Some remarks on contact Calabi–Yau 7-manifolds

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(Joint work with Henrique Sá Earp and Julieth Saavedra)

Calabi–Yau and G_2 geometry: circle bundles

6D: Calabi–Yau $\leftrightsquigarrow$ **7D: G_2 geometry**

$(Z^6, h, \omega, \Upsilon)$

$$\begin{tikzcd} \mathcal{S}^1_\epsilon \arrow[r] & (M^7, g, \eta, \varphi) \arrow[d] \\ & Z \end{tikzcd}$$

metric h
Kähler form ω
holomorphic
volume form Υ

metric $g = \epsilon^2\eta^2 + h$
connection 1-form η
G_2-structure
$\varphi = \epsilon\eta \wedge \omega + \mathrm{Re}\,\Upsilon$

Limit $\epsilon \to 0$

- Geometry: collapsing/ “nearly collapsed”
- Analysis: series expansion in ϵ
- Physics: String Theory $\leftrightsquigarrow$ M-theory

Contact Calabi–Yau 7-manifolds

Example: $M^7 = \{(z_0, z_1, z_2, z_3, z_4) \in \mathbb{C}^5 \,:\, \sum_{k=0}^4 z_k^5 = 0\} \cap \mathcal{S}^9$

- $Z = \{[z_0, z_1, z_2, z_3, z_4] \in \mathbb{CP}^4 \,:\, \sum_{k=0}^4 z_k^5 = 0\}$ Fermat quintic
- connection 1-form η such that $\mathrm{d}\eta = \omega$

Definition

(M^7, g, η, Υ) *contact Calabi–Yau 7-manifold:*

- (M^7, g) *Sasakian with contact form* η
- $E = \ker \eta$ *with transverse Calabi–Yau structure* $(h, \omega = \mathrm{d}\eta, \Upsilon)$

$\rightsquigarrow$ G_2-structure $\varphi = \epsilon\eta \wedge \omega + \mathrm{Re}\,\Upsilon$ with $g = \epsilon^2\eta^2 + h$

$\rightsquigarrow$ $*\varphi = \frac{1}{2}\omega^2 - \epsilon\eta \wedge \mathrm{Im}\,\Upsilon \Rightarrow \mathrm{d} * \varphi = 0$

$\rightsquigarrow$ φ coclosed with $\mathrm{d}\varphi = \epsilon\omega^2$

Geometric flows

Laplacian flow

$$\frac{\partial \varphi_t}{\partial t} = \Delta_t \varphi_t = (\mathrm{d}\mathrm{d}_t^* + \mathrm{d}_t^*\mathrm{d})\varphi_t$$

Laplacian coflow

$$\frac{\partial *_t \varphi_t}{\partial t} = \Delta_t *_t \varphi_t = \mathrm{d}\mathrm{d}_t^* *_t \varphi_t$$

- Critical pt: torsion-free $\varphi \rightsquigarrow$ Ricci-flat g with $\mathrm{Hol}(g) \subseteq \mathrm{G}_2$
- Laplacian (co)flow restricted to (co)closed G_2-structures $=$ gradient flow of Hitchin volume functional on $[\varphi_t]$ ($[*_t\varphi_t]$)

Theorem (L.–Sá Earp–Saavedra)

(M^7, g, η, Υ) contact Calabi–Yau (with $\omega = \mathrm{d}\eta$)

$$\varphi_0 = \epsilon\eta \wedge \omega + \mathrm{Re}\,\Upsilon \quad \textit{and} \quad *_0\,\varphi_0 = \tfrac{1}{2}\omega^2 - \epsilon\eta \wedge \mathrm{Im}\,\Upsilon$$

- *Laplacian flow has finite-time singularity at* $t = \frac{1}{8\epsilon^2}$
- *Laplacian coflow exists* $\forall t > 0$ *and has infinite-time singularity*

M compact $\Rightarrow \mathrm{Vol}(M, g_t) \to \infty$

Heterotic G_2 system (or G_2-Hull–Strominger system)

G_2-instanton: connection A on $(M^7, \varphi) \Leftrightarrow F_A \wedge \varphi = - * F_A$
$\Leftrightarrow F_A \wedge *\varphi = 0$

Definition

φ G_2-structure, A connection on E, B connection on TM, $\alpha' > 0$
$\rightsquigarrow (\varphi, (A, E), B, \alpha')$ solution to **heterotic G_2 system**:

- *φ coclosed $\rightsquigarrow \mathrm{d}\varphi = \frac{7}{3}\lambda * \varphi - *H$ $(H \wedge \varphi = 0,\ H \wedge *\varphi = 0)$*
- *A G_2-instanton and B G_2-instanton (up to $O(\alpha')^2$)*
- *anomaly-free condition: $\mathrm{d}H = \frac{\alpha'}{4}(\operatorname{tr} F_A^2 - \operatorname{tr} F_B^2)$ (H flux)*

Theorem (L.–Sá Earp)

$\forall \alpha' > 0$ (M^7, g, η, Υ) contact Calabi–Yau admits solution $(\varphi, (A, \ker \eta), B, \alpha')$ to heterotic G_2 system where

$$\varphi = \epsilon \eta \wedge \omega + \operatorname{Re} \Upsilon, \qquad \mathrm{d}H \neq 0 \quad \textit{and} \quad \epsilon \to 0 \textit{ as } \alpha' \to 0$$

Flow ansatz

Laplacian flow

$$\frac{\partial \varphi_t}{\partial t} = \Delta_t \varphi_t$$

Laplacian coflow

$$\frac{\partial *_t \varphi_t}{\partial t} = \Delta_t *_t \varphi_t$$

(M^7, g, η, Υ) contact Calabi–Yau, $\omega = \mathrm{d}\eta \rightsquigarrow$ consider

$$\varphi_t = f_t h_t^2 \eta \wedge \omega + h_t^3 \operatorname{Re} \Upsilon \quad \text{and} \quad *_t \varphi_t = \frac{1}{2} h_t^4 \omega^2 - f_t h_t^3 \eta \wedge \operatorname{Im} \Upsilon$$

$$\Delta_t \varphi_t = \frac{4 f_t^3}{h_t^2} \eta \wedge \omega \quad \text{and} \quad \Delta_t *_t \varphi_t = 2 f_t^2 \omega^2$$

$\rightsquigarrow$ ansatz preserved, $\mathrm{d} *_t \varphi_t = 0 \ \forall t$ and flows $\leftrightarrow$ ODE systems

Laplacian coflow: $(f_t, h_t) = (\epsilon(1 + 10\epsilon^2 t)^{-3/10}, (1 + 10\epsilon^2 t)^{1/10})$

- $[*_t \varphi_t] = [*_0 \varphi_0] \ \forall t$

Laplacian flow: $(f_t, h_t) = (\epsilon(1 - 8\epsilon^2 t)^{-1/2}, 1)$

- (in general) $[*_t \varphi_t] \neq [*_0 \varphi_0]$ and detects singular time

Singularity analysis 1

Recall:

- $g_0 = \epsilon^2\eta^2 + h$ on M^7
- $\nabla_t\varphi_t$ encoded by torsion 2-tensor T_t

Singularities:

- Laplacian flow for $t \in (-\infty, \frac{1}{8\epsilon^2})$ (ancient solution) $\rightsquigarrow$ finite-time singularity
- Laplacian coflow for $t \in (-\frac{1}{10\epsilon^2}, \infty)$ (immortal solution) $\rightsquigarrow$ does not converge

M compact $\rightsquigarrow$ (L.–Wei, G. Chen) suggest singularity formation controlled by

$$\Lambda(t) = \sup_M(|Rm_t|_t^2 + |T_t|_t^4 + |\nabla_t T_t|_t^2)^{1/2}$$

Singularity analysis 2

$$\Lambda(t) = \sup_M (|Rm_t|_t^2 + |T_t|_t^4 + |\nabla_t T_t|_t^2)^{1/2}$$

Laplacian flow:

- $\Lambda(t) \to \infty$ as $t \to \frac{1}{8\epsilon^2}$ but $\lim_{t \to \frac{1}{8\epsilon^2}} (\frac{1}{8\epsilon^2} - t)\Lambda(t) < \infty$ ("Type I"/rapidly forming)
- volume normalized flow $\rightsquigarrow$ converges to $\mathbb{R}$ as $t \to \frac{1}{8\epsilon^2}$ and to $\mathbb{C}^3$ as $t \to -\infty$

Laplacian coflow (with non-flat h):

- $\Lambda(t) \to 0$ as $t \to \infty$ but $\lim_{t\to\infty} t\Lambda(t) = \infty$ ("Type IIb"/slowly forming)
- volume normalized flow $\rightsquigarrow$ converges to $\mathbb{C}^3$ as $t \to \infty$ and to $\mathbb{R}$ as $t \to -\frac{1}{10\epsilon^2}$

Hitchin flow

Hitchin flow for coclosed G_2 structures on M^7:

$$\frac{\partial *_t \varphi_t}{\partial t} = \mathrm{d}\varphi_t$$

$\rightsquigarrow$ torsion-free $\mathrm{Spin}(7)$-structure on $I_t \times M^7$ for $I_t \subseteq \mathbb{R}$:

$$\Phi = \mathrm{d}t \wedge \varphi_t + *_t\varphi_t$$

$\rightsquigarrow$ Ricci-flat g with $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$

Theorem (L.–Sá Earp–Saavedra)

(M^7, g, η, Υ) contact Calabi–Yau (with $\omega = \mathrm{d}\eta$)

$$\varphi_0 = \epsilon\eta \wedge \omega + \mathrm{Re}\,\Upsilon \quad \textit{and} \quad *_0\,\varphi_0 = \frac{1}{2}\omega^2 - \epsilon\eta \wedge \mathrm{Im}\,\Upsilon$$

Hitchin flow coincides with Laplacian coflow (up to reparametrization of time) and defines Calabi–Yau structure on $I_t \times M^7$ $\rightsquigarrow$ (incomplete) Ricci-flat Kähler metric on $I_t \times M^7$

Solving the heterotic G_2 system 1

(M^7, g, η, Υ) contact Calabi–Yau

Want: $(\varphi, (A, E), B, \alpha')$

- $\alpha' > 0$ arbitrary ✓
- $\epsilon = \epsilon(\alpha') > 0 \rightsquigarrow$

$$\varphi_\epsilon = \epsilon\eta \wedge \omega + \operatorname{Re}\Upsilon \quad \text{and} \quad *\varphi_\epsilon = \frac{1}{2}\omega^2 - \epsilon\eta \wedge \operatorname{Im}\Upsilon$$

$\rightsquigarrow \mathrm{d} * \varphi_\epsilon = 0$ ✓

- flux $H_\epsilon = -\epsilon^2\eta \wedge \omega + \epsilon \operatorname{Re}\Upsilon \rightsquigarrow \mathrm{d}H_\epsilon = -\epsilon^2\omega^2 \neq 0$ ✓
- $E = \ker\eta$, A transverse connection $\rightsquigarrow$

$$F_A \wedge \omega^2 = 0, \quad F_A \wedge \Upsilon = 0 \quad \rightsquigarrow \quad F_A \wedge *\varphi_\epsilon = 0 \ ✓$$

- **need** B on TM so that

$$F_B \wedge *\varphi_\epsilon = O(\alpha')^2 \quad \text{and} \quad \mathrm{d}H_\epsilon = -\epsilon^2\omega^2 = \frac{\alpha'}{4}(\operatorname{tr} F_A^2 - \operatorname{tr} F_B^2)$$

Solving the heterotic G_2 system 2

Recall: "Bismut connection" B^+_ϵ for $\varphi_\epsilon \rightsquigarrow$

$$\nabla_{B^+_\epsilon} g_\epsilon = 0 \qquad \nabla_{B^+_\epsilon} \varphi_\epsilon = 0 \qquad \text{totally skew torsion } H_\epsilon$$

(Metric connection B^-_ϵ with torsion $-H_\epsilon$ is "Hull connection")

Idea: let $\kappa = \kappa(\alpha') > 0$ and $B = B^+_{\kappa\epsilon}$: Bismut connection for $\varphi_{\kappa\epsilon}$

Take $\kappa^2 = (\alpha')^{-3}$ and $\epsilon^2 = 2(\alpha')^5 \rightsquigarrow$

- $F_B \wedge *\varphi_\epsilon = O(\alpha')^2$ ✓
- $\mathrm{d}H_\epsilon = \frac{\alpha'}{4}(\operatorname{tr} F_A^2 - \operatorname{tr} F_B^2)$ ✓

Note: we can also

- modify Hull and Levi-Civita connections for $\varphi_{\kappa\epsilon}$
- make B approximate G_2-instanton to order $O(\alpha')^k$ for any $k \geq 2$

Summary and questions

Summary: M^7 contact Calabi–Yau $\rightsquigarrow$ coclosed G_2-structures

- Laplacian flow $\rightsquigarrow$ rapidly forming finite-time singularity, collapsing to $\mathbb{R}$
- Laplacian coflow $\rightsquigarrow$ slowly forming infinite-time singularity, collapsing to $\mathbb{C}^3$
- solution to heterotic G_2 system for any $\alpha' > 0$

Questions

- gauge theory on contact Calabi–Yau 7-manifolds? (cf. Sá Earp et. al., Y. Wang)
- other $\mathcal{S}^1$-bundles/$\mathcal{S}^1$-symmetry? (cf. Apostolov–Salamon, Foscolo–Haskins–Nordström, Fowdar)
- coclosed G_2-structures and geometric flows?
- conditions for (exact) solutions/numerical approximations to heterotic G_2 system?