohomogeneity one manifold $M = \mathbb{R}^4 \times S^3$ with group diagram $\mathrm{SU}(2)^2 \supset \Delta\mathrm{SU}(2) \supset \{1\}$, there is a family of SU(2)-invariant coclosed G_2-structures which is given by three positive smooth functions satisfying certain boundary conditions around the singular orbit, and a non-zero parameter. Moreover, any SU(2)-invariant coclosed G_2-structure constructed from a half flat SU(3)-structure is in this family.

This family includes a $\mathrm{SU}(2)^3$-invariant subfamily of explicit structures. It also includes some known families of torsion-free G_2-structures as particular cases: the Bryant-Salamon G_2-holonomy metric [BS89] and the 1-parameter family of complete $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$-invariant G_2-metrics of Brandhuber et al. [BGGG01] and Bogoyavlenskaya [Bog13], also known as the $\mathbb{B}_7$ family. We observe that the existence of a class of coclosed G_2-structures, not necessarily torsion-free, of Theorem C contrasts with Theorem B, as in the seven dimensional analogue, the cohomogeneity one hypothesis does not force coclosed G_2-structures to be torsion-free.

On $S^4 \times S^3$, we prove a non-existence result.

Theorem D. [Alo22] On the cohomogeneity one manifold $M = S^4 \times S^3$ with group diagram $\mathrm{SU}(2)^2 \supset \Delta\mathrm{SU}(2), \Delta\mathrm{SU}(2) \supset \{1\}$, there are no $\mathrm{SU}(2)^2$-invariant coclosed G_2-structures constructed from half-flat SU(3)-structures.

G_2-instantons on $\mathbb{R}^4 \times S^3$

Chapter 4 of this thesis continues the work of the previous one, and is devoted to the construction of G_2-instantons on the cohomogeneity one manifold $\mathbb{R}^4 \times S^3$, with the coclosed G_2-structures form the previous chapter. We study the existence of G_2-instantons on the two $\mathrm{SU}(2)^3$-invariant $\mathrm{SU}(2)$-bundles on $\mathbb{R}^4 \times S^3$:

$$P_1 = \mathrm{SU}(2)^2 \times_{(\Delta\mathrm{SU}(2),1)} \mathrm{SU}(2), \quad P_{\mathrm{id}} = \mathrm{SU}(2)^2 \times_{(\Delta\mathrm{SU}(2),\mathrm{id})} \mathrm{SU}(2),$$

corresponding to the trivial and identity isotropy homomorphisms, respectively. The main result of this work is the following theorem.

Theorem E. Let $M = \mathbb{R}^4 \times S^3$, with a $\mathrm{SU}(2)^3$-invariant coclosed G_2-structure given by A_1 and $b_0 > 0$ as in Proposition 3.3.9. There exists two 1-parameter families of smooth $\mathrm{SU}(2)^3$-invariant G_2-instantons with gauge group $\mathrm{SU}(2)$: θ^{x_1}, $x_1 \in [0, \infty)$ on the bundle P_1; and θ_{y_0}, $y_0 \in [-1/b_0, 1/b_0]$ on the bundle P_{id}.

Theorem E generalises the previous existence results of G_2-instantons on $\mathbb{R}^4 \times S^3$ [Cla14, LO18, ST23] by now considering coclosed but not necessarily torsion-free G_2-structures.

We also show that the first family of G_2-instantons of Theorem E presents a "bubbling" behaviour and removable singularity phenomenon, and we describe the relation between all G_2-instantons encountered. At the end of the chapter we provide existence results for locally defined $\mathrm{SU}(2)^2$-invariant G_2-instantons.

Chapter 1

Preliminaries

The first chapter of this thesis addresses the following questions:

- What are the geometric structures which we study in this thesis?
- Why do we care about them?
- How do we study them?

Therefore, we will start by giving a preliminary background on the special geometric structures that we will consider in the following chapters: SU(3)-structures and G_2-structures, as well as G_2-instantons. Then, we will formally introduce the main motivation of the work presented here: heterotic systems, with a special focus on the six-dimensional version, the Hull–Strominger system, and the seven-dimensional version, the heterotic G_2-system. Finally, we will describe our technique for this study: cohomogeneity one. We will also establish the notation used throughout the thesis.

The layout of this chapter is as follows. In Section 1.1.1 we will recall some background on Hermitian geometry, including balanced manifolds. We then introduce SU(3)-structures, define and characterise stable forms, and describe Hitchin's construction. In Section 1.1.2 we will give a review of the concept of G_2-structures, the associated splitting of differential forms and the torsion classes of the structure. We also explain how to construct a G_2-structure from a half-flat SU(3)-structure. Finally, we introduce G_2-instantons and discuss known examples.

In Section 1.2.1, we will talk about the motivation of the work presented in Chapter 2 of this thesis: the Hull–Strominger system. The desire to find solutions to this system of partial differential equations, coming from heterotic string theory, will provide the setting in which we will be working in Chapter 2 of the thesis. We will give a definition of this system and a short summary of what we know about the existence of solutions to it. Then, in Section 1.2.2, we present the seven-dimensional analogue of this system, known as heterotic G_2 system, as it appears in [LE21], and discuss its appearance in the mathematics literature. This system will provide the setting in which we will be working in Chapter 3 and Chapter 4 of the thesis.

In Section 1.3, we will develop the relevant mathematical background on cohomogeneity one manifolds with two, one or zero ends, including the description of a cohomogeneity one manifold in terms of its group diagram.

1.1 Special geometric structures

Since the principal bundle formulation of Yang–Mills theory in the 1970s, there has been a substantial interaction between various areas of physics and differential geometry, via gauge theory. These approaches require one to consider manifolds endowed with specific geometric structures, such as metrics with holonomy $\mathrm{SU}(n)$ or G_2.

1.1.1 SU(3)-structures and stability of forms

Let X be a complex manifold of dimension n with complex structure J and with underlying smooth manifold M.

Definition 1.1.1 (Hermitian metric)**.** A *Hermitian metric* is a Riemannian metric g on M such that

$$g(J\cdot, J\cdot) = g(\cdot, \cdot).$$

Definition 1.1.2 (Hermitian form)**.** Let g be an Hermitian metric on M, the *Hermitian form* ω of g is defined by

$$\omega(\cdot, \cdot) = g(J\cdot, \cdot).$$

The Hermitian form is a $(1,1)$-form which is also known as the *fundamental form.* Given any two elements of (g, J, ω), the third one can be reconstructed from these two. There are different types of Hermitian structures.

Definition 1.1.3 (Kähler metrics)**.** A Hermitian metric g on X is *Kähler* if $d\omega = 0$. We say that a complex manifold is kählerian if it admits a Kähler metric.

Equivalently, g is a Kähler metric if $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.

There is another special type of Hermitian structure, weaker than the Kähler condition.

Definition 1.1.4 (balanced metrics)**.** A Hermitian metric g on X is *balanced* if $d\omega^{n-1} = 0$, We say that a complex manifold is balanced if it admits a balanced metric.

An immediate consequence of this definition is that a Kähler metric is also balanced. Recall that $d^* = -{*}d{*}$ is the adjoint of the exterior differential d operator for the hermitian metric g, where $*$ denotes the Hodge star operator of g. Then, as $*\omega = \omega^{n-1}/(n-1)!$, the balanced condition is equivalent to $d^*\omega = 0$.

Definition 1.1.5 (Calabi–Yau n-fold)**.** A *Calabi–Yau n-fold* is a pair (X, Ω) where X is a simply connected complex manifold of dimension n and Ω is a non-vanishing holomorphic global section of the canonical bundle $K_X = \Lambda^n T^* X$.

Remark 1.1.6. There are different definitions of Calabi–Yau manifolds, and some of them are inequivalent. Simple connectedness can be included or not in the definition, and so can compactness. Other definitions require X to have vanishing first real Chern class instead of the existence of a non-vanishing holomorphic global section of the canonical bundle. By the Calabi conjecture (see [Joy00, Chapter 6]), for a compact complex Kähler manifold, this is equivalent to admitting a Kähler metric with zero Ricci form (see [Bes87, Chapter 11]).

Definition 1.1.7 (Norm of Ω)**.** We define the *norm of* Ω, $||\Omega||_\omega$ by the equation

$$||\Omega||^2_\omega \frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \bar{\Omega}. \tag{1.1}$$

Definition 1.1.8 (U(n)-structure)**.** A *U(n)-structure* (g, J) on a $2n$-dimensional smooth manifold M consists of a Riemannian metric g and a g-orthogonal almost complex structure J. The pair (g, J) is also known as an *almost Hermitian structure* on M.

When J is integrable, i.e. (M, J) is a complex manifold, (g, J) defines a Hermitian structure on M.

Definition 1.1.9 (SU(n)-structure)**.** An *SU*(n)*-structure* (g, J, Ψ) on a $2n$-dimensional smooth manifold M is a U(n)-structure (g, J) on M together with an $(n, 0)$-form $\Psi = \psi_+ + i\psi_-$ of non-zero constant norm satisfying the *normalisation condition*

$$\Psi \wedge \bar{\Psi} = (-1)^{n(n+1)/2}(2i)^n \frac{\omega^n}{n!}, \tag{1.2}$$

where $\omega := g(J\cdot, \cdot)$.

Note that (1.2) is equivalent to $||\Psi||^2_\omega = 2^n$. Our case of interest is $n = 3$, where $\Psi = \psi_+ + i\psi_-$ is a $(3, 0)$-form of non-zero constant norm.

Definition 1.1.10 (SU(3)-structure)**.** An *SU*(3)*-structure* on a six-dimensional smooth manifold M is the data of a Riemannian metric g, a g-orthogonal almost complex structure J, and a $(3, 0)$-form $\Psi = \psi_+ + i\psi_-$ of non-zero constant norm satisfying the normalisation condition

$$\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3, \tag{1.3}$$

where $\omega := g(J\cdot, \cdot)$.

One can show that the SU(3)-structure actually depends only on the pair $(\omega, \psi_+) \in \Omega^2(M) \times \Omega^3(M)$. Before getting into the proof of this fact, we first need to explain the concept of stability of forms in vector spaces and in manifolds.

Definition 1.1.11 (stability of forms)**.** Let V real six-dimensional vector space. We say that $\alpha \in \Omega^k(V^*)$ is *stable* if its orbit under the action of $\mathrm{GL}(V)$ is open in $\Omega^k(V^*)$. We say that $\alpha \in \Omega^k(M)$ is *stable* if α_p is a stable form on the vector space T_pM for all $p \in M$.

Fix an orientation $\Omega \in \Omega^6(V^*)$. Consider the isomorphism

$$\begin{aligned} A : \Omega^5(V^*) &\to V \otimes \Omega^6(V^*); \\ \alpha &\mapsto A(\alpha) = v \otimes \Omega, \end{aligned} \tag{1.4}$$

where v is such that $\iota_v \Omega = \alpha$, and ι_v denotes the contraction by the vector v. If $\psi \in \Omega^3(V^*)$, then $\iota_v \psi \wedge \psi \in \Omega^5(V^*)$, so we can define a linear map

$$\begin{aligned} K_\psi : V &\to V \otimes \Omega^6(V^*); \\ v &\mapsto A(\iota_v \psi \wedge \psi), \end{aligned} \tag{1.5}$$

and a non-linear map

$$\begin{aligned} P : \Omega^3(V^*) &\to \Omega^6(V^*)^2; \\ \psi &\mapsto \frac{1}{6}\mathrm{tr}(K_\psi^2). \end{aligned} \tag{1.6}$$

With these functions we write the following definition.

Definition 1.1.12. Define $\lambda : \Omega^3(V^*) \to \mathbb{R}$ by

$$\lambda(\psi) = \iota_{\Omega \otimes \Omega} P(\psi).$$

Note that A and the sign of λ do not depend on the choice of orientation (only λ depends on the scale). The following Proposition characterises the stability of 2-forms and 3-forms.

Proposition 1.1.13. ([Hit00, Rei07]) Let V be an oriented, six-dimensional real vector space. Then

(i) a 2-form $\omega \in \Omega^2(V^*)$ is stable if and only if it is non-degenerate,

(ii) a 3-form $\psi \in \Omega^3(V^*)$ is stable if and only if $\lambda(\psi) \neq 0$.

We not are interested in stable 3-forms with positive λ as they do not define almost complex structures. We denote by $\Omega^3_+(V^*)$ the open orbit of stable 3-forms satisfying $\lambda < 0$. The $\mathrm{GL}_+(V)$-stabilizer of a 3-form lying in this orbit is isomorphic to $\mathrm{SL}(3, \mathbb{C})$. As a consequence, every $\psi \in \Omega^3_+(V^*)$ gives rise to a complex structure

$$J_\psi := -\frac{1}{\sqrt{|P(\psi)|}} K_\psi, . \tag{1.7}$$

which depends only on ψ and on the volume form Ω. From the definition of K_ψ, we have $K_\psi^2 = \lambda(\psi)I$, so $J_\psi^2 = -I$.

Let $(\omega, \psi_+) \in \Omega^2(M) \times \Omega^3_+(M)$ be a pair of stable forms, satisfying the *compatibility condition*

$$\omega \wedge \psi_+ = 0. \tag{1.8}$$

Let $J = J_{\psi_+}$. Define

$$\psi_- := J\psi_+ = \psi_+(J\cdot, J\cdot, J\cdot), \tag{1.9}$$

and define $\Psi := \psi_+ + i\psi_-$, which can be shown to be a nowhere vanishing $(3, 0)$-form. We observe that $\psi_- = -\psi_+(J\cdot, \cdot, \cdot)$. Finally, setting $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$ we get an $\mathrm{SU}(3)$-structure (g, J, Ψ) provided that g is a Riemannian metric and the normalisation condition is satisfied, just from the data (ω, ψ_+). We also note that

$$\psi_+ \wedge \psi_- = P(\psi_+)$$

up to a constant, so the normalisation condition (2.6) can be written purely in terms of ω and ψ_+.

Conversely, given an SU(3)-structure (g, J, Ψ) on M, the pair (ω, ψ_+) given by

$$\omega \coloneqq g(J\cdot, \cdot), \quad \psi_+ \coloneqq \mathrm{Re}(\Psi)$$

satisfies the compatibility condition $\omega \wedge \psi_+ = 0$ and the stability condition $\lambda(\psi_+) < 0$.

From [FTUV09], we define balanced SU(3)-structures as follows:

Definition 1.1.14 (balanced SU(3)-structure)**.** We say an SU(3)-structure (g, J, Ψ) on a six-dimensional manifold M is *balanced* if the Kähler form ω satisfies $d\omega^2 = 0$ and the complex volume $(3, 0)$-form Ψ is closed.

We also introduce another condition for SU(3)-structures, which is weaker than being balanced.

Definition 1.1.15 (half-flat SU(3)-structure)**.** We say that an SU(3)-structure given by the pair (ω, ψ_+) is *half-flat* if

$$\mathrm{d}\psi_+ = 0, \quad \mathrm{d}\omega^2 = 0.$$

1.1.2 The geometry of G_2-structures

We will give a summary of the geometry of G_2-structures, from a Riemannian-geometric point of view, including a discussion of the torsion. For more details we refer the reader to [Kar19, Section 4] and references therein.

Consider $\mathbb{R}^7$ with the standard euclidean metric g_0, for which the standard basis $e_1, ..., e_7$ is orthonormal. Let $\mu_0 = e^1 \wedge ... \wedge e^7$ be the standard volume form associated to g_0 and the standard orientation. Define the "associative" 3-form φ_0 by

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{257} - e^{356} - e^{246} - e^{347}, \tag{1.10}$$

where $e^1, ..., e^7$ is the standard dual basis of $(\mathbb{R}^7)^*$ and we write $e^{ijk} = e^i \wedge e^j \wedge e^k$. Define the "coassociative" 4-form ψ_0 by

$$\psi_0 = e^{4567} + e^{2367} + e^{2345} + e^{1346} - e^{1247} - e^{1357} - e^{1256}. \tag{1.11}$$

The order of $e^1, ..., e^7$ used might change between references. One can show using equation (1.10) that

$$(a \lrcorner \varphi_0) \wedge (b \lrcorner \varphi_0) \wedge \varphi_0 = -6 g_0(a,b)\mu_0.$$

Let $*_0$ be the Hodge star operator induced from (g_0, μ_0). Then

$$\psi_0 = *_0\varphi_0. \tag{1.12}$$

We also have

$$\|\varphi_0\|^2 = \|\psi_0\|^2 = 7, \tag{1.13}$$

$$\varphi_0 \wedge \psi_0 = 7\mu_0. \tag{1.14}$$

Definition 1.1.16 (Group G_2)**.** The group G_2 is the subgroup of $\mathrm{GL}(7,\mathbb{R})$ that preserves the standard G_2-package on $\mathbb{R}^7$, that is,

$$G_2 = \{A \in \mathrm{GL}(7,\mathbb{R}) : A^*g_0 = g_0, A^*\mu_0 = \mu_0, A^*\varphi_0 = \varphi_0\}. \tag{1.15}$$

Since elements of the group G_2 preserve the standard Euclidean metric and orientation on $\mathbb{R}^7$, G_2 is also a subgroup of $\mathrm{SO}(7)$.

Theorem 1.1.17. *([Bry87]) If $A \in GL(7,\mathbb{R})$ preserves φ_0, then it also automatically preserves g_0 and μ_0.*

Corollary 1.1.18. The group G_2 can be viewed explicitly as the subgroup of $\mathrm{SO}(7)$ consisting of those elements $A \in \mathrm{SO}(7)$ of the form

$$A = (f_1 | f_2 | f_1 \times_0 f_2 | f_4 | f_1 \times_0 f_4 | f_2 \times_0 f_4 | (f_1 \times_0 f_2) \times_0 f_4)), \tag{1.16}$$

where $f_1, \ldots, f_7$ are column vectors in $\mathbb{R}^7$, $\times_0$ is the cross product operation on $\mathbb{R}^7$ and $\{f_1, f_2, f_4\}$ is an orthonormal triple satisfying $\varphi_0(f_1, f_2, f_4) = 0$.

Bryant shows that G_2 is a compact, connected, simply connected Lie group of dimension 14 and rank 2.

Definition 1.1.19 (G_2-structure)**.** Let M be a smooth seven-manifold. A G_2*-structure* on M is a smooth 3-form φ on M such that, at every $p \in M$, there exists a linear isomorphism $T_pM \cong \mathbb{R}^7$ with respect to which $\varphi_p \in \Lambda^3(T_p^*M)$ corresponds to $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$.

A G_2-structure φ on M induces a Riemannian metric g_φ and associated Riemannian volume form μ_φ by

$$(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = -6 g_\varphi(X, Y)\mu_\varphi, \tag{1.17}$$

for all $X, Y \in \mathcal{X}(M)$. Let $*_\varphi$ be the Hodge star operator induced from (g_φ, μ_φ). Then the coassociative form is given by

$$\psi = *_\varphi \varphi.$$

Only certain smooth manifolds admit G_2-structures.

Proposition 1.1.20. ([Gra69])A smooth seven-manifold M admits a G_2-structure if and only if M is both orientable and spinnable. This is equivalent to the vanishing of the first two Stiefel–Whitney classes $w_1(TM)$ and $w_2(TM)$.

When the conditions of the previous Proposition hold, for any metric g the Riemannian manifold (M, g) admits a compatible G_2-structure $\varphi \in \Omega^3_+(M)$ satisfying the compatibility condition

$$(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = -6 g(X, Y)\mathrm{vol}_g,$$

for all $X, Y \in \mathcal{X}(M)$.

Let $\Omega^k = \Gamma(\Lambda^k(T^*M))$ be the space of smooth k-forms on M. We can write a decomposition of $\Omega^\bullet$ into irreducible G_2 representations. Any subspaces of Ω^k defined using

φ, ψ, g and $*$ will be G_2 representations. Let Ω^k_l be the irreducible representation of G_2 of (pointwise) rank l. We have

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14},$$

$$\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27},$$

where

$$\begin{aligned}
\Omega^2_7 &= \{\beta \in \Omega^2 | *(\varphi \wedge \beta) = -2\beta\}, \\
\Omega^2_{14} &= \{\beta \in \Omega^2 | *(\varphi \wedge \beta) = \beta\} = \{\beta \in \Omega^2 | \beta \wedge \psi = 0\}, \\
\Omega^3_1 &= \{f\varphi | f \in \Omega^0\}, \\
\Omega^3_7 &= \{X \lrcorner \psi | X \in \Gamma(TM)\}, \\
\Omega^3_{27} &= \{\gamma \in \Omega^3 | \gamma \wedge \varphi = 0, \gamma \wedge \psi = 0\}.
\end{aligned} \tag{1.18}$$

Let (M, φ) be a manifold with a G_2-structure and let ∇ be the Levi-Civita covariant derivative associated to the Riemannian metric coming from φ.

Definition 1.1.21 (full torsion tensor)**.** We can write

$$\nabla_X \varphi = T(X) \lrcorner \psi,$$

for some vector field $T(X)$ on M. We call $T \in \Gamma(T^*M \otimes T^*M)$ the *full torsion tensor* of φ.

The following proposition explains how the torsion of φ is completely described by four quantities.

Proposition 1.1.22. ([Bry06, Proposition 1]) For any G_2-structure $\varphi \in \Omega^3_+(M)$, there exist unique differential forms $\tau_0 \in \Omega^0(M)$, $\tau_1 \in \Omega^1$, $\tau_2 \in \Omega^2_{14}$ and $\tau_3 \in \Omega^3_{27}$ so that the following equations hold:

$$d\varphi = \tau_0 \psi + 3\tau_1 \wedge \varphi + *\tau_3, \tag{1.19}$$

$$d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi. \tag{1.20}$$

Definition 1.1.23 (torsion forms)**.** The quadruple of forms $(\tau_0, \tau_1, \tau_2, \tau_3)$ defined by (1.19) and (1.20) will be referred to as the *intrinsic torsion forms* of φ.

Definition 1.1.24 (torsion-free)**.** The G_2-structure φ is called *torsion-free* if $\nabla\varphi = 0$. We say (M, φ) is a *G_2-manifold* if φ is a torsion-free G_2-structure on M.

Hence φ is torsion-free if and only if $T = 0$. From (1.19) and (1.20) we also have that φ is torsion-free if and only if both $d\varphi = 0$ and $d\psi = 0$. When φ is torsion-free, g_φ is Ricci-flat and its holonomy is contained in G_2. For compact manifolds, the following theorem tells us when the holonomy is full.

Proposition 1.1.25. ([Joy00, Chapter 11]) Let (M, φ, g) be a compact G_2-manifold. Then $\mathrm{Hol}(g) = G_2$ if and only if $\pi_1(M)$ is finite.

The group G_2 is one of the two exceptional holonomy groups in the Berger's classification [Ber55].

Definition 1.1.26 (Associative submanifold)**.** An oriented compact three-dimensional submanifold $L \subset M$ is called an *associative* if

$$\varphi|_L = \mathrm{vol}_L.$$

We can classify G_2-structures into different types, depending on which components of the torsion vanish:

(i) torsion-free ($\nabla\varphi = 0$): $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$;

(ii) closed ($d\varphi = 0$): $\tau_0 = \tau_1 = \tau_3 = 0$;

(iii) coclosed ($d\psi = 0$): $\tau_1 = \tau_2 = 0$;

(iv) nearly-parallel ($d\varphi = \lambda\psi$, $\lambda \neq 0$): $\tau_1 = \tau_2 = \tau_3 = 0$.

A G_2-structure is of *pure type* if all but one of the torsion components vanish. We will be interested in coclosed G_2-structures, as for our manifold to admit a solution to the heterotic G_2 system, we need that (possibly after a conformal transformation) both τ_1 and τ_2 vanish. As proved by Crowley and Nordström in [CN15], coclosed G_2-structures exist on any oriented spin seven-manifold.

There is a relation between G_2-structures and SU(3)-structures, following from the fact that there is a Lie group inclusion $\mathrm{SU}(3) \subset G_2$. For the rest of this section and from Chapter 3, to avoid confusion with the coassociative form, we will write $\Omega_1 = \psi_+$ and $\Omega_2 = \psi_-$ for an SU(3)-structure $(g, J, \Psi = \psi_+ + i\psi_-)$.

Recall that an SU(3)-structure given by the pair (ω, Ω_2) is *half-flat* if

$$\mathrm{d}\Omega_1 = 0, \quad \mathrm{d}\omega^2 = 0.$$

We consider the case where $M = I \times N$, where M and N are smooth manifolds of dimensions 7 and 6 respectively and I is an interval with coordinate $t \in \mathbb{R}$. Let $(\omega(t), \Omega_2(t))$ be a 1-parameter family of SU(3)-structures on N parameterised by $t \in I$. Then, the following forms give a G_2-structure on M:

$$\begin{aligned} \varphi &= \mathrm{d}t \wedge \omega(t) + \Omega_1(t), \\ \psi &= \frac{\omega^2(t)}{2} - \mathrm{d}t \wedge \Omega_2(t), \end{aligned} \tag{1.21}$$

where $\Omega_1(t) = J\Omega_2(t)$ and J is the almost complex structure determined by $\Omega_2(t)$ using Hitchin's construction. Every G_2-structure on M can be constructed from an SU(3)-structure on N. Assume that $(\omega(t), \Omega_2(t))$ are half-flat. The G_2-structure φ is closed if the 1-parameter family $(\omega(t), \Omega_2(t))$ is a solution of

$$\dot{\Omega}_1 = \mathrm{d}\omega.$$

The dot denotes the derivative with respect to the parameter t. The G_2-structure is coclosed if $(\omega(t), \Omega_2(t))$ is a solution of

$$\omega \wedge \dot{\omega} = -\mathrm{d}\Omega_2.$$

Hence the G_2-structure is torsion-free if $(\omega(t), \Omega_2(t))$ is a solution of both equations. This is subject to the SU(3)-structure being half-flat.

Remark 1.1.27. Note that a coclosed G_2-structure whose restriction to each principal orbit is half-flat only requires $d\Omega_1 = 0$ as an extra condition, as $d\omega^2 = 0$ automatically holds.

Let M be a seven-dimensional manifold and let φ be a G_2-structure. We further assume that the G_2-structure is coclosed. Let $P \to M$ be a principal bundle with structure group G, which we assume to be a compact Lie group. Let A be a connection over the principal bundle P.

Definition 1.1.28 (G_2-instanton)**.** We say A is a G_2*-instanton* if

$$F_A \wedge \psi = 0,$$

where F_A is the curvature of A. We refer to this equation as the G_2*-instanton equation.*

Equivalently, G_2-instantons can be defined as the solutions to the G_2-analogue of the "anti-self-dual" condition on four-dimensional manifolds:

$$F_A \wedge \varphi = -{*F_A}.$$

The study of G_2-instantons is a research topic of interest for the gauge theory community, due to Donaldson–Thomas' suggestion [DT98] that it may be possible to use G_2-instantons to define invariants for G_2-manifolds. Some details of this idea were later worked out by Donaldson and Segal [DS11]. Although the naïve count of G_2-instantons on a compact G_2-manifold cannot produce a deformation-invariant, one may hope that by counting G_2-instantons together with Seiberg–Witten monopoles on associative submanifolds the mutual degenerations will solve the issue, while giving a relation between G_2-instantons and Seiberg–Witten monopoles [Hay17]. G_2-instantons also play a key role in heterotic string theory, as we will see in the next section.

Remark 1.1.29. For the definition of G_2-instantons, we do need to ask for the structure to be coclosed in order not to get an extra condition when differentiating $F_A \wedge \psi = 0$, as the equation would be over-determined. We do not need to ask for the structure to be closed.

On a compact G_2-manifold, G_2-instantons minimise the Yang–Mills energy functional

$$\mathcal{YM}(A) = \int_M |F_A|^2 = \int_M \operatorname{tr}(F_A \wedge *F_A)$$

on the space of finite-energy connections on P.

Example 1.1.30. There are different examples of G_2-instantons over both compact and non-compact G_2-manifolds. Trivial examples are flat connections, i.e. with $F_A = 0$. More interesting examples are the following:

(i) Over Joyce's compact G_2-manifolds ([Joy96]) constructed by desingularising G_2-orbifolds T^7/Γ (where Γ is a finite group of G_2-involutions) "Generalised Kummer Construction"; Walpuski constructed non-trivial examples of G_2-instantons, with a method based on gluing anti-self-dual instantons over asymptotically locally euclidean (ALE) spaces to flat bundles [Wal13].

(ii) Over the compact G_2-manifolds constructed by generalising the Generalised Kummer Construction (Joyce and Karigiannis [JK21]); Platt [Pla22] constructed G_2-instantons by generalising Walpuski's method from [Wal13]. These manifolds are resolutions of orbifolds of the form Y/Γ, where Y is a manifold with holonomy contained in G_2, but not necessarily flat.

(iii) Over compact G_2-manifolds constructed by the "twisted connected sum" method ([Kov03, CHNP15]); Sá Earp and Walpuski [EW15] gave an abstract construction of G_2-instantons, and one example [Wal16]. Later, Menet–Sá Earp–Nordström constructed other examples of G_2-instantons over these manifolds on [MNE21].

(iv) Over $\mathbb{R}^4 \times S^3$ with the Bryant–Salamon metric [BS89]; Clarke [Cla14] constructed a 1-parameter family of G_2-instantons. Lotay–Oliveira [LO18] gave existence, non-existence and classification results for $\mathrm{SU}(2)^3$-invariant and $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$-invariant G_2-instantons (including on the $\mathbb{B}_7$ family of G_2-metrics on $\mathbb{R}^4 \times S^3$, which contains the BGGG metric). Very recently, Stein and Turner [ST23] completed the study of $\mathrm{SU}(2)^3$-invariant G_2-instantons over the spinor bundle of S^3 [LO18] with the Bryant–Salamon metric by constructing a new 1-parameter family of examples.

(v) Foscolo, Haskins and Nordström constructed infinitely many 1-parameter families of complete asymptotically locally conical (ALC) G_2-metrics [FHN21b]. Over the asymptotically conical limit of the $\mathbb{C}_7$ family of G_2-metrics (this is a manifold which is diffeomorphic to a G_2-cone outside of a compact set, and whose metric is in some sense asymptotic to a conical metric), Matthies, Nordström and Turner constructed a 1-parameter family of $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$-invariant G_2-instantons with gauge group $\mathrm{SU}(2)$ in [MNT22]. This family is not explicit.

1.2 Heterotic systems

Ten-dimensional heterotic supergravity is a theory that includes a supersymmetric gauge theory. It is formulated on a ten dimensional spin manifold N (i.e. oriented with vanishing second Stiefel–Whitney class) and with a choice of an element in $H^1(N, \mathbb{Z}_2)$. To study this theory in lower dimensions, we impose the following ansatz:

$$N = M^{10-D} \times Y^D,$$

where Y^D is a D-dimensional Lorentzian manifold and M^{10-D} is a Riemannian spin manifold which encodes the extra dimensions of a supersymmetric vacuum and is known as the *internal space*. This mechanism is called *compactification* and reduces the equations the fields must satisfy to a system on the compact smooth manifold. Following [CGFT20], the *Killing spinor equations in (heterotic) supergravity*, for a Riemannian metric g, a spinor ψ, a function f (the dilaton), a 3-form H (the NS-flux), a g-compatible connection ∇ with skew-symmetric torsion H, and a connection A with curvature F_A on a principal K-bundle P_K over M^{10-D}, can be written as

$$\begin{aligned} &\nabla\psi = 0, \\ &(df - \frac{1}{4}H) \cdot \psi = 0, \\ &F_A \cdot \psi = 0. \end{aligned}$$

These equations, together with the instanton condition

$$R_\nabla \cdot \psi = 0,$$

and the *Bianchi identity*

$$dH = \alpha \left(\mathrm{tr}(R_\nabla \wedge R_\nabla) - \mathrm{tr}(F_A \wedge F_A) \right),$$

give the equations of motion.

The equations that we are going to study are a particular instance of this system of equations. In the case where $D = 4$, we get the Hull–Strominger system (with three complex dimensions). When $D = 3$, we get the heterotic G_2 system.

1.2.1 The Hull–Strominger system

The motivation for the work presented in Chapter 2 is finding new solutions to the Hull–Strominger system. This system was introduced in 1986 independently by Strominger in [Str86] and Hull in [Hul86] and has its origins in the low-energy limit of heterotic string theory. For details of the physical origins of the Hull–Strominger system, we refer the reader to [GF18a]; here we just present a brief discussion of them. The relation between the heterotic supergravity equations and the Hull–Strominger system arises via Strominger–Hull compactifications, where the internal space M is a compact smooth oriented spin six-dimensional manifold. It also requires M admitting a complex structure and being Calabi–Yau.

The Hull–Strominger system is the natural generalization of the Calabi problem (Ricci-flat equation) for non-kählerian complex manifolds. It is believed (Fu and Yau in [FY08]) that metrics motivated by theoretical physics should have good properties. Although it is well-motivated and studied both by mathematicians and physicists, the paucity of non-trivial examples of solutions to the Strominger system means that there is a clear gap in our understanding of the system.

The setup for the Hull–Strominger system is the following:

- A Calabi–Yau manifold (X, Ω) of dimension n: X is a complex manifold of complex dimension n and Ω is a non-vanishing holomorphic global section of the canonical bundle K_X. We denote the underlying smooth manifold by M and the almost complex structure by J;
- a hermitian metric g on (X, Ω);
- an holomorphic vector bundle ε over X, with underlying smooth complex vector bundle E;
- a hermitian metric h on ε;
- a unitary connection A on (E, h) (with curvature F_A);
- an integrable Dolbeault operator $\bar{\partial}_T$ on (TM, J);
- a unitary connection ∇ on $(TM, J, \bar{\partial}_T)$ (with curvature R_∇);

- a non-vanishing real constant α, which is proportional to the slope parameter in string theory.

Definition 1.2.1 (Hull–Strominger system)**.** In the above setup, the Hull–Strominger system for g, A and ∇ is the following system of coupled non-linear differential equations:

$$\begin{aligned} &\Lambda F_A = 0, \quad F_A^{0,2} = 0, \\ &\Lambda R_\nabla = 0, R_\nabla^{0,2} = 0, \\ &d^*\omega - d^c \log ||\Omega||_\omega = 0, \\ &dd^c\omega - \alpha(\mathrm{tr} R_\nabla \wedge R_\nabla - \mathrm{tr} F_A \wedge F_A) = 0, \end{aligned} \tag{1.22}$$

where Λ denotes the contraction operator by ω.

The first two equations are the Hermite–Einstein condition for the connections A and ∇, with Einstein factor $\gamma = 0$. The third one is called the *dilatino equation.* It is equivalent to the *conformally balanced equation*:

$$d(||\Omega||_\omega \omega^{n-1}) = 0. \tag{1.23}$$

The fourth one is called the *anomaly equation* or *Bianchi identity* and couples both curvatures. It is called an anomaly because it is condition necessary for the consistency of the quantised theory. It is the most difficult to understand equation of the Hull–Strominger system.

Remark 1.2.2. Historically, the name "Hull–Strominger system" or "Strominger system" has been used for a system of coupled non-linear differential equations of mixed order for g, h and $\bar\partial_T$:

$$\begin{aligned} &\Lambda F_h = 0, \\ &\Lambda R_g = 0, \\ &d^*\omega - d^c \log ||\Omega||_\omega = 0, \\ &dd^c\omega - \alpha(\mathrm{tr} R_g \wedge R_g - \mathrm{tr} F_h \wedge F_h) = 0, \end{aligned} \tag{1.24}$$

where $F_h \in \Omega^{1,1}(\mathrm{End}(E,h))$ is the curvature of the Chern connection on (ε, h) and R_g is the curvature of the Chern connection of g, regarded as a hermitian metric on the holomorphic vector bundle $T = (TM, J, \bar{\partial}_T)$. However, Martelli and Sparks [MS11] showed that the Chern connection is never an instanton.

For (X, Ω, E) to admit a solution of the Hull–Strominger system there are some cohomological obstructions on the Chern classes:

$$\deg(E) = 0, \tag{1.25}$$

and also

$$c_2(E) = c_2(X), \tag{1.26}$$

where c_2 denotes the second Chern class.

By the (set theoretical) Kobayashi–Hitchin correspondence, for a complex manifold to admit solutions to the Hull–Strominger system, the holomorphic bundles $\varepsilon = (E, \bar{\partial}_A)$ and $T = (TM, J, \bar{\partial}_\nabla)$ must be polystable. On general grounds, an effective check of any of the two equivalent conditions in the Kobayashi–Hitchin correspondence is a difficult problem.

The next result tells us that for M to admit a solution of the Hull–Strominger system, it has to be balanced. Its proof can be found in [GF18a] and uses the equivalence of the dilatino equation and the conformally balanced equation.

Proposition 1.2.3. [GF18a, Proposition 3.3] Let σ be a hermitian conformal class on (X, Ω). Then,

(i) if $n = 2$, σ admits a solution of the dilatino equation if and only if all $g \in \sigma$ is a solution, if and only if there exists a Kähler Ricci-flat metric on σ;

(ii) if $n \geq 3$, σ admits a solution of the dilatino equation if and only if σ admits a balanced metric.

If X is compact, then there exists at most one balanced metric in σ up to homothety.

Our case of interest would be $n = 3$, where the problem of existence and uniqueness is still open. Therefore, in Chapter 2 of this thesis we will look for balanced manifolds.

Known solutions to the Hull–Strominger system with $n = 3$ can be classified into three main groups depending on the method used to find them. The first ones were obtained in non-kählerian threefolds by Fu and Yau in [FY08], on suitable torus fibrations over a K3 surface, i.e. a compact connected complex manifold of dimension 2 with trivial canonical bundle and $\dim H^1(X, \Omega^0) = 0$. This reduction method of Fu and Yau was based on the non-kählerian fibred threefolds constructed by Goldstein and Prokushkin in [GP04]. More recently, Fino et al. [FGV19a] generalised Fu–Yau solution to torus bundles over K3 orbifolds.

The second group of solutions to the Hull–Strominger system have been found in non-kählerian homogeneous spaces, especially on nilmanifolds. For example, in [OUV17], the invariant solutions are found on three different compact non-Kähler homogeneous spaces which are obtained as the quotient by a lattice of maximal rank of a nilpotent Lie group (the nilmanifold $\mathfrak{h}_3$), the semisimple group $\mathrm{SL}(2, \mathbb{C})$ and a solvable Lie group (the solvmanifold $\mathfrak{g}_7$). Further solutions in non-kählerian homogeneous spaces have been found in [Gra11, FIUV09].

The third group of solutions follows from the work of Björn Andreas and Mario García Fernández [AGF12]: they prove that a given Calabi–Yau threefold endowed with a Kähler Ricci-flat metric and with a stable holomorphic vector bundle can be perturbed to a solution of the Strominger system provided the topological condition (1.26) is satisfied. This theorem assumes that the manifold can be endowed with a Kähler Ricci-flat metric, which in particular means that it is kählerian. In order to look for different solutions to the Hull–Strominger system, we will focus on manifolds which are balanced but non-Kähler.

In [FY15] a class of invariant solutions to the Hull–Strominger system on complex Lie groups was provided; these solutions extend to solutions on all compact complex parallelizable manifolds, by Wang's classification theorem [Wan54]. Moreover, in [FGV19b], it was shown that a compact complex homogeneous space with invariant complex volume admitting a balanced metric is necessarily a complex parallelizable manifold, so the complex compact homogeneous case is exhausted by the invariant solutions given in [FY15]. Then, the natural next step in complexity is to study the cohomogeneity one case.

1.2.2 The heterotic G_2 system

The heterotic G_2 system, as it appears in [LE21], is the following system, whose equations come from [dlOLS18, Section 2].

Definition 1.2.4. The *heterotic G_2 system* on a seven-manifold M with G_2-structure φ and coassociative form ψ is comprised of the following degrees of freedom.

- Geometric fields:
  - *scalar field* $\lambda \in \mathbb{R}$;
  - *dilaton* $\mu \in C^\infty(M)$;
  - *flux* $H \in \Omega^3(M)$.
- Gauge fields:
  - $A \in \mathcal{A}(E)$, where $E \to M$ is a vector bundle and A is a G_2-instanton, i.e. $F_A \wedge \psi = 0$ for F_A the curvature of A;
  - $\theta \in \mathcal{A}(TM)$ such that θ is a G_2-instanton, i.e. $R_\theta \wedge \psi = 0$, for R_θ the curvature of θ.

Let $\alpha' \neq 0$ be a (small) real constant, related to the string scale. The heterotic G_2 system consists of the following relations between the geometric fields and the intrinsic torsion forms:

$$\begin{cases} \tau_0 = \frac{3}{7}\lambda; \\ \tau_1 = \frac{1}{2}d\mu; \\ \tau_2 = 0; \\ H = \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3; \end{cases}$$

together with the *anomaly free condition* or *heterotic Bianchi identity* that relates the curvatures of the gauge fields:

$$dH = \frac{\alpha'}{4}(\mathrm{tr} F_A \wedge F_A - \mathrm{tr} R_\theta \wedge R_\theta). \tag{1.27}$$

Note that the Bianchi identity can only happen if

$$p_1(E) = p_1(M) \in H^4_{dR}(M). \tag{1.28}$$

After a conformal transformation, we can assume that $\tau_1 = 0$. Hence we are looking for G_2-structures such that

$$d\psi = 0.$$

This is analogous to the balanced condition of the Hull–Strominger system in three complex dimensions.

There has been substantial interest in the heterotic G_2 system, both from mathematicians and from physicists. The heterotic G_2 system was first studied in the physics literature in [GN95, GMWK01, FI02, FI03, GMPW04, GMW04, II05, LM11, GLL12]. It appeared in the mathematics literature for the first time in [FIUV11], where Fernández et al. constructed the first explicit compact solutions, with non-zero field strength, non-flat instanton and constant dilaton, based on nilmanifolds. Then in [FIUV15] smooth solutions with non-vanishing flux, non-trivial instanton and non-constant dilaton based on the quaternionic Heisenberg group are constructed. Clarke et al. [CGFT16, CGFT20] studied the moduli space of solutions and prove that the space of infinitesimal deformations, modulo automorphisms, is finite dimensional. In [CGFT20] they also provide a new family of solutions to this system, on T^3 bundles over K3 surfaces. The only constructive solutions of the heterotic G_2 system with non-zero τ_0 currently available in the literature are found in [LE21], over contact Calabi–Yau seven-manifolds and in [dlOG21], on squashed homogeneous 3-Sasakian manifolds (the seven-sphere and the Aloff–Wallach space). In [LE21], the connections over the tangent bundle from the authors' solutions are approximate G_2-instantons (they satisfy the instanton condition only to first order in α').

Torsion-free G_2-manifolds have seen plenty of study in the context of the heterotic G_2-system. Moreover, the torsion-free condition is very restrictive, as many interesting solutions to the equations arise in manifolds with non-vanishing torsion. Hence, we will focus on the case where the G_2-structure is coclosed but not necessarity closed.

1.3 Cohomogeneity one manifolds

We will give a description of the structure of cohomogeneity one manifolds and their metrics. We will use [BB82, Zil09] as our main references.

Definition 1.3.1 (cohomogeneity one manifold)**.** A Riemannian manifold M is of *cohomogeneity one* for the action of the compact Lie group G if:

(i) G is a closed subgroup of the isometry group of M,

(ii) and G has an orbit of codimension one.

We denote the action by $\alpha : G \times M \to M$, and by $\tilde{\alpha} : G \to \text{Diff}(M)$ the Lie group homomorphism induced by the action

$$\begin{aligned} \tilde{\alpha} : G &\to \text{Diff}(M); \\ g &\mapsto \tilde{\alpha}(g) : m \mapsto gm. \end{aligned} \tag{1.29}$$

Definition 1.3.2 (almost effective action)**.** We say that the action α is *almost effective* if $\ker \tilde{\alpha}$ is discrete, i.e. the set of points of G whose action is trivial is discrete.

We will assume that the manifold M and the group G are connected. Let $\pi : M \to M/G$ be the canonical projection onto the orbit space M/G. M/G is a connected riemannian manifold of dimension one (with or without boundary), which has the quotient topology relative to π. Connected riemannian manifolds of dimension one must be of one of the following options (up to isometries): S^1, $[0, L]$, $(-\infty, \infty)$, $(0, \infty)$, $(0, L)$, $[0, \infty)$, $[0, L)$, where $L > 0$. From [BB82], we get the following result:

Proposition 1.3.3. [BB82, Proposition 2.5] In the previous situation:

(i) M is compact if and only if M/G is compact;

(ii) M is complete if and only if M/G is complete.

In particular, this means that if the manifold M is compact then M/G has to be a closed interval or S^1. We continue by introducing the two different types of orbits that we can find in a cohomogeneity one manifold.

Definition 1.3.4 (principal orbits, singular orbits)**.** The inverse images of the interior points of the orbit space M/G are known as *principal orbits*.
The inverse images of the boundary points, if they exist, are called *singular orbits*.

A cohomogeneity one manifold then can have zero, one or two singular orbits. We call the *principal part* the union of all principal orbits, which is an open dense subset of M, and denote it by M^{princ}. We denote by $G_p = \{g \in G | g \cdot p = p\}$ the *isotropy group* at a point $p \in M$.

We will parameterise M with the help of M/G and a principal orbit. Let m be a point in one of the principal orbits of M. Let $\gamma : I \to M$ be the orthogonal geodesic through the orbit of m. This geodesic will be orthogonal to the orbit of G at any of its points.

Definition 1.3.5 (normal geodesic)**.** We call a geodesic orthogonal to every principal orbit a *normal geodesic*.

If $M/G = S^1$, then all orbits are principal and $\pi : M \to M/G$ is a bundle map. Also, using the homotopy sequence we have that $\pi_1(M)$ is infinite, so M is not simply connected. This will not be our case of interest, so from the rest of the section we will focus on the case where M/G is not a circle.

If M/G is not a circle, then $\pi \circ \gamma : I \to M/G$ is a global isometry. Let K be the isotropy group of m by the action of G. We parameterise M with the map

$$F : G/K \times I \to M;$$
$$(gK, t) \mapsto g\gamma(t),$$

which is well defined as K preserves γ pointwise. F is surjective and differentiable, so it induces a diffeomorphism

$$G/K \times \mathring{I} \cong M^{\text{princ}}.$$

This diffeomorphism is G-equivariant under the action of G which is trivial on I and the usual action on G/K:

$$F(g(g'K, t)) = F((gg')K, t) = (gg')\gamma(t) = g(g'\gamma(t)) = gF(g'K, t).$$

We will consider first the case where M is compact and simply connected, so I would be a closed interval, say $[-1, 1]$. We denote the singular isotropy groups by $H_- = G_{\gamma(-1)}$ and $H_+ = G_{\gamma(1)}$.

Proposition 1.3.6. The spaces H_-/K and H_+/K are spheres. They are called *normal spheres.*

Proof. We identify $H_\pm/K$ with the unit sphere in $T_{\gamma(\pm1)}(G/H_\pm)^\perp$. □

We denote the singular orbits by $O_- = \pi^{-1}(-1)$ and $O_+ = \pi^{-1}(1)$. Let $\gamma : [-1, 1] \to M$ be a normal geodesic, which we assume is of minimal length, and suppose that $\pi \circ \gamma = \mathrm{Id}_{[-1,1]}$. There exists a subgroup K of G such that $G_{\gamma(t)} = K$ for all $t \in (-1, 1)$ and K is a subgroup of $G_{\gamma(-1)}$ and $G_{\gamma(1)}$. Up to conjugation along the orbit, we have three possible isotropy groups: $H_- = G_{\gamma(-1)}$, $H_+ = G_{\gamma(1)}$ and $K = G_{\gamma(t)}$, $t \in (-1, 1)$. Then $O_- \cong G/H_-$, $O_+ \cong G/H_+$ and for all $t \in (-1, 1)$, $G \cdot \gamma(t) = \pi^{-1}(t) \cong G/K$.

Denote the tubular neighbourhoods of the singular orbits by $D(O_-) = \pi^{-1}([-1, 0])$ and $D(O_+) = \pi^{-1}([0, 1])$. Let $D_\pm$ be unit disks in $T_{\gamma(\pm1)}(G/H_\pm)^\perp$. By the slice theorem, the tubular neighbourhoods can be described by

$$D(O_\pm) = G \times_{H_\pm} D_\pm,$$

where $G \times_{H_\pm} D_\pm$ is the quotient space $G \times D_\pm / H_\pm$ of the action of $H_\pm$ on $G \times D_\pm$ given by

$$k \star (g, p) = (gk^{-1}, kp).$$

In the last equation k acts on $D_\pm$ via the slice representation, i.e. the restriction of the isotropy representation to $T_{\gamma(\pm 1)}(G/H_\pm)^\perp$. Hence we have the decomposition

$$M = D(O_-) \cup_{G/K} D(O_+).$$

The principal orbit G/K is canonically identified with $\partial D(O_\pm) = G \times_{H_\pm} S_\pm$ via the maps

$$gK \mapsto [(g, \dot\gamma(-1))],$$
$$gK \mapsto [(g, -\dot\gamma(1))],$$

where $S_\pm = \partial D_\pm = H_\pm/K$. In conclusion we can recover M from the groups G, H_-, H_+ and K.

Definition 1.3.7 (group diagram)**.** The collection of G with its isotropy groups $G \supset H_-, H_+ \supset K$ is called a *group diagram.*

Conversely, we can start with a group diagram $G \supset H_-, H_+ \supset K$, where $H_\pm/K \cong S^{l_\pm}$ are $l_\pm$-dimensional spheres, and build a cohomogeneity one manifold. We know that a transitive action of a compact Lie group on a sphere is conjugate to a linear action. Hence we can assume that $H_\pm$ acts linearly on $S^{l_\pm}$ with isotropy group $K \subset H_\pm$ at some point $p_\pm \in S^{l_\pm - 1}$. It hence extends to a linear action on the bounding disk $D^{l_\pm}$ and we can thus define a manifold

$$M = (G \times_{H_-} D_-) \cup_{G/K} (G \times_{H_+} D_+),$$

where we glue the two boundaries by sending $[(g, p_-)]$ to $[(g, p_+)]$. The action of G on M on each half is

$$g \star [(g', p)] = [(gg', p)].$$

We can check that this is a cohomogeneity one manifold with the desired group diagram, that the gluing is G-equivariant, and that the action has isotropy groups $H_\pm$ at $[(e, 0)]$ and K at $[(e, p_\pm)]$.

For M non-compact, either M/G is homeomorphic to an open interval or to an interval with one closed end. In the former case, M is a product manifold

$$M \cong I \times G/K.$$

In the latter case, there exists exactly one singular orbit, and $M/G \cong I$ where $I = [0, L)$ and L is either infinity or 1. Analogously to the compact case, there exists a normal geodesic $\gamma : [0, L) \to M$ such that $\gamma(0) \in \pi^{-1}(0)$ and we can suppose $\pi \circ \gamma = \mathrm{Id}_{[0,L)}$. In addition, there exists a subgroup K of G such that $G_{\gamma(t)} = K$ for all $t \in (0, L)$ and if $H := G_{\gamma(0)}$, K is a subgroup of H. Up to conjugation along the orbit, we have two possible isotropy groups: $H = G$ and K, $t \in \mathring{I}$. Then $G \cdot \gamma(0) = \pi^{-1}(0) \cong G/H$ and for all $t \in \mathring{I}$, $G \cdot \gamma(t) = \pi^{-1}(t) \cong G/K$.

If M is a non-compact cohomogeneity one manifold with one singular orbit we define the *group diagram* of M by the collection of G and the isotropy groups, $G \supset H \supset K$, where the homogeneous space H/K will be a sphere. In this case, as before we have the homotopy equivalence

$$M = G \times_H D,$$

where D is a unit disc in $T_{\gamma(0)}(G/H)^{\perp}$ and H acts on $G \times D$ as in the compact situation. The converse is also true: the group diagram $G \supset H \supset K$ where H/K is an l-dimensional sphere defines a non-compact cohomogeneity one manifold.

Borel classified transitive effective actions of compact Lie groups on spheres. The classification or Borel list is summarised in Table 1.1.

H	K	$S^l = H/K$
$\mathrm{SO}(n)$	$\mathrm{SO}(n-1)$	S^{n-1}
$\mathrm{U}(n)$	$\mathrm{U}(n-1)$	S^{2n-1}
$\mathrm{SU}(n)$	$\mathrm{SU}(n-1)$	S^{2n-1}
$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$\mathrm{Sp}(n-1)\mathrm{Sp}(1)$	S^{4n-1}
$\mathrm{Sp}(n)\mathrm{U}(1)$	$\mathrm{Sp}(n-1)\mathrm{U}(1)$	S^{4n-1}
$\mathrm{Sp}(n)$	$\mathrm{Sp}(n-1)$	S^{4n-1}
G_2	$\mathrm{SU}(3)$	S^6
$\mathrm{Spin}(7)$	G_2	S^7
$\mathrm{Spin}(9)$	$\mathrm{Spin}(7)$	S^{15}

Table 1.1: Transitive effective actions of compact Lie groups on spheres

Remark 1.3.8. If a cohomogeneity one manifold M has group diagram $G \supset H_-, H_+ \supset K$ or $G \supset H \supset K$, there are some operations that result in a G-equivariantly diffeomorphic manifold:

(i) switching H_+ and H_-,

(ii) conjugating each group in the diagram by the same element of G,

(iii) replacing $H_\pm$ (respectively H) with $aH_\pm a^{-1}$ (respectively aHa^{-1}) for $a \in N(K)_0$, where $N(K)_0$ is the identity component of the normaliser of K.

We can fix a coordinate system such that on the principal part the metric is determined by

$$g = dt^2 + g_t, \tag{1.30}$$

where dt^2 is the $(0,2)$-tensor corresponding to the vector field $\xi := \gamma'(t)$ at $\gamma(t)$ and g_t is a G-invariant metric on the homogeneous orbit $G \cdot \gamma(t)$. Since the regular points are dense on M, this expression also describes the metric on M. Fix a bi-invariant inner product B on $\mathfrak{g}$. We can write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ where the splitting is B-orthogonal and $\mathfrak{m}$ is $\mathrm{Ad}(K)$-invariant. Let $\mathfrak{m} = \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_r$ be the B-orthogonal decomposition into $\mathrm{Ad}(K)$-invariant irreducible subspaces. If $\mathfrak{m}_1$, ..., $\mathfrak{m}_r$ are inequivalent to each other, then we can write the metric g_t as

$$g_t = f_1^2(t)B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + ... + f_r^2(t)B|_{\mathfrak{m}_r \times \mathfrak{m}_r}$$

for some functions $f_1, ..., f_r$. But if M has one or more singular orbits, in order for g to be smoothly extending to these orbits, $f_1, ..., f_r$ have to satisfy certain smoothness conditions at them. Writing these conditions and checking whether they hold will be a crucial step to prove the results on the next chapters of this thesis (see Sections 2.4, 3.2.2, 4.1.3).

Definition 1.3.9 (equivalence of cohomogeneity one manifolds)**.** Let M_i be cohomogeneity one manifolds with respect to the action of Lie groups G_i, $i = 1, 2$. We say that the action of G_1 on M_1 is equivalent to the action of G_2 on M_2 if there exists a Lie group isomorphism $\phi : G_1 \to G_2$ and an equivariant diffeomorphism $f : M_1 \to M_2$ with respect to the isomorphism ϕ.

We shall study cohomogeneity one manifolds up to this type of equivalence.

Chapter 2

Balanced metrics on six-dimensional cohomogeneity one manifolds

In this chapter, we study manifolds that can admit balanced non-kähler SU(3)-structures, and that are of cohomogeneity one under the action of a compact connected Lie group G. Such structures are of interest for both Hermitian geometry and string theory, since they provide the ideal setting for the Hull–Strominger system. Balanced metrics have been extensively studied in [BV17, FGV19a, FV15, FV16, FLY12, Mic82, PPZ19].

As we explained on Section 1.2.1, a manifold M with a solution to the Hull–Strominger system has to be conformally balanced. When one assumes all structures to be invariant under the smooth action of a certain Lie group G, the aforementioned condition reduces to the balanced equation $d\omega^{n-1} = 0$. We also need M to be endowed with an invariant nowhere-vanishing holomorphic $(3,0)$-form Ψ satisfying a normalisation condition. In these cases, (g, J, Ψ) is a balanced SU(3)-structure on M, up to a suitable uniform scaling of Ψ. Moreover, we require M to be simply connected.

The layout of this chapter is as follows. Our strategy will be to classify all possible principal parts (up to G-equivariant diffeomorphisms). If we denote the principal isotropy group by K, then the principal part of the manifold is completely determined by the pair (G, K), or, up to finite quotients, by their Lie algebras $(\mathfrak{g}, \mathfrak{k})$. In Section 2.1 we will obtain a list of possible principal parts of a manifold with an SU(3)-structure which is invariant under a cohomogeneity one action. In Section 2.2 we will use the assumption that M is simply connected to reduce the list to only three possibilities. In Section 2.3 we will state and prove the first main result of this chapter, which is a local result for the existence of balanced non-Kähler SU(3)-structures, by working on M^{princ}. We first establish the general setup that we will use later for studying the remaining cases, and we will do so carrying out a case-by-case analysis in Sections 2.3.2, 2.3.1 and 2.3.3. Finally, in Section 2.4 we prove our main theorem, which shows that when K is connected none of these local solutions can be extended to a global one. We will also comment on the consequences of this result for the problem of existence of balanced SU(3)-structures on $S^3 \times S^3$.

2.1 SU(3)-structures on cohomogeneity one manifolds

Let (g, J, ψ) be an SU(3)-structure on a simply connected cohomogeneity one manifold M of complex dimension three for the almost effective action of a compact connected Lie group G. The group G preserves the SU(3)-structure. In particular, for any $p \in M$, the principal isotropy group K acts on T_pM preserving (g_p, J_p, ψ_p), which means that K is a subgroup of SU(3). Recall that $\xi = \gamma'(t) \in T_{\gamma(t)}M$ is the vector field normal to the orbits. The K-action is J-invariant, so it fixes the subspaces $\langle \xi|_p \rangle$ and also $\langle J\xi|_p \rangle$ of T_pM, as $K(J\xi|_p) = J(K\xi|_p) \in \langle J\xi|_p \rangle$. We can write

$$T_pM = \langle \xi|_p \rangle \oplus \langle J\xi|_p \rangle \oplus V, \tag{2.1}$$

where V is the 4-dimensional g_p-orthogonal complement of $\langle \xi|_p, J\xi|_p \rangle$ in T_pM. Then V is J_p-invariant and K-invariant. This means that for every element of K, its action on

$T_pM = \langle \xi|_p \rangle \oplus \langle J\xi|_p \rangle \oplus V$ is described by a 6×6 matrix of the form

$$\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \\ \hline & & \\ & & A \\ & & \end{array}\right).$$

We had that $K \subset \mathrm{SU}(3)$, so the matrix A is in $\mathrm{SU}(2)$. This means that we can identify K with a subgroup of $\mathrm{SU}(2)$. If we denote $\mathfrak{k} = \mathrm{Lie}(K)$, it is a subalgebra of $\mathfrak{su}(2)$, and hence isomorphic to $\{0\}$, $\mathbb{R}$ or $\mathfrak{su}(2)$, as $\mathfrak{su}(2)$ has no 2-dimensional subalgebras. We can then classify all possible pairs $(\mathfrak{g}, \mathfrak{k})$ which may admit an $\mathrm{SU}(3)$-structure in cohomogeneity one. The only possible decompositions into irreducibles for any principal point p, as $\dim(\mathfrak{g}) - \dim(\mathfrak{k}) = \dim(G \cdot p) = 5$, are:

(a) if $\mathfrak{k} = \{0\}$, then

   (1) $\mathfrak{g} = \mathfrak{su}(2) \oplus 2\mathbb{R}$,

   (2) $\mathfrak{g} = 5\mathbb{R}$,

(b) if $\mathfrak{k} = \mathbb{R}$, then

   (1) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$,

   (2) $\mathfrak{g} = \mathfrak{su}(2) \oplus 3\mathbb{R}$,

   (3) $\mathfrak{g} = 6\mathbb{R}$,

(c) if $\mathfrak{k} = \mathfrak{su}(2)$, then

   (1) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus 2\mathbb{R}$,

   (2) $\mathfrak{g} = \mathfrak{su}(2) \oplus 5\mathbb{R}$,

(3) $\mathfrak{g} = \mathfrak{su}(3)$.

We will discard some pairs from this list using the simple connectedness of M.

2.2 Ruling out cases using simple connectedness

The hypothesis of simple connectedness of M will let us discard most of the cases that appear in the previous list. We will have to consider the compact and the non-compact situations separately. The van Kampen theorem for cohomogeneity one manifolds that are compact appears in [Hoe10a]. It has has already been used to rule out some cases in this list in [PS10]. However, the non-compact situation has not been explored in this way before. The following proposition tells us how to compute the fundamental group of a compact cohomogeneity one manifold using the group diagram.

Proposition 2.2.1 (van Kampen, compact case)**.** [Hoe10a, Proposition 1.8] Let M be the compact cohomogeneity one manifold given by the group diagram $G \supset H_-, H_+ \supset K$ with $H_\pm/K = S^{l_\pm}$ and assume $l_\pm \geq 1$. Then $\pi_1(M) \cong \pi_1(G/K)/N_-N_+$ where

$$N_\pm = \ker\{\pi_1(G/K) \to \pi_1(G/H_\pm)\} = \mathrm{Im}\{\pi^1(H_\pm/K) \to \pi^1(G/K)\}.$$

In particular M is simply connected if and only if the images of $\pi_1(H_\pm/K) = \pi_1(S^{l_\pm})$ generate $\pi_1(G/K)$ under the natural inclusions.

Following the proof from [Hoe10a], we can adapt it to the case where M is non-compact with $M/G \cong [-1,\infty)$, and has one orbit by decomposing M as $\pi^{-1}([-1,0]) \cup \pi^{-1}([0,\infty))$. Now $\pi^{-1}([0,\infty))$ deformation retracts to $\pi^{-1}(0) = G \cdot x_0 \cong G/K$, and $\pi^{-1}([-1,0])$ deformation retracts to $\pi^{-1}(-1) \cong G/H$. The technical part of the proof would be identical except for the substitution of $[0,1]$ by $[0,\infty)$ and H_+ by K. Finally, the equivalent 'N_+' will be $N_+ = \ker\{\pi_1(G/K) \to \pi_1(G/K)\} = \{1\}$. Then $\pi_1(M) \cong$

$\pi_1(G/K)/N$, where

$$N = \ker\{\pi_1(G/K) \to \pi_1(G/H)\} = \mathrm{Im}\{\pi_1(H/K) \to \pi_1(G/K)\}.$$

Moreover, we have $M \simeq G/H$, so $\pi_1(M) \cong \pi_1(G/H)$. Hence we have proved the following proposition.

Proposition 2.2.2 (van Kampen, non-compact case)**.** Let M be the non-compact cohomogeneity one manifold given by the group diagram $G \supset H \supset K$ with $H/K = S^l$, and $l \geq 1$. Then, $\pi_1(M) \cong \pi_1(G/H)$. In particular M is simply connected if and only if the image of $\pi_1(H/K) = \pi_1(S^l)$ generates $\pi_1(G/K)$ under the natural inclusions.

We know that $\pi_1(S^l)$ is either $\{0\}$ (if $l > 1$) or $\mathbb{Z}$ (if $l = 1$). Now we observe that, for cases (a.1) and (c.1), $\pi_1(G/K) = \mathbb{Z}^2$, for cases (a.2), (b.3) and (c.2), $\pi_1(G/K) = \mathbb{Z}^5$ and for case (b.2), $\pi_1(G/K)$ is either $\mathbb{Z}^2$ or $\mathbb{Z}^3$, depending on the immersion of $\mathfrak{k}$ in $\mathfrak{g}$. If M is non-compact and has no singular orbits, we had that $M \cong I \times G/K$ for some open interval I, so we have $\pi_1(M) = \pi_1(G/K)$. As a consequence of this fact and the previous analysis together with Proposition 2.2.2, we can discard the pairs (a.1), (a.2), (b.2), (b.3), (c.1) and (c.2) when M is non-compact. If M is compact, by Proposition 2.2.1 we can easily discard the pairs, (a.2), (b.2) when $\pi_1(G/K) = \mathbb{Z}^3$, (b.3) and (c.2) as $\pi_1(M)$ would be infinite.

We can also discard cases (b.2) when $\pi_1(G/K) = \mathbb{Z}^2$ and (c.1) in the compact case using the classification of groups acting transitively on spheres. Let now H denote either H_+ or H_- for the compact case, or H for the non-compact case. As H/K has to be a sphere and H acts transitively on H/K, we can study the different options for pairs (H, K). The cohomogeneity one condition gives $\dim(G/K) = 5$, so $\dim(H/K)$ has to be 1,2,3,4 or 5. We will write the options using the Borel list for writing S^n, $n = 1, ..., 5$, but without the cases where $\mathfrak{k} \neq \{0\}, \mathbb{R}, \mathfrak{su}(2)$:

$$S^1 = \frac{\mathrm{SO}(2)}{\{1\}} = \frac{\mathrm{U}(1)}{\{1\}}; \qquad S^2 = \frac{\mathrm{SO}(3)}{\mathrm{SO}(2)};$$
$$S^3 = \frac{\mathrm{SO}(4)}{\mathrm{SO}(3)} = \frac{\mathrm{U}(2)}{\mathrm{U}(1)} = \frac{\mathrm{SU}(2)}{\{1\}} = \frac{\mathrm{Sp}(1)\mathrm{Sp}(1)}{\{1\}\mathrm{Sp}(1)} = \frac{\mathrm{Sp}(1)\mathrm{U}(1)}{\{1\}\mathrm{U}(1)}; \qquad S^5 = \frac{\mathrm{SU}(3)}{\mathrm{SU}(2)}.$$

Alternatively, we can classify them by the different options for K:

(a) Case $\mathfrak{k} = \{0\}$ and $K = \{1\}$: we can have either

$H = \mathrm{U}(1) = \mathrm{SO}(2)$ (corresponds to $\mathfrak{h} = \mathbb{R}$ and normal sphere S^1),

or $H = \mathrm{Sp}(1)$ (corresponds to $\mathfrak{h} = \mathfrak{su}(2)$ and normal sphere S^3).

(b) Case $\mathfrak{k} = \mathbb{R}$ and $K = \mathrm{U}(1) = \mathrm{SO}(2)$: we can have either

$H = \mathrm{SO}(3)$ (corresponds to $\mathfrak{h} = \mathfrak{su}(2)$ and normal sphere S^2),

$H = \mathrm{U}(2)$ (corresponds to $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathbb{R}$ and normal sphere S^3),

or $H = \mathrm{Sp}(1)\mathrm{U}(1)$ (corresponds to $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathbb{R}$ and normal sphere S^3).

(c) Case $\mathfrak{k} = \mathfrak{su}(2)$: we can have either

$K = \mathrm{SU}(2) = \mathrm{Sp}(1)$ and $H = \mathrm{SU}(3)$ (corresponds to $\mathfrak{h} = \mathfrak{su}(3)$ and normal sphere S^5),

$H = \mathrm{Sp}(1)\mathrm{Sp}(1)$ (corresponds to $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and normal sphere S^3),

or $K = \mathrm{SO}(3)$ and $H = \mathrm{SO}(4)$ (corresponds to $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and normal sphere S^3).

After this analysis, we can make the following observation.

Observation 2.2.3. Studying the classification of effective transitive actions of groups on spheres, we see that only when $\mathfrak{k} = \{0\}$ (case (a)) we could have $l = 1$, for $H/K = S^l$.

In particular, from this observation we get that when M is compact and $\mathfrak{k} = \mathfrak{su}(2)$, if $H^{\pm}/K = S^{l_{\pm}}$, then $l_{\pm} > 1$. This lets us discard case (c.1) also for the compact case.

Therefore, the possible pairs which may admit a balanced SU(3)-structure on a simply connected manifold of cohomogeneity one under the almost effective action of a compact connected Lie group G are (a.1) (only when M is compact), (b.1), and (c.3).

To summarise, the remaining options are:

(a.1) Case $\mathfrak{k} = \{0\}$ and $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R} \oplus \mathbb{R}$.

(b.1) Case $\mathfrak{k} = \mathbb{R}$ and $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

(c.3) Case $\mathfrak{k} = \mathfrak{su}(2)$ and $\mathfrak{g} = \mathfrak{su}(3)$.

We will study each of them in the following sections.

2.3 Proof of Theorem A

We are going to fix the notation that will be used for the rest of this chapter. We denote by

- $\mathcal{B}$ the negative of the Killing form on $\mathfrak{g}$;
- $\{\tilde{e}_i\}_{i=1,2,3}$ the generic basis for $\mathfrak{su}(2)$, which is given by

$$\tilde{e}_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \tilde{e}_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \tilde{e}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

- $\{f_i\}_{i=1,\dots,m}$ the generic basis for $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, with $\mathfrak{k} = \langle f_1, \dots, f_k \rangle$ and $\mathfrak{m} = \langle f_{k+1}, \dots, f_m \rangle$, where $k = \dim \mathfrak{k}$ and $m = \dim \mathfrak{g}$;
- $e_1 := \xi \cong \frac{\partial}{\partial t}$;
- $e_i := \widehat{f}_{\dim \mathfrak{k} - 1 + i}$ the Killing vector fields on M^{princ} induced by the G-action, for $i = 2, \dots, 6$;
- e^i the dual 1-forms to e_i.

Therefore, $\{e_i\}_{i=1,\dots,6}$ will be vectors on M^{princ} which provide a basis for T_pM at each point $p = \gamma(t) \in M^{\text{princ}}$, where $\gamma\colon \mathring{I} \to M$ is a normal geodesic through the point p. Since $g \cdot \gamma_p = \gamma_{g \cdot p}$, e_1 is invariant under the adjoint action of every element in $\mathfrak{g}$.

Every k-form α on M^{princ} is of the form

$$\alpha = \sum_{1 \leq i_1 < \ldots < i_k \leq 6} \alpha_{i_1 \ldots i_k} e^{i_1 \ldots i_k},$$

where we use the notation $e^{i_1 \cdots i_k} = e^{i_1} \wedge \ldots \wedge e^{i_k}$. It is G-invariant if and only if α_p is $\mathrm{Ad}(K)$-invariant for all $p \in M^{\text{princ}}$. Moreover, if α is a G-invariant k-form on M and $X_1, \ldots, X_k$ are G-invariant vector fields on M, then $\alpha\left(X_1, \ldots, X_k\right)|_p$ is constant along the G-orbit through p, for each $p \in M$. The general form of a pair of forms on M^{princ} of degrees two and three is

$$\omega = \sum_{1 \leq i < j \leq 6} h_{ij} e^{ij}, \qquad \psi_+ = \sum_{1 \leq i < j < k \leq 6} p_{ijk} e^{ijk} \tag{2.2}$$

with coefficients $h_{ij}, p_{ijk} \in C^\infty(\mathring{I})$. For (ω, ψ_+) to be G-invariant, we have to ask for $(\omega_p, \psi_+|_p)$ to be $\mathrm{Ad}(K)$-invariant for all $p \in M^{\text{princ}}$. Recall that we can always we recover the whole SU(3)-structure from a pair of $\mathrm{Ad}(K)$-invariant stable forms (ω, ψ_+) of degrees two and three respectively.

We are now ready to state Theorem A.

Theorem A. Let M be a six-dimensional simply connected cohomogeneity one manifold under the almost effective action of a connected Lie group G, and let K be the principal isotropy group. Then, the principal part M^{princ} admits a G-invariant balanced non-Kähler SU(3)-structure (g, J, Ψ) if and only if $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathbb{R})$ and $\mathbb{R}$ is diagonally embedded in $\mathfrak{g}$ or M is compact and $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus 2\mathbb{R}, \{0\})$.

In order for the pair (ω, ψ_+) to define a G-invariant balanced non-Kähler SU(3)-structure on M^{princ} we have to impose the following conditions, that we explained at the beginning of the chapter:

(1) the stability conditions:

- $\omega^3 \neq 0$,

- $\lambda := \lambda(\psi_+) < 0$,

(2) the compatibility conditions $\psi_\pm \wedge \omega = 0$,

(3) the normalisation condition: $\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3$,

(4) $d\psi_\pm = 0$,

(5) the balanced condition $d\omega^2 = 0$,

(6) the non-Kähler condition $d\omega \neq 0$,

(7) the positive-definiteness of the induced symmetric bilinear form $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ on M^{princ}.

From the above discussion, the only possible pairs allowing M^{princ} to support a balanced SU(3)-structure are (a.1) with M compact, (c.3), and (b.1). We start with the case with larger $\mathfrak{k}$.

2.3.1 Case (c.3)

For this section, will assume that $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(3), \mathfrak{su}(2))$. We consider the usual $\mathcal{B}$-orthogonal basis of $\mathfrak{g} = \mathfrak{su}(3)$, which is given by

$$f_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$
$$f_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad f_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad f_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad f_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}.$$

We can assume that the embedding of $\mathfrak{k} = \mathfrak{su}(2)$ in $\mathfrak{g}$ is such that $\mathfrak{k} = \langle f_1, f_2, f_3 \rangle$. We denote $\mathfrak{a} := \langle f_8 \rangle$ and $\mathfrak{n} := \langle f_4, f_5, f_6, f_7 \rangle$, so that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

The $\mathrm{Ad}(K)$-invariant irreducible modules in the decomposition of $\mathfrak{g}$ are pairwise inequivalent, so metric g on M^{princ} has the form

$$g = dt^2 + h(t)^2 \mathcal{B}|_{\mathfrak{a}\times\mathfrak{a}} + f(t)^2 \mathcal{B}|_{\mathfrak{n}\times\mathfrak{n}},$$

where $h, f \in C^\infty(\mathring{I})$. The structure equations with respect to the frame $\{e_i\}_{i=1,\dots,6}$ of M^{princ} are:

$$\begin{aligned}
de^1 &= 0, & de^2 &= -\sqrt{3}e^{36}, & de^3 &= \sqrt{3}e^{26}, \\
de^4 &= -\sqrt{3}e^{56}, & de^5 &= \sqrt{3}e^{46}, & de^6 &= -\sqrt{3}(e^{23} + e^{45}).
\end{aligned}$$

The general form of $\mathrm{Ad}(K)$-invariant forms ω and ψ_+ is:

$$\omega = h_1 e^{16} + h_2(e^{23} + e^{45}) + h_3(e^{24} - e^{35}) + h_4(e^{25} + e^{34}),$$

$$\psi_+ = p_1(e^{123}+e^{145})+p_2(e^{124}-e^{135})+p_3(e^{246}-e^{135})+p_4(e^{236}+e^{456})+p_5(e^{125}+e^{134})+p_6(e^{256}+e^{346}).$$

The form ψ_- should be $\mathrm{Ad}(K)$-invariant as well, so we can also write it as

$$\psi_- = q_1(e^{123}+e^{145})+q_2(e^{124}-e^{135})+q_3(e^{246}-e^{135})+q_4(e^{236}+e^{456})+q_5(e^{125}+e^{134})+q_6(e^{256}+e^{346}),$$

where coefficients $q_1, \dots, q_6$ will depend on coefficients $p_1, \dots, p_6$. Then the conditions for the coefficients $h_1, \dots, h_4$ and $p_1, q_1, \dots, p_6, q_6$ obtained from equations (1)–(6) for this case will be the following (we do not need to use (7)).

(1) Stability condition:

$$0 \neq \omega^3 = 6h_1(h_2^2 + h_3^2 + h_4^2)e^{123456}. \tag{2.3}$$

(2) The compatibility condition $\psi_+ \wedge \omega = 0$ is equivalent to the following two algebraic equations:

$$\begin{aligned} p_1 h_2 + p_5 h_4 + p_2 h_3 &= 0, \\ p_3 h_3 + p_4 h_2 + p_6 h_4 &= 0. \end{aligned} \tag{2.4}$$

Analogously, the compatibility condition $\psi_- \wedge \omega = 0$ is equivalent to:

$$\begin{aligned} q_1 h_2 + q_5 h_4 + q_2 h_3 &= 0, \\ q_3 h_3 + q_4 h_2 + q_6 h_4 &= 0. \end{aligned} \tag{2.5}$$

(3) The normalisation condition $\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3$ can be written as

$$p_2 q_3 - p_3 q_2 + p_5 q_6 - p_6 q_5 = 2h_1(h_2^2 + h_3^2 + h_4^2). \tag{2.6}$$

(4) From $d\psi_+ = 0$, we get ODE's:

$$\begin{aligned} p_5 &= -\tfrac{1}{2\sqrt{3}} p_3', \\ p_2 &= \tfrac{1}{2\sqrt{3}} p_6', \\ p_4 &= 0. \end{aligned} \tag{2.7}$$

Analogously, condition $d\psi_- = 0$ yields:

$$\begin{aligned} q_5 &= -\tfrac{1}{2\sqrt{3}} q_3', \\ q_2 &= \tfrac{1}{2\sqrt{3}} q_6', \\ q_4 &= 0. \end{aligned} \tag{2.8}$$

(5) The balanced condition $d\omega^2 = 0$ becomes the ODE:

$$0 = h_2(\sqrt{3}h_1 + h_2') + h_4'h_4 + h_3'h_3. \tag{2.9}$$

(6) Non-Kähler condition:

$$\begin{aligned} 0 \neq d\omega = \ & (\sqrt{3}h_1 + h_2')(e^{123} + e^{145}) + h_3'(e^{124} - e^{135}) + h_4'(e^{125} + e^{134}) \\ & + 2\sqrt{3}h_3(e^{346} + e^{256}) + 2\sqrt{3}h_4(e^{356} - e^{246}). \end{aligned} \tag{2.10}$$

We start by imposing the stability of ψ_+. We will assume that $p_4 = 0$, since we got this from the condition of ψ_+ being closed. A straightforward computation implies that

$$\lambda(\psi_+) = -4(p_2p_6 - p_3p_5)^2 - 4p_1^2(p_6^2 + p_3^2)$$

This is negative if $p_2p_6 - p_3p_5 \neq 0$ or $p_1^2(p_6^2 + p_3^2) \neq 0$.

Recall that $\psi_- = J\psi_+ = -\frac{1}{\sqrt{\lambda(\psi_+)}}\psi_+(K_{\psi_+}\cdot, \cdot, \cdot)$. Hence the coefficients for ψ_- will satisfy

$$\begin{aligned} -\sqrt{-\lambda(\psi_+)}q_1 &= 2p_1(p_2p_3 + p_5p_6), \\ -\sqrt{-\lambda(\psi_+)}q_2 &= 2p_2p_5p_6 - 2p_1^2p_3 - 2p_3p_5^2, \\ -\sqrt{-\lambda(\psi_+)}q_3 &= 2p_6(p_2p_6 - p_3p_5), \\ -\sqrt{-\lambda(\psi_+)}q_4 &= 2p_1(p_3^2 + p_6^2), \\ -\sqrt{-\lambda(\psi_+)}q_5 &= 2p_2p_3p_5 - 2p_1^2p_6 - 2p_2^2p_6, \\ -\sqrt{-\lambda(\psi_+)}q_6 &= 2p_3(p_3p_5 - p_2p_6). \end{aligned}$$

Condition $d\psi_- = 0$ implies $q_4 = 0$, which in turn implies either $p_1 = 0$ or $p_3 = p_6 = 0$. The second situation would also imply that $p_2 = p_5 = 0$ (from equation (2.7)), and from normalisation conditions (2.6) and (2.3), that $\omega^3 = 0$. But that would mean that ω is not stable, so from now on we assume $p_1 = 0$. Multiplying by $-(-\lambda(\psi_+))^{-1/2} = -|2(p_2p_6 - p_3p_5)|^{-1}$, and recalling that for ψ_+ to be stable, $p_2p_6 - p_3p_5$ must be non-zero, we have two situations:

(i) Case $p_2p_6 - p_3p_5 < 0$:

$$q_1 = 0, q_2 = p_5, q_3 = p_6, q_4 = 0, q_5 = -p_2, q_6 = -p_3.$$

(ii) Case $p_2p_6 - p_3p_5 > 0$:

$$q_1 = 0, q_2 = -p_5, q_3 = -p_6, q_4 = 0, q_5 = p_2, q_6 = p_3.$$

We can also check that the rest of the equations in (2.8) that we got from $d\psi_- = 0$ are a consequence of (2.7) in either of these cases.

Compatibility conditions (2.4) and (2.5) are now in both cases

$$\begin{aligned}
&p_3h_3 + p_6h_4 = 0,\\
&-p_6h_3 + p_3h_4 = 0,\\
&p_6'h_3 - p_3'h_4 = 0,\\
&p_3'h_3 + p_6'h_4 = 0,
\end{aligned}$$

The first two of these imply that either $p_3 = p_6 = 0$ or $h_3 = h_4 = 0$. We already discussed that if $p_3 = p_6 = 0$ we do not have a solution. If $h_3 = h_4 = 0$, from the balanced condition (2.9) we get that either $h_2 = 0$ or $\sqrt{3}h_1 + h_2' = 0$. But if $h_2 = 0$ then $\omega^3 = 0$ by (2.3), so ω would not be stable. On the other hand, if $\sqrt{3}h_1 + h_2' = 0$, by (2.10) ω would be Kähler.

Hence, we can conclude that the system of equations coming from conditions (1) to (7) is incompatible.

2.3.2 Case (b.1)

For this section, will assume that $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathbb{R})$.

Remark 2.3.1. We shall need to divide the discussion depending on the embeddings of $\mathfrak{k} = \mathbb{R}$ in $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ which, up to isomorphism, are all generated by an element of the form

$$\left(\begin{pmatrix} ip & 0 \\ 0 & -ip \end{pmatrix}, \begin{pmatrix} iq & 0 \\ 0 & -iq \end{pmatrix}\right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2),$$

with fixed $p, q \in \mathbb{N}$. We can assume either $(p, q) = (1, 0)$ or p, q to be coprime if neither is zero. Notice that when $(p, q) = (1, 1)$ or $(p, q) = (1, 0)$, $\mathfrak{k}$ induces a decomposition of $\mathfrak{g}$ into $\mathrm{Ad}(K)$-modules, some of which are equivalent. In the former case, we shall say that $\mathfrak{k}$ is diagonally embedded in $\mathfrak{g}$, while in the latter $\mathfrak{k}$ is said to be trivially embedded in one of the two $\mathfrak{su}(2)$-factors of $\mathfrak{g}$. When instead p, q are different and non-zero, the $\mathrm{Ad}(K)$-modules are pairwise inequivalent.

In the notation of Remark 2.3.1, let us first suppose p, q non-zero and coprime with $(p, q) \neq (1, 1)$. Consider the $\mathcal{B}$-orthonormal basis of $\mathfrak{g}$ given by

$$\begin{aligned} f_1 &= \frac{1}{2\sqrt{2(p^2+q^2)}}(p\tilde{e}_1, q\tilde{e}_1), & f_2 &= \frac{1}{2\sqrt{2(p^2+q^2)}}(q\tilde{e}_1, -p\tilde{e}_1), \\ f_3 &= \frac{1}{2\sqrt{2}}(\tilde{e}_3, 0), & f_4 &= \frac{1}{2\sqrt{2}}(0, \tilde{e}_3), \\ f_5 &= \frac{1}{2\sqrt{2}}(\tilde{e}_2, 0), & f_6 &= \frac{1}{2\sqrt{2}}(0, \tilde{e}_2). \end{aligned} \tag{2.11}$$

Take $\mathfrak{k} = \langle f_1 \rangle$. Notice that, since $\mathrm{rk}(\mathfrak{su}(2)) = 1$, this assumption is not restrictive. The decomposition of $\mathfrak{g}$ into irreducible $\mathrm{Ad}\,(K)$-modules is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2,$$

where $\mathfrak{a} := \langle f_2 \rangle$ is $\mathrm{Ad}(K)$-fixed, $\mathfrak{b}_1 := \langle f_3, f_5 \rangle$ and $\mathfrak{b}_2 := \langle f_4, f_6 \rangle$, and hence $\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2$. Fix the orientation given by $\Omega = e^{1\ldots 6}$ and consider the general G-invariant 3-form ψ_+ on M^{princ},

$$\psi_+ := p_1\, e^{135} + p_2\, e^{146} + p_3\, e^{235} + p_4\, e^{246},$$

where $p_j \in C^\infty(\mathring{I})$, $j = 1, \ldots, 4$ and e^i are defined at the beginning of Section 2.3. A simple calculation (done with the help of mathematical software) shows that $\lambda(\psi_+) =$

$(p_1p_4 - p_2p_3)^2 \geq 0$, so the stability condition $\lambda(\psi_+) < 0$ never holds. Alternatively, by [Hit00, Proposition 2] we can directly see that λ is non-negative, as $\psi_+ = (p_1\, e^1 + p_3\, e^2) \wedge e^{35} + (p_2\, e^1 + p_4\, e^2) \wedge e^{46}$ can be written as a sum of real decomposable forms.

Now let $(p, q) = (1, 0)$ and consider the $\mathcal{B}$-orthogonal basis of $\mathfrak{g}$ given by (2.11) when $(p, q) = (1, 0)$ and assume $\mathfrak{k} = \langle f_1 \rangle$ as before. Then, the decomposition of $\mathfrak{g}$ into irreducible $\mathrm{Ad}(K)$-modules is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3,$$

where $\mathfrak{b}_1 := \langle f_3, f_5 \rangle$, $\mathfrak{a}_1 := \langle f_2 \rangle$, $\mathfrak{a}_2 := \langle f_4 \rangle$ and $\mathfrak{a}_3 := \langle f_6 \rangle$. Observe that the $\mathfrak{a}_i$'s are equivalent. Consider the general G-invariant 3-form ψ_+ on M^{princ}, which is of the form

$$\psi_+ := p_1\, e^{124} + p_2\, e^{126} + p_3\, e^{135} + p_4\, e^{146} + p_5\, e^{235} + p_6\, e^{246} + p_7\, e^{345} + p_8\, e^{356},$$

where $p_j \in C^\infty(\mathring{I})$, $j = 1, \ldots, 8$. It is straightforward to show that $\lambda(\psi_+) = (p_1p_8 + p_2p_7 - p_3p_6 + p_4p_5)^2 \geq 0$. Similarly, ψ_+ can be written as a sum of real 3-forms $(p_3\, e^1 + p_5\, e^2 - p_7\, e^4 + p_8\, e^6) \wedge e^{35}$ and $p_1\, e^{124} + p_2\, e^{126} + p_4\, e^{146} + p_6\, e^{246}$, both of which are decomposable.

Remark 2.3.2. By the previous discussion we have that when $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathbb{R})$ with $\mathfrak{k}$ not diagonally embedded in $\mathfrak{g}$, M admits no G-invariant $\mathrm{SL}(3, \mathbb{C})$-structures, i.e. G-invariant stable 3-forms inducing an almost complex structure on M.

Finally, let us consider the case where $\mathfrak{k}$ is diagonally embedded in $\mathfrak{g}$. Without loss of generality, we can assume $(p, q) = (1, 1)$. We consider the $\mathcal{B}$-orthonormal basis of $\mathfrak{g}$ given by (2.11) when $(p, q) = (1, 1)$. The decomposition of $\mathfrak{g}$ into irreducible $\mathrm{Ad}\,(K)$-modules is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2,$$

where $\mathfrak{k} = \langle f_1 \rangle$, $\mathfrak{a} := \langle f_2 \rangle$ is $\mathrm{Ad}(K)$-fixed, $\mathfrak{b}_1 := \langle f_3, f_5 \rangle$ and $\mathfrak{b}_2 := \langle f_4, f_6 \rangle$. Then, $\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2$. Unlike the case $p \neq q$ both non-zero, here the equivalence of the $\mathfrak{b}_i$-modules implies that the metric g on $M^{\rm princ}$ is not necessarily diagonal but of the form

$$g = \mathrm{d}t^2 + f(t)^2\mathcal{B}|_{\mathfrak{a}\times\mathfrak{a}} + h_1(t)^2\mathcal{B}|_{\mathfrak{b}_1\times\mathfrak{b}_1} + h_2(t)^2\mathcal{B}|_{\mathfrak{b}_2\times\mathfrak{b}_2} + \mathcal{Q}|_{\mathfrak{b}_1\times\mathfrak{b}_2},$$

for some $f, h_1, h_2 \in C^\infty(\mathring{I})$, where $\mathcal{Q}$ denotes a symmetric quadratic form on the isotypic component $\mathfrak{b}_1 \oplus \mathfrak{b}_2$. In particular, the metric coefficients $g_{ij} \coloneqq g(e_i, e_j)$ must satisfy

$$\begin{aligned} &g_{1i} = g_{i1} = 0, \quad i = 2, \dots, 6, \\ &g_{2i} = g_{i2} = 0, \quad i = 3, \dots, 6, \\ &g_{33} = g_{55}, \quad g_{35} = g_{53} = 0, \\ &g_{44} = g_{66}, \quad g_{46} = g_{64} = 0, \end{aligned} \tag{2.12}$$

where e_i, $i = 1, \dots, 6$, are the vector fields defined in the usual way. Fix the orientation given by $\Omega \coloneqq e^{1\dots6}$, and consider a pair of G-invariant forms (ω, ψ_+) of degree two and three, given respectively by

$$\begin{aligned} \omega \coloneqq& h_1\, e^{12} + h_2\, e^{35} + h_3\, e^{46} + h_4(e^{34} + e^{56}) + h_5(e^{36} + e^{45}), \\ \psi_+ \coloneqq& p_1\, e^{135} + p_2\, e^{146} + p_3(e^{134} + e^{156}) + p_4(e^{136} + e^{145}) \\ &+ p_5\, e^{235} + p_6\, e^{246} + p_7(e^{234} + e^{256}) + p_8(e^{236} + e^{245}), \end{aligned}$$

where $h_i, p_j \in C^\infty(\mathring{I})$, $i = 1, \dots, 5$, $j = 1, \dots, 8$. Moreover, the structure equations are given by

$$\mathrm{d}e^1 = 0,\ \mathrm{d}e^2 = \frac{1}{2}\left(e^{35} - e^{46}\right),\ \mathrm{d}e^3 = -\frac{1}{2}e^{25},\ \mathrm{d}e^4 = \frac{1}{2}e^{26},\ \mathrm{d}e^5 = \frac{1}{2}e^{23},\ \mathrm{d}e^6 = -\frac{1}{2}e^{24}.$$

In order to find a G-invariant balanced non-Kähler SU(3)-structure on $M^{\rm princ}$, we have to impose the conditions (1) to (7) listed at the beginning of this section, together with (2.12). We shall show that this system of equations is incompatible. This implies there are no G-invariant balanced non-Kähler SU(3)-structures on the corresponding M. In order to see this, we write conditions in terms of the coefficients h_i, p_j of (ω, ψ_+), for $i = 1, \ldots, 5$, $j = 1, \ldots, 8$.

(1) The first stability condition $\omega^3 \neq 0$ is:

$$\omega^3 = -6h_1(h_2h_3 - h_4^2 - h_5^2)e^{123456} \neq 0$$

In particular, $h_1 \neq 0$. For the second stability condition $\lambda < 0$ we compute:

$$\begin{aligned}\lambda = \ & p_2^2p_5^2 - 2p_1p_2p_5p_6 + 4p_3^2p_5p_6 + 4p_4^2p_5p_6 + p_1^2p_6^2 \\ & -4p_2p_3p_5p_7 - 4p_1p_3p_6p_7 + 4p_1p_2p_7^2 - 4p_4^2p_7^2 \\ & -4p_2p_4p_5p_8 - 4p_1p_4p_6p_8 + 8p_3p_4p_7p_8 + 4p_1p_2p_8^2 \\ & -4p_3^2p_8^2\end{aligned}$$

(2) Compatibility condition $\psi_+ \wedge \omega$:

$$\begin{cases} -h_3p_1 - h_2p_2 + 2h_4p_3 + 2h_5p_4 = 0, \\ -h_3p_5 - h_2p_6 + 2h_4p_7 + 2h_5p_8 = 0. \end{cases} \tag{2.13}$$

Compatibility condition $\psi_- \wedge \omega$:

$$\begin{cases} -h_3q_1 - h_2q_2 + 2h_4q_3 + 2h_5q_4 = 0, \\ -h_3q_5 - h_2q_6 + 2h_4q_7 + 2h_5q_8 = 0. \end{cases} \tag{2.14}$$

(3) Normalisation condition $\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3$. We can write

$$\psi_+ \wedge \psi_- = -6\sqrt{-\lambda}e^{123456},$$

so the normalisation condition becomes

$$-6\sqrt{-\lambda} = -4h_1(h_2h_3 - h_4^2 - h_5^2). \tag{2.15}$$

(4) A computation shows $\mathrm{d}\psi_+ = 0$ if and only if

$$\begin{cases} p_8' - p_3 = 0, \\ p_7' + p_4 = 0, \\ p_5 = p_6, \\ p_6' = 0. \end{cases} \tag{2.16}$$

Let us suppose that ψ_+ is stable with $\lambda < 0$, and consider the induced almost complex structure J on M^{princ}. Recall that, by G-invariance, $\psi_- = J\psi_+$ needs to be of the same general form of ψ_+, namely

$$\begin{aligned} \psi_- = &q_1 e^{135} + q_2 e^{146} + q_3(e^{134} + e^{156}) + q_4(e^{136} + e^{145}) \\ &+ q_5 e^{235} + q_6 e^{246} + q_7(e^{234} + e^{256}) + q_8(e^{236} + e^{245}), \end{aligned}$$

where the q_i's are functions of $\{p_j\}_{j=1,\dots,8}$ for $i = 1, \dots, 8$. Therefore, $\mathrm{d}\psi_- = 0$ if and only if

$$\begin{cases} q_8' - q_3 = 0, \\ q_7' + q_4 = 0, \\ q_5 = q_6, \\ q_6' = 0. \end{cases} \tag{2.17}$$

(5) Balanced condition: one has that $\mathrm{d}\omega^2 = 0$ if and only if

$$\frac{h_1}{2}(h_3 - h_2) - \left(h_2 h_3 - h_4^2 - h_5^2\right)' = 0. \tag{2.18}$$

(6) Non-kähler condition $d\omega \neq 0$. We have $\mathrm{d}\omega = 0$ if and only if

$$\begin{cases} -\dfrac{h_1}{2} + h_2' = 0, \\ (h_2 + h_3)' = 0, \\ h_4 = h_5 = 0. \end{cases}$$

(7) We would also need to check that the metric g is positive definite.

Moreover, the only non-redundant equation from (2.12) is $g_{12} = 0$, which is equivalent to

$$p_1 p_6 + p_2 p_6 - 2p_3 p_7 - 2p_4 p_8 = 0, \tag{2.19}$$

where we have already assumed $p_5 = p_6$ from (2.16). Since $p_6' = 0$ and all the conditions for the G-invariant balanced non-Kähler SU(3)-structure involve only homogeneous polynomials, we can assume either $p_6 = 0$ or $p_6 = 1$, up to scalings.

We use K to compute ψ_- and get

$$\begin{aligned}
-\sqrt{-\lambda} q_1 &= -3(-p_1 p_2 p_5 + 2p_3^2 p_5 + 2p_4^2 p_5 + p_1^2 p_6 - 2p_1 p_3 p_7 - 2p_1 p_4 p_8) \\
-\sqrt{-\lambda} q_2 &= -3(p_2^2 p_5 - p_1 p_2 p_6 + 2p_3^2 p_6 + 2p_4^2 p_6 - 2p_2 p_3 p_7 - 2p_2 p_4 p_8) \\
-\sqrt{-\lambda} q_3 &= -3(p_2 p_3 p_5 + p_1 p_3 p_6 - 2p_1 p_2 p_7 + 2p_4^2 p_7 - 2p_3 p_4 p_8) \\
-\sqrt{-\lambda} q_4 &= -3(p_2 p_4 p_5 + p_1 p_4 p_6 - 2p_3 p_4 p_7 - 2p_1 p_2 p_8 + 2p_3^2 p_8) \\
-\sqrt{-\lambda} q_5 &= +3(p_2 p_5^2 - p_1 p_5 p_6 - 2p_3 p_5 p_7 + 2p_1 p_7^2 - 2p_4 p_5 p_8 + 2p_1 p_8^2) \\
-\sqrt{-\lambda} q_6 &= -3(p_2 p_5 p_6 - p_1 p_6^2 + 2p_3 p_6 p_7 - 2p_2 p_7^2 + 2p_4 p_6 p_8 - 2p_2 p_8^2) \\
-\sqrt{-\lambda} q_7 &= -3(2p_3 p_5 p_6 - p_2 p_5 p_7 - p_1 p_6 p_7 + 2p_4 p_7 p_8 - 2p_3 p_8^2) \\
-\sqrt{-\lambda} q_8 &= -3(2p_4 p_5 p_6 - 2p_4 p_7^2 - p_2 p_5 p_8 - p_1 p_6 p_8 + 2p_3 p_7 p_8)
\end{aligned}$$

We now note that as $p_5 = p_6$, the condition $q_5 = q_6$ implies

$$(p_1 - p_2)(-p_6^2 + p_7^2 + p_8^2) = 0 \tag{2.20}$$

We will show that both for $p_6 = 0$ and for $p_6 = 1$, (1)–(7) requires $p_1 = p_2$. If $p_6 = 0$, then

$$\lambda = 4p_1p_2(p_7^2 + p_8^2) - 4(p_4p_7 - p_3p_8)^2.$$

From equation (2.20) we must have $p_1 = p_2$, as if it were not true, we would have $p_7 = p_8 = 0$ and therefore $\lambda = 0$.

Suppose now that $p_6 = 1$. Equation (2.20) implies that we have two possibilities: either $p_7^2 + p_8^2 = 1$ or $p_1 = p_2$. We first assume $p_7^2 + p_8^2 = 1$. Differentiating this expression and using (2.16), we get

$$p_4p_7 - p_3p_8 = 0.$$

Hence,

$$\lambda \;\; = 4(p_3^2 + p_4^2 - (p3p7 + p4p8)^2)$$

By Cauchy–Schwarz,

$$(p3p7 + p4p8)^2 \leq (p_3^2 + p_4^2)(p_7^2 + p_8^2),$$

and since $p_7^2 + p_8^2 = 1$, we get $\lambda \geq 0$ which gives a contradiction. Hence from now on we assume $p_1 = p_2$.

We now provide an example of a solution to (1)–(7) on the M^{princ}. To do so, we set $p_1 = p_2 = p_3 = p_5 = p_6 = p_8 = h_4 = h_5 = 0$. In particular, the compatibility conditions will then be automatically satisfied. A solution is given by setting

$$\begin{aligned} h_2 = &\;\; -3t, \\ h_3 = &\;\; t, \\ h_1 = &\;\; -3, \\ p_4 = &\;\; -\frac{3\sqrt{2}}{2}t^{1/2}, \\ p_7 = &\;\; \sqrt{2}t^{3/2}, \end{aligned}$$

for $t > 0$ and all other coefficients equal to 0. This satisfies conditions (1)–(7) and hence gives G-invariant balanced non-Kähler SU(3)-structure on M^{princ}. Then, by performing the change of variable

$$\tilde{t}(t) = \sqrt{6}t^{3/2},$$

we get

$$\begin{aligned}\omega &= -3\,e^{12} + 6^{-1/3}\tilde{t}^{2/3}(-3e^{35} + e^{46}),\\ \psi_+ &= -2^{-2/3}3^{5/6}\tilde{t}^{1/3}(e^{136} + e^{145}) + 3^{-1/2}\tilde{t}(e^{234} + e^{256}),\end{aligned}$$

and the metric on M^{princ} with respect to the $\tilde{t}$ parameter is then represented by the matrix (so the basis of the dual space taken is $\{d\tilde{t}, e^2, e^3, e^4, e^5, e^6\}$)

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & \tilde{t}^2 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{3}{2}\tilde{t}^2 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}\tilde{t}^2 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{3}{2}\tilde{t}^2 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\tilde{t}^2 \end{pmatrix}.$$

It follows from [VZ20] or [FH17] that this solution does not extend to a singular orbit at $t = 0$ to give a smooth metric on the whole manifold. In Section 2.4 we will study the extension of the structure to the singular orbits.

2.3.3 Case (a.1)

Through this section, we will assume that $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus 2\mathbb{R}, \{0\})$. We recall that this is only possible when M is compact, as M non-compact implies M not simply connected, so $I = [-1, 1]$ for this section. As $\mathfrak{k} = \{0\}$, any generic forms on M^{princ} are $\text{Ad}(K)$-invariant, so the most general way of writing the pair (ω, ψ_+) is:

$$\omega = \sum_{1\le i<j\le 6} h_{ij}e^{ij}, \qquad \psi_+ = \sum_{1\le i<j<k\le 6} p_{ijk}e^{ijk}, \tag{2.21}$$

where $h_{ij}, p_{ijk} \in C^\infty((-1,1))$. We choose a $\mathcal{B}$-orthogonal basis of $\mathfrak{su}(2)$ with vectors of constant norm

$$f_i = (\tilde{e}_i, 0, 0),\ i = 1, 2, 3,$$

and extend it to a basis $\{f_i\}_{i=1,\dots,5}$ of $\mathfrak{g}$. The structure equations with respect to $\{e_i\}_{i=1,\dots,6}$ of M^{princ} will be given by

$$de^1 = 0, \quad de^2 = -2e^{34}, \quad de^3 = 2e^{24}, \quad de^4 = -2e^{23}, \quad de^5 = 0, \quad de^6 = 0.$$

We fix the volume form $\Omega := -e^{1\dots6}$. If we set

$$\begin{aligned}
&p_{134} = p_{234} = 1,\\
&p_{136} = p_{235} = p_{246} = -p_{145} = e^{2t},\\
&h_{12} = \frac{3}{2}\frac{e^{4t}}{\sqrt{9+3e^{6t}}},\\
&h_{34} = -\frac{1}{3}\left(-3+\sqrt{9+3e^{6t}}\right)e^{-2t},\\
&h_{35} = h_{36} = h_{46} = -h_{45} = 1,\\
&h_{56} = 2e^{2t},
\end{aligned}$$

for each $t \in (-1,1)$, and all other coefficients equal to zero. Then, by performing the change of variable

$$\tilde{t}(t) := \int_0^t a(s)\mathrm{d}s, \quad a(s) = \sqrt{\frac{3}{2}}(9+3\,e^{6t})^{-\frac{1}{4}}e^{2t},$$

one can easily check that the resulting pair (ω, ψ_+) defines a G-invariant balanced non-Kähler SU(3)-structure on the corresponding M^{princ}. With respect to the t parameter, the resulting pair (ω, ψ_+) given by (2.21) is

$$\omega = \frac{3}{2}\frac{e^{4t}}{\sqrt{9+3e^{6t}}}e^{12} - \frac{1}{3}\left(-3+\sqrt{9+3e^{6t}}\right)e^{-2t}e^{34} + e^{35} + e^{36} - e^{45} + e^{46} + e^{2t}e^{56},$$

$$\psi_+ = e^{123} + e^{234} + e^{2t}(e^{136} - e^{145} + e^{235} + e^{246}),$$

and the metric on M^{princ} is represented by the matrix

$$(g_{ij}) = \begin{pmatrix} \frac{3}{2}\frac{e^{4t}}{\sqrt{9+3e^{6t}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2}\frac{e^{4t}}{\sqrt{9+3e^{6t}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3+\sqrt{9+3e^{6t}}}{3e^{2t}} & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{3+\sqrt{9+3e^{6t}}}{3e^{2t}} & 1 & 1 \\ 0 & 0 & 1 & 1 & 2e^{2t} & 0 \\ 0 & 0 & -1 & 1 & 0 & 2e^{2t} \end{pmatrix}.$$

This concludes the proof of Theorem A.

However, using the results in [VZ20] we can check that this example cannot be extended to the singular orbits to give a smooth metric on the whole manifold. Indeed, suppose that g can be extended smoothly to the singular orbit $O_- = \pi^{-1}(-1)$. By Lemma 3.8 in [VZ20], g_{55} and g_{66} are even functions on $t+1$. But the exponential function is not even, so we get to a contradiction. The question of whether any solution to (1)–(7) could be smoothly extended to the singular orbits will be answered in the next section. Because of the existence of the previous example, the methods used for other cases would not work for this situation. The aim is then to use a different approach to tackle the problem. As the computations become too complicated as a consequence of $\mathfrak{k}$ being trivial, we will instead look into topological arguments.

2.4 Proof of Theorem B

We will finally prove the main theorem of this chapter.

Theorem B. Let M be a six-dimensional simply connected cohomogeneity one manifold under the almost effective action of a connected Lie group G, and let K be the principal isotropy group. Assume $(\mathfrak{g}, \mathfrak{k}) \neq (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \Delta\mathbb{R})$. Then M admits no G-invariant balanced non-Kähler SU(3)-structures.

By Theorem A, we only need to discuss if there exist balanced non-Kähler SU(3)-structures of cohomogeneity one arising as the compactification of the principal part determined by the pair $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus 2\mathbb{R}, \{0\})$. We leave the study of the case where $(\mathfrak{g}, \mathfrak{k}) \neq (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \Delta\mathbb{R})$ for future work.

By [Hoe10b], a six-dimensional compact simply connected cohomogeneity one manifold M whose corresponding principal part is given by the pair $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus 2\mathbb{R}, \{0\})$ at the Lie algebra level, is G-equivariantly diffeomorphic to the product of two 3-dimensional spheres, i.e.

$$M \cong S^3 \times S^3.$$

If we denote by $H_\pm$ the singular isotropy groups for the G-action on M and by $\mathfrak{h}_\pm = \mathrm{Lie}(H_\pm)$ their Lie algebras, from Section 2.2, we get that if M is simply connected, then $\pi_1(H_\pm/K) = \mathbb{Z}$ and we have that both $\mathfrak{h}_+$ and $\mathfrak{h}_-$ are isomorphic to $\mathbb{R}$ so that both the singular orbits of M are 4-dimensional compact submanifolds of M. We now recall a result due to Michelsohn, that we will refer to as Michelsohn's obstruction.

Proposition 2.4.1. [Mic82, Corollary 1.7] Suppose X is an n-dimensional complex manifold which admits a balanced metric. Then every compact complex subvariety of dimension $n-1$ in X represents a non-zero class in $H_{2n-2}(X;\mathbb{R})$.

Proof. Let V be a compact complex submanifold of dimension $n-1$ and let $i : V \hookrightarrow M$ be the inclusion. Then

$$\frac{1}{(n-1)!}\int_V i^*(\omega^{n-1}) = \mathrm{vol}(V) \neq 0$$

Suppose V represents a zero class in $H_{2n-2}(X;\mathbb{R})$. Then every closed form defined on V is exact. Let ω be a fundamental form such that $d\omega^{n-1} = 0$. Since d and the pullback of i commute,

$$0 = i^*(d\omega^{n-1}) = d(i^*\omega^{n-1}).$$

Hence, there exists a form α such that

$$i^*\omega^{n-1} = d\alpha.$$

Therefore by Stokes Theorem

$$\int_V i^*(\omega^{n-1}) = \int_V d\alpha = 0,$$

which is a contradiction. □

By Michelsohn's obstruction, and as $H_4(S^3 \times S^3; \mathbb{R}) = 0$, if M admitted any 4-dimensional compact complex submanifold S, then M would not admit a balanced metric. Therefore, we can make a few considerations by focusing on a tubular neighborhood of one singular orbit G/H at a time. In particular we divide the discussion depending on the immersion of $\mathfrak{h} \subset \mathfrak{g}$. Let S be the singular orbit given by the group diagram $G \supset H \supset K$. We notice that, if S is J-invariant, a complex structure on M would induce a complex structure on S, so we can discard all these cases by Michelsohn's obstruction. We have that $T_qM = T_qS \oplus V$ where $V = T_qS^\perp$ is the slice at $q \in S$; since S is 4-dimensional, V is always a 2-plane. We recall that the H-action on T_qS is given by the adjoint representation while the H-action on V is given via the slice representation (which is determined from the embedding $K \subset H$), and since V is two-dimensional, this action is just a rotation on V of a certain weight, say, a. Let us start by considering the case where $\mathfrak{h}$ is contained in the center of $\mathfrak{g}$, $\xi(\mathfrak{g})$. In this case the H-action on T_qS is trivial. Therefore T_qS and V are inequivalent modules of the H-action on T_qM and, since J commutes with the H-action, J preserves T_qS for each $q \in S$, i.e. S is an almost complex manifold and we may apply Michelsohn's obstruction to discard this case. Therefore we may suppose that $\mathfrak{h}$ has a non-trivial component in the $\mathfrak{su}(2)$-factor of $\mathfrak{g}$. In particular, since $\operatorname{rank}(\mathfrak{su}(2)) = 1$ and the adjoint action ignores components in the center of $\mathfrak{g}$, we will assume for our discussion that $\mathfrak{h} = \langle f_1 \rangle$. Moreover, if we denote by $\mathfrak{m}$ the tangent space to S via the usual identification, the decomposition of $\mathfrak{m}$ in irreducible H-modules is given by

$$\mathfrak{m} = l_0 \oplus l_1,$$

where H acts on l_0 trivially and on l_1 via rotation of speed d. The assumption $\mathfrak{h} = \langle f_1 \rangle$ does not change our discussion, since more generally if $h = \langle f_1 + X \rangle$, where $X \in \mathbb{R} \oplus \mathbb{R}$, $\{f_1 + X, f_2, f_3, f_4, f_5\}$ are again a basis for $\mathfrak{g}$ and $\mathfrak{m} \cong \langle f_2, f_3, f_4, f_5 \rangle$ so the H-action on $\hat{\mathfrak{m}}|_q$ is again the adjoint representation $ad_{\mathfrak{h}}$ which splits T_qS as sum of l_0 and l_1 as before.

If the integer a is different from d the modules l_0, l_1 and V are inequivalent for the H-action and again, since J commutes with the H-action, it cannot exchange two different modules. In particular $J(T_qS) \subseteq T_qS$ and we may apply Michelsohn's obstruction as before. For the remaining case $a = d$ we have that the two modules l_1 and V are equivalent, hence $J(l_1 \oplus V) \subseteq l_1 \oplus V$ but not necessarily $J(l_1) \subseteq l_1$. In particular, when this case occurs, the orbit S is not J-invariant and we do not have obstructions to the existence of balanced metrics. Therefore, from now on, we assume this is the case.

Let $\partial/\partial x$ be a vector field such that $(\xi|_q, \partial/\partial x|_q)$ is an orthonormal basis for the slice V and $T_q^*M = \langle e^1|_q, dx|_q, e^3|_q, e^4|_q, e^5|_q, e^6|_q \rangle$. Let $\varphi : \mathfrak{h} \to \mathrm{End}(T_qM)$ be the $\mathfrak{h}$-action on T_qM. Then, in order to have l_1 and V $\mathfrak{h}$-equivalent, $\varphi(f_1)^*$ acts on 1-forms given with the previous basis as

$$\varphi(f_1)^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Fix the volume form $\Omega = e^{1\dots 6}$ and consider the 3-form

$$\begin{aligned}\psi_+ \quad &:= p_1 e^{123} + p_2 e^{124} + p_3 e^{125} + p_4 e^{126} + p_5 e^{134} + p_6 e^{135} + p_7 e^{136} + p_8 e^{145} + p_9 e^{146} + p_{10} e^{156} \\ &+ p_{11} e^{234} + p_{12} e^{235} + p_{13} e^{236} + p_{14} e^{245} + p_{15} e^{246} + p_{16} e^{256} \\ &+ p_{17} e^{345} + p_{18} e^{346} + p_{19} e^{356} + p_{20} e^{456},\end{aligned}$$

where $p_j \in C^\infty((-1,1))$ for any $j = 1, \dots, 20$.

The condition $d\psi_+ = 0$ is equivalent to the following ODE system:

$$\begin{cases} p'_{11} = 0, \\ p'_{12} + 2p_8 = 0, \\ p'_{13} + 2p_9 = 0, \\ p'_{14} - 2p_6 = 0, \\ p'_{15} - 2p_7 = 0, \\ p'_{17} + 2p_3 = 0, \\ p'_{18} + 2p_4 = 0, \\ p_{16} = p_{19} = p_{20} = 0. \end{cases} \tag{2.22}$$

From now on, we will assume $p_{16} = p_{19} = p_{20} = 0$.

Let the slice be $V \cong \mathbb{R}^2$ so that the singular point $q \in \mathcal{O}_-$ is identified with $0 \in \mathbb{R}^2$, and let $r : V \to \mathbb{R}$ be the radial distance, such that for $v = (v_1, v_2) \in V$, $r(v) = |v| = \sqrt{v_1^2 + v_2^2}$. Then r is not in $C^\infty(V)$, and neither are the odd powers of r. Via the exponential map, we can identify $t + 1$ with the radial distance r.

Let α be a G-invariant 1-form on M. Then

$$\alpha(t) = \sum_{i=1}^{6} \alpha_i(t) e^i,$$

for $t \in (-1, 1)$ and some smooth functions α_i, $i = 1, ..., 6$. This expression has to extend smoothly to $t=-1$. In particular, the Taylor expansion of $\alpha_k(t)$ around $t=-1$ for $k\geq 2$ only has even powers of $t+1$:

$$\alpha_k(t)\sim\sum_{n>1}a_{k,2n}(t+1)^{2n}.$$

Now, for $2\leq i<j<k\leq 6$ fixed, the e^{ijk}-coefficients extend smoothly to $t=-1$. Hence,

$$p_{12}(t)\sim\sum_{n>1}a_{2n}(t+1)^{2n},$$

and similarly for the Taylor expansions of $p_{13}(t), p_{14}(t)$ and $p_{15}(t)$ around $t=-1$. Therefore $\lim_{t\to-1}p'_{12}(t)=\lim_{t\to-1}p'_{13}(t)=\lim_{t\to-1}p'_{14}(t)=\lim_{t\to-1}p'_{15}(t)=0$. From (2.22), we obtain that $\lim_{t\to-1}p_6(t)=\lim_{t\to-1}p_7(t)=\lim_{t\to-1}p_8(t)=\lim_{t\to-1}p_9(t)=0$.

The 3-form ψ_+ at $t=0$ has to be H-invariant, and hence can be written as

$$\begin{aligned}\rho= \;& c_3e^1\wedge dx\wedge e^5+c_4e^1\wedge dx\wedge e^6+c_6e^{135}+c_7e^{136}\\ &+c_8e^{145}+c_9e^{146}-c_8dx\wedge e^{35}-c_9dx\wedge e^{36}\\ &+c_6dx\wedge e^{45}+c_7dx\wedge e^{46}+c_{17}e^{345}+c_{18}e^{346},\end{aligned}$$

for some $c_3,c_4,c_6,c_7,c_8,c_9,c_{17},c_{18}\in\mathbb{R}$. But $c_i=\lim_{t\to-1}p_i(t)=0$ for $i=6,7,8,9$. Therefore, we can easily compute that

$$\lambda|_{t=-1}=(c_{18}c_3-c_{17}c_4)^2\geq 0.$$

This concludes case (a.1).

We note that it is possible to reach a contradiction by just studying the behaviour around one of the singular orbits. However, if we do not use the information coming from Michelsohn's obstruction, the computations get significantly more complicated. The main point is that from $d\psi_-=0$ and using the stability condition $\lambda<0$, we get $p_{10}=0$. If we assume this too, the 3-form ψ_+ at $t=-1$ can be written as

$$\begin{aligned}\rho = \;& c_1 e^1\wedge dx\wedge e^3 + c_2 e^1\wedge dx\wedge e^4 + c_3 e^1\wedge dx\wedge e^5 + c_4 e^1\wedge dx\wedge e^6\\ &+c_5 e^{134} + c_6 e^{135} + c_7 e^{136} + c_8 e^{145} + c_9 e^{146}\\ &+c_{11} dx\wedge e^{34} + c_{12} dx\wedge e^{35} + c_{13} dx\wedge e^{36} + c_{14} dx\wedge e^{45} + c_{15} dx\wedge e^{46}\\ &+c_{17} e^{345} + c_{18} e^{346},\end{aligned}$$

for some $c_i \in \mathbb{R}$, $i = 1, ..., 18, i \neq 10, 16$. Then, once again we find that $\lambda|_{t=-1} = (c_{18}c_3 - c_{17}c_4)^2 \geq 0$, which finishes the case.

We may also reaching a contradiction by studying the possible H-actions on T_qS and V and showing that the weights a and d cannot be equal.

Remark 2.4.2. We also note that in case (a.1) and when $\mathfrak{h} = \mathbb{R}$, we can remove the hypothesis of simple connectedness from the non-compact case and still get a non-existence result. Let M be a six-dimensional non-compact cohomogeneity one manifold under the almost effective action of a connected Lie group G and let K, H be the principal and singular isotropy groups, respectively, with $(\mathfrak{g}, \mathfrak{h}, \mathfrak{k}) = (\mathfrak{su}(2) \oplus 2\mathbb{R}, \mathbb{R}, \{0\})$. Then M admits no G-invariant balanced non-Kähler SU(3)-structures.

In [FLY12], balanced metrics were constructed on the connected sum of $k \geq 2$ copies of $S^3 \times S^3$. However, it is not known whether $S^3 \times S^3$ admits balanced structures. By [AI01, Remark 1], in a manifold with six real dimensions, there is no non-Kähler Hermitian metric which is simultaneously balanced and strong Kähler-with-torsion, i.e. $\partial\bar{\partial}\omega = 0$, also known as SKT. In [FV15] the authors conjectured that on non-Kähler compact complex manifolds it is never possible to find an SKT metric and also a balanced metric. In [GGP08] an example of an SKT structure on $S^3 \times S^3$ is provided. One of the key cases that needs to be tackled in Theorem B is precisely $S^3 \times S^3$. From Theorem B we get the following Corollary.

Corollary 2.4.3. There is no non-Kähler balanced SU(3)-structure on $S^3 \times S^3$ which is invariant under a cohomogeneity one action.

Chapter 3

Coclosed G_2-structures on cohomogeneity one manifolds

The aim of this chapter is to find new coclosed G_2-structures on manifolds of cohomogeneity one. A coclosed G_2-structure in a smooth seven-manifold M is a positive 3-form φ such that $d{*_\varphi}\varphi = 0$, where $*_\varphi$ is the Hodge star operator associated to the metric and volume form on M defined by φ. We are interested in manifolds with coclosed G_2-structures as they can admit solutions to the heterotic G_2-system. They have also attracted substantial interest in M-theory. In heterotic string theory they give rise to three-dimensional vacua, while in M-theory to four-dimensional $N = 1$ Minkowski vacua.

Our strategy is to work out which Lie groups G could act with cohomogeneity one on a seven-manifold M, and then look for G-invariant coclosed G_2-structures on M. We are interested in cohomogeneity one coclosed G_2-structures as this is the next step of complexity after Reidegeld studied homogeneous coclosed G_2-structures in [Rei09]. Since the situation of simple G was already considered in [CS02], we focus on the case where there is an action of $G = \mathrm{SU}(2)^2$. We will also consider the case where the symmetry group enhances to $\mathrm{SU}(2)^3$.

The main results of this chapter are Theorem C and Theorem D. We find that for the SU(2)-symmetric cohomogeneity one manifolds $M = \mathbb{R}^4 \times S^3$ and $M = S^4 \times S^3$

respectively, and three given smooth functions satisfying certain conditions around a unique singular orbit S^3 for the non-compact case, and around both singular orbits, both diffeomorphic to S^3, for the compact case, and an initial non-zero parameter, there is a unique G_2-structure extending smoothly to the singular orbit (respectively, singular orbits).

On the manifold $M = \mathbb{R}^4 \times S^3$, there are two explicit complete G_2-holonomy metrics: the Bryant–Salamon (BS) metric [BS89] and the Brandhuber et al. (BGGG) metric [BGGG01]. The second one is a member of a family of complete $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$-invariant G_2-holonomy metrics found by Bogoyavlenskaya in [Bog13], and known as the $\mathbb{B}_7$ family. More recently, Foscolo, Haskins and Nordström constructed infinitely many new 1-parameter families of simply connected complete noncompact G_2–manifolds [FHN21b], which are also $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$-invariant. In [Pod21], Podestà proved the existence of a one-parameter family of nearly parallel G_2-structures, which are mutually non-isomorphic and invariant under the cohomogeneity one action of the group $\mathrm{SU}(2)^3$.

The layout of this chapter is as follows. In Section 3.1.1, we classify all possible principal orbit types of cohomogeneity one almost effective actions that preserve the G_2-structure of a simply connected manifold. We will use the same strategy as we did for SU(3)-structures on six-dimensional cohomogeneity one manifolds in the previous chapter to look for possible solutions of the Hull–Strominger system. In sections 3.1.2 and 3.1.3 we discuss Cleyton and Swann's results on cohomogeneity one G_2-structures with simple acting group.

In Section 3.2.1 we give a description of a G_2-structure on a manifold which is invariant under the cohomogeneity one action of $\mathrm{SU}(2)^2$. In Section 3.2.2 we discuss the conditions for the metric to be extended to a singular orbit. In Sections 3.2.4 and 3.2.5 we obtain the systems of equations for the G_2-structure to be closed and coclosed. Then we consider some special cases of coclosed G_2-structures in 3.2.6.

In Section 3.3 we find a class of coclosed G_2-structures on a given seven-dimensional simply connected manifold under the cohomogeneity one action of $\mathrm{SU}(2)^2$. In Section

3.3.1 we will provide existence and uniqueness results for the existence of a coclosed G_2-structure (constructed from a half-flat SU(3)-structure) in the principal part of a seven-dimensional simply connected manifold M under the cohomogeneity one action of $\mathrm{SU}(2)^2$. We use a technical result (Theorem 3.3.1) which provides local solutions to singular initial value problems. In Sections 3.3.2 and 3.3.3 we will study the extension of the corresponding metric to singular orbit(s). Finally, in Section 3.3.4 we make a few concluding remarks.

3.1 Cohomogeneity one G_2-structures

Let M be a seven-dimensional simply connected manifold of cohomogeneity one for the almost effective action of a compact connected Lie group G. Let φ be a G_2-structure on M which is preserved by the action, and denote $\psi = *\varphi$. Let K be the principal isotropy group.

3.1.1 Principal orbit structure

We require that the action of G preserves the G_2-structure, so the principal isotropy group K acts on T_pM with $K \subset G_2 \subset \mathrm{SO}(7)$, for any $p \in M$. If we denote $\mathfrak{k} = \mathrm{Lie}(K)$, then $\mathfrak{k} \subset \mathfrak{g}_2$. Let $\xi = \partial/\partial t$, where t is the parameter normal to the orbits. The subgroup of G_2 that fixes the subspace $\langle \xi|_p \rangle$ of T_pM is SU(3). Hence, the requirement that G acts on M with cohomogeneity one preserving φ implies that the representation of the isotropy group $K = K_p$ on the tangent space of the principal orbit is a subgroup of SU(3). Therefore at the Lie algebra level $\mathfrak{k} \coloneqq \mathrm{Lie}\,(K)$ is $\{0\}$, $\mathbb{R}$, $\mathfrak{su}\,(2)$, $2\mathbb{R}$, $\mathfrak{u} = \mathfrak{su}(2) \oplus \mathbb{R}$ or $\mathfrak{su}(3)$. Note that $\mathfrak{su}(2)$ has two different embeddings in $\mathfrak{su}(3)$.

As $\dim \mathfrak{g} - \dim \mathfrak{k} = 6$, the only possible decompositions of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ into simple summands are:

(a) if $\mathfrak{k} = \{0\}$, then

(1) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$,

(2) $\mathfrak{g} = \mathfrak{su}(2) \oplus 3\mathbb{R}$,

(3) $\mathfrak{g} = 6\mathbb{R}$,

(b) if $\mathfrak{k} = \mathbb{R}$, then

(1) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$,

(2) $\mathfrak{g} = \mathfrak{su}(2) \oplus 4\mathbb{R}$,

(3) $\mathfrak{g} = 7\mathbb{R}$,

(c) if $\mathfrak{k} = \mathfrak{su}(2)$, then

(1) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$,

(2) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus 3\mathbb{R}$,

(3) $\mathfrak{g} = \mathfrak{su}(2) \oplus 6\mathbb{R}$,

(4) $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathbb{R}$.

(d) if $\mathfrak{k} = 2\mathbb{R}$, then

(1) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus 2\mathbb{R}$,

(2) $\mathfrak{g} = \mathfrak{su}(2) \oplus 5\mathbb{R}$,

(3) $\mathfrak{g} = 8\mathbb{R}$,

(4) $\mathfrak{g} = \mathfrak{su}(3)$.

(e) if $\mathfrak{k} = \mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathbb{R}$, then

(1) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$,

(2) $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus 4\mathbb{R}$,

(3) $\mathfrak{g} = \mathfrak{su}(2) \oplus 7\mathbb{R}$,

(4) $\mathfrak{g} = \mathfrak{su}(3) \oplus 2\mathbb{R}$,

(5) $\mathfrak{g} = \mathfrak{sp}(2)$,

(f) if $\mathfrak{k} = \mathfrak{su}(3)$, then

(1) $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$,

(2) $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus 3\mathbb{R}$,

(3) $\mathfrak{g} = \mathfrak{su}(3) \oplus 6\mathbb{R}$,

(4) $\mathfrak{g} = \mathfrak{g}_2$.

By the hypothesis of simple connectedness, using Proposition 2.2.1 for the compact case and Proposition 2.2.2 for the non-compact case, we can easily rule out many of these cases: (a.2), (a.3), (b.2), (b.3), (c.2), (c.3), (d.2), (d.3), (e.2), (e.3), (f.2) and (f.3). Also, in case (c.4), for M to be simply connected we would need that for one singular isotropy group H, $H/K \cong S^1$, and for this to happen that $\mathfrak{k} = \{0\}$, which is not true in (c), so we can rule it out as well. The same happens in case (b.1) when $\mathfrak{k}$ is not embedded in the $\mathbb{R}$ component of $\mathfrak{g}$, (d.1) when $\mathfrak{k}$ is not embedded in the $2\mathbb{R}$ component of $\mathfrak{g}$, (e.1) when the $\mathbb{R}$ component of $\mathfrak{k}$ is not embedded in the $\mathbb{R}$ component of $\mathfrak{g}$ and (e.4) when the $\mathbb{R}$ component of $\mathfrak{k}$ is not embedded in one of the $\mathbb{R}$ components of $\mathfrak{g}$.

If $\mathfrak{k} \subset \mathfrak{g}$ is an ideal, then the principal isotropy would act trivially on M and the action would not be almost effective. This allows us to rule out case (f.1) as well.

Then, up to finite quotients, the principal orbits are one of the following types:

(1) $S^3 \times S^3 = \mathrm{SU}(2)^2 = \dfrac{\mathrm{SU}(2)^2 \times \mathrm{U}(1)}{\mathrm{U}(1)} = \dfrac{\mathrm{SU}(2)^3}{\mathrm{SU}(2)}$;

(2) $S^5 \times S^1 = \dfrac{\mathrm{SU}(3) \times \mathrm{U}(1)}{\mathrm{SU}(2)}$;

(3) $F_{1,2} = \dfrac{\mathrm{SU}(3)}{T^2}$ ($\mathfrak{k} = 2\mathbb{R}$, $\mathfrak{g} = \mathfrak{su}(3)$);

(4) $\mathbb{C}P(3) = \dfrac{\mathrm{Sp}(2)}{\mathrm{SU}(2)\mathrm{U}(1)}$ ($\mathfrak{k} = \mathfrak{su}(2) \oplus \mathbb{R}$, $\mathfrak{g} = \mathfrak{sp}(2)$);

(5) $S^6 = \dfrac{G_2}{\mathrm{SU}(3)}$ ($\mathfrak{k} = \mathfrak{su}(3)$, $\mathfrak{g} = \mathfrak{g}_2$).

For the last three cases, and up to a finite quotient, the group G acting is SU(3), Sp(2) and G_2, respectively. They have been studied in [CS02], where Cleyton and Swann classified all solutions with holonomy G_2 and weak holonomy G_2. In this paper, they started considering G_2-structures with a cohomogeneity one action of a compact Lie group G, then wrote the connected simple groups G that can act. The simple groups in question are: SU(3), G_2 and Sp(2). Later they studied the coclosed (or cosymplectic, as they call them) structures and determined the topological types of manifolds admitting such structures. They also found new examples of compact manifolds with coclosed G_2-structures. On account of this, our focus will be on case (1), where there is an invariant action of $\mathrm{SU}(2)^2$ symmetry. We will also consider situations where there are extra symmetries, such as extra SU(2).

In the next two sections, we present some of the results found in [CS02] about the existence of coclosed G_2-structures in the last three cases.

3.1.2 Case $\mathfrak{k} = \mathfrak{su}(3)$ and $\mathfrak{g} = \mathfrak{g}_2$

For this section $\mathfrak{k} = \mathfrak{su}(3)$ and $\mathfrak{g} = \mathfrak{g}_2$. Hence, up to finite quotients, the principal orbits would be

$$\frac{G}{K} = \frac{G_2}{\mathrm{SU}(3)} = S^6.$$

From [CS02] there are precisely two singular orbit types: $\mathbb{R}P(6)$ and a point $\{*\}$. Therefore, there are two spaces with base homeomorphic to the non-compact interval $[0, \infty)$:

(i) $G/H = \mathbb{R}P(6)$. Then M is the canonical line bundle over $\mathbb{R}P(6)$.

(ii) $G/H = \{*\}$. Then $M = \mathbb{R}^7$ viewed as a 7-dimensional vector bundle over a point.

There is another non-compact case, $M = \mathbb{R} \times S^6$, but we are not interested in it as it is just a product manifold.

In the compact case (base $B = [0, 1]$) there are three possibilities:

(i) $G/H_{\pm} = \{*\}$. Then $M = S^7$.

(ii) $G/H_- = \{*\}$, $G/H_+ = \mathbb{R}P(6)$. Then $M = \mathbb{R}P(7)$.

(iii) $G/H_{\pm} = \mathbb{R}P(6)$. Then $M = \mathbb{R}P(7)\#\mathbb{R}P(7)$.

There is another compact case, $M = S^1 \times S^6$, but it is not simply connected so we are not interested in it.

From [CS02], the space of invariant 3-forms is is two dimensional, spanned by 3-forms α and β. Let g_0 be the canonical metric on $S^6 = G_2/\mathrm{SU}(3)$ with sectional curvature one. Then let ω, α and β be such that

$$\begin{aligned}
\omega^3 &= 6\mathrm{vol}_0, \\
d\omega &= 3\alpha, \\
*_0\alpha &= \beta, \\
d\beta &= -2\omega^2.
\end{aligned}$$

Let γ be a geodesic through p orthogonal to the principal orbit and parameterise it by the arc length $t \in I \subset \mathbb{R}$. On the union of principal orbits $M^{\mathrm{princ}} = I \times G_2/\mathrm{SU}(3)$ there are smooth functions $f, \theta : I \to \mathbb{R}$ such that

$$\begin{aligned}
g &= dt^2 + f^2 g_0, \\
\mathrm{vol} &= f^6 \mathrm{vol}_0 \wedge dt, \\
\varphi &= f^2 \omega \wedge dt + f^3(\cos\theta\alpha + \sin\theta\beta).
\end{aligned}$$

Then

$$\begin{aligned}
\psi &= \frac{1}{2} f^4 \omega^2 + f^3(\cos\theta\beta - \sin\theta\alpha) \wedge dt, \\
d\varphi &= (3f^2 - (f^3\cos\theta)^{\cdot})\alpha \wedge dt - (f^3\sin\theta)^{\cdot}\beta \wedge dt - 2f^3\sin\theta\omega^2
\end{aligned}$$

The coclosed equations required from the heterotic G_2 system, $d{*}\varphi = 0$, are equivalent to

$$\dot{f} = \cos\theta. \tag{3.1}$$

Here the dot denotes derivative respect to the variable t. Locally, these are described by one arbitrary function θ. If (3.1) holds, then

$$d\varphi = f^2 \sin\theta(3\sin\theta + f\dot{\theta})\alpha \wedge dt - f^2 \cos\theta(3\sin\theta + f\dot{\theta})\beta \wedge dt - 2f^3 \sin\theta\omega^2.$$

There exist examples of both compact and non-compact complete manifolds with a coclosed G_2-structure preserved by an action of G_2 of cohomogeneity one.

3.1.3 Cases $\mathfrak{k} = 2\mathbb{R}$, $\mathfrak{g} = \mathfrak{su}(3)$ and $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathbb{R}$, $\mathfrak{g} = \mathfrak{sp}(2)$

First, we assume $\mathfrak{k} = 2\mathbb{R}$ and $\mathfrak{g} = \mathfrak{su}(3)$. Hence, up to finite quotients, the principal orbits would be

$$\frac{G}{K} = \frac{\mathrm{SU}(3)}{T^2} = F_{1,2}.$$

From [CS02], any SU(3)-invariant G_2-structure on $M^{\mathrm{princ}} = I \times \mathrm{SU}(3)/T^2$ has

$$g = dt^2 + f_1^2 g_1 + f_2^2 g_2 + f_3^2 g_3, \tag{3.2}$$

$$\mathrm{vol} = f_1^2 f_2^2 f_3^2 \mathrm{vol}_0 \wedge dt, \tag{3.3}$$

where $t \in I \subset \mathbb{R}$ is the arc-length parameter of an orthogonal geodesic and $f_1, f_2, f_3 : I \to \mathbb{R}$ are non-vanishing functions. The corresponding invariant 3-form is

$$\varphi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge dt + f_1 f_2 f_3(\cos\theta\alpha + \sin\theta\beta), \tag{3.4}$$

for some function $\theta : I \to \mathbb{R}$. The G_2-structure is coclosed if

$$(f_1^2 f_2^2)^{\cdot} = (f_3^2 f_1^2)^{\cdot} = (f_2^2 f_3^2)^{\cdot} = 2f_1 f_2 f_3 \cos\theta.$$

These equations were solved in [CS02]. They found that the structure may be determined by the function f_1.

Theorem 3.1.1. *([CS02, Theorem 6.1]) Consider a coclosed G_2-structure preserved by an action of $SU(3)$ of cohomogeneity one. Then the metric is given by equation (3.2). Arrange the coefficients so that $f_3^2 \geq f_2^2 \geq f_1^2$. Then*

$$|\dot{f}_1| \leq \sqrt{\Xi(f_1, \mu, \nu)}, \tag{3.5}$$

for some constants $\nu \geq \mu \geq 0$ and

$$\Xi(f_1, \mu, \nu) = \frac{f_1^8 + 2(2\nu^2 - \mu^2)f_1^4 + \mu^4}{2f_1^4\left(f_1^4 + 2\nu^2 - \mu^2 + \sqrt{f_1^8 + 2(2\nu^2 - \mu^2)f_1^4 + \mu^4}\right)}.$$

Conversely, any smooth function f_1 satisfying the differential inequality (3.5) gives a coclosed G_2-structure with f_2 determined by

$$f_2^2 - \nu^2 f_2^{-2} = f_1^2 - \mu^2 f_1^{-2}, \tag{3.6}$$

f_3 determined by

$$f_3^2 - (\nu^2 - \mu^2)f_3^{-2} = f_1^2 + \mu^2 f_1^{-2}, \tag{3.7}$$

and θ by

$$f_3 \cos\theta = (f_1 f_2)^{\cdot}. \tag{3.8}$$

The case $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathbb{R}$ and $\mathfrak{g} = \mathfrak{sp}(2)$, where up to finite quotients we have

$$\frac{G}{K} = \frac{\mathrm{Sp}(2)}{\mathrm{U}(1) \times \mathrm{SU}(2)} = \mathbb{C}P(3),$$

can be regarded as a special case of the previous case. Sp(2)-invariant G_2-structures are given by equations (3.2) and (3.4) with $f_2 = f_3$.

There exist examples of both compact and non-compact complete manifolds with a coclosed G_2-structure preserved by an action of SU(3) of cohomogeneity one.

3.2 $\mathrm{SU}(2)^2$-invariant equations

For the rest of this chapter, let M be a seven-dimensional simply connected cohomogeneity one manifold under the action of the compact connected Lie group G such that, up to finite quotients, the principal orbits are

$$\frac{G}{K} \cong S^3 \times S^3.$$

In this case there is an invariant action of $\mathrm{SU}(2)^2$. We will also consider situations where there are extra symmetries.

3.2.1 G_2-structures on cohomogeneity one manifolds

In this section we construct an $\mathrm{SU}(2)^2$-invariant half-flat SU(3)-structure on the principal part $I_t \times N$ of a cohomogeneity one manifold M. We will describe the SU(3)-structure, the associated G_2-structure given by equation (1.21) and the corresponding metric on M in terms of six real valued functions, by using an orthonormal basis given by a Milnor frame. This follows from the work by Schulte-Hegensbach in his PhD thesis [SH10] and Madsen and Salamon in [MS13]. Here we will follow the reformulation by Lotay and Oliveira from [LO18, Section 2].

We can construct a basis of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ written as $\mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{su}^+(2) \oplus \mathfrak{su}^-(2)$. Let $\{T_i\}_{i=1}^3$ be a basis for $\mathfrak{su}(2)$ such that $[T_i, T_j] = 2\epsilon_{ijk}T_k$. Then

$$T_i^+ = (T_i, T_i), \quad T_i^- = (T_i, -T_i),$$

define a basis for $\mathfrak{su}^+(2)$ and $\mathfrak{su}^-(2)$ respectively. Let $\{\eta_i^+\}_{i=1}^3$ and $\{\eta_i^-\}_{i=1}^3$ be dual basis to $\{T_i^+\}_{i=1}^3$ and $\{T_i^-\}_{i=1}^3$ respectively. Then the structure equations are

$$\begin{aligned} \mathrm{d}\eta_i^+ &= -\epsilon_{ijk}(\eta_j^+ \wedge \eta_k^+ + \eta_j^- \wedge \eta_k^-), \\ \mathrm{d}\eta_i^- &= -2\epsilon_{ijk}\eta_j^- \wedge \eta_k^+. \end{aligned}$$

We will denote $\eta_{ij}^{\pm} = \eta_i^{\pm} \wedge \eta_j^{\pm}$, and $\eta_{123}^{\pm} = \eta_1^{\pm} \wedge \eta_2^{\pm} \wedge \eta_3^{\pm}$. A generic $\mathrm{SU}(2)^2$-invariant half-flat $\mathrm{SU}(3)$-structure on the principal bundle is given by

$$\begin{aligned} \omega &= 4\textstyle\sum_{i=1}^{3} A_i B_i \eta_i^- \wedge \eta_i^+, \\ \Omega_1 &= 8B_1B_2B_3\eta_{123}^- - 4\textstyle\sum_{i,j,k=1}^{3} \epsilon_{ijk} A_i A_j B_k \eta_i^+ \wedge \eta_j^+ \wedge \eta_k^-, \\ \Omega_2 &= -8A_1A_2A_3\eta_{123}^+ + 4\textstyle\sum_{i,j,k=1}^{3} \epsilon_{ijk} B_i B_j A_k \eta_i^- \wedge \eta_j^- \wedge \eta_k^+, \end{aligned} \tag{3.9}$$

for real-valued functions $A_i, B_i : I_t \to \mathbb{R}$, $i = 1, 2, 3$, $A_i(t), B_i(t) \neq 0$ for t in the interior of I_t. The compatible metric determined by this $\mathrm{SU}(3)$-structure on $\{t\} \times M$ is ([MS13]):

$$g_t = \sum_{i=1}^{3} \left((2A_i)^2 \eta_i^+ \otimes \eta_i^+ + (2B_i)^2 \eta_i^- \otimes \eta_i^- \right),$$

and the resulting metric on $\mathbb{R}_t \times M$, compatible with the G_2-structure $\varphi = \mathrm{d}t \wedge \omega + \Omega_1$, is given by

$$g = \mathrm{d}t^2 + g_t. \tag{3.10}$$

Hence, we can see the functions $A_i(t)$ and $B_i(t)$ as describing deformations of the standard cone metric.

Remark 3.2.1. For a G_2-structure given by a half-flat $\mathrm{SU}(3)$-structure as before, using the previous expressions and equation (1.21), we have that $\psi \lrcorner d\varphi = 0$, so

$$\tau_0 = 0.$$

This means that for all of the G_2-structures considered, the scalar curvature is zero.

Remark 3.2.2. Recall that in seven dimensional heterotic string theory, we say that the *flux* is a 3-form given by

$$H = \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3.$$

One of the equations of the heterotic G_2 system, the *heterotic Bianchi identity* or *anomaly free condition*, relates the exterior differential of the flux to the curvatures of two gauge fields. If the G_2-structure is coclosed as before, and using that $\tau_0 = 0$ from the previous remark, we get

$$
\begin{aligned}
dH = \ & \tfrac{1}{6}\prod_{i=1}^{3}|A_iB_i| \\
& [16(-4A_1B_1 + (B_1B_2B_3)^{\cdot} - (A_2A_3B_1)^{\cdot} + (A_3A_1B_2)^{\cdot} + (A_1A_2B_3)^{\cdot})\eta_{23}^{-} \wedge \eta_{23}^{+} \\
& +16(-4A_2B_2 + (B_1B_2B_3)^{\cdot} + (A_2A_3B_1)^{\cdot} - (A_3A_1B_2)^{\cdot} + (A_1A_2B_3)^{\cdot})\eta_{13}^{-} \wedge \eta_{13}^{+} \\
& +16(-4A_3B_3 + (B_1B_2B_3)^{\cdot} + (A_2A_3B_1)^{\cdot} + (A_3A_1B_2)^{\cdot} - (A_1A_2B_3)^{\cdot})\eta_{12}^{-} \wedge \eta_{12}^{+}].
\end{aligned}
$$

This expression will be relevant when looking for solutions of the heterotic G_2 system for the $\mathrm{SU}(2)^2$-invariant cohomogeneity one manifolds and the described coclosed G_2-structures. The author is still working on this.

3.2.2 Extension to the singular orbits

The union of the principal orbits M^{princ} of the manifold M is a dense subset of M. Hence, it is possible to extend the metric on the principal part to the singular orbit(s) to give a metric on the manifold M. However, there are some extra conditions that we need to impose in order to ensure that this extension is smooth. These conditions follow from a method developed by Eschenburg and Wang [EW00] to find when a metric (or more generally, a tensor) extends smoothly to a singular orbit. In [VZ20], Verdiani and Ziller gave an efficient way of checking the conditions for a smooth extension of the metric. To use this method, we need to fix our cohomogeneity one manifold. We will consider two situations, one of a compact and one of a non-compact example.

As explained in Section 1.3, a non-compact cohomogeneity one manifold is given by a homogeneous vector bundle, while a compact one by the union of two homogeneous disc bundles. We are interested in the smoothness conditions near a singular orbit, so we restrict ourselves to only one such bundle.

The non-compact manifold that we are going to consider is $M = \mathbb{R}^4 \times S^3$, seen as a cohomogeneity one manifold with group diagram $\mathrm{SU}(2) \times \mathrm{SU}(2) \supset \Delta\mathrm{SU}(2) \supset \{1\}$. In particular, we consider the embedding $\mathbb{R}^4 \times S^3 \hookrightarrow \mathbb{H} \times \mathbb{H}$ and let $\mathrm{SU}(2) \times \mathrm{SU}(2)$ act via

$$(a_1, a_2) \cdot (p, q) = (a_1 p, a_1 q \bar{a}_2).$$

One of the reasons why we are interested in this manifold is that it admits torsion-free G_2-structures [BS89, BGGG01] and G_2-instantons [LO18]. $\mathbb{R}^4 \times S^3$ is diffeomorphic to the total space of the spinor bundle $\mathcal{S} \to S^3$ over the 3-sphere, which can be described as the quotient

$$\frac{\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{R}^4}{\Delta \mathrm{SU}(2)},$$

where $\mathrm{SU}(2)$ is acting on the right diagonally.

The next Lemma tells us the conditions on the functions from the previous section A_i, B_i of $t \in \mathbb{R}^+$, $i = 1, 2, 3$, for the metric to extend to the singular orbit $Q = \mathrm{SU}(2)^2/\Delta\mathrm{SU}(2) \cong S^3$.

Lemma 3.2.3. [LO18, Lemma 8] The metric g in (3.10) extends smoothly (as a metric) over the singular orbit $Q = \mathrm{SU}(2)^2/\Delta\mathrm{SU}(2)$ if and only if A_i, B_i are non-zero for $t > 0$ and:

(i) the A_i's are odd with $\dot{A}_i(0) = 1/2$;

(ii) the B_i's are even with $B_1(0) = B_2(0) = B_3(0) \neq 0$ and $\ddot{B}_1(0) = \ddot{B}_2(0) = \ddot{B}_3(0)$.

Note that condition (i) says that the metrics have to be almost round near to the singular orbit, while condition (ii) guarantees that the singular orbit is totally geodesic.

For a compact manifold, we consider the cohomogeneity one manifold $M = S^4 \times S^3$ with group diagram $\mathrm{SU}(2) \times \mathrm{SU}(2) \supset \Delta\mathrm{SU}(2), \Delta\mathrm{SU}(2) \supset \{1\}$. We consider the embedding $S^3 \times S^4 \hookrightarrow \mathbb{H} \times (\mathbb{H} \times \mathbb{R})$, then $S^3 \times S^3$ acts on M via

$$(a_1, a_2) \cdot (p, q, t) = (a_1 p a_2^{-1}, a_2 q, t).$$

The metric g in (3.10) extends smoothly (as a metric) over the singular orbits $Q_{1,2} =$

$SU(2)^2/\Delta SU(2)$ if the conditions for A_i, B_i from the previous lemma are satisfied around both singular orbits.

Remark 3.2.4. For the two previous situations, the corresponding normal sphere(s) H/K have dimension 3. It would be interesting to explore another example where there is a normal sphere with dimension 1, such as the Bazaikin-Bogoyavlenskaya manifolds [BB13], which have group diagram $SU(2)^2 \supset U(1) \supset \mathbb{Z}/4$.

3.2.3 Torsion-free ODEs

The general ODEs describing $SU(2)^2$-invariant G_2-manifolds of cohomogeneity one are [MS13]:

$$\begin{aligned} \dot{A}_i &= \frac{1}{2}\left(\frac{A_i^2}{A_jA_k} - \frac{A_i^2}{B_jB_k} - \frac{A_j^2+A_k^2}{A_jA_k} + \frac{B_j^2+B_k^2}{B_jB_k}\right), \\ \dot{B}_i &= \frac{1}{2}\left(\frac{A_j^2+B_k^2}{A_jB_k} + \frac{A_k^2+B_j^2}{A_kB_j} - \frac{B_i^2}{A_jB_k} - \frac{B_i^2}{A_kB_j}\right), \end{aligned} \tag{3.11}$$

where $\{i,j,k\}$ denotes a cyclic permutation of $\{1,2,3\}$. In every example solution of these equations there is an extra $U(1)$-symmetry: this $U(1)$ acts diagonally on $S^3 \times S^3$ with infinitesimal generator T_1^+. As a consequence, we have $A_2 = A_3$ and $B_2 = B_3$, and (3.11) becomes

$$\begin{aligned} \dot{A}_1 &= \frac{1}{2}\left(\frac{A_1^2}{A_2^2} - \frac{A_1^2}{B_2^2}\right), \\ \dot{A}_2 &= \frac{1}{2}\left(\frac{B_1^2+B_2^2-A_2^2}{B_1B_2} - \frac{A_1}{A_2}\right), \\ \dot{B}_1 &= \frac{A_2^2+B_2^2-B_1^2}{A_2B_2}, \\ \dot{B}_2 &= \frac{1}{2}\left(\frac{A_2^2+B_1^2-B_2^2}{A_2B_1} + \frac{A_1}{A_2}\right). \end{aligned} \tag{3.12}$$

These equations appeared in Brandhuber et al. [BGGG01] for the first time.

Example 3.2.5. One of the first examples of a torsion-free G_2-structure with a complete metric is the one that gives the Bryant–Salamon metric on $\mathbb{R}^4 \times S^3$ from [BS89]. The Bryant–Salamon metric is actually $SU(2)^3$-invariant, and it can be realised as a cohomogeneity one manifold with group diagram $SU(2)^3 \supset SU(2)^2 \supset SU(2)$ (where $SU(2)$ is embedded in $SU(2)^3$ as $1 \times 1 \times SU(2)$ and $SU(2)^2$ as $\Delta_{1,2}SU(2) \times SU(2)$), as well as with an action of $\mathrm{SU}(2)^2$ in multiple inequivalent ways. The extra symmetry means that $A_1 = A_2 = A_3$ and $B_1 = B_2 = B_3$, and the torsion-free equations reduce to

$$\dot{A}_1 = \frac{1}{2}\left(1 - \frac{A_1^2}{B_1^2}\right), \qquad \dot{B}_1 = \frac{A_1}{B_1}.$$

We find a solution

$$A_1 = \frac{r}{3}\sqrt{1 - r^{-3}}, \qquad B_1 = \frac{r}{\sqrt{3}},$$

where $r \in [1, +\infty)$ is a coordinate defined implicitly by

$$t(r) = \int_1^r \frac{ds}{\sqrt{1 - s^{-3}}},$$

and t denotes the arc length parameter. When $t \to \infty$, $A_1(t) \sim t/3$ and $B_1(t) \sim t/\sqrt{3}$. The metric will then be

$$g = dt^2 + \sum_{i=1}^{3}\left(\frac{4r^2}{9}(1 - r^{-3})\eta_i^+ \otimes \eta_i^+ + \frac{4r^2}{3}\eta_i^- \otimes \eta_i^-\right).$$

This metric is asymptotically conical, and the asymptotic cone is the standard homogeneous nearly Kähler structure on $S^3 \times S^3$.

Example 3.2.6. There is another example of an explicit complete holonomy G_2 metric on $\mathbb{R}^4 \times S^3$, by Brandhuber, Gomis, Gubser and Gukov [BGGG01]. In this example the symmetry is enhanced and the G_2-structures are $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$-invariant, the extra $\mathrm{U}(1)$ meaning that $A_2 = A_3$, $B_2 = B_3$. Brandhuber et al. obtain a torsion-free G_2-structure, which is given by

$$\begin{aligned} A_1 &= \frac{\sqrt{(r - 9/4)(r + 9/4)}}{\sqrt{(r - 3/4)(r + 3/4)}}, \quad A_2 = A_3 = \sqrt{\frac{(r - 9/4)(r + 3/4)}{3}}, \\ B_1 &= \frac{2r}{3}, \quad B_2 = B_3 = \sqrt{\frac{(r - 3/4)(r + 9/4)}{3}}, \end{aligned}$$

where $r \in [9/4, +\infty)$ is a coordinate defined implicitly by

$$t(r) = \int_{9/4}^{r} \frac{\sqrt{(s-3/4)(s+3/4)}}{\sqrt{(s-9/4)(s+9/4)}} ds.$$

3.2.4 Closed equations

In this section we look for the equations that the functions A_i, B_i have to satisfy such that the G_2-structure obtained from them is closed. Recall from Section 1.1.1 that they will be derived from

$$\dot{\Omega}_1 = \mathrm{d}\omega. \tag{3.13}$$

A straightforward computation using the expressions for ω and Ω_1 from (3.9) gives us the following Proposition.

Proposition 3.2.7. Let M be a seven-dimensional simply connected cohomogeneity one manifold under the action of $\mathrm{SU}(2)^2$, with a G_2-structure coming from a half-flat SU(3)-structure. Let the SU(3)-structure $(\omega, \Omega_1, \Omega_2)$ be written as in (3.9). Then, the equations for the G_2-structure to be closed are:

$$\begin{cases} (B_1B_2B_3)^{\cdot} = +A_1B_1 + A_2B_2 + A_3B_3, \\ (A_2A_3B_1)^{\cdot} = -A_1B_1 + A_2B_2 + A_3B_3, \\ (A_3A_1B_2)^{\cdot} = +A_1B_1 - A_2B_2 + A_3B_3, \\ (A_1A_2B_3)^{\cdot} = +A_1B_1 + A_2B_2 - A_3B_3. \end{cases}$$

3.2.5 Coclosed equations

We derive equations for the functions A_i, B_i such that the G_2-structure obtained via (3.9) is coclosed. Recall from Section 1.1.1 that they will be derived from

$$\omega \wedge \dot{\omega} = -\mathrm{d}\Omega_2. \tag{3.14}$$

Using the expression for ω and Ω_2 from equation (3.9), we obtain the system of ODEs that will give us the coclosed conditions. We present it in the following Proposition.

Proposition 3.2.8. Let M be a seven-dimensional simply connected cohomogeneity one manifold under the action of $\mathrm{SU}(2)^2$, with a G_2-structure coming from a half-flat SU(3)-structure. Let the SU(3)-structure $(\omega, \Omega_1, \Omega_2)$ be written as in (3.9). Then, the equations for the G_2-structure to be coclosed are:

$$\begin{cases} A_2B_2(\dot{A}_3B_3 + A_3\dot{B}_3) + A_3B_3(\dot{A}_2B_2 + A_2\dot{B}_2) = A_1A_2A_3 - B_2B_3A_1 + B_1B_3A_2 + B_1B_2A_3, \\ A_1B_1(\dot{A}_3B_3 + A_3\dot{B}_3) + A_3B_3(\dot{A}_1B_1 + A_1\dot{B}_1) = A_1A_2A_3 + B_2B_3A_1 - B_1B_3A_2 + B_1B_2A_3, \\ A_1B_1(\dot{A}_2B_2 + A_2\dot{B}_2) + A_2B_2(\dot{A}_1B_1 + A_1\dot{B}_1) = A_1A_2A_3 + B_2B_3A_1 + B_1B_3A_2 - B_1B_2A_3. \end{cases} \tag{3.15}$$

We will assume that for t in the interior of I_t, all A_i's are sign definite. Hence, this system will be well defined in interior of I_t and we will study the behaviour for the boundary points, where the A_i's are zero. If we define

$$D_1 = A_2B_2A_3B_3,$$
$$D_2 = A_1B_1A_3B_3,$$
$$D_3 = A_1B_1A_2B_2,$$

then we can write (3.15) as a system of ODEs for the functions D_1, D_2, D_3, where the coefficients depend on free functions A_1, A_2, A_3, given that they satisfy some properties to ensure that the G_2-structure could be extended to singular orbits. Note that the initial conditions from Lemma 3.2.3 correspond to

$$D_i(t) = \frac{b_0^2}{4}t^2 + O(t^4), \quad t \in [0, L), \tag{3.16}$$

and D_i even, $i = 1, 2, 3$. The system is

$$\begin{cases} \dot{D}_1 = A_1A_2A_3 - \dfrac{A_1^2 D_1}{A_1A_2A_3} + \dfrac{A_2^2 D_2}{A_1A_2A_3} + \dfrac{A_3^2 D_3}{A_1A_2A_3}, \\ \dot{D}_2 = A_1A_2A_3 + \dfrac{A_1^2 D_1}{A_1A_2A_3} - \dfrac{A_2^2 D_2}{A_1A_2A_3} + \dfrac{A_3^2 D_3}{A_1A_2A_3}, \\ \dot{D}_3 = A_1A_2A_3 + \dfrac{A_1^2 D_1}{A_1A_2A_3} + \dfrac{A_2^2 D_2}{A_1A_2A_3} - \dfrac{A_3^2 D_3}{A_1A_2A_3}. \end{cases} \tag{3.17}$$

Writing $D = (D_1, D_2, D_3)^T$ the system of equations now becomes

$$\dot{D} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} A_1A_2A_3 + \frac{1}{A_1A_2A_3} \begin{pmatrix} -A_1^2 & A_2^2 & A_3^2 \\ A_1^2 & -A_2^2 & A_3^2 \\ A_1^2 & A_2^2 & -A_3^2 \end{pmatrix} D. \tag{3.18}$$

We will write it with matrix notation. Let

$$M = \frac{1}{A_1A_2A_3} \begin{pmatrix} -A_1^2 & A_2^2 & A_3^2 \\ A_1^2 & -A_2^2 & A_3^2 \\ A_1^2 & A_2^2 & -A_3^2 \end{pmatrix}, \quad N = A_1A_2A_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{3.19}$$

Then our system is

$$\dot{D} = M(t)D + N(t). \tag{3.20}$$

Remark 3.2.9. We would like to recover the B_i's from the D_i's and $b_0 \neq 0$ with:

$$B_i(t) = \operatorname{sign}(b_0)\sqrt{\frac{D_jD_k}{D_iA_i^2}}, t > 0, \quad B_i(0) = b_0, \tag{3.21}$$

where $\ddot{D}_i(0) = b_0^2/2$ and $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. We can only do that if $D_iD_j/D_k > 0$ for $t > 0$.

3.2.6 Examples of coclosed G_2-structures

We now study some special cases of coclosed G_2-structures.

3.2.6.1 Case $A_1 = A_2 = A_3$ and $B_1 = B_2 = B_3$.

In this section, we will see what happens when $A_1 = A_2 = A_3$ and $B_1 = B_2 = B_3$. We only have one equation:

$$(A_1 B_1)^{\cdot} = \frac{1}{2} \frac{A_1^2 + B_1^2}{B_1}.$$

If we define $D = A_1^2 B_1^2$, we get

$$\dot{D} = A_1^3 + \frac{D}{A_1}. \tag{3.22}$$

Example 3.2.10. If we take $A_1 = t/2$, our equation is

$$\dot{D} = \frac{t^3}{8} + \frac{2D}{t}.$$

This has solutions depending on one constant c:

$$D = ct^2 + \frac{t^4}{16}.$$

Hence renaming $4c$ by c:

$$B = \sqrt{\frac{t^2}{4} + c}.$$

This is an even function, and $B_1(0) \neq 0$ if and only if $c \neq 0$. For $c > 0$, since it is well defined for $t \in [0, \infty)$, we can smoothly extend the metric to the singular orbit at $t = 0$. The corresponding metric is

$$g = dt^2 + t^2 \sum_{i=1}^{3} \eta_i^+ \otimes \eta_i^+ + (t^2 + c) \sum_{i=1}^{3} \eta_i^- \otimes \eta_i^-.$$

In the limit $c \to 0$, the metric is conical.

The general solution to equation (3.22) is

$$D = ce^{\int_{1/2}^{t} \frac{1}{A_1(\xi)} d\xi} + e^{\int_{1/2}^{t} \frac{1}{A_1(\xi)} d\xi} \int_0^t A_1^3(\eta) e^{-\int_{1/2}^{\eta} \frac{1}{A_1(\xi)} d\xi} d\eta. \tag{3.23}$$

As $\lim_{t\to 0} D(t)t^2 = 4c$, and we are only interested in positive solutions, we rename c by $b_0^2/16$ so that the behaviour of D for small t agrees with (3.16). Let f_{A_1} be the function defined by

$$f_{A_1} : t \mapsto e^{\int_{1/2}^{t} \frac{1}{A_1(\xi)} d\xi},$$

we conclude that $D = b_0^2/16 f_{A_1}(t) + f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta$ and

$$B_1 = \sqrt{A_1^{-2}(t) \left(\frac{b_0^2}{16} f_{A_1}(t) + f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \right)}.$$

For these solutions, there is an extra SU(2) symmetry, and hence they are not only $\mathrm{SU}(2)^2$-invariant but actually $\mathrm{SU}(2)^3$-invariant.

If we consider $\mathbb{R}^4 \times S^3$ and assume a linear asymptotic behaviour of A_1, i.e.

$$A_1(t) \sim at,$$

$a > 0$, when $t \to \infty$, we have linear asymptotic behaviour of B_1 (as in the Bryant–Salamon metric, for which $a = 1/3$) where $a > 1/4$ (but only in this case). Specifically,

$$B_1(t) \sim \sqrt{\frac{a^2}{4a-1}} t.$$

3.2.6.2 Case $A_1 = A_2 = A_3$.

In this section, we will see what happens when $A_1 = A_2 = A_3$. Writing $D = (D_1, D_2, D_3)^T$ the system of equations (3.14) is

$$\dot{D} = \frac{1}{A_1} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} D + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} A_1^3.$$

We will solve the system for a generic smooth $A_1(t)$, with $A_1(t) \neq 0$ if $t \neq 0$. By first solving the homogeneous system and then using variation of parameters for the the general solution to the in-homogeneous system, we find that the general solution is

$$D = \begin{pmatrix} -f_{A_1}^{-2}(t) & -f_{A_1}^{-2}(t) & f_{A_1}(t) \\ 0 & -f_{A_1}^{-2}(t) & f_{A_1}(t) \\ -f_{A_1}^{-2}(t) & 0 & f_{A_1}(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \\ f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \\ f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \end{pmatrix},$$

for some real constants c_1, c_2, c_3. We observe that although $1/A_1(t)$ might not be locally integrable in a neighborhood of 0, the exponential of the integral $\pm \int_{1/2}^t A_1^{-1}(\xi)d\xi$ need not present a singularity at 0 (see examples below). Similarly for
$f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta$.

As motivated from the conditions from Lemma 3.2.3, we now suppose a Taylor expansion of $A_1(t)$ of the following form

$$A_1(t) = \frac{t}{2} + a_{1,3}t^3 + a_{1,5}t^5 + ...$$

We have then

$$\int_{1/2}^t \frac{1}{A_1(\xi)} d\xi = \int_{1/2}^t \frac{2}{\xi} d\xi + \int_{1/2}^t \frac{a_{1,3}\xi + a_{1,5}\xi^3 + ...}{\frac{1}{4} + \frac{a_{1,3}}{2}\xi^2 + \frac{a_{1,5}}{2}\xi^4 + ...} d\xi = 2 \ln t + f(t)$$

The second term is the integral of a smooth function, and we have denoted it by $f(t)$. Hence

$$f_{A_1}(t) = e^{f(t)} t^2, \quad f_{A_1}^{-2}(t) = e^{-2f(t)} t^{-4}.$$

Imposing $D_i(0) = 0$ so that conditions from Lemma 3.2.3 can be satisfied, we see that we must have $c_1 = c_2 = 0$, so $B_1 = B_2 = B_3$ and we are in special case of 3.2.6.1. Renaming

c_3 as $b_0^2/16$ as before, the general solution becomes

$$D = \begin{pmatrix} \frac{b_0^2}{16} f_{A_1}(t) + f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \\ \frac{b_0^2}{16} f_{A_1}(t) + f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \\ \frac{b_0^2}{16} f_{A_1}(t) + f_{A_1}(t) \int_0^t A_1^3(\eta) f_{A_1}^{-1}(\eta) d\eta \end{pmatrix}.$$

3.3 Class of $\mathbf{SU}(2)^2$-invariant coclosed G_2-structures

We are now ready to solve the coclosed equations (3.17). We use a technical result which provides solutions to singular initial value problems, and then prove that we can extend this solutions to our manifolds.

3.3.1 Existence and uniqueness results

The main tool that we are going to use to get existence and uniqueness results for coclosed G_2-structures is the following theorem ([Mal74, Theorem 7.1] and [FH17] for this statement).

Theorem 3.3.1. *[FH17, Theorem 4.7] Consider the singular initial value problem*

$$\dot{y} = \frac{1}{t} M_{-1}(y) + M(t, y), \quad y(0) = y_0, \tag{3.24}$$

where y takes values in $\mathbb{R}^k$, $M_{-1} : \mathbb{R}^k \to \mathbb{R}^k$ is a smooth function of y in a neighbourhood of y_0 and $M : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is smooth in t, y in a neighbourhood of $(0, y_0)$. Assume that

(i) $M_{-1}(y_0) = 0$;

(ii) $hId - d_{y_0} M_{-1}$ is invertible for all $h \in \mathbb{N}$, $h \geq 1$.

Then there exists a unique solution $y(t)$ of (3.24). Furthermore y depends continuously on y_0 satisfying (i) and (ii).

Note that this result only gives a short-time solution. However, we will be able to further extend the solution (see Remark 3.3.3).

We will write our system as in this theorem. Let $A_1, A_2, A_3 : [0, L) \to \mathbb{R}$ be smooth functions, where we can assume that L is either infinity or 1, that satisfy the corresponding conditions from Lemma 3.2.3, i.e. with $A_i(t) > 0$ for $t \in (0, L)$, and such that

(i) A_i's are odd;

(ii) $\dot{A}_i(0) = 1/2$.

Let $D_0 = (0, 0, 0)^T$ as $D_i(0)$ must be 0 if we want an extension to a singular orbit in $t = 0$. Then we can write (3.18) as

$$\dot{D} = \frac{1}{t} M_{-1}(D) + M(t, D),$$

where

$$M_{-1} = 2 \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3,$$

and

$$M(t, D) = A_1 A_2 A_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{A_1 A_2 A_3} \begin{pmatrix} -A_1^2 & A_2^2 & A_3^2 \\ A_1^2 & -A_2^2 & A_3^2 \\ A_1^2 & A_2^2 & -A_3^2 \end{pmatrix} D - \frac{1}{t} M_{-1}(D)$$

is smooth in t, D. The condition (i) from Theorem 3.3.1 holds as $M_{-1}(D_0) = 0$. However, (ii) does not hold as $d_{y_0} M_{-1}$ has one positive eigenvalue: $d_{y_0} M_{-1} - hId$ is not invertible for $h = 2$:

$$d_{y_0} M_{-1} = 2 \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

However, if we divide by t^2 we will be able to use the Theorem. The following example illustrates what we will do for a simple case.

Example 3.3.2. Consider the special case from Section 3.2.6.1. In order to reduce the singular system (3.22) to a non-singular one, we have to do an intermediate step. Suppose that $A(t) = t/2 + O(t^2)$. We can write the coclosed ODE as

$$\dot{D} = \frac{D}{A} + A^3 = \frac{1}{t}2D + \left(\frac{1}{A} - \frac{2}{t}\right) D + A^3,$$

with initial value $D_0 = 0$. Let $M_{-1} = 2 : \mathbb{R} \to \mathbb{R}$ and

$$M(t, D) = \left(\frac{1}{A} - \frac{2}{t}\right) D + A^3,$$

which is smooth in t, D. The condition (ii) from Theorem 3.3.1 does not hold for the system $\dot{M} = M_{-1}(D)/t + M(t, D)$, as $h\mathrm{Id} - d_{D_0}M_{-1} = 0$ for $h = 2$. However, if we divide the equation by t^2, we get

$$\frac{d}{dt}\left(\frac{D}{t^2}\right) = M'\left(t, \frac{D}{t^2}\right),$$

where

$$M'(t, E) = \left(\frac{1}{A} - \frac{2}{t}\right) E + \frac{A^3}{t^2}$$

is a smooth function in t, E. For every $E_0 \in \mathbb{R}$, there exists a solution $E(t) = D(t)/t^2$ of the initial value problem $\dot{E} = M'(E, t)$, $E_0 = E(0)$. We further assume $E_0 > 0$ and write $E_0 = b_0^2/4$. Hence, there is a 1-parameter family of solutions D_i to the system (3.22), with $D_i(0) = \dot{D}_i(0) = 0$ and $\ddot{D}_i(0) = b_0^2/2$.

We will do the same in the general case. Dividing (3.18) by t^2, we get

$$\frac{\dot{D}}{t^2} = \frac{1}{t}M_{-1}\left(\frac{D}{t^2}\right) + M'\left(t, \frac{D}{t^2}\right),$$

where M_{-1} is as before and

$$M'(t,y) = \frac{1}{t^2}A_1A_2A_3\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{A_1A_2A_3}\begin{pmatrix}-A_1^2 & A_2^2 & A_3^2\\ A_1^2 & -A_2^2 & A_3^2\\ A_1^2 & A_2^2 & -A_3^2\end{pmatrix}y - \frac{1}{t}2\begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix}y$$

which is smooth in (t, y) in a neighborhood of $(0, D_0)$. Then

$$\begin{aligned}\frac{d}{dt}\left(\frac{D}{t^2}\right) &= \frac{\dot{D}}{t^2} - \frac{2D}{t^3}\\ &= \frac{1}{t}(M_{-1} - 2Id)\left(\frac{D}{t^2}\right) + M'\left(t, \frac{D}{t^2}\right)\\ &= \frac{1}{t}M'_{-1}\left(\frac{D}{t^2}\right) + M'\left(t, \frac{D}{t^2}\right),\end{aligned}$$

where

$$M'_{-1} = 2\begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3.$$

Therefore $d_{y_0}M'_{-1}$ has eigenvalues $0, -6, -6$. Fix

$$E_0 = \begin{pmatrix}b_0^2/4\\ b_0^2/4\\ b_0^2/4\end{pmatrix},$$

for some $b_0 \neq 0$. Note that $M'_{-1}(E_0) = 0$. The new singular initial value problem for $E(t) = D(t)/t^2$ is:

$$\dot{E} = \frac{1}{t}M'_{-1}(E) + M'(t, E), \quad E(0) = E_0. \tag{3.25}$$

It satisfies the conditions from Theorem 3.3.1, meaning there exists a unique solution $E(t)$ in a neighbourhood of 0. Furthermore it depends continuously on E_0. Then we can recover D_1, D_2, D_3 by $D(t) = t^2E(t)$.

Remark 3.3.3. As for every $i = 1, 2, 3$ we have that A_i is smooth and positive on $(0, L)$, the function $(D,t) \mapsto M(t)D + N(t)$ (with M and N are as in (3.19)) is Lipschitz in D on any closed interval contained in $(0, L)$. Hence, using Picard–Lindelöf Theorem we know that we can extend the solution in a neighbourhood of 0 to $[0, L)$.

We have proved the next Proposition.

Proposition 3.3.4. Let $A_1, A_2, A_3 : [0, L) \to \mathbb{R}$ be smooth functions with $A_i(t) > 0$ for $t \in (0, L)$, where L is either infinity or 1, such that

(i) A_i's are odd;

(ii) $\dot{A}_i(0) = 1/2$.

Let $b_0 \neq 0$. Consider the singular initial value problem

$$\begin{aligned}
\dot{D}_1 &= A_1A_2A_3 - \frac{A_1^2 D_1}{A_1A_2A_3} + \frac{A_2^2 D_2}{A_1A_2A_3} + \frac{A_3^2 D_3}{A_1A_2A_3},\\
\dot{D}_2 &= A_1A_2A_3 + \frac{A_1^2 D_1}{A_1A_2A_3} - \frac{A_2^2 D_2}{A_1A_2A_3} + \frac{A_3^2 D_3}{A_1A_2A_3},\\
\dot{D}_3 &= A_1A_2A_3 + \frac{A_1^2 D_1}{A_1A_2A_3} + \frac{A_2^2 D_2}{A_1A_2A_3} - \frac{A_3^2 D_3}{A_1A_2A_3},
\end{aligned}$$

with

$$D_i(0) = \dot{D}_i(0) = 0, \quad \ddot{D}_i(0) = b_0^2/4.$$

Then there exists a unique solution on $[0, L)$ for this system of ODEs, that depends continuously on b_0. □

We need to check whether we can recover B_1, B_2, B_3 using equation (3.21) to obtain a coclosed G_2-structure in the principal part. We can only do that if $D_1D_2D_3 > 0$. The following Lemma guarantees that this is true.

Lemma 3.3.5. Let $D_1, D_2, D_3 : [0, L) \to \mathbb{R}$ be the unique solutions from Proposition 3.3.4, for some L which is either infinity or 1. Then $D_i(t) > 0$ for $t \in (0, L)$.

Proof. By construction $D_i(t) = \frac{b_0^2}{4}t^2 + O(t^3)$, so in a neighborhood around 0 they are all positive. Suppose the statement is not true, and let $t' > 0$ be the smallest point where one of the D_i's is 0. Then

$$\dot{D}_i(t') = A_1(t')A_2(t')A_3(t') + \frac{A_j^2(t')D_j(t')}{A_1(t')A_2(t')A_3(t')} + \frac{A_k^2(t')D_k(t')}{A_1(t')A_2(t')A_3(t')} > 0$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. However, D_i cannot be increasing at t', so this is a contradiction. □

Corollary 3.3.6. Let $A_1, A_2, A_3 : [0, L) \to \mathbb{R}$ be smooth functions with $A_i(t) > 0$ for $t \in (0, L)$, where L is either infinity or 1, such that

(i) A_i's are odd;

(ii) $\dot{A}_i(0) = 1/2$.

Let $b_0 \neq 0$. Then there exist unique functions $B_1, B_2, B_3 : [0, L) \to \mathbb{R}$ that, together with A_1, A_2, A_3, will give a solution for the system of ODEs (3.15) with $B_i(0) = b_0$, and the B_i's depend continuously on b_0.

The next step is checking the extension of this structure to a singular orbit $Q = \mathrm{SU}(2)^2/\Delta\mathrm{SU}(2)$.

3.3.2 Extension on $\mathbb{R}^4 \times S^3$

Recall the expression of the B_i's:

$$B_i(t) = \mathrm{sign}(b_0)\sqrt{\frac{D_j D_k}{D_i A_i^2}}, t \in (0, L), \quad B_i(0) = b_0,$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. Note that $B_i(t)$ is continuous at $t = 0$. We check whether they satisfy the conditions from Lemma 3.2.3 for the metric to be extended to a singular orbit $Q = \mathrm{SU}(2)^2/\Delta\mathrm{SU}(2) \cong S^3$. First, a direct consequence from Lemma 3.3.5 is that B_i's are sign definite for $t > 0$. Second, we need the functions B_i to be even.

Lemma 3.3.7. The functions B_1, B_2, B_3 from Corollary 3.3.6 are even.

Proof. Recall that we can write (3.15) in matrix form as $dD/dt = M(t)D + N(t)$, where $D = (D_1, D_2, D_3)^T$ and both M, N are matrices of smooth odd functions. Let $D(t)$ be the unique solution of this equation in a neighbourhood of 0. Then the equation $dD(-t)/d(-t) = M(-t)D(-t) + N(-t)$ also holds. As both M, N are odd, this equation is $dD(-t)/dt = M(t)D(-t) + N(t)$, so by uniqueness $D(-t) = D(t)$. Finally as D_i's are even, B_i's are even too. □

Finally, in the next Lemma we show that with the previous hypothesis, $\ddot{B}_1(0) = \ddot{B}_2(0) = \ddot{B}_3(0)$.

Lemma 3.3.8. The functions B_1, B_2, B_3 from Corollary 3.3.6 satisfy $\ddot{B}_1(0) = \ddot{B}_2(0) = \ddot{B}_3(0)$.

Proof. We denote by $a_{i,3}$ the coefficient accompanying t^3 in the Taylor expansion of $A_i(t)$, $b_{i,2}$ the coefficient accompanying t^2 in the Taylor expansion of $B_i(t)$ and $d_{i,4}$ the parameter accompanying t^4 in the Taylor expansion of $D_i(t)$:

$$\begin{aligned}
A_i(t) &= \frac{t}{2} + a_{i,3}t^3 + O(t^5),\\
B_i(t) &= b_0 + b_{i,2}t^2 + O(t^4),\\
D_i(t) &= \frac{b_0^2}{4}t^2 + d_{i,4}t^4 + O(t^6).
\end{aligned}$$

Let $\{i, j, k\}$ be a cyclic permutation of $\{1, 2, 3\}$. As $D_i(t) = A_j(t)B_j(t)A_k(t)B_k(t)$:

$$d_{i,4} = \frac{b_0^2}{2}(a_{j,3} + a_{k,3}) + \frac{b_0}{4}(b_{j,2} + b_{k,2}).$$

Then

$$\begin{aligned}
b_{1,2} = &\ \frac{2}{b_0}(-d_{1,4} + d_{2,4} + d_{3,4} - b_0^2 a_{1,3})\\
b_{2,2} = &\ \frac{2}{b_0}(+d_{1,4} - d_{2,4} + d_{3,4} - b_0^2 a_{2,3})\\
b_{3,2} = &\ \frac{2}{b_0}(+d_{1,4} + d_{2,4} - d_{3,4} - b_0^2 a_{3,3}).
\end{aligned} \tag{3.26}$$

Note that parameters $d_{i,4}$ depend on $a_{i,3}$. We can deduce this dependence from the ODEs for $D(t)$ by considering the Taylor expansion of both sides of

$$A_1A_2A_3\dot{D}_i = (A_1A_2A_3)^2 - A_i^2D_i + A_j^2D_j + A_k^2D_k.$$

Looking at the coefficients accompanying t^6 on both sides we obtain:

$$\frac{1}{2}d_{i,4} + \frac{b_0^2}{8}(a_{1,3}+a_{2,3}+a_{3,3}) = \frac{1}{64} - \frac{1}{4}d_{i,4} + \frac{1}{4}d_{j,4} + \frac{1}{4}d_{k,4} + \frac{b_0^2}{4}(-a_{i,3}+a_{j,3}+a_{k,3})$$

We get

$$3d_{i,4} - d_{j,4} - d_{k,4} = \frac{1}{16} + \frac{b_0^2}{2}(-3a_{i,3}+a_{j,3}+a_{k,3}).$$

The three equations corresponding to $\{i,j,k\}$ being each cyclic permutation of $\{1,2,3\}$ gives that for $i = 1,2,3$,

$$d_{i,4} = \frac{1}{16} - \frac{1}{2}b_0^2a_{i,3}. \tag{3.27}$$

Introducing this equation into equation (3.26), $b_{1,2} = b_{2,2} = b_{3,2}$ is automatically satisfied, so condition $\ddot{B}_1(0) = \ddot{B}_2(0) = \ddot{B}_3(0)$ is always true. In particular, if we denote this constant by b_2, then

$$b_2 = \frac{1}{8b_0} - b_0(a_{1,3}+a_{2,3}+a_{3,3}).$$

□

We can now write the following Proposition.

Proposition 3.3.9. Let $M = \mathbb{R}^4 \times S^3$ be a seven-dimensional non-compact simply connected cohomogeneity one manifold with group diagram $\mathrm{SU}(2)^2 \supset \Delta\mathrm{SU}(2) \supset \{1\}$, with a G_2-structure coming from a half-flat SU(3)-structure which is invariant under the cohomogeneity one action. Let the SU(3)-structure $(\omega, \Omega_1, \Omega_2)$ be written as in (3.9). Let $A_1, A_2, A_3 : [0,\infty) \to \mathbb{R}$ be smooth functions with $A_i(t) > 0$ for $t \in (0, L)$, where L is either infinity or 1, such that

(i) A_i's are odd;

(ii) $\dot{A}_i(0) = 1/2$.

Let $b_0 \neq 0$. Then there exist unique functions $B_1, B_2, B_3 : [0, \infty) \to \mathbb{R}$ that, together with A_1, A_2, A_3, will give a solution to (3.15) with $B_i(0) = b_0$, and the B_i's depend continuously on $b_0 \neq 0$. The metric g given by (3.10) extends smoothly over the singular orbit.

Proof. By Corollary 3.3.6, there exists unique functions $B_1, B_2, B_3 : [0, \infty) \to \mathbb{R}$ that, together with A_1, A_2, A_3, will give a solution to (3.15) with $B_i(0) = b_0$, and the B_i's depend continuously on $b_0 \neq 0$. Note that the B_i's are smooth functions in $[0, \infty)$, and $B_1(0) = B_2(0) = B_3(0) \neq 0$. By Lemma 3.3.7, the functions B_i are even, and by Lemma 3.3.8, $\ddot{B}_1(0) = \ddot{B}_2(0) = \ddot{B}_3(0)$. They are also sign definite. Hence, by Lemma 3.2.3, as the conditions for the functions A_i are satisfied by construction, the metric g extends smoothly over the singular orbit. □

The next Theorem summarizes the results from the previous Proposition.

Theorem C. On the cohomogeneity one manifold $M = \mathbb{R}^4 \times S^3$ with group diagram $\mathrm{SU}(2)^2 \supset \Delta\mathrm{SU}(2) \supset \{1\}$, there is a family of $\mathrm{SU}(2)^2$-invariant coclosed G_2-structures which is given by three positive smooth functions $A_1, A_2, A_3 : [0, \infty) \to \mathbb{R}$ satisfying the boundary conditions at $t = 0$

$$A_i(t) = \frac{t}{2} + O(t^3),$$

and a non-zero parameter. Moreover, any $\mathrm{SU}(2)^2$-invariant coclosed G_2-structure constructed from a half flat $\mathrm{SU}(3)$-structure is in this family.

Remark 3.3.10. The volume of the singular orbit at $t = 0$ is proportional to b_0^3.

3.3.3 Extension on $S^4 \times S^3$

In this section we consider the seven-dimensional compact simply connected cohomogeneity one manifold $M = S^4 \times S^3$ with group diagram $\mathrm{SU}(2)^2 \supset \Delta\mathrm{SU}(2), \Delta\mathrm{SU}(2) \supset \{1\}$.

From Section 3.3.1, we know that on the principal part of that manifold there is a family of coclosed G_2-structures constructed from a half-flat SU(3)-structure. In the last section, we saw that it is possible to extend the structure to one singular orbit $\mathrm{SU}(2)^2/\Delta\mathrm{SU}(2)$.

The next Theorem shows that the previous structure cannot be smoothly extended to two singular orbits of type $\mathrm{SU}(2)^2/\Delta\mathrm{SU}(2)$.

Theorem D. On the cohomogeneity one manifold $M = S^4 \times S^3$ with group diagram $\mathrm{SU}(2)^2 \supset \Delta\mathrm{SU}(2), \Delta\mathrm{SU}(2) \supset \{1\}$, there are no $\mathrm{SU}(2)^2$-invariant coclosed G_2-structures constructed from half-flat SU(3)-structures.

Proof. Suppose that M has a G_2-structure constructed from a half-flat SU(3)-structure on its principal part, which is invariant under the cohomogeneity one action. Then the SU(3)-structure $(\omega, \Omega_1, \Omega_2)$ can be written as in (3.9), and there are $A_1, A_2, A_3 : [0,1] \to \mathbb{R}$ smooth functions with $A_i(t) > 0$ for $t \in (0,1)$ such that

(i) A_i's are odd around $t = 0$;

(ii) $\dot{A}_i(0) = 1/2$.

Let $b_0 \neq 0$. By Proposition 3.3.4 we know that there exists a unique solution of the system (3.17), $D_i : [0,1) \to \mathbb{R}$ with initial conditions $D_i(0) = \dot{D}_i(0) = 0, \ddot{D}_i(0) = b_0^2/4$, for $i = 1,2,3$. Then, there exist unique functions $B_1, B_2, B_3 : [0,1) \to \mathbb{R}$ that, together with A_1, A_2, A_3, will solve (3.15). Also, by Proposition 3.3.9 the metric (3.10) can be extended to the singular orbit at $t = 0$. Hence, this solution can be extended to the interval $[0,1)$. It remains to check whether we can extend the solutions to $t = 1$. In equation (3.17), we add

$$\frac{d}{dt}(D_1 + D_2 + D_3) = 3A_1A_2A_3 + \frac{A_1^2 D_1 + A_2^2 D_2 + A_3^2 D_3}{A_1A_2A_3},$$

In Lemma 3.3.5 we proved that for $t \in (0,1)$, $D_i(t) > 0$, $i = 1,2,3$ so $d(D_1+D_2+D_3)/dt > 0$. In particular, as $D_1 + D_2 + D_3$ vanishes at $t = 0$, it cannot be 0 at $t = 1$. Therefore, there cannot be an extension to the singular orbit at $t = 1$. This is because the B_i functions obtained from the D_i's blow up at this orbit. □

3.3.4 Conclusions

The first thing that we observe is that on $M = \mathbb{R}^4 \times S^3$ there are many more G_2-structures constructed from half-flat SU(3)-structures that are coclosed than that are torsion-free (in general, one can perturb a coclosed G_2-structure by adding a small exact 4-form to construct another coclosed G_2-structure). In particular, Lotay and Oliveira give a two-parameter family of torsion-free G_2-structures constructed from half-flat SU(3)-structures [LO18, Remark 21]. The coclosed family described in this chapter depends on three smooth functions satisfying certain boundary conditions and a non-zero parameter. More generally, known examples of torsion-free G_2-structures constructed from half-flat SU(3)-structures have an extra U(1)-symmetry. With our previous notation, this means that $A_2 = A_3$ and $B_2 = B_3$. We showed that if we relax the torsion-free condition to coclosed, the A_i's only need to be equal at orders lower or equal than 1 in the Taylor expansion of the cohomogeneity one parameter around the singular orbit, and the B_i's at orders lower or equal than 2. Our families of structures contain the Bryant-Salamon G_2-holonomy metric [BS89] and the 1-parameter family of complete $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$-invariant G_2-metrics of Brandhuber et al. [BGGG01] and Bogoyavlenskaya [Bog13], also known as the $\mathbb{B}_7$ family. In particular, we deduce from [FHN21b, Theorem 6.16] that if we have an extra U(1)-symmetry and impose that the metric is G_2 and complete, we get precisely this family.

In the previous chapter of this thesis, we showed that if M is a six-dimensional simply connected cohomogeneity one manifold under the almost effective action of a connected Lie group G and $(\mathfrak{g}, \mathfrak{k}) \neq (\mathfrak{su}(2) \oplus \mathfrak{su}(2), \Delta\mathbb{R})$, then M admits no G-invariant balanced non-Kähler SU(3)-structures. As in this chapter, the search for balanced SU(3)-structures was also motivated by heterotic string theory; in particular, the Hull–Strominger system. This means that on most cases (and possibly always), the existence of a balanced structure (which can be seen as the six-dimensional analogue to a coclosed G_2-structure) in the cohomogeneity one setting forces the structure to be Kähler. We observe that the existence of a class of coclosed G_2-structures, not necessarily torsion-free, in the cohomogeneity one setting contrasts with this result, as in the 7 dimensional analogue, the cohomogeneity one hypothesis does not force coclosed G_2-structures to be torsion-free.

Given the structures found in this chapter, the question that arises is what are the G_2-instantons over $\mathbb{R}^4 \times S^3$ with these structures. Another question is whether it is possible to find solutions to the heterotic G_2 system over them. The next chapter of this thesis deals with the first of these questions.

Chapter 4

$\mathrm{SU}(2)^2$-invariant G_2-instantons

In this chapter, we study the existence of $\mathrm{SU}(2)^2$-invariant G_2-instantons on $\mathbb{R}^4 \times S^3$ with the coclosed G_2-structures found in Chapter 3. G_2-instantons are special kind of connections on Riemannian seven-manifolds. They play a key role in high dimensional gauge theory: they are analogues of anti-self-dual connections over four-manifolds and flat connections over three-manifolds, in the sense that they are critical points of a Chern–Simons type functional.

On noncompact complete holonomy G_2-manifolds, the first examples of G_2-instantons where found on the spinor bundle of S^3 with the Bryant–Salamon metric [BS89] by Clarke in [Cla14]; these examples were later generalized in [Oli14,LO18]. Very recently, Stein and Turner [ST23] completed the study of $\mathrm{SU}(2)^3$-invariant G_2-instantons over the spinor bundle of S^3 [LO18] with the Bryant–Salamon metric by constructing a new 1-parameter family of examples. In [MNT22], Matthies, Nordström and Turner construct a 1-parameter family of G_2-instantons on the asymptotically conical limit of the $\mathbb{C}^7$ family of G_2-metrics of [FHN21b]. All of these constructions used the fact that the manifolds had a cohomogeneity one structure. Hence, it is natural to continue exploiting cohomogeneity one symmetries to construct more examples of G_2-instantons.

In the previous chapter, a large family of $\mathrm{SU}(2)^2$-invariant coclosed G_2-structures was constructed over the manifold $M = \mathbb{R}^4 \times S^3$. Hence, it is a natural question to ask what are the G_2-instantons for the G_2-structures of these families. In this paper, we will construct and classify G_2-instantons for the G_2-structures of these families. This expands the study of $\mathrm{SU}(2)^2$-invariant G_2-instantons initiated in [LO18] by considering coclosed but not necessarily torsion-free G_2-structures. Moreover, $\mathrm{SU}(2)^2$-invariant G_2-structures and G_2-instantons considered previously had an extra SU(2) or U(1) symmetry.

On $M = \mathbb{R}^4 \times S^3$, for every $\mathrm{SU}(2)^3$-invariant coclosed G_2-structure from Section 3.3.2, we prove an existence result of two 1-parameter families of G_2-instantons with larger symmetry group ($\mathrm{SU}(2)^3$-invariant), extending smoothly to the singular orbit. These two families appear on two different principal bundles. We also provide existence results for locally defined $\mathrm{SU}(2)^2$-invariant G_2-instantons.

In Section 4.1 we present a summary of the setup of the problem of finding G_2-instantons on a vector bundle over a cohomogeneity one manifold. We state the general equations for an $\mathrm{SU}(2)^2$-invariant connection to be a G_2-instanton, the conditions for this connection to extend smoothly to a singular orbit, and consider the particular case where the structure group of the instanton is abelian.

In Section 4.2 we study $\mathrm{SU}(2)^3$-invariant G_2-instantons on $\mathbb{R}^4 \times S^3$. First we present the ODEs for $\mathrm{SU}(2)^3$-invariant G_2-instantons in 4.2.1. In sections 4.2.2 we review known results over the Bryant–Salamon manifold $\mathbb{R}^4 \times S^3$, when the G_2-structures are torsion-free, and explain the bubbling behaviour, removable singularity phenomenon and conservation of energy of some particular examples, the G_2-instantons of Clarke [Cla14]. In 4.2.3 we present our main results, on the existence of $\mathrm{SU}(2)^3$-invariant G_2-instantons with coclosed G_2-structures, dividing our discussion depending on the two available choices of principal SU(2)-bundle. In 4.2.4 we analyze the behaviour of sequences of instantons found, which present a "bubbling" behaviour, and the relation between all G_2-instantons encountered.

In Section 4.3 we study the most general situation of $\mathrm{SU}(2)^2$-invariant G_2-instantons. We derive a system of six ordinary differential equations for a connection to be a G_2-instanton and the conditions for it to extend smoothly to a singular orbit P_1 or P_{id} in 4.3.1. In 4.3.2 we give an existence result of a 3-parameter family of G_2-instantons in a neighbourhood of the singular orbit P_1. In 4.3.3 we study the existence of G_2-instantons with singular orbit P_{id}, and find another 3-parameter family of G_2-instantons.

4.1 G_2-instantons on cohomogeneity one manifolds

A G_2-instanton on a principal bundle over a cohomogeneity one manifold will be an extension of a connection on a principal bundle that is a homogeneous bundle on the principal orbit G/K.

4.1.1 Homogeneous bundles and isotropy homomorphism

Definition 4.1.1. We say that a principal H-bundle P on a homogeneous manifold G/K is *G-homogeneous* if the action of G on G/K lifts to a G-action on P which commutes with the action of H.

These bundles are determined by their isotropy homomorphism, which we will now define. Let u_0 be an arbitrary point of P. For any $k \in K$, we have that ku_0 is a point in P, which lies in the same fibre as u_0. Hence, there exists some $h \in H$ such that $ku_0 = u_0 h$. We define the *isotropy homomorphism* as

$$\begin{aligned} \lambda : K &\to H; \\ k &\mapsto h. \end{aligned} \tag{4.1}$$

Then λ is indeed a homomorphism. Conversely, given a homomorphism $\lambda : K \to H$, the associated bundle

$$P_\lambda = G \times_{(K,\lambda)} H \tag{4.2}$$

is a homogeneous bundle over G/K whose isotropy homomorphism is λ.

The Lie algebra $\mathfrak{g}$ of G has a reductive splitting with respect to K, which we write as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. We call the *canonical invariant connection* the connection on the bundle $G \to G/K$ whose horizontal space is $\mathfrak{m}$. Its connection form $A_\lambda^{\mathrm{can}} \in \Omega^1(G, \mathfrak{h})$ is the left-invariant translation of $d\lambda \oplus 0 : \mathfrak{k} \oplus \mathfrak{m} \to \mathfrak{h}$. This invariant connection induces a corresponding canonical connection on any P_λ. Wang's theorem classifies other invariant connections, which are in correspondence with morphisms of K-representations:

Theorem 4.1.2. *[Wan58, Theorem 1] There is a 1-1 correspondence between G-invariant connections on P_λ and K-morphisms:*

$$\Lambda : (\mathfrak{m}, Ad) \to (\mathfrak{h}, Ad \circ \lambda). \tag{4.3}$$

We see that any invariant connection will differ from the canonical invariant connection by a morphism Λ and the horizontal space of such a connection is given by the kernel of this morphism. This allows us to parameterise all invariant connections on a bundle P_λ.

4.1.2 G_2-instanton equations

Let $M = I_t \times N$ be a seven-dimensional manifold with a G_2-structure coming from a half-flat SU(3)-structure. Let P be a principal H-bundle on M, then P is a pullback of a bundle on N. We will work in temporal gauge, so we can assume that a connection on P is of the form $A = a(t)$, where $a(t)$ is a one-parameter family of connections on the bundle over N. The curvature of A is given by

$$F_A = dt \wedge \dot{a} + F_a(t), \tag{4.4}$$

where $F_a(t)$ is the curvature of the connection $a(t)$. Hence, A is a G_2-instanton if and only if the following equation for $a(t)$ is satisfied:

$$\dot{a} \wedge \frac{1}{2}\omega^2 - F_a \wedge \Omega_2 = 0, \quad F_a \wedge \frac{1}{2}\omega^2 = 0. \tag{4.5}$$

Lemma 4.1.3. [LO18, Lemma 1] Let $M = I_t \times N$ be equipped with a G_2-structure φ as in (1.21) satisfying $\omega \wedge d\omega = 0$ and $\omega \wedge \dot{\omega} = -d\Omega_2$, which is equivalent to $d\psi = 0$. Then,

G_2-instantons A for φ are in one-to-one correspondence with one-parameter families of connections $\{a(t)\}, t \in I_t$ solving the equation

$$J_t\dot{a} = -*_t(F_a \wedge \Omega_2), \tag{4.6}$$

subject to the constraint $\Lambda_t F_a = 0$, where Λ_t denotes the metric dual of the operation of wedging with $\omega(t)$. Moreover, this constraint is compatible with the evolution: more precisely, if it holds for some $t_0 \in I_t$, then it holds for all $t \in I_t$.

The most general $\mathrm{SU}(2)^2$-invariant connection on any $\mathrm{SU}(2)^2$-homogeneous H-bundle $P_\lambda = \mathrm{SU}(2)^2 \times_{(K,\lambda)} H$ over $N = \mathrm{SU}(2)^2/K$, where $\lambda : K \to H$ is a group homomorphism, can be written as

$$a = \sum_{i=1}^{3} a_i^+ \otimes \eta_i^+ + a_i^- \otimes \eta_i^-, \tag{4.7}$$

where $a_i^\pm \in \mathfrak{h}$ are constant on each principal orbit.

Lemma 4.1.4. [LO18, Lemma 2] In the previous situation, the curvature of the connection $a(t)$ on $\{t\} \times N$ is given by

$$\begin{aligned} F_a = \; & \textstyle\sum_{i=1}^3 [a_i^+, a_i^-] \otimes \eta_i^+ \wedge \eta_i^- \\ & + \textstyle\sum_{i=1}^3 ((-2a_i^+ + [a_j^+, a_k^+]) \otimes \eta_j^+ \wedge \eta_k^+ + (-2a_i^+ + [a_j^-, a_k^-]) \otimes \eta_j^- \wedge \eta_k^-) \\ & + \textstyle\sum_{i=1}^3 ((-2a_i^- + [a_j^-, a_k^+]) \otimes \eta_j^- \wedge \eta_k^+ + (-2a_i^- + [a_j^+, a_k^-]) \otimes \eta_j^+ \wedge \eta_k^-), \end{aligned} \tag{4.8}$$

where in the summation above $\{j, k\}$ is such that $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$.

Lemma 4.1.5. [LO18, Lemma 3] Let $\{i, j, k\}$ be a cyclic permutation of $\{1, 2, 3\}$. The equations (4.5) for $\mathrm{SU}(2)^2$-invariant instantons a on $\mathbb{R}_t^+ \times N$ are

$$\begin{aligned} \frac{B_i}{A_i}\dot{a}_i^+ + \left(\frac{B_i}{B_jB_k} - \frac{B_i}{A_jA_k}\right) a_i^+ &= \frac{B_i}{2B_jB_k}[a_j^-, a_k^-] - \frac{B_i}{2A_jA_k}[a_j^+, a_k^+], \\ \frac{A_i}{B_i}\dot{a}_i^- + \left(\frac{A_i}{B_jA_k} + \frac{A_i}{A_jB_k}\right) a_i^- &= \frac{A_i}{2B_jA_k}[a_j^-, a_k^+] + \frac{A_i}{2A_jB_k}[a_j^+, a_k^-], \end{aligned} \tag{4.9}$$

together with the constraint

$$\sum_{i=1}^{3} \frac{1}{A_i B_i}[a_i^+, a_i^-] = 0. \tag{4.10}$$

4.1.3 Extension to a singular orbit

In this section, we present the conditions for the extension of a $\mathfrak{g}$-valued 1-form to a singular orbit $Q = \mathrm{SU}(2)^2/\Delta\mathrm{SU}(2) \cong S^3$ in $\mathbb{R}^4 \times S^3$. This conditions are obtained using the Eschenburg–Wang method [EW00], which for this particular situation was worked out in [LO18, Appendix A].

We first consider the case where $H = \mathrm{U}(1)$.

Lemma 4.1.6. [LO18, Lemma 9] The 1-form b

$$b = \sum_{i=1}^{3} b_i^+ \otimes \eta_i^+ + \sum_{i=1}^{3} b_i^- \otimes \eta_i^-,$$

extends over the singular orbit $Q = \mathrm{SU}(2)^2/\Delta\mathrm{SU}(2)$ if and only if the $b_i^{\pm}$'s are even and $b_i^{\pm}(0) = 0$ for $i = 1, 2, 3$.

For most of our analysis, we will take $H = \mathrm{SU}(2)$. The conditions for the extension to the singular orbit will now depend on the choice of bundle. Over the principal orbits $G/K_0 \cong \mathrm{SU}(2)^2/\{0\}$, the only $\mathrm{SU}(2)$-bundle is the trivial one $P = \mathrm{SU}(2)^2 \times \mathrm{SU}(2)$. The singular orbit in the manifold considered is $\mathrm{SU}(2)^2/\Delta\mathrm{SU}(2)$. Up to an isomorphism of homogeneous bundles, as there are only two possible homomorphisms $\lambda : \mathrm{SU}(2) \to \mathrm{SU}(2)$, the trivial one and the identity. Hence, for each singular orbit there are two choices of bundle:

$$P_1 = \mathrm{SU}(2)^2 \times_{(\Delta\mathrm{SU}(2),1)} \mathrm{SU}(2), \quad P_{\mathrm{id}} = \mathrm{SU}(2)^2 \times_{(\Delta\mathrm{SU}(2),\mathrm{id})} \mathrm{SU}(2).$$

Therefore, we have two possible bundles over $\mathbb{R}^4 \times S^3$, that we will also denote P_1 and P_{id} when it does not lead to confusion. Although these two bundles are trivial, they have inequivalent group actions and only P_1 is equivariantly trivial. Recall that, as it was described in Section 3.2.2, $\mathbb{R}^4 \times S^3$ is diffeomorphic to the total space of the spinor bundle $\mathcal{S} \to S^3$ over the 3-sphere. Hence, the bundles P_1 and P_{id} are pull-backs of the bundles from S^3.

Lemma 4.1.7. [LO18, Lemma 10] Let b be an $\mathfrak{su}(2)$-valued 1-form

$$b = \sum_{i=1}^{3} b_i^+ \otimes \eta_i^+ + \sum_{i=1}^{3} b_i^- \otimes \eta_i^-.$$

Write $b_i^\pm = \sum_{j=1}^{3} b_{ij}^\pm T_j$, where $\{T_i\}_{i=1}^3$ is the standard basis for $\mathfrak{su}(2)$. Then the 1-form b extends over the singular orbit $Q = \mathrm{SU}(2)^2/\Delta\mathrm{SU}(2)$ if:

(i) On the bundle P_{id}: for $i = 1, 2, 3$, $b_{ii}^\pm$'s are even and there are $c_0^-, c_2^\pm \in \mathbb{R}$ such that

$$b_{ii}^+ = c_2^+ t^2 + O(t^4), \quad b_{ii}^- = c_0^- + c_2^- t^2 + O(t^4)$$

and for $i \neq j$, $b_{ij}^\pm = O(t^4)$ are even.

(ii) On the bundle P_1: $b_{ij}^\pm$'s are even with $b_{ij}^\pm(0) = 0$.

4.1.4 Abelian instantons

Suppose the Lie algebra structure of the gauge group is trivial. Then equations (4.9) reduce to

$$\begin{aligned} \dot{a}_i^+ + \left(\frac{A_i}{B_j B_k} - \frac{A_i}{A_j A_k} \right) a_i^+ &= 0 \\ \dot{a}_i^- + \left(\frac{B_i}{B_j A_k} - \frac{B_i}{A_j B_k} \right) a_i^- &= 0, \end{aligned} \tag{4.11}$$

and we have the following Proposition, which is similar to [LO18, Proposition 4] but for $\mathbb{R}^4 \times S^3$ with a coclosed G_2-structure constructed from a half-flat SU(3)-structure.

Proposition 4.1.8. Let θ be an $\mathrm{SU}(2)^2$-invariant G_2-instanton on a U(1)-bundle, or equivalently a complex line bundle, on $\mathbb{R}^4 \times S^3$ with a $\mathrm{SU}(2)^2$-invariant coclosed G_2-structure as in Proposition 3.3.9. Then θ lies in a 3-parameter family; in particular it can be written as

$$\theta = \sum_{i=1}^{3} a_i^+(t_0) t_0^{-2} \exp\left(- \int_{t_0}^{t} \left(\frac{A_i}{B_j B_k} - \frac{A_i}{A_j A_k} \right) ds \right) \eta_i^+, \tag{4.12}$$

for some fixed $t_0 \in \mathbb{R}^+$ and $a_i^+(t_0) \in \mathbb{R}$ for $i = 1, 2, 3$ where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$.

Proof. We observe that the only principal U(1)-bundle is the trivial one, as the only possible isotropy isomorphism $\lambda : \Delta\mathrm{SU}(2) \to \mathrm{U}(1)$ is trivial. We compute the coefficients $a_i^\pm$ from (4.7) integrating in (4.11), and knowing the Taylor expansions around $t = 0$ of the term inside the parenthesis, we have

$$a_i^+(t) = a_i^+(t_0) \exp\left(- \int_{t_0}^{t} \left(\frac{A_i}{B_j B_k} - \frac{A_i}{A_j A_k} \right) ds \right) = a_i^+(t_0) t_0^{-2} t^2 + O(t^4),$$

$$a_i^-(t) = a_i^-(t_0) \exp\left(- \int_{t_0}^{t} \left(\frac{B_i}{B_j A_k} + \frac{B_i}{A_j B_k} \right) ds \right) = a_i^+(t_0) t_0^4 t^{-4} + O(t^{-2}),$$

and both of them are even. By Lemma 4.1.6, the corresponding instantons do extend smoothly to the singular orbit at $t = 0$ if and only if $a_i^-(t_0) = 0$ for $i = 1, 2, 3$. Then, $a_i^+(t_0) \in \mathbb{R}$, $i = 1, 2, 3$ give the 3-parameter family of G_2-instantons. This finishes the proof. □

If we specialise Proposition 4.1.8 to the Bryant Salamon metric, we get the following Corollary for G_2-instantons with gauge group U(1).

Corollary 4.1.9. [LO18, Corollary 1 (a)] Any $\mathrm{SU}(2)^2$-invariant G_2-instanton A with gauge group U(1) over the Bryant Salamon G_2-manifold $\mathbb{R}^4 \times S^3$ can be written as

$$\theta = \frac{r^3 - 1}{r} \sum_{i=1}^{3} x_i \eta_i^+,$$

for some $x_1, x_2, x_3 \in \mathbb{R}$, where $r \in [1, +\infty)$ is a coordinate defined implicitly by $t(r) = \int_1^r \frac{ds}{\sqrt{1 - s^{-3}}}$.

4.2 $\mathrm{SU}(2)^3$-invariant instantons

We start our discussion by considering the case where the G_2-structure enjoys an extra $\mathrm{SU}(2)$-symmetry, i.e. $A_1 = A_2 = A_3$ and $B_1 = B_2 = B_3$.

4.2.1 $\mathrm{SU}(2)^3$-invariant ODEs

As we already discussed abelian G_2-instantons in Section 4.1.4, we now consider a non-abelian gauge group: SU(2). The next Proposition simplifies the ODEs and constraints in Lemma 4.1.5 to this case.

Proposition 4.2.1. Let θ be an $\mathrm{SU}(2)^3$-invariant G_2-instanton with gauge group SU(2) on $\mathbb{R}^4 \times S^3$. There is a standard basis $\{T_i\}$ of $\mathfrak{su}(2)$ such that (up to an equivariant gauge transformation) we can write

$$\theta = A_1 x \left(\sum_{i=1}^{3} T_i \otimes \eta_i^+ \right) + B_1 y \left(\sum_{i=1}^{3} T_i \otimes \eta_i^- \right), \tag{4.13}$$

with $x, y : (0, \infty) \to \mathbb{R}$ satisfying

$$\dot{x} = \left(-\frac{\dot{A}_1}{A_1} + \frac{1}{A_1} - \frac{A_1}{B_1^2} \right) x + y^2 - x^2, \tag{4.14}$$

$$\dot{y} = \left(-\frac{\dot{B}_1}{B_1} - \frac{2}{A_1} + 2x \right) y. \tag{4.15}$$

Proof. By the same argument as in [LO18, Proposition 5], we may always write θ as in (4.13). Then, the constraints from Lemma 4.1.5 hold, and the ODEs may be written in two different ways. First, we observe that if we write

$$\theta = x^+ \left(\sum_{i=1}^{3} T_i \otimes \eta_i^+ \right) + x^- \left(\sum_{i=1}^{3} T_i \otimes \eta_i^- \right), \tag{4.16}$$

then we obtain the following ODEs:

$$\dot{x}^+ = \frac{x^+}{A_1}\left(1 - \frac{A_1^2}{B_1^2} - x^+\right) + \frac{A_1}{B_1^2}(x^-)^2, \tag{4.17}$$

$$\dot{x}^- = \frac{2x^-}{A_1}(x^+ - 1). \tag{4.18}$$

This way of writing the equations will be useful in the future. Now from the relation

$$x^+ = xA_1, \quad x^- = yB_1,$$

the system (4.35), (4.36) becomes (4.14), (4.15). □

The next Lemma, whose prove uses Lemma 4.1.7, tells us when the G_2-instantons extend smoothly over the singular orbit $S^3 = \mathrm{SU}(2)^2/\Delta\mathrm{SU}(2)$. The conditions for the smooth extension will depend on the choice of bundle.

Lemma 4.2.2. [LO18, Lemma 4] The connection θ in equation (4.13) extends smoothly over the singular orbit S^3 if $x(t)$ is odd, $y(t)$ is even, and their Taylor expansions around $t = 0$ are

- either $x(t) = x_1 t + x_3 t^3 + ...$, $y(t) = y_2 t^2 + ...$, in which case θ extends smoothly as a connection on P_1;
- or $x(t) = \frac{2}{t} + x_1 t + ...$, $y(t) = y_0 + y_2 t^2 + ...$, in which case θ extends smoothly as a connection on P_{id}.

4.2.2 Known examples on $\mathbb{R}^4 \times S^3$

In [LO18], Lotay and Oliveira studied and classified the $\mathrm{SU}(2)^3$-invariant G_2-instantons over the manifold $\mathbb{R}^4 \times S^3$ with the Bryant–Salamon G_2-metric [BS89]. In this section we present a summary of the results that they found. Some of these results were previously found by Clarke in [Cla14].

The next theorem classifies and explicitly describes the G_2-instantons with gauge group SU(2) smoothly extending to a singular orbit on P_1. They are precisely the 1-parameter family of G_2-instantons found in [Cla14] on the Bryant–Salamon $\mathbb{R}^4 \times S^3$.

Theorem 4.2.3. *[LO18, Theorem 4] Let A be an $SU(2)^3$-invariant G_2-instanton with gauge group $SU(2)$ on the Bryant–Salamon G_2-manifold $\mathbb{R}^4 \times S^3$, which smoothly extends over the singular orbit P_1. Then, A is one of Clarke's examples [Cla14], in which case there is $x_1 \in \mathbb{R}$ such that, in the notation of Proposition 4.2.1,*

$$x(t) = \frac{2x_1 A_1(t)}{1 + x_1(B_1^2(t) - 1/3)} \quad \text{and } y(t) = 0.$$

Given such an $x_1 \in \mathbb{R}$ we shall denote the resulting instanton by A^{x_1}. Observe that A^{x_1} is defined globally on $\mathbb{R}^4 \times S^3$ if and only if $x_1 \geq 0$ and that A^0 is the trivial flat connection.

The next proposition gives locally defined solutions around the singular orbit in the case where $P = P_{\text{id}}$.

Proposition 4.2.4. [LO18, Proposition 6] Let S^3 be the singular orbit in the Bryant–Salamon G_2-manifold $\mathbb{R}^4 \times S^3$. There is a one-parameter family of $\text{SU}(2)^3$-invariant G_2-instantons, with gauge group SU(2), defined on a neighbourhood of S^3 and smoothly extending over S^3 on P_{id}. The instantons are parameterised by $y_0 \in \mathbb{R}$ and satisfy, in the notation of Proposition 4.2.1,

$$x(t) = \frac{2}{t} + \frac{y_0^2 - 1}{4} t + O(t^3) \quad \text{and } y(t) = y_0 + \frac{y_0}{2}\left(\frac{y_0^2}{2} - 3\right) t^2 + O(t^4).$$

For certain values of the parameter, we can extend the instanton away from the singular orbit.

Theorem 4.2.5. *[LO18, Theorem 5] The G_2-instanton arising from the case where $y_0 = 0$ in Proposition 4.2.4 extends to the Bryant–Salamon G_2-manifold $\mathbb{R}^4 \times S^3$, and is* *given by*

$$A^{lim} = \frac{A_1^2(t)}{1/2(B_1^2(t) - 1/3)} \sum_{i=1}^{3} T_i \otimes \eta_i^+.$$

Theorem 4.2.6. *[ST23, Theorem 3.7] The G_2-instantons from Proposition 4.2.4 extend to the Bryant–Salamon G_2-manifold $\mathbb{R}^4 \times S^3$ with quadratic curvature decay if and only if $y_0 \in [-\sqrt{3}, \sqrt{3}]$. Moreover, the instantons are flat if and only if $y_0 = \pm\sqrt{3}$, and all non-flat instantons on this manifold are asymptotic to A^{nK} with rate -3, where A^{nK} is the nearly Kähler instanton*

$$A^{nK} = \frac{2}{3} \sum_{i=1}^{3} T_i \otimes \eta_i^+.$$

We see that as $x_1 \to \infty$, Clarke's G_2-instantons "bubble off" an ASD connection along the normal bundle to the associative $S^3 = \{0\} \times S^3 \subset \mathbb{R}^4 \times S^3$. More precisely, consider the following re-scaling: for $p \in S^3$ and $\delta > 0$ we define

$$s_\delta^p : B_1 \subset \mathbb{R}^4 \to B_\delta \times \{p\} \subset \mathbb{R}^4 \times S^3;$$
$$x \mapsto (\delta x, p).$$

The basic ASD instanton on $\mathbb{R}^4$ with scale $\delta > 0$ can be written as

$$A_\lambda^{\mathrm{ASD}} = \frac{\lambda t^2}{1 + \lambda t^2} \sum_{i=1}^{3} T_i \otimes \eta_i^+. \tag{4.19}$$

Then, we have the following theorem.

Theorem 4.2.7. *[LO18, Theorem 6] Let $\{A^{x_1}\}$ be a sequence of Clarke's G_2-instantons with $x_1 \to \infty$.*

(i) Given any $\lambda > 0$, there is a sequence of positive real numbers $\delta = \delta(x_1, \lambda) \to 0$ as $x_1 \to \infty$ such that: for all $p \in S^3$, $(s_\delta^p)^* A^{x_1}$ converges uniformly with all derivatives to the basic ASD instanton A_λ^{ASD} on $B_1 \subset \mathbb{R}^4$ as in (4.19).

(ii) The connections A^{x_1} converge uniformly with all derivatives to A^{lim} on every compact subset of $(\mathbb{R}^4 \setminus \{0\}) \times S^3$ as $x_1 \to \infty$.

We can interpret (ii) as a "removable singularity" phenomenon since A^{lim} is a smooth connection on $\mathbb{R}^4 \times S^3$. We observe the expected energy concentration along the associative S^3.

Corollary 4.2.8. [LO18, Corollary 2] The function $|F_{A^{x_1}}|^2 - |F_{A^{\mathrm{lim}}}|^2$ is integrable for all $x_1 > 0$. Moreover, as $x_1 \to \infty$ it converges to $8\pi^2 \delta_{\{0\}\times S^3}$ as a current, i.e. for all compactly supported functions f we have

$$\lim_{x_1 \to \infty} \int_{\mathbb{R}^4 \times S^3} f(|F_{A^{x_1}}|^2 - |F_{A^{\mathrm{lim}}}|^2)\mathrm{dvol}_g = 8\pi^2 \int_{\{0\}\times S^3} f \mathrm{dvol}_{g|_{\{0\}\times S^3}}.$$

The sequence of instantons A^{x_1} determines a constant Fueter section (see for example [DS11]), taking value at the basic ASD instanton on $\mathbb{R}^4$.

4.2.3 Coclosed $\mathbf{SU}(2)^3$-invariant G_2-instantons

In this section we focus on the case where the G_2-structures are $\mathrm{SU}(2)^3$-invariant and coclosed but not necessarily torsion-free. Let $A_1 : [0, \infty) \to \mathbb{R}$ be a smooth function with $A_1(t) > 0$ for $t \in (0, \infty)$, such that

(i) A_1 is odd around $t = 0$;

(ii) $\dot{A}_1(0) = 1/2$.

We denote

$$A_1(t) = \frac{t}{2} + a_{1,3}t^3 + O(t^5).$$

Consider the coclosed G_2-structure found in Proposition 3.3.9 for $A_1 = A_2 = A_3$. By Section 3.2.6.1, we have that the functions B_1, B_2, B_3 that define the G_2-structure are all equal, so the G_2-structure presents an extra $\mathrm{SU}(2)$-symmetry.

Remark 4.2.9. There is a complicated but explicit expression of B_1 in terms of A_1 and

b_0 (see Section 3.2.6.1):

$$B_1(t) = \sqrt{A_1^{-2}(t)\left(\frac{b_0^2}{16}e^{\int_{1/2}^t \frac{1}{A_1(\xi)}d\xi} + e^{\int_{1/2}^t \frac{1}{A_1(\xi)}d\xi}\int_0^t A_1^3(\eta)e^{-\int_{1/2}^\eta \frac{1}{A_1(\xi)}d\xi}d\eta\right)}. \tag{4.20}$$

We recall that

$$B_1(t) = b_0 + b_2t^2 + O(t^4).$$

We need to divide our discussion depending on which choice of bundle over a singular orbit we take: P_1 or P_{id} (see Section 4.1.3).

4.2.3.1 Extension on P_1

For the manifold $\mathbb{R}^4 \times S^3$ with any of the coclosed G_2-structures from Chapter 3, we obtain a 1-parameter family of G_2-instantons extending over the singular orbit P_1.

Theorem 4.2.10. *Let $M = \mathbb{R}^4 \times S^3$, with a $SU(2)^3$-invariant coclosed G_2-structure given by A_1 and $b_0 > 0$ as in Proposition 3.3.9. There is an explicit 1-parameter family of $SU(2)^3$-invariant G_2-instantons with gauge group $SU(2)$ on the bundle P_1, given by*

$$\theta^{x_1} = \frac{x_1 A_1 e^{\int_{1/2}^t F(\xi)d\xi}}{1 + x_1\int_0^t e^{\int_{1/2}^\eta F(\xi)d\xi}d\eta}\sum_{i=1}^3 T_i \otimes \eta_i^+, \tag{4.21}$$

where

$$F(t) = -\frac{\dot{A}_1}{A_1} + \frac{1}{A_1} - \frac{A_1}{B_1^2},$$

and $x_1 \in [0,\infty)$. Given such x_1 we denote the resulting instanton by θ^{x_1}, and θ^0 is the trivial flat connection. Moreover, any $SU(2)^3$-invariant G_2-instantons with gauge group $SU(2)$ on the bundle P_1 are in this family.

Proof. From Lemma 4.2.2 we see that for the connection A to smoothly extend to the singular orbit on P_1, we need $u, v : (0,\infty) \to \mathbb{R}$ real analytic even functions such that

$$x(t) = x_1 t + t^3 u(t),$$
$$y(t) = t^2 v(t).$$

Here we are using the notation of Proposition 4.2.1. Then the system (4.14), (4.15) gives

$$\begin{aligned} \dot{u} &= \frac{-2u - x_1^2 - x_1(8a_{1,3} + 1/2b_0^2)}{t} + f_1(t,u,v), \\ \dot{v} &= \frac{-6v}{t} + f_2(t,u,v), \end{aligned} \tag{4.22}$$

where $f_1, f_2 : (0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are some real analytic functions. Theorem 3.3.1 guarantees the existence and uniqueness of solutions to this system in a neighbourhood of the initial value $t = 0$ provided that

$$\begin{aligned} u(0) &= \frac{-x_1^2}{2} - x_1\left(4a_{1,3} + \frac{1}{4b_0^2}\right), \\ v(0) &= 0. \end{aligned} \tag{4.23}$$

Suppose $y = 0$. Then equation (4.14) becomes

$$\dot{x} = \left(-\frac{\dot{A}_1}{A_1} + \frac{1}{A_1} - \frac{A_1}{B_1^2}\right)x - x^2. \tag{4.24}$$

We would like to give an explicit solution to this equation. Let

$$F(t) = -\frac{\dot{A}_1}{A_1} + \frac{1}{A_1} - \frac{A_1}{B_1^2},$$

which has the following Taylor expansion at 0:

$$F(t) = \frac{1}{t} + \left(-8a_{1,3} - \frac{1}{2b_0^2}\right)t + O(t^3). \tag{4.25}$$

The equation

$$\dot{x} = F(t)x - x^2 \tag{4.26}$$

is a Bernoulli differential equation, and hence can be solved by making the change of variables $z = x^{-1}$ and later using an integrating factor. By a straightforward computation we get that all the solutions to (4.26) are $x \equiv 0$ or

$$x(t) = \frac{e^{\int_{1/2}^{t} F(\xi)d\xi}}{k + \int_0^t e^{\int_{1/2}^{\xi} F(\eta)d\eta} d\xi}, \tag{4.27}$$

for some choice of real constant k. Note that choosing k is the same as choosing the limits of the integrals. Suppose that $k \neq 0$. Then we can write

$$x(t) = \frac{x_1 e^{\int_{1/2}^{t} F(\xi)d\xi}}{1 + x_1 \int_0^t e^{\int_{1/2}^{\eta} F(\xi)d\xi} d\eta}, \tag{4.28}$$

for some real x_1. This expression together with $y = 0$ agrees with the one corresponding to the unique solution of (4.22) in a neighbourhood of $t = 0$, as a straightforward computation shows that (4.23) holds. If we take $x_1 \geq 0$, this explicit expression is well defined for $t > 0$, so the resulting instantons are defined globally on $\mathbb{R}^4 \times S^3$. It remains to check that (4.28) extends smoothly to the singular orbit at $t = 0$ as a connection on P_1. Let $0 < t \ll 1$, and fix $a \ll 1$, $a > t$. We can write

$$x(t) = \frac{x_1 \exp\left(\int_{1/2}^{a} F(\xi)d\xi\right) \exp\left(\int_a^t F(\xi)d\xi\right)}{1 + x_1 \int_0^t \exp\left(\int_{1/2}^{a} F(\xi)d\xi\right) \exp\left(\int_a^{\eta} F(\xi)d\xi\right) d\eta}.$$

We observe that $\int_{1/2}^{a} F(\xi)d\xi$ is constant, and denote it by c. Then for $\xi \in [t, a]$, we can approximate $F(\xi) \cong \xi^{-1}$. Then

$$x(t) \cong \frac{x_1 e^c \exp\left(\int_a^t \xi^{-1} d\xi\right)}{1 + x_1 e^c \int_0^t \exp\left(\int_a^{\eta} \xi^{-1} d\xi\right) d\eta} = \frac{x_1 e^c a^{-1} t}{1 + x_1 e^c a^{-1} t^2/2}.$$

Hence, $\lim_{t\to 0} x t^{-1} \in \mathbb{R}$. Finally, the fact that x is odd follows from $F(t) = 1/t + O(t)$ and F odd. □

Remark 4.2.11. We recover Clarke's instantons [Cla14]

$$\theta^{x_1} = \frac{2x_1 A_1(t)^2}{1 + x_1(B_1(t)^2 - 1/3)} \sum_{i=1}^{3} T_i \otimes \eta_i^+, \tag{4.29}$$

when $A_1 = \frac{r}{3}\sqrt{1-r^{-3}}$, $b_0 = \frac{1}{\sqrt{3}}$ and $r \in [1, +\infty)$ is a coordinate defined implicitly by $t(r) = \int_1^r \frac{ds}{\sqrt{1-s^{-3}}}$. These were the first examples of G_2-instantons on $\mathbb{R}^4 \times S^3$ with the Bryant Salamon G_2-holonomy metric.

One may wonder what is the asymptotic behaviour of θ^{x_1} when $t \to \infty$. However, this would depend on the asymptotic behaviour of the data A_1, and we have not introduced any a priori restrictions on it.

We can compute the curvature of this instantons using (4.4) and [LO18, Lemma 2]:

$$F_{\theta^{x_1}} = T_i \otimes \left(\frac{d}{dt}(A_1 x) dt \wedge \eta_i^+ + A_1 x (A_1 x - 1) \epsilon_{ikj} \eta_j^+ \wedge \eta_k^+ - A_1 x \epsilon_{ijk} \eta_j^- \wedge \eta_k^- \right).$$

We deduce that

$$\lim_{t \to 0} F_{\theta^{x_1}} = \lim_{t \to 1} F_{\theta^{x_1}} = -\epsilon_{ijk} T_i \otimes \eta_j^- \wedge \eta_k^-.$$

In particular, the curvature is bounded at the singular orbits.

4.2.3.2 Extension on P_{id}

We now study the existence of smooth G_2-instantons on a bundle P_{id} at a singular orbit, and its extension on $\mathbb{R}^4 \times S^3$.

Proposition 4.2.12. Let S^3 be the singular orbit in $\mathbb{R}^4 \times S^3$, with coclosed G_2-structure given by $A_1 = A_2 = A_3$ as in Proposition 3.3.9. There is exactly a 1-parameter family of $\mathrm{SU}(2)^3$-invariant G_2-instantons, with gauge group $\mathrm{SU}(2)$, defined in a neighbourhood of S^3 and smoothly extending over S^3 on P_{id}.

Proof. We see from Lemma 4.2.2 that for the connection θ to smoothly extend to the singular orbit at $t = 0$, we need $u, v : [0, \infty) \to \mathbb{R}$ real analytic functions, such that

$$\begin{aligned} x(t) &= \frac{2}{t} + tu(t), \\ y(t) &= y_0 + t^2 v(t). \end{aligned} \tag{4.30}$$

Then the system (4.14), (4.15) gives

$$\begin{aligned} \dot{u} &= \frac{-4u + y_0^2 - 16a_{1,3} - 1/b_0^2}{t} + f_1(t, u, v), \\ \dot{v} &= \frac{-2v + 2y_0 u + y_0(8a_{1,3} - 2b_2/b_0)}{t} + f_2(t, u, v), \end{aligned} \tag{4.31}$$

where $f_1, f_2 : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ are some real analytic functions. Theorem [FH17, Theorem 4.7] guarantees the existence and uniqueness of solutions to this system in a neighbourhood of the initial value $t = 0$ provided that

$$\begin{aligned} u(0) &= \frac{y_0^2}{4} - 4a_{1,3} - \frac{1}{4b_0^2}, \\ v(0) &= \frac{y_0^3}{4} - y_0 \left(\frac{1}{4b_0^2} - \frac{b_2}{b_0} \right). \end{aligned} \tag{4.32}$$

Therefore $(x(t), y(t))$ given by (4.30) provide a solution of (4.14) and (4.15). Both $F(t)$ and $G(t) = -\dot{B}_1(t)/B_1(t) - 2/A_1(t)$ are odd. We deduce that $(\tilde{x}(t), \tilde{y}(t)) := (-x(-t), y(-t))$ is also a solution of (4.14), (4.15). Note that although F and G are only defined for $t > 0$, we can extend them on $t < 0$ as odd functions. We can write

$$\begin{aligned} \tilde{x}(t) &= \frac{2}{t} + t\tilde{u}(t), \\ \tilde{y}(t) &= y_0 + t^2 \tilde{v}(t), \end{aligned}$$

for real analytic $\tilde{u}, \tilde{v}$ with $\tilde{u}(0) = u(0)$ and $\tilde{v}(0) = v(0)$, and by uniqueness $u' = u$, $v' = v$. Therefore, we can smoothly extend (x, y) to $t < 0$ by $(x(-t), y(-t)) = (-x(t), y(t))$ and they still solve (4.14) and (4.15), which gives that x is odd, y is even as desired. Hence, for each y_0 we have a smooth G_2-instanton in a neighbourhood of the singular orbit. □

If $y_0 = 0$, we can get an explicit expression of the G_2-instanton.

Theorem 4.2.13. *There is an explicit $SU(2)^3$-invariant G_2-instanton with gauge group $SU(2)$ on $M = \mathbb{R}^4 \times S^3$ with coclosed G_2-structure given by $A_1 = A_2 = A_3$ as in Proposition 3.3.9 corresponding to $y_0 = 0$ in Proposition 4.2.12. It is given by*

$$\theta_0 = \frac{A_1(t)e^{\int_{1/2}^t F(\xi)d\xi}}{\int_0^t e^{\int_{1/2}^\eta F(\xi)d\xi}d\eta} \sum_{i=1}^3 T_i \otimes \eta_i^+. \tag{4.33}$$

where

$$F(t) = -\frac{\dot{A}_1}{A_1} + \frac{1}{A_1} - \frac{A_1}{B_1^2}.$$

It smoothly extends on the bundle P_{id} over $\mathbb{R}^4 \times S^3$.

Proof. Here we are also using the notation of Proposition 4.2.1. We first see that $y = 0$ gives a solution, which corresponds to the value of the parameter $y_0 = 0$, by taking the solution x_0 from (4.27) with $k = 0$:

$$x_0(t) = \frac{e^{\int_{1/2}^t F(\xi)d\xi}}{\int_0^t e^{\int_{1/2}^\eta F(\xi)d\xi}d\eta}.$$

We will prove that it satisfies that $x_0(0)A_1(0) = 1$. Let $0 < t \ll 1$, and fix $a \ll 1$, $a > t$. We can write

$$x_0(t)A_1(t) = \frac{A_1(t)\exp\left(\int_{1/2}^a F(\xi)d\xi\right)\exp\left(\int_a^t F(\xi)d\xi\right)}{\int_0^t \exp\left(\int_{1/2}^a F(\xi)d\xi\right)\exp\left(\int_a^\eta F(\xi)d\xi\right)d\eta} = \frac{A_1(t)\exp\left(\int_a^t F(\xi)d\xi\right)}{\int_0^t \exp\left(\int_a^\eta F(\xi)d\xi\right)d\eta},$$

Then for $\xi \in [t, a]$, we can approximate $F(\xi)$ by ξ^{-1}, so

$$x_0(t)A_1(t) \cong \frac{\frac{t}{2}\exp\left(\int_a^t \xi^{-1}d\xi\right)}{\int_0^t \exp\left(\int_a^\eta \xi^{-1}d\xi\right)d\eta} = \frac{t^2a^{-1}/2}{a^{-1}\int_0^t \eta d\eta} = \frac{t^2a^{-1}/2}{t^2a^{-1}/2} = 1.$$

Another computation shows that (4.32) holds. Furthermore, $x_0(t)$ being odd follows from $F(t) = 1/t + O(t)$ and F odd. Hence, as the explicit expression is well defined for all $t \in [0, \infty)$, and the instanton smoothly extends to the singular orbit at $t = 0$, the instanton (4.33) extends to the whole manifold. □

Remark 4.2.14. When $M = \mathbb{R}^4 \times S^3$, $A_1 = \frac{r}{3}\sqrt{1-r^{-3}}$, $b_0 = \frac{1}{\sqrt{3}}$ and $r \in [1, +\infty)$ is a coordinate defined implicitly by $t(r) = \int_1^r \frac{ds}{\sqrt{1-s^{-3}}}$, we recover the G_2-instanton $A^{\lim}$ from [LO18, Theorem 5]:

$$A^{\lim} = \frac{A_1(t)^2}{1/2(B_1(t)^2 - 1/3)} \sum_{i=1}^{3} T_i \otimes \eta_i^+. \tag{4.34}$$

We now move on to study the extension of instantons on P_{id} for other values of the parameter y_0. Unlike when $y_0 = 0$, we will not have explicit expressions. Recall that if we perform the change $x^+ = xA_1$, $x^- = yB_1$, then (4.14) and (4.15) become

$$\dot{x}^+ = \frac{x^+}{A_1}\left(1 - \frac{A_1^2}{B_1^2} - x^+\right) + \frac{A_1}{B_1^2}(x^-)^2, \tag{4.35}$$

$$\dot{x}^- = \frac{2x^-}{A_1}(x^+ - 1). \tag{4.36}$$

We observe that the critical points of this system of ordinary differential equations are $(0,0)$, $(1,1)$ and $(1,-1)$. The point $(0,0)$ corresponds to the flat connection $\theta = 0$. The other points, $(1,1)$ and $(1,-1)$, correspond to connections defined over $\mathbb{R}^4 \times S^3$ and which are smooth on P_{id}, and that in the notation of Proposition 4.2.12 correspond to values of the parameter y_0 of $1/b_0$ and $-1/b_0$, respectively. The corresponding connections, which we denote by θ_{1/b_0} and θ_{-1/b_0}, are:

$$\theta_{1/b_0} = \sum_{i=1}^{3} T^i \otimes \eta_i^+ + \sum_{i=1}^{3} T^i \otimes \eta_i^-, \quad \theta_{-1/b_0} = \sum_{i=1}^{3} T^i \otimes \eta_i^+ - \sum_{i=1}^{3} T^i \otimes \eta_i^-. \tag{4.37}$$

A quick computation shows that both of these connections are flat. We will show that other values of the parameter y_0 also give smooth instantons on our manifolds of interest, although no longer explicit.

We say that a subset $R \subset \mathbb{R}^n$ is *forward-invariant* for an ODE system $\dot{x} = F(x,t)$ if for a solution $x(t)$ and a non-singular time t_0 (i.e. the ODE is regular at t_0) such that $x(t_0) \in R$, then $x(t) \in R$ for any $t > t_0$ such that $x(t)$ exists. We will use forwardinvariance of sets to show that the non-autonomous system of ODEs (4.35), (4.36), with initial point $(x^+(0), x^-(0)) = (1, y_0 b_0)$, has a solution away from the singular orbit.

Proposition 4.2.15. For values of the parameter y_0 in $[-1/b_0, 1/b_0]$, the G_2-instantons from Proposition 4.2.12 extend to the manifold $\mathbb{R}^4 \times S^3$. We denote these instantons by θ_{y_0}.

Proof. We already considered the case $y_0 = 0$ on Theorem 4.2.13, and $y_0 = \pm 1/b_0$ corresponds to the flat instantons (4.37). Separating variables in (4.36), we can solve it to get

$$x^-(t) = x^-(0)\exp\left(\int_0^t \frac{2}{A_1(\xi)}(x^+(\xi) - 1)d\xi\right). \tag{4.38}$$

Therefore, away from the singular orbits, y is either always positive, or always negative, or identically 0. We observe that $(x, y) \mapsto (x, -y)$ is a symmetry of equations (4.14) and (4.15). Hence, we may assume $y > 0$ or equivalently $x^- > 0$ and study the case $y_0 \in (0, 1/b_0)$. The remaining case $y_0 \in (-1/b_0, 0)$ will follow by changing the signs of y. We will show that the following set is forward-invariant:

$$R := \{(x^+, x^-) \in \mathbb{R}^2 | 0 < x^+ < 1, 0 < x^- < 1\}.$$

There exists $t_0 > 0$ such that the solution (x^+, x^-) of (4.35) and (4.36) is defined on $[0, t_0]$ and $(x^+(t_0), x^-(t_0)) \in R$. First, we see that $\dot{x}^- < 0$ when $x^+ < 1$, and $\{(x_+, 0) | 0 \leq x^+ \leq 1\}$ is an invariant line for the flow. We also observe that for $t > 0$

$$\dot{x}^+|_{x^+=1} = \frac{A_1}{B_1^2}((x^-)^2 - 1) < 0,$$

and

$$\dot{x}^+|_{x^+=0} = \frac{A_1}{B_1^2}(x^-)^2 > 0,$$

when $0 < x^- < 1$, as $A_1 > 0$. Hence, the set R is forward-invariant and (x^+, x^-) cannot blow-up, so the instanton given by (x^+, x^-) is defined for every $t \in [0, \infty)$. □

Remark 4.2.16. Once again, taking $A_1 = \frac{r}{3}\sqrt{1-r^{-3}}$, $b_0 = \frac{1}{\sqrt{3}}$ and $r \in [1, +\infty)$ a coordinate defined implicitly by $t(r) = \int_1^r \frac{ds}{\sqrt{1-s^{-3}}}$, we can recover the corresponding 1-parameter family G_2-instantons on $\mathbb{R}^4 \times S^3$ with the Bryant–Salamon metric; for this case its existence was shown in [ST23, Theorem 3.7], where it is denoted as $T'_{\gamma'}$, $\gamma' \in [-1, 1]$.

Remark 4.2.17. We can go from θ_{y_0} to θ_{-y_0} by exchanging the factors of $\mathrm{SU}(2)^2$.

Putting everything together, we get the following theorem.

Theorem E. Let $M = \mathbb{R}^4 \times S^3$, with a $\mathrm{SU}(2)^3$-invariant coclosed G_2-structure given by A_1 and $b_0 > 0$ as in Proposition 3.3.9. There exists two 1-parameter families of smooth $\mathrm{SU}(2)^3$-invariant G_2-instantons with gauge group $\mathrm{SU}(2)$: θ^{x_1}, $x_1 \in [0, \infty)$ on the bundle P_1; and θ_{y_0}, $y_0 \in [-1/b_0, 1/b_0]$ on the bundle P_{id}.

One may hope to make this result into a classification result, by studying the remaining possible situation, corresponding the values of the parameter y_0 with $|y_0| > 1/b_0$. In [ST23] it is shown that for $(\mathbb{R}^4 \times S^3, g_{BS})$, solutions corresponding to these initial parameter do not produce uniformly bounded instantons.

4.2.4 Behaviour of solutions

We observe the expected bubbling behaviour of the sequence of instantons θ^{x_1}: they "bubble off" an ASD connection along the normal bundle to the singular orbit $S^3 = \{0\} \times S^3 \subset \mathbb{R}^4 \times S^3$, which is an associative submanifold.

Theorem 4.2.18. *Let θ^{x_1}, $x_1 \geq 0$ be the sequence of instantons from Theorem 4.2.10, given by (4.21). Then*

(i) For any $\lambda > 0$ there is a sequence $\delta = \delta(x_1, \lambda) > 0$ converging to 0 when $x_1 \to \infty$ such that: for all $p \in S^3$, and if we define

$$s_\delta^p : B_1 \subset \mathbb{R}^4 \to B_\delta \times \{p\} \subset \mathbb{R}^4 \times S^3;$$

$$x \mapsto (\delta x, p),$$

then $(s^p_\delta)^*\theta^{x_1}$ *converges uniformly with all derivatives to the basic ASD instanton* θ^{ASD}_λ *with scale* λ *on* $B_1 \subset \mathbb{R}^4$*:*

$$\theta^{ASD}_\lambda = \frac{\lambda t^2}{1+\lambda t^2}\sum_{i=1}^{3} T_i \otimes \eta_i^+.$$

(ii) Suppose that θ_0 *is bounded. The connections* θ^{x_1} *converge to* θ_0 *given in Theorem 4.2.13 on every compact subset of* $(\mathbb{R}^4 \setminus \{0\}) \times S^3$ *when* $x_1 \to \infty$*.*

Proof. (i) Recall that near $t = 0$, $A_1(t) = t/2 + O(t^3)$ and $F(t) = 1/t + O(t^3)$, so we also have $e^{\int F(t)dt} = t + O(t^3)$. Then we can compute

$$\begin{aligned} (s^p_\delta)^*\theta^{x_1} &= A_1(\delta t)x(\delta t)\textstyle\sum_{i=1}^3 T_i \otimes \eta_i^+ \\ &= \frac{x_1\delta^2 t^2/2 + O(x_1\delta^4 t^4)}{1 + x_1\delta^2 t^2/2 + O(x_1\delta^4 t^4)}\textstyle\sum_{i=1}^3 T_i \otimes \eta_i^+ \end{aligned}$$

By taking $\delta = \delta(x_1, \lambda) = \sqrt{2\lambda/x_1} > 0$, we have that $\delta(x_1, \lambda) \to 0$ when $x_1 \to \infty$ and for every $k \in \mathbb{N} \cup \{0\}$, there is a $c_k > 0$, not depending on λ, x_1, such that

$$||(s^p_\delta)^*\theta^{x_1} - \theta^{\mathrm{ASD}}_\lambda||_{C^k(B_1)} \leq c_k \frac{\lambda^2}{x_1}.$$

The uniform convergence with all derivatives follows.

(ii) θ^{x_1} converges pointwise to θ_0. We have

$$\begin{aligned} |\theta^{x_1} - \theta_0| &= A_1 e^{\int_{1/2}^t F(\xi)d\xi}\left|\frac{x_1}{1 + x_1\int_0^t e^{\int_{1/2}^\eta F(\xi)d\xi}d\eta} - \frac{1}{\int_0^t e^{\int_{1/2}^\eta F(\xi)d\xi}d\eta}\right|\left|\textstyle\sum_{i=1}^3 T_i \otimes \eta_i^+\right| \\ &= A_1 e^{\int_{1/2}^t F(\xi)d\xi}\left|\frac{1}{(1 + x_1\int_0^t e^{\int_{1/2}^\eta F(\xi)d\xi}d\eta)\int_0^t e^{\int_{1/2}^\eta F(\xi)d\xi}d\eta}\right|\left|\textstyle\sum_{i=1}^3 T_i \otimes \eta_i^+\right| \\ &= |\theta_0|\left|\frac{1}{1 + x_1\int_0^t e^{\int_{1/2}^\eta F(\xi)d\xi}d\eta}\right|, \end{aligned}$$

where we denote $|\theta_0| = |A_1(t)e^{\int_{1/2}^t F(\xi)d\xi}/\int_0^t e^{\int_{1/2}^\eta F(\xi)d\xi}d\eta|$. Let $c_1 > 0$ be a bound for $|\theta_0|$. A quick computation (similar to the one at the end of the proof of Theorem 4.2.10) shows that on every compact subset of $(\mathbb{R}^4 \setminus \{0\}) \times S^3$, there exists a constant $c_2 > 0$ such that

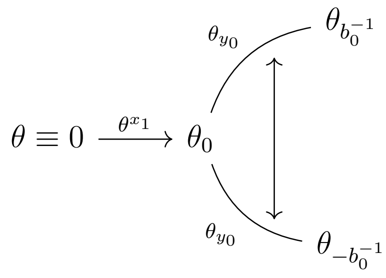

Figure 4.1: Representation of families of G_2-instantons on $\mathbb{R}^4 \times S^3$ and their relations.

$c_2 \leq \int_0^t e^{\int_{1/2}^{\eta} F(\xi)d\xi} d\eta$. Then

$$|\theta^{x_1} - \theta_0| \leq \frac{c_1}{1 + x_1 c_2}.$$

Therefore $\theta^{x_1} - \theta_0$ converges uniformly to zero when $x_1 \to \infty$. Similarly the derivatives of $\theta^{x_1} - \theta_0$ converge uniformly to zero when $x_1 \to \infty$. □

Remark 4.2.19. We are interested in connections that have, at the very least, bounded curvature. Hence the condition that θ_0 is bounded on (ii) of the previous theorem is reasonable, as we are already looking for A_1 functions that make our instantons bounded.

Example 4.2.20. For $A_1(t) = t/2$, we get

$$\theta^{x_1} = \frac{x_1 t/2}{t^2/4 + c} \frac{1}{1 + 2x_1 \log(t^2/4 + c)} \sum_{i=1}^{3} T^i \otimes \eta_i^+, \quad \theta_0 = \frac{t^2/4}{t^2/4 + c} \frac{1}{\log(t^2/4 + c)} \sum_{i=1}^{3} T^i \otimes \eta_i^+.$$

Then $\theta^{x_1} \to 0$, $\theta_0 \to 0$ when $t \to \infty$.

Figure 1 represents the instantons found on $\mathbb{R}^4 \times S^3$ and their relations.

4.3 SU(2)^2-invariant G_2-instantons

In this section we consider the most general case, when the coclosed G_2-structures are SU(2)^2-invariant. We provide an existence and classification results for G_2-instantons on neighbourhoods of a singular orbit of our manifolds of interest. Let $A_1, A_2, A_3 : [0, L) \to \mathbb{R}$ be smooth functions with $A_i(t) > 0$ for $t \in (0, L)$, where L is either infinity or 1, such that

(i) A_i's are odd;

(ii) $\dot{A}_i(0) = 1/2$.

We denote

$$A_i(t) = \frac{t}{2} + a_{i,3}t^3 + O(t^5).$$

Consider the coclosed G_2-structure found in Proposition 3.3.9 for A_1, A_2, A_3.

4.3.1 $\mathrm{SU}(2)^2$-invariant ODEs and boundary conditions

We define

$$c_i^+ = \frac{a_i^+}{A_i}, \quad c_i^- = \frac{a_i^-}{B_i}.$$

Then the general $\mathrm{SU}(2)^2$-invariant G_2-instanton equations (4.9) for A from Lemma 4.1.5 are:

$$\begin{aligned} \dot{c}_i^+ + \left(\frac{\dot{A}_i}{A_i} + \frac{A_i}{B_jB_k} - \frac{A_i}{A_jA_k}\right)c_i^+ &= \frac{1}{2}[c_j^-, c_k^-] - \frac{1}{2}[c_j^+, c_k^+], \\ \dot{c}_i^- + \left(\frac{\dot{B}_i}{B_i} + \frac{B_i}{B_jA_k} + \frac{B_i}{A_jB_k}\right)c_i^- &= \frac{1}{2}[c_j^-, c_k^+] + \frac{1}{2}[c_j^+, c_k^-], \end{aligned} \tag{4.39}$$

together with the constraint

$$\sum_{i=1}^{3}[c_i^+, c_i^-] = 0. \tag{4.40}$$

Proposition 4.3.1. Let θ be an $\mathrm{SU}(2)^2$-invariant G_2-instanton on $\mathbb{R}^4 \times S^3$ with gauge group $\mathrm{SU}(2)$. There is a standard basis $\{T_i\}$ of $\mathfrak{su}(2)$ such that (up to an invariant gauge transformation) we can write

$$\theta = \sum_{i=1}^{3} A_i f_i^+ T_i \otimes \eta_i^+ + \sum_{i=1}^{3} B_i f_i^- T_i \otimes \eta_i^-. \tag{4.41}$$

with $f_i^{\pm} : [0,\infty) \to \mathbb{R}$ satisfying

$$\begin{aligned}
\dot{f}_1^+ + \left(\frac{\dot{A}_1}{A_1} + \frac{A_1}{B_2B_3} - \frac{A_1}{A_2A_3}\right) f_1^+ &= f_2^- f_3^- - f_2^+ f_3^+, \\
\dot{f}_2^+ + \left(\frac{\dot{A}_2}{A_2} + \frac{A_2}{B_1B_3} - \frac{A_2}{A_1A_3}\right) f_2^+ &= f_1^- f_3^- - f_1^+ f_3^+, \\
\dot{f}_3^+ + \left(\frac{\dot{A}_3}{A_3} + \frac{A_3}{B_1B_2} - \frac{A_3}{A_1A_2}\right) f_3^+ &= f_1^- f_2^- - f_1^+ f_2^+, \\
\dot{f}_1^- + \left(\frac{\dot{B}_1}{B_1} + \frac{B_1}{B_2A_3} + \frac{B_1}{A_2B_3}\right) f_1^- &= f_2^- f_3^+ + f_2^+ f_3^-, \\
\dot{f}_2^- + \left(\frac{\dot{B}_2}{B_2} + \frac{B_2}{B_1A_3} + \frac{B_2}{A_1B_3}\right) f_2^- &= f_3^- f_1^+ + f_3^+ f_1^-, \\
\dot{f}_3^- + \left(\frac{\dot{B}_3}{B_3} + \frac{B_3}{B_1A_2} + \frac{B_3}{A_1B_2}\right) f_3^- &= f_2^- f_1^+ + f_2^+ f_1^-.
\end{aligned} \tag{4.42}$$

Proof. We must consider $\mathrm{SU}(2)^2$-homogeneous $\mathrm{SU}(2)$-bundles over a slice $S^3 \times S^3 \cong \mathrm{SU}(2)^2/\{1\}$. Such bundles are parameterised by the trivial isotropy homomorphism $1 : \{1\} \to \mathrm{SU}(2)$. By Wang's Theorem (Theorem 4.1.2)), invariant connections on the bundle $\mathrm{SU}(2)^2 \times_{(\{1\},1)} \mathrm{SU}(2)$ can be written as a left-invariant extension $\Lambda : (\mathfrak{m}, \mathrm{Ad}) \to (\mathfrak{su}(2), \mathrm{Ad} \circ 1)$. Here $\mathfrak{m}$ splits into irreducibles as

$$\mathfrak{m} = \underbrace{\mathbb{R} \oplus \ldots \oplus \mathbb{R}}_{6 \text{ times}}$$

Therefore, we can apply a gauge transformation so that

$$a = \sum_{i=1}^{3} A_i f_i^+ T_i \otimes \eta_i^+ + \sum_{i=1}^{3} B_i f_i^- T_i \otimes \eta_i^-,$$

where $f_i^{\pm}$, $i = 1,2,3$, are constants. We then extend this connection to $M = \mathbb{R}^4 \times S^3$ or $S^4 \times S^3$ and get

$$a(t) = \gamma \left(\sum_{i=1}^{3} A_i f_i^+ T_i \otimes \eta_i^+ + \sum_{i=1}^{3} B_i f_i^- T_i \otimes \eta_i^- \right) \gamma^{-1},$$

for $\gamma : [0, \infty) \to \mathrm{SU}(2)$ and $f_i^\pm : [0, \infty) \to \mathbb{R}$. The constraint (4.10) is satisfied, and the symmetry of (4.9) means that

$$A_i f_i^+ [\gamma^{-1}\dot{\gamma}, T_j] = 0, \quad B_i f_i^- [\gamma^{-1}\dot{\gamma}, T_j] = 0,$$

$i, j = 1, 2, 3$. Finally, if $\theta \neq 0$ then $\dot{\gamma} = 0$, so we may always find a gauge transformation such that θ is written as in (4.41). Substituting this expression into (4.9) finishes the proof. □

The next Lemma deals with the conditions for the extension to the singular orbit, which is of the form $S^3 \subset M$.

Lemma 4.3.2. The connection θ extends smoothly over the singular orbit S^3 if and only if f_i^+ are odd, f_i^- are even, and their Taylor expansions around 0 are

- either

$$f_i^+ = f_{i,1}^+ t + O(t^3), \quad f_i^- = f_{i,2}^- t^2 + O(t^4),$$

  for $i = 1, 2, 3$, in which case θ extends smoothly as a connection on P_1;

- or

$$f_i^+ = \frac{2}{t} + (b_2^+ - 4a_{i,3})t + O(t^3), \quad f_i^- = b_0^- + b_2^- t^2 + O(t^4),$$

  for $i = 1, 2, 3$, in which case θ extends smoothly as a connection on P_{id}.

Proof. For P_1, we apply [LO18, Lemma 10] to θ and get the first set of expressions above. For P_{id}, we apply [LO18, Lemma 10] to

$$\theta - \theta^{\mathrm{can}} = \sum_{i=1}^{3} (A_i f_i^+ - 1) T_i \otimes \eta_i^+ + \sum_{i=1}^{3} B_i f_i^- T_i \otimes \eta_i^- .$$

We obtain the second set of expressions above. □

4.3.2 Extension on P_1

We start by studying the existence of G_2-instantons around a singular orbit extending as a connection on P_1. We find a 3-parameter family of G_2-instantons defined in a neighbourhood of the singular orbit.

Proposition 4.3.3. Let $M = \mathbb{R}^4 \times S^3$, with coclosed G_2-structure given by A_1, A_2, A_3 as in Proposition 3.3.9, and let $S^3 \subset M$ be the singular orbit. There is a 3-parameter family of $\mathrm{SU}(2)^2$-invariant G_2-instantons with gauge group $\mathrm{SU}(2)$ in a neighbourhood of S^3 which smoothly extends in P_1. In the notation of Proposition 4.3.1, these instantons have $f_i^- = 0$, $i = 1, 2, 3$ and $f_1^+, f_2^+, f_3^+ : (0, L) \to \mathbb{R}$ (where L is either infinity or 1) solve the ODEs

$$\begin{aligned}
\dot{f}_1^+ + \left(\frac{\dot{A}_1}{A_1} + \frac{A_1}{B_2B_3} - \frac{A_1}{A_2A_3}\right) f_1^+ &= -f_2^+ f_3^+, \\
\dot{f}_2^+ + \left(\frac{\dot{A}_2}{A_2} + \frac{A_2}{B_1B_3} - \frac{A_2}{A_1A_3}\right) f_2^+ &= -f_1^+ f_3^+, \\
\dot{f}_3^+ + \left(\frac{\dot{A}_3}{A_3} + \frac{A_3}{B_1B_2} - \frac{A_3}{A_1A_2}\right) f_3^+ &= -f_1^+ f_2^+.
\end{aligned} \tag{4.43}$$

subject to $f_i^+ = f_{i,1}^+ t + t^3 u_i(t)$, $i = 1, 2, 3$, where $f_{i,1}^+ \in \mathbb{R}$ and the $u_i : (0, \infty) \to \mathbb{R}$ are real analytic functions such that

$$\begin{aligned}
u_1(0) &= -\left(\frac{1}{4b_0^2} + 2a_{2,3} + 2a_{3,3}\right) f_{1,1}^+ - f_{2,1}^+ f_{3,1}^+, \\
u_2(0) &= -\left(\frac{1}{4b_0^2} + 2a_{1,3} + 2a_{3,3}\right) f_{2,1}^+ - f_{1,1}^+ f_{3,1}^+, \\
u_3(0) &= -\left(\frac{1}{4b_0^2} + 2a_{1,3} + 2a_{2,3}\right) f_{3,1}^+ - f_{1,1}^+ f_{2,1}^+.
\end{aligned} \tag{4.44}$$

Proof. By Lemma 4.3.2, we write

$$f_i^+ = f_{i,1}^+ t + t^3 u_i(t), \quad f_i^- = t^2 v_i(t),$$

for some real analytic functions $u_i, v_i : (0, L) \to \mathbb{R}$, $i = 1, 2, 3$. The new initial value problem for $X(t) = (u_1(t), u_2(t), u_3(t), v_1(t), v_2(t), v_3(t))^T$ can be written as a singular IVP:

$$\frac{dX}{dt} = \frac{M_{-1}(X)}{t} + M(t, X), \quad X(0) = X_0, \tag{4.45}$$

where $M(t, X)$ is real analytic on the first coordinate and

$$M_{-1}(X) = \begin{pmatrix} -2u_1 - \left(\frac{1}{2b_0^2} + 4a_{2,3} + 4a_{3,3}\right) f^+_{1,1} - f^+_{2,1} f^+_{3,1} \\ -2u_2 - \left(\frac{1}{2b_0^2} + 4a_{1,3} + 4a_{3,3}\right) f^+_{2,1} - f^+_{1,1} f^+_{3,1} \\ -2u_3 - \left(\frac{1}{2b_0^2} + 4a_{1,3} + 4a_{2,3}\right) f^+_{3,1} - f^+_{1,1} f^+_{2,1} \\ -6v_1 \\ -6v_2 \\ -6v_3 \end{pmatrix}.$$

To show existence and uniqueness of solutions, we use Theorem 3.3.1. This theorem guarantees the existence and uniqueness of short-time solutions to (4.45) in a neighbourhood of the singular orbit if $M_{-1}(0) = 0$ and $h\mathrm{Id} - d_{X_0} M_{-1}$ is invertible for all integers $h \geq 1$. We see that $dM_{-1}(X(0)) = \mathrm{diag}(-2, -2, -2, -6, -6, -6)$, so the second condition applies. The first condition implies that $v_1(0) = v_2(0) = v_3(0) = 0$ and that

$$\begin{aligned} u_1(0) &= -\left(\frac{1}{4b_0^2} + 2a_{2,3} + 2a_{3,3}\right) f^+_{1,1} - \frac{f^+_{2,1} f^+_{3,1}}{2}, \\ u_2(0) &= -\left(\frac{1}{4b_0^2} + 2a_{1,3} + 2a_{3,3}\right) f^+_{2,1} - \frac{f^+_{1,1} f^+_{3,1}}{2}, \\ u_3(0) &= -\left(\frac{1}{4b_0^2} + 2a_{1,3} + 2a_{2,3}\right) f^+_{3,1} - \frac{f^+_{1,1} f^+_{2,1}}{2}. \end{aligned} \tag{4.46}$$

If the previous equation holds, then the Theorem guarantees the existence of a solution on a neighbourhood of the singular orbit. The solution of (4.43) and $f_1^- = f_2^- = f_3^- = 0$ solves the IVP, so by uniqueness all f_i^- must vanish. Remains to show that all f_i^+ are odd. We argue that

$$F_i(t) = \frac{\dot{A}_i}{A_i} + \frac{A_i}{B_j B_k} - \frac{A_i}{A_j A_k} \tag{4.47}$$

is odd, from where we deduce that $(-f_1^+(-t), -f_2^+(-t), -f_3^+(-t))$ is a solution of (4.43). We write

$$-f_i^+(-t) = f_{i,1}^+ t + t^3 \tilde{u}_i(t),$$

and find $\tilde{u}_i$ with $\tilde{u}_i(0) = u_i(0)$ solving (4.43), so we can smoothly extend f_i^+ to $t < 0$ by $f_i^+(-t) = -f_i^+(t)$ also solving (4.43), giving the desired parity conditions which guarantee the smooth extension of the instantons. The G_2-instantons obtained are then parameterised by three constants $f_{1,1}^+, f_{2,1}^+, f_{3,1}^+ \in \mathbb{R}$. □

4.3.3 Extension on $P_{\mathbf{id}}$

Proposition 4.3.4. Let $M = \mathbb{R}^4 \times S^3$, with coclosed G_2-structure given by A_1, A_2, A_3 as in Proposition 3.3.9, and let $S^3 \subset M$ be the singular orbit. There is a 3-parameter family of $\mathrm{SU}(2)^2$-invariant G_2-instantons with gauge group SU(2) in a neighbourhood of S^3 on the bundle P_{id}. In the notation of Proposition 4.3.1, these instantons have $f_i^\pm$ solving the system of ODEs (4.42).

Proof. By Lemma 4.3.2, we write

$$f_i^+ = \frac{2}{t} + (b_2^+ - 4a_{i,3})t + t^3 u_i, \quad f_i^- = b_0^- + t^2 v_i,$$

for some real analytic $u_i, v_i : (0, L) \to \mathbb{R}$, $i = 1, 2, 3$. Note that $v_1(0) = v_2(0) = v_3(0)$. We can write equations (4.42) as a system of equations for

$X(t) = (u_1(t), u_2(t), u_3(t), v_1(t), v_2(t), v_3(t))^T$ which takes the form of an initial value problem:

$$\frac{dX}{dt} = \frac{M_{-3}(b_0^-, b_2^+)}{t^3} + \frac{M_{-1}(X)}{t} + M(t, X).$$

We can compute

$$M_{-3}(b_0^-, b_2^+) = \left(-4b_2^+ + (b_0^-)^2 - \frac{1}{b_0^2}, -4b_2^+ + (b_0^-)^2 - \frac{1}{b_0^2}, -4b_2^+ + (b_0^-)^2 - \frac{1}{b_0^2}, 0, 0, 0\right)^T.$$

We must require $M_{-3}(b_0^-, b_2^+) = 0$. We achieve this by imposing the extra condition

$$4b_2^+ = (b_0^-)^2 - \frac{1}{b_0^2}.$$

We take b_2^+ such that this holds. We now want to apply [FH17, Theorem 4.7] to the IVP

$$\frac{dX}{dt} = \frac{M_{-1}(X)}{t} + M(t, X).$$

The eigenvalues of dM_{-1} are -8, -8, -6, -2, 0, 0, so condition (ii) of 3.3.1 holds. We also need condition (i) to hold, so we impose that $M_{-1}(X(0)) = 0$. We have

$$M_{-1}(X(0)) = \begin{pmatrix} -2(u_1(0) + u_2(0) + u_3(0)) + b_0^-(v_2(0) + v_3(0)) \\ -2(u_1(0) + u_2(0) + u_3(0)) + b_0^-(v_1(0) + v_3(0)) \\ -2(u_1(0) + u_2(0) + u_3(0)) + b_0^-(v_1(0) + v_2(0)) \\ -6v_1(0) + 2v_2(0) + 2v_3(0) \\ +2v_1(0) - 6v_2(0) + 2v_3(0) \\ +2v_1(0) + 2v_2(0) - 6v_3(0) \end{pmatrix} + K,$$

where K is a constant. The computation of K is more involved that in previous situations, as we need to consider Taylor expansions up to a higher order. We consider the Taylor expansion of $A_i(t)$ up to order 5

$$A_i(t) = \frac{t}{2} + a_{i,3}t^3 + a_{i,5}t^5 + O(t^7),$$

and then use it to compute

$$\frac{\dot{A}_1}{A_1} + \frac{A_1}{B_2 B_3} - \frac{A_1}{A_2 A_3} = -\frac{1}{t} + X_{1,1}t + X_{1,3}t^3 + O(t^5),$$

where

$$X_{1,1} = 4a_{2,3} + 4a_{3,3} + \frac{1}{2b_0^2},$$

$$X_{1,3} = \frac{a_{1,3}}{b_0} - \frac{b_2}{b_0^3} + 4(a_{1,5} + a_{2,5} + a_{3,5}) - 8(a_{1,3}^2 + a_{2,3}^2 + a_{3,3}^2) + 8(-a_{2,3}a_{3,3} + a_{1,3}a_{3,3} + a_{1,3}a_{2,3}).$$

Similarly for the other permutations of $\{1, 2, 3\}$. Recall that both b_2, b_2^+ can be written in terms of the data $b_0, a_{i,3}, a_{i,5}$: in particular, $b_2 = 1/8b_0 - b_0(a_{1,3} + a_{2,3} + a_{3,3})$ (see 3.3.8). The expression for K that we obtain is the following:

$$K = \begin{pmatrix} -(b_2^+)^2 - \frac{b_2^+}{2b_0^2} + \frac{2b_2}{b_0^3} - 8(a_{1,5} + a_{2,5} + a_{3,5}) + 16(a_{1,3}^2 + a_{2,3}^2 + a_{3,3}^2) \\ -(b_2^+)^2 - \frac{b_2^+}{2b_0^2} + \frac{2b_2}{b_0^3} - 8(a_{1,5} + a_{2,5} + a_{3,5}) + 16(a_{1,3}^2 + a_{2,3}^2 + a_{3,3}^2) \\ -(b_2^+)^2 - \frac{b_2^+}{2b_0^2} + \frac{2b_2}{b_0^3} - 8(a_{1,5} + a_{2,5} + a_{3,5}) + 16(a_{1,3}^2 + a_{2,3}^2 + a_{3,3}^2) \\ 2b_0^- b_2^+ - \frac{2b_2 b_0^-}{b_0} \\ 2b_0^- b_2^+ - \frac{2b_2 b_0^-}{b_0} \\ 2b_0^- b_2^+ - \frac{2b_2 b_0^-}{b_0} \end{pmatrix}.$$

The solutions $X(0) = (u_1(0), u_2(0), u_3(0), v_1(0), v_2(0), v_3(0))^T$ to the non-homogeneous system of equations $M_{-1}(X(0)) = 0$ are given by

$$v_1(0) = v_2(0) = v_3(0) = b_0^- b_2^+ - \frac{b_2 b_0^-}{b_0}, \tag{4.48}$$

and

$$\begin{aligned} u_1(0) + u_2(0) + u_3(0) = \; & -\frac{(b_2^+)^2}{2} - \frac{b_2^+}{4b_0^2} + \frac{b_2}{b_0^3} + (b_0^-)^2 b_2^+ - \frac{b_2 (b_0^-)^2}{b_0} \\ & -4(a_{1,5} + a_{2,5} + a_{3,5}) + 8(a_{1,3}^2 + a_{2,3}^2 + a_{3,3}^2). \end{aligned} \tag{4.49}$$

Hence, we can fix any $u_2(0), u_3(0) \in \mathbb{R}$ and then $u_1(0)$ will be uniquely determined by equation (4.49), so they lie in a 2-parameter family. Therefore, for each $b_0^-, u_2(0), u_3(0) \in \mathbb{R}$ there is a unique solution $X(t)$ giving a G_2-instanton. To guarantee the smoothness of found instantons, it remains to check the parity of $f_i^\pm$ at $t = 0$, i.e. that f_i^+ are odd and

f_i^- are even. Again, we argue that $F_i(t)$ from (4.47) and

$$G_i(t) = \frac{\dot{A}_i}{A_i} + \frac{A_i}{B_j B_k} - \frac{A_i}{A_j A_k}$$

are odd, from where we deduce that
$(-f_1^+(-t), -f_2^+(-t), -f_3^+(-t), f_1^-(-t), f_2^-(-t), f_3^-(-t))$ is a solution of (4.42). We write

$$-f_i^+(-t) = \frac{2}{t} + (b_2^+ - 4a_{i,3})t + t^3\tilde{u}_i(t), \quad f_i^-(-t) = b_0^- + b_2^- t^2 \tilde{v}_i(t),$$

and find $\tilde{u}_i, \tilde{v}_i$ with $\tilde{v}_i(0) = u_i(0)$, $\tilde{v}_i(0) = v_i(0)$, $i = 1, 2, 3$, solving (4.42), so we can smoothly extend f_i^+ to $t < 0$ by $f_i^+(-t) = -f_i^+(t)$, and f_i^- to $t < 0$ by $f_i^-(-t) = f_i^-(t)$ also solving (4.42), giving the desired parity conditions which guarantee the smooth extension of the instantons. We have found three parameters $b_0^-, u_2(0), u_3(0) \in \mathbb{R}$ giving a smooth G_2-instanton in a neighbourhood of S^3. □

Corollary 4.3.5. Let $M = \mathbb{R}^4 \times S^3$, with coclosed G_2-structure given by A_1, A_2, A_3 as in Proposition 3.3.9, and let $S^3 \subset M$ be the singular orbit. Suppose there is an extra U(1) symmetry, meaning that $A_2 = A_3$. Then there is a 2-parameter family of $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$-invariant G_2-instantons with gauge group SU(2) in a neighbourhood of S^3 which smoothly extends in P_{id}.

Proof. By [LO18, Proposition 8], the extra symmetry means that $f_2^\pm = f_3^\pm$. The result follows by having $u_2(0) = u_3(0)$ in the previous theorem. □

Remark 4.3.6. There is a mistake in [LO18, Proposition 8], which claims that in the case of $\mathbb{R}^4 \times S^3$ and when the G_2-structure is also torsion-free, there is only a 1-parameter family (instead of 2-parameter family) of $(\mathrm{SU}(2)^2 \times \mathrm{U}(1))$-invariant G_2-instantons with gauge group SU(2) in a neighbourhood of the singular orbit smoothly extending over P_{id}.

We leave the study of the extension of these families of instantons away from a singular orbit to a future work.

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