Collapse in Riemannian Geometry

Fabian Lehmann, Joe Swinson and Albert Wood
supervised by Mark Haskins and Jason Lotay

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Abstract

This text is divided into four main parts, with the underlying theme of Collapse in Riemannian Geometry.

Section 1 (authored by Albert Wood) is devoted to a short introduction of the basic notions of Riemannian geometry, including the covariant derivative, curvature, and Riemannian submersions.

Sections 2 & 3 (authored by Joe Swinson) provide an exposition of the basics of Gromov and Cheeger's theory of collapse with bounded curvature. The emphasis is on motivation and basic examples, with indications of the more general theory towards the end of Section 3.

Section 4 (authored by Albert Wood) provides a digression concerning differential analysis of Riemannian manifolds, in particular the theory of elliptic PDEs. We give a sufficient condition for solvability of elliptic PDE problems, after looking at the motivational example of Poisson's equation.

Finally, in Section 5 (authored by Fabian Lehmann) we provide an exposition of Foscolo's recent paper ([12]) on collapsing hyperkähler metrics on a K3 surface.

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1 Riemannian Geometry

In this section, we aim to give an introduction to the main characters of the story, namely Riemannian manifolds and maps between them, and various related notions of curvature.

1.1 A Very Brief Introduction to Riemannian Geometry

Riemannian geometry is the study of manifolds (Hausdorff and paracompact topological spaces which are locally homeomorphic to $\mathbb{R}^n$) equipped with a **Riemannian metric**, which is a tool that allows us to calculate distances and angles. Explicitly, a Riemannian metric g on a manifold M is a smooth choice of inner product on every tangent space T_pM (It is usually denoted $\langle \cdot, \cdot \rangle_g$, $\langle \cdot, \cdot \rangle$, or $g(\cdot, \cdot)$.

The intuition behind defining a metric on the *tangent* spaces of a manifold instead of trying to create a distance directly on the surface of the manifold is that the tangent spaces are where the velocity vectors lie, and in Euclidean space, the length and direction of the velocity vectors (e.g. of a smooth path) tell us everything. For example, the length of a smooth path $\gamma : [0, 1] \to \mathbb{R}^n$ is found by integrating the size of the velocity,

$$\mathcal{L}(\gamma) = \int_0^1 |\gamma'(t)| dt,$$

and the (extrinsic) curvature is given by

$$\kappa = \frac{|T'(t)|}{|\gamma'(t)|},$$

where $T(t)$ is the unit tangent vector to the curve.

The directional derivative of a function $f : M \to \mathbb{R}$ makes sense once one has made rigorous the notion of tangent spaces to a manifold. However, problems arise when one attempts to take a *second* derivative, as different tangent spaces are not as easily identified as in Euclidean space, and so we cannot directly compare derivative vectors at different points on the manifold. The solution to this is the concept of a *connection*, which creates a notion of the derivative of a vector field $X \in \Gamma(TM)$:

Definition 1.1. *A* ***connection*** ∇ *on a Riemannian manifold* M *is a mapping*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

that is $\mathbb{R}$*-bilinear, satisfying the following properties:*

- $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$,
- $\nabla_X (fY) = f \nabla_X Y + X(f) Y$.

In fact we can demand two extra properties of the connection, and if we do, there is a *unique* choice:

Theorem 1.2. *For a Riemannian manifold* (M, g)*, there is a unique connection, the* ***Levi-Civita*** *connection, satisfying the following extra properties:*

- $[X, Y] - (\nabla_X Y - \nabla_Y X) = 0$ *(Torsion-free)* ,
- $X \cdot \langle Y, Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + \langle X, \nabla_Y Z\rangle_g$ *(Compatibility with g)* .

Proof. We follow the proof in [3]. Suppose initially the existence of such a connection, ∇. Then:

$$X\langle Y, Z\rangle_g = \langle \nabla_X Y, Z\rangle_g, \langle Y + \nabla_X Z\rangle_g,$$

along with similar identities with the letters permuted. Then, using the torsion-free property:

$$\begin{aligned}
&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\
&= \langle \nabla_Y Z - \nabla_Z Y, X\rangle + \langle \nabla_X Z - \nabla_Z X, Y\rangle + \langle \nabla_X Y + \nabla_Y X, Z\rangle \\
&= \langle [Y, Z], X\rangle + \langle [X, Z], Y\rangle + \langle 2\nabla_X Y - [X, Y], Z\rangle,
\end{aligned}$$

and this implies the *Koszul Formula* for the Levi-Civita connection:

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle.$$

This proves uniqueness, since this expression is uniquely determined. But working backwards, if we define ∇ by this formula, it will have all the properties required. □

From now on, unless specified, we will be working with the Levi-Civita connection exclusively. A remark: a vector field Y on $M \subset \mathbb{R}^n$ may be considered simply as an $\mathbb{R}^n$-valued function on M with $Y(p)$ tangential to M at each $p \in M$. It turns out that for vector fields X, Y on $M \subset \mathbb{R}^n$, $\nabla_X Y$ equals the orthogonal projection to M of the componentwise time-derivative of Y along a flow of X. For this reason, ∇ is often called the **covariant derivative.**

A final comment is that it will be useful to work with tensors other than vector fields, and to take derivatives of them as well. The connection provides us with a unique way to do that, if we simply demand that the covariant derivative *commutes with tensor contraction*:

$$\begin{aligned}
\nabla_X(T(X)) &= \nabla_X(\text{cont.}(T \otimes X)) \\
&= \text{cont.}(\nabla_X(T \otimes X)),
\end{aligned}$$

and satisfies the *tensor Leibniz rule*:

$$\nabla_X(T_1 \otimes T_2) = \nabla_X T_1 \otimes T_2 + T_1 \otimes \nabla_X T_2.$$

As an example, it will often be valuable to think of the covariant derivative as an operator ∇, taking vector fields to $(1,1)$ tensors: $\nabla X(Y) := \nabla_Y(X)$, analogously to the exterior derivative, or Jacobean matrix of Euclidean space. Then, the derivative $\nabla_X(\nabla Z)$ should be defined as follows:

$$\begin{aligned}\nabla_X(\nabla Z)(Y) &= \text{cont.}(\nabla_X(\nabla Z) \otimes Y) \\ &= \text{cont.}(\nabla_X(\nabla Z \otimes Y) - \nabla Z \otimes \nabla_X Y) \\ &= \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z,\end{aligned}$$

and then the *second* covariant derivative, $\nabla\nabla$, is a $(1,2)$-tensor defined by:

$$\nabla\nabla Z(X,Y) := \nabla_X(\nabla Z)(Y) = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z.$$

1.2 Curvature

Now we have our main geometric tools, it is time to define the various notions of *curvature* of a manifold. This is a concept that has gone through many iterations, and the finished product (the Riemannian curvature tensor) seems initially very abstract and removed from the intuition. It has the advantage of being a strong concept, in that many other familiar and useful intrinsic curvatures can be derived from it.

1.2.1 The Curvature Tensor and Sectional Curvature

Definition 1.3. *The* ***Riemannian curvature tensor*** *R associated to a Riemannian manifold (M,g) is a correspondence associating a vector field to every triple of vector fields:*

$$R : \Gamma(TM)^3 \to \Gamma(TM),$$

given explicitly by

$$R(X,Y)Z := \big(\nabla_{[X,Y]} - (\nabla_X \nabla_Y - \nabla_Y \nabla_X)\big)Z.$$

We often make R into a $(0,4)$-tensor using the inner product:

$$R(X,Y,Z,T) := \langle R(X,Y)Z, T\rangle_g.$$

The ***sectional curvature*** *is a function*

$$K : G^2(M) \to \mathbb{R},$$

(where $G^2(M)$ is the 2-Grassmannian of M), defined on a pair of vectors at a point by:

$$K(x, y) := \frac{R_m(x, y, x, y)}{\langle x \wedge y, x \wedge y \rangle_g} = \frac{R_m(x, y, x, y)}{\langle x, x \rangle_g \langle y, y \rangle_g - \langle x, y \rangle_g^2}.$$

Both K_p and R_p only depend on the values of the input vector fields at the point p. It turns out that $R(x, y)z$ describes the "infinitesimal holonomy" of Z around a parallelogram spanned by x, y at nearby points (for more information on this viewpoint, see for example [26]), while the sectional curvature $K_p(x, y)$ equals the Gaussian curvature at p of the plane swept out by geodesics starting in directions $ax + by$ at p, for $a, b \in \mathbb{R}$.

This notion gives us exactly what we would expect for the classical examples - e.g the sphere $S^n \subset \mathbb{R}^n$ has constant curvature 1 and the hyperbolic plane $\mathcal{H}^n$ has constant curvature -1 (for a proof, see [28], chapter 3). As a simple example of working with curvature, we look at the curvature tensor of a product manifold.

Theorem 1.4. *Let $M = M_1 \times M_2$ be the Riemannian product of M_1 and M_2, i.e if g_1, g_2 are the metrics on M_1, M_2 respectively, and $X \in \Gamma(TM)$ decomposes into $X = X_1 + X_2$ for projections $X_i \in \Gamma(TM_i)$, then*

$$g(X_1 + X_2, Y_1 + Y_2) := g_1(X_1, Y_1) + g_2(X_2, Y_2).$$

Then, the Riemannian curvature tensor of M is given by

$$R(X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2, T_1 + T_2) = R_1(X_1, Y_1, Z_1, T_1) + R_2(X_2, Y_2, Z_2, T_2),$$

for $X_i, Y_i, Z_i, T_i \in \Gamma(TM_i)$.

Proof. From the definition of a Riemannian product, we have the formula

$$\langle X_1 + X_2, Y_1 + Y_2 \rangle_g = \langle X_1, Y_1 \rangle_{g_1} + \langle X_2, Y_2 \rangle_{g_2}.$$

Using a mixture of local coordinates and the Koszul formula from the proof of Theorem 1.2, it is then easy to establish the intermediary results:

- $[X_1 + X_2, Y_1 + Y_2] = [X_1, Y_1]_1 + [X_2, Y_2]_2$
- $\nabla_{X_1+X_2}(Y_1 + Y_2) = \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2$.

The result follows from direct calculation:

$$
\begin{aligned}
&R(X_1+X_2, Y_1+Y_2, Z_1+Z_2, T_1+T_2)\\
&= \langle R(X_1+X_2, Y_1+Y_2)(Z_1+Z_2), T_1+T_2\rangle_g\\
&= \langle \nabla_{[X_1+X_2, Y_1+Y_2]}(Z_1+Z_2) - \nabla_{X_1+X_2}\nabla_{Y_1+Y_2}(Z_1+Z_2) + \nabla_{Y_1+Y_2}\nabla_{X_1+X_2}(Z_1+Z_2), T_1+T_2\rangle_g\\
&= \langle \nabla_{[X_1,Y_1]+[X_2,Y_2]}(Z_1+Z_2) - \nabla_{X_1+X_2}(\nabla_{Y_1}Z_1 + \nabla_{Y_2}Z_2) + \nabla_{Y_1+Y_2}(\nabla_{X_1}Z_1 + \nabla_{X_2}Z_2), T_1+T_2\rangle_g\\
&= \langle \nabla_{[X_1,Y_1]}Z_1 + \nabla_{[X_2,Y_2]}Z_2 - \nabla_{X_1}\nabla_{Y_1}Z_1 - \nabla_{X_2}\nabla_{Y_2}Z_2 + \nabla_{Y_1}\nabla_{X_1}Z_1 + \nabla_{Y_2}\nabla_{X_2}Z_2, T_1+T_2\rangle_g\\
&= R_1(X_1, Y_1, Z_1, T_1) + R_2(X_2, Y_2, Z_2, T_2).
\end{aligned}
$$

□

A basic application of this theorem is that the n-torus $(S^1)^n$, equipped with the product metric, is flat, since S^1 is flat. As a more complicated application, let us examine the sectional curvatures of $S^2 \times S^2$. So as to distinguish between the spaces, let M_1 and M_2 be the two distinct copies of S^2.

From the above theorem, given $X, Y = X_1 + X_2, Y_1 + Y_2$ orthonormal in $\Gamma(TM)$, the sectional curvature is given by:

$$
\begin{aligned}
K(X,Y) &= R(X,Y,X,Y)\\
&= R_1(X_1,Y_1,X_1,Y_1) + R_2(X_2,Y_2,X_2,Y_2)\\
&= K(X_1,Y_1)\cdot g(X_1\wedge Y_1, X_1\wedge Y_1) + K(X_2,Y_2)\cdot g(X_2\wedge Y_2, X_2\wedge Y_2)\\
&= g(X_1\wedge Y_1, X_1\wedge Y_1) + g(X_2\wedge Y_2, X_2\wedge Y_2)\\
&= \sum_{i=1}^{2}\left(\|X_i\|_g\|Y_i\|_g - \langle X_i, Y_i\rangle_g^2\right),
\end{aligned}
$$

as the sphere has constant curvature 1. Since $\|X\|_g = 1$, we must have

$$\|X_1\|_g = \cos(\phi_X), \|X_2\|_g = \sin(\phi_X),$$

and similarly for Y. Then,

$$K(X,Y) = \cos(\phi_X)\cos(\phi_Y) + \sin(\phi_X)\sin(\phi_Y) - (\langle X_1, Y_1\rangle_g^2 + \langle X_2, Y_2\rangle_g^2).$$

Since $g(X_i\wedge Y_i, X_i\wedge Y_i)$ must be positive, it follows that $K(X,Y)$ is between 0 and 1. To demonstrate that these bounds are sharp,

- Consider the situation where $\phi_X = 0$ and $\phi_Y = 0$ - this corresponds to $X, Y \in \Gamma(TM_1)$. Then $\cos(\phi_X) = \cos(\phi_Y) = 1$, so

$$K(X,Y) = 1.$$

- Consider the situation where $\phi_X = 0$ and $\phi_Y = \frac{\pi}{2}$ - this corresponds to $X \in \Gamma(TM_1)$ and $Y \in \Gamma(TM_2)$. Then $\sin(\phi_X) = \cos(\phi_Y) = 0$ and $X_2 = Y_1 = 0$, so

$$K(X, Y) = 0.$$

We will later look at more intricate examples of calculating sectional curvature, for example in Section 2.2 when we calculate the curvature of the *Berger sphere*, and in section 2.3 when we calculate the curvature of the *Heisenberg group*.

1.2.2 Ricci and Scalar Curvature

Definition 1.5. *The* ***Ricci*** *curvature is defined as a trace of the curvature tensor:*

$$Ric(x, y) = Tr(R(x, \cdot)y).$$

Componentwise, we have a $(0, 2)$*-tensor:*

$$Ric = R_{ij} dx^i \wedge dx^j,$$

where

$$R_{ik} = \sum_{j=1}^{n} R^j_{ijk}.$$

Often we take the quadratic form viewpoint, and say that the Ricci curvature of X *is* $Ric(X, X)$*.*

The Riemannian curvature tensor has a high number of symmetries, and in fact the Ricci curvature is the *only* (nonzero) quadratic form that one can wring from it. Another important fact about Ricci curvature is that it is simply the sum of the sectional curvatures through the vector in question:

Lemma 1.6. *If* $u \in T_pM$*, and* $\{u, e_2, \ldots, e_n\}$ *is the completion of* u *to an orthonormal basis of* T_pM*, then:*

$$Ric(u, u) = \sum_{i=2}^{n} K(u, e_i).$$

Proof. This follows directly from the definition. Since, for orthonormal e_i, e_j, we have

$$R(e_i, e_j, e_i, e_j) = K(e_i, e_j),$$

it then follows that

$$\begin{aligned} \mathrm{Ric}(u, u) &= R(u, u, u, u) + \sum_{i=2}^{n} R(u, e_i, u, e_i) \\ &= \sum_{i=2}^{n} K(u, e_i). \end{aligned}$$

□

Therefore we can think of Ricci curvature as $(n-1)$ times the *average* sectional curvature of planes through the vector u.

There is another geometric interpretation of the Ricci curvature. Near any point m in a Riemannian manifold, we can pick local coordinates such that geodesics through m correspond to straight lines through the origin - these are *geodesic normal coordinates.* With respect to these coordinates, taking a nearby point $x = (x^1, x^2, \ldots, x^n)$, we can expand the metric tensor, we find that there is no first order term, and the second order term is controlled by the curvature:

Theorem 1.7. *With respect to geodesic normal coordinates around $m \in M$, the coefficients of the Taylor expansion of $g_{ij}(x)$ around $x = 0$ may be expressed using components of the curvature tensor and its covariant derivatives. Explicitly, up to order 2,*

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}R_{ijkl}(m)x^k x^l + O\big(|x|^3\big).$$

It follows that the volume element of the manifold is locally given by

$$dvol_g = \Big(1 + \frac{1}{6}R_{jk}x^j x^k + O\big(|x|^3\big)\Big)dx^1 \wedge \ldots \wedge dx^n.$$

Proof. To prove this statement we require the theory of *Jacobi Fields*, which are the variation vector fields of geodesics. For a full treatment of Jacobi fields, see [3] or [28]. All we need is the following: If $\gamma(t)$ is a geodesic, and $\gamma_s(t)$ a geodesic variation, such that $\gamma_0(t) = \gamma(t)$. Then:

$$J(t) := \partial_s \gamma_s(t)$$

is the **Jacobi field** associated to the variation, and it satisfies the following differential equation, the **Jacobi equation**:

$$\nabla^2_{\partial_s} J(t) + R(\dot\gamma(t), J(t))\dot\gamma(t) = 0.$$

Now, we Taylor expand g_{ij}. Take a point $x = (x^1, \ldots, x^n) \in M$ near m, and set

$$g_{ij}(t) := g_{ij}(tx^1, \ldots, tx^n).$$

Note that

$$\gamma(t) := (tx^1, \ldots, tx^n)$$

is a geodesic since we are in geodesic normal coordinates, and

$$\gamma_s(t) := (tx^1, \ldots, t(x^i + s), \ldots, tx^n)$$

is a geodesic variation. The variation vector field

$$Y_i(t) := \partial_s \gamma_s(t) = t\partial_i|_{\gamma(t)} \tag{1}$$

is a Jacobi field, so it satisfies the Jacobi equation:

$$\nabla^2_{\partial_t} Y_i(t) + R(\dot{\gamma}(t), J(t))\dot{\gamma}(t) = 0. \tag{2}$$

Also, using (1),

$$t^2 g_{ij}(t) = \langle Y_i(t), Y_j(t)\rangle_g. \tag{3}$$

Now, if we differentiate (2) and (3), we inductively obtain:

$$k(k-1)g_{ij}^{(k-2)}(t) + 2ktg_{ij}^{(k-1)}(t) + t^2 g_{ij}^{(k)}(t) = \sum_{l=0}^{k} \binom{k}{l} \langle \nabla^{k-l}_{\partial_t} Y_i(t), \nabla^{l}_{\partial_t} Y_j(t)\rangle_g \tag{4}$$

and

$$\nabla^k_{\partial_t} Y_i(t) + \sum_{l=0}^{k-2} \binom{k-2}{l} (\nabla^{k-2-l}_{\partial_t} R)(\dot{\gamma}(t), \nabla^l_{\partial_t} Y_i(t))\dot{\gamma}(t) = 0, \tag{5}$$

the first equation following from the compatibility of the connection with the metric, and the second following from the definition of covariant derivative of general tensors, and the fact that $\nabla_{\partial_t}(\dot{\gamma}(t)) = 0$, since γ is a geodesic.

Now, we already know that

$$\begin{aligned} Y_i(0) &= 0, \\ \nabla_{\partial_t} Y_i(0) &= \partial_i|_{\gamma(t)}. \end{aligned}$$

Using (5),

$$\begin{aligned} \nabla^2_{\partial_t} Y_i(0) &= -R(\dot{\gamma}(0), Y_i(0))\dot{\gamma}(0) = 0, \\ \nabla^3_{\partial_t} Y_i(0) &= -\sum_{l=0}^{1} \binom{1}{l} (\nabla^{1-l}_{\partial_t} R)(\dot{\gamma}(0), \nabla^l_{\partial_t} Y_i(t)|_{t=0})\dot{\gamma}(0) \\ &= -\nabla_{\partial_t} R(\dot{\gamma}(0), Y_i(0))\dot{\gamma}(0) - R(\dot{\gamma}(0), \nabla_{\partial_t} Y_i(t)|_{t=0})\dot{\gamma}(t) \\ &= -\sum_{k,l} R(x^k \partial_k|_m, \partial_i|_m) x^l \partial_l|_m \\ &= -R^m_{kil} x^k x^l \partial_m. \end{aligned}$$

We can then use (4) to calculate the derivatives of g_{ij} with respect to t (everything is evaluated at $t = 0$):

$$\begin{aligned}
g_{ij}(t) &= \delta_{ij}, \\
6g_{ij}^{(1)}(t) &= -3tg_{ij}^{(2)}(t) - t^2 g_{ij}^{(3)}(t) + \sum_{l=0}^{3} \binom{3}{l} \langle \nabla_{\partial_t}^{3-l} Y_i(t), \nabla_{\partial_t}^{l} Y_j(t) \rangle_g \\
&= 0, \\
12g_{ij}^{(2)}(t) &= 4\langle \nabla_{\partial_t}^3 Y_i(t), \partial_j \rangle_g + 4\langle \partial_i, \nabla_{\partial_t}^3 Y_j(t) \rangle_g \\
&= 4\langle R_{kil}^m x^k x^l \partial_m, \partial_j \rangle_g + 4\langle \partial_i, R_{kjl}^m x^k x^l \partial_m \rangle_g \\
&= 4R_{jkil} x^k x^l + 4R_{ikjl} x^k x^l \\
&= 8R_{jkil} x^k x^l.
\end{aligned}$$

Finally, we can plug these values into a Taylor expansion of $g_{ij}(t)$ to obtain the result:

$$g_{ij}(x) = g_{ij}(1) = \delta_{ij} + \frac{1}{3} R_{jkil} x^k x^l + O\big(|x|^3\big).$$

As for the volume form, for a Riemannian manifold , the volume form is given by

$$\text{dvol} = \sqrt{det(g_{ij})} dx^1 \wedge \ldots \wedge dx^n,$$

so this result follows immediately.

□

Therefore, Ricci curvature "controls the local growth of balls" within the manifold.

The final important notion of curvature is the (quite weak) **Scalar** curvature. This is the trace of the Ricci curvature:

$$\text{Scal}_p(M) := \sum_{i=1}^{n} \text{Ric}(e_i, e_i),$$

where $\{e_i\}$ is an orthonormal basis for T_pM. Intuitively, the scalar curvature gives a number representing the "average" curvature - this is literally true if we divide by $(n-1)$ (to turn Ricci into an average of sectional curvatures) and also by n to average out these averages. We give an example of calculating the Ricci and Scalar curvatures in the next section.

1.3 Riemannian Submersions

Intuitively, if we collapse a manifold by a quotient construction (in a metric-preserving way), the curvature can only increase. We formalise and prove this statement, by making specific what is meant by 'metric-preserving' and giving a precise formula for curvature of the resulting manifold.

If $p : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a mapping of Riemannian manifolds, we say it is a **Riemannian submersion** if, for all $\tilde{m} \in \widetilde{M}$:

- p is surjective,
- $dp_{\tilde{m}} : \ker(dp_{\tilde{m}})^{\perp} \to T_{p(\tilde{m})}(M)$ is an isometry.

Intuitively, a Riemannian submersion is locally an orthogonal projection, killing a 'vertical' subspace of every tangent space and preserving a 'horizontal' subspace. More precisely,

$$T_{\tilde{m}}\widetilde{M} \cong \mathcal{H}_{\tilde{m}}\widetilde{M} \oplus \mathcal{V}_{\tilde{m}}\widetilde{M},$$

where we define the **horizontal subspace**

$$\mathcal{H}_{\tilde{m}}\widetilde{M} := \ker(dp_{\tilde{m}})^{\perp}$$

and the **vertical subspace**

$$\mathcal{V}_{\tilde{m}}\widetilde{M} := \ker(dp_{\tilde{m}}).$$

A useful alternative characterisation of the vertical subspace is the following:

Lemma 1.8. *A vector field $U \in \Gamma(T\widetilde{M})$ is vertical if and only if*

$$U(f \circ p) = 0$$

for any function f on M.

A very important facet of this construction is that any vector field $X \in \Gamma(TM)$ can be *lifted* to a unique horizontal vector field

$$\widetilde{X} \in \Gamma(T\widetilde{M}),\ \widetilde{X}^{\mathcal{V}} = 0,$$

where $\widetilde{X}^{\mathcal{V}}$ denotes the **vertical component** of $\widetilde{X}$.

1.3.1 Examples of Riemannian Submersions

Since a Riemannian submersion is 'locally a product', an obvious example is the projection from a product manifold down to one of its factors:

$$\pi_1 : M_1 \times M_2 \to M_1.$$

More complicated examples are given by quotient constructions arising from Lie group actions:

Theorem 1.9 (Quotient of a Manifold by a Lie Group Action)**.** *Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold, and G be a Lie group of isometries acting freely* $(\forall x \in \widetilde{M},\ \ Stab(x) = \{e\})$

and properly (for compact $K \subset \widetilde{M}$, $\{g \in G : gK \cap K \neq \varnothing\}$ is precompact).

Then $M := \frac{\widetilde{M}}{G}$ is a manifold,

$$p : \widetilde{M} \to M$$

is a fibration, and there exists a unique Riemannian metric g on M such that p is a Riemannian submersion.

As a key example of the above that we will discuss in detail, consider the complex projective plane $\mathbb{CP}^n$, defined as a quotient

$$\mathbb{CP}^n := \frac{S^{2n+1}}{S^1},$$

where we consider S^{2n+1} as the norm-1 subset of $\mathbb{C}^{n+1}$ and the S^1 action is given by

$$e^{i\theta} \cdot (z_1, \dots z_{n+1}) := (e^{i\theta} z_1, \dots, e^{i\theta} z_{n+1}).$$

This satisfies all the criteria for Theorem 1.9, since the circle action is free and S^1 is a compact Lie group. Therefore the quotient is a Riemannian submersion. The horizontal and vertical space are given by:

$$\begin{aligned} \mathcal{H}_{\tilde{m}} S^{2n+1} &= \{x \in \mathbb{C}^{n+1} \ : \ \langle \tilde{m}, x \rangle_{\mathbb{R}} = \langle i \cdot \tilde{m}, x \rangle_{\mathbb{R}} = 0\} \\ &= \{x \in \mathbb{C}^{n+1} \ : \ \langle \tilde{m}, x \rangle_{\mathbb{C}} = 0\}, \\ \mathcal{V}_{\tilde{m}} S^{2n+1} &= \{ki \cdot \tilde{m} \ : \ k \in \mathbb{R}\}, \end{aligned}$$

where the real inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ treats $x \in \mathbb{C}^n$ as an element of $\mathbb{R}^{2n}$, and the complex inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is defined in the usual way:

$$\text{For } x, y \in \mathbb{C}^n, \ \langle x, y \rangle_{\mathbb{C}} := \bar{x}^T y.$$

1.3.2 Curvature of a Riemannian Submersion

We now would like to describe the curvature of submersions in terms of the covering manifold. We start with the most simple example - a Riemannian product.

We have seen from the previous chapter that the sectional curvature of a product manifold $M = M_1 \times M_2$ is given by

$$K(X_1 + X_2, Y_1 + Y_2) = K_1(X_1, Y_1) \cdot g_1(X_1 \wedge Y_1) + K_2(X_2, Y_2) \cdot g_2(X_2 \wedge Y_2).$$

Now consider the Riemannian submersion given by the projection map:

$$\pi_1 : M \to M_1.$$

Then the horizontal lift $\widetilde{X_1}$ of a vector field $X_1 \in \Gamma(TM_1)$ is simply X_1, considered as a vector field in $\Gamma(TM)$. Then using the above formula, we see that the sectional curvature is unchanged by a horizontal lift:

$$K(\widetilde{X_1}, \widetilde{Y_1}) = K_1(X_1, Y_1).$$

Therefore the curvature can be completely recovered from the original space.
It is in general untrue that given a submersion $p : \widetilde{M} \to M$, the curvature can be found just by lifting, i.e. it isn't generally the case that

$$\widetilde{K}(\widetilde{X}, \widetilde{Y}) = K(X, Y).$$

To find the true formula, we must investigate the behaviour of horizontal and vertical vector fields. In particular, the following facts follow from the definitions and direct calculation:

Lemma 1.10. *Let $p : \widetilde{M} \to M$ be a Riemannian submersion, and let $\widetilde{X}, \widetilde{Y}$ be the horizontal lifts of vector fields $X, Y \in \Gamma(TM)$. Then:*

1. $[\widetilde{X}, \widetilde{Y}] - \widetilde{[X, Y]}$ *is vertical,*
2. *If V is vertical, $[\widetilde{X}, V]$ is vertical,*
3. $\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}$.
4. *If V is vertical, then $\langle \widetilde{\nabla}_V \widetilde{X}, \widetilde{Y} \rangle_{\tilde{g}} = -\frac{1}{2}\langle [\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}, V \rangle_{\tilde{g}}$.*

For proofs of these, see [28]. These lemmas give us all we need to know to resolve $K(\widetilde{X}, \widetilde{Y})$. Lets assume our X and Y are orthonormal. Then,

$$\begin{aligned}
\widetilde{K}(\widetilde{X}, \widetilde{Y}) &= R(\widetilde{X}, \widetilde{Y}, \widetilde{X}, \widetilde{Y}) \\
&= \langle R(\widetilde{X}, \widetilde{Y})\widetilde{X}, \widetilde{Y} \rangle_{\tilde{g}} \\
&= \langle \widetilde{\nabla}_{[\widetilde{X}, \widetilde{Y}]} \widetilde{X} - (\widetilde{\nabla}_{\widetilde{X}} \widetilde{\nabla}_{\widetilde{Y}} \widetilde{X} - \widetilde{\nabla}_{\widetilde{Y}} \widetilde{\nabla}_{\widetilde{X}} \widetilde{X}), \widetilde{Y} \rangle_{\tilde{g}}.
\end{aligned}$$

We tackle each bit separately:

- $\widetilde{\nabla}_{\widetilde{X}} \widetilde{X} = \widetilde{\nabla_X X} + \frac{1}{2}[\widetilde{X}, \widetilde{X}]^{\mathcal{V}} = \widetilde{\nabla_X X}$,
- $\widetilde{\nabla}_{\widetilde{Y}} \widetilde{X} = \widetilde{\nabla_Y X} + \frac{1}{2}[\widetilde{Y}, \widetilde{X}]^{\mathcal{V}}$,
- $[\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]} + U$,

where $U := [\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}$. Therefore:

$$
\begin{aligned}
\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}X &= \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla_X X} \\
&= \widetilde{\nabla_Y\nabla_X}X + \tfrac{1}{2}[\widetilde{Y}, \widetilde{\nabla_X X}]^{\mathcal{V}} \\
&= \widetilde{\nabla_Y\nabla_X}X + vert., \\
\widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} &= \widetilde{\nabla}_{\widetilde{X}}(\widetilde{\nabla_Y X}) + \widetilde{\nabla}_{\widetilde{X}}(\tfrac{1}{2}[\widetilde{Y}, \widetilde{X}]^{\mathcal{V}}) \\
&= \widetilde{\nabla_X\nabla_Y}X + \tfrac{1}{2}[\widetilde{X}, \widetilde{\nabla_X Y}]^{\mathcal{V}} + \widetilde{\nabla}_{\widetilde{X}}(\tfrac{1}{2}[\widetilde{Y}, \widetilde{X}]^{\mathcal{V}}) \\
&= \widetilde{\nabla_X\nabla_Y}X + vert. + \widetilde{\nabla}_{\widetilde{X}}(\tfrac{1}{2}[\widetilde{Y}, \widetilde{X}]^{\mathcal{V}}), \\
\widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{X} &= \widetilde{\nabla}_{\widetilde{[X,Y]}}\widetilde{X} + \widetilde{\nabla}_U\widetilde{X} \\
&= \widetilde{\nabla_{[X,Y]}}X + \tfrac{1}{2}[[\widetilde{X}, \widetilde{Y}], \widetilde{X}]^{\mathcal{V}} + \widetilde{\nabla}_U\widetilde{X} \\
&= \widetilde{\nabla_{[X,Y]}}X + vert. + \widetilde{\nabla}_U\widetilde{X},
\end{aligned}
$$

and putting it all together, labelling $U' = \frac{1}{2}[\widetilde{Y}, \widetilde{X}]^{\mathcal{V}}$ and remembering that the torsion-free property of ∇ gives us $\nabla_X Y - \nabla_Y X - [X, Y] = 0$:

$$
\begin{aligned}
\widetilde{K}(\widetilde{X}, \widetilde{Y}) &= \langle \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{X} - (\widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{X}), \widetilde{Y}\rangle_{\tilde{g}} \\
&= \langle \widetilde{\nabla_{[X,Y]}}X - (\widetilde{\nabla_X\nabla_Y}X - \widetilde{\nabla_Y\nabla_X}X), \widetilde{Y}\rangle_{\tilde{g}} + \langle -\widetilde{\nabla}_{\widetilde{X}}U' + \widetilde{\nabla}_U\widetilde{X}, \widetilde{Y}\rangle_{\tilde{g}} \\
&= K(X, Y) + \langle -\widetilde{\nabla}_{\widetilde{X}}U' + \widetilde{\nabla}_U\widetilde{X}, \widetilde{Y}\rangle_{\tilde{g}} \\
&= K(X, Y) + \langle -\widetilde{\nabla}_{U'}\widetilde{X}, \widetilde{Y}\rangle_{\tilde{g}} + \langle \widetilde{\nabla}_U\widetilde{X}, \widetilde{Y}\rangle_{\tilde{g}} \\
&= K(X, Y) + \tfrac{1}{2}\langle [\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}, U'\rangle_{\tilde{g}} - \tfrac{1}{2}\langle [\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}, U\rangle_{\tilde{g}} \\
&= K(X, Y) - \tfrac{1}{4}|[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}|^2 - \tfrac{1}{2}|[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}|^2 \\
&= K(X, Y) - \tfrac{3}{4}|[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}|^2.
\end{aligned}
$$

We have achieved the following formula:

Theorem 1.11 (O'Neill's Formula). *Let $p : \widetilde{M} \to M$ be a Riemannian submersion, and let $\widetilde{X}, \widetilde{Y}$ be the horizontal lifts of vector fields $X, Y \in \Gamma(TM)$. Then:*

$$K(X, Y) = \widetilde{K}(\widetilde{X}, \widetilde{Y}) + \tfrac{3}{4}|[\widetilde{X}, \widetilde{Y}]^{\mathcal{V}}|^2.$$

1.3.3 Curvature of $\mathbb{CP}^n$

As an application of this formula, let us calculate the curvature of $\mathbb{CP}^n$, which is given by a quotient of S^{2n+1} by a circle action. Remembering from before that the horizontal and vertical spaces are given by

$$\begin{aligned}\mathcal{H}_{\tilde{m}}S^{2n+1} &= \{x \in \mathbb{C}^{n+1} \;:\; \langle \tilde{m}, x\rangle_{\mathbb{R}} = \langle i \cdot \tilde{m}, x\rangle_{\mathbb{R}} = 0\}\\ &= \{x \in \mathbb{C}^{n+1} \;:\; \langle \tilde{m}, x\rangle_{\mathbb{C}} = 0\},\\ \mathcal{V}_{\tilde{m}}S^{2n+1} &= \{ki \cdot \tilde{m} \;:\; k \in \mathbb{R}\},\end{aligned}$$

we see that, if $\widetilde{X}, \widetilde{Y}$ are horizontal lifts of $X, Y \in \Gamma(T\mathbb{CP}^n)$,

$$\begin{aligned}&\langle p, \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X}|_p\rangle_{\mathbb{C}} = -\langle \widetilde{Y}|_p, \widetilde{X}|_p\rangle_{\mathbb{C}}\\ \implies &\langle p, \widetilde{\nabla_Y X}|_p + \tfrac{1}{2}[\widetilde{Y}, \widetilde{X}]^{\mathcal{V}}\rangle_{\mathbb{C}} = -\langle \widetilde{Y}|_p, \widetilde{X}|_p\rangle_{\mathbb{C}}\\ \implies &\langle p, [\widetilde{Y}, \widetilde{X}]^{\mathcal{V}}\rangle_{\mathbb{C}} = -2\langle \widetilde{Y}|_p, \widetilde{X}|_p\rangle_{\mathbb{C}}.\end{aligned}$$

Since the vertical space at p is just $\{kip : k \in \mathbb{R}\}$, it follows that

$$[\widetilde{Y}, \widetilde{X}]^{\mathcal{V}} = -2p\langle \widetilde{Y}, \widetilde{X}\rangle_{\mathbb{C}}.$$

Applying Theorem 1.11 then gives us the curvature:

$$\begin{aligned}K(X, Y) &= \widetilde{K}(\widetilde{X}, \widetilde{Y}) + \tfrac{3}{4}|2p\langle \widetilde{X}, \widetilde{Y}\rangle_{\mathbb{C}}|^2\\ &= 1 + 3|\langle \widetilde{X}, \widetilde{Y}\rangle_{\mathbb{C}}|^2,\end{aligned}$$

using the fact that the sphere has constant curvature 1. Since we demanded that $\widetilde{X}$, $\widetilde{Y}$ were orthonormal, this value can be at most 4, and at least 1.

More explicity, take a vector $\tilde{v} \in T_{\tilde{m}}S^{2n+1}$. Any vector in $T_{\tilde{m}}S^{2n+1}$ orthogonal to $\tilde{v}$ can be written as:

$$\tilde{u} = \cos(\phi)\tilde{w} + \sin(\phi)i \cdot \tilde{v},$$

where $\langle \tilde{v}, \tilde{w}\rangle_{\mathbb{C}} = 0$. Then:

$$\begin{aligned}\langle \cos(\phi)\tilde{w} + \sin(\phi)i \cdot \tilde{v}, \tilde{v}\rangle_{\mathbb{C}} &= \langle \cos(\phi)\tilde{w}, \tilde{v}\rangle_{\mathbb{C}} + \langle \sin(\phi)i\tilde{v}, \tilde{v}\rangle_{\mathbb{C}}\\ &= \sin(\phi)i\langle \tilde{v}, \tilde{v}\rangle_{\mathbb{C}}\\ &= i\sin(\phi).\end{aligned}$$

Therefore the sectional curvature of the plane in $T_m\mathbb{CP}^n$ spanned by the vectors u, v (which lift to $\tilde{u}, \tilde{v}$ respectively) is

$$K[u, v] = K[\tilde{u}, \tilde{v}] + 3|i\sin(\phi)|^2 = 1 + 3\sin(\phi)^2.$$

Finally, we find the Ricci curvature and scalar curvature of $\mathbb{CP}^n$. If we pick a vector $u \in T_p\mathbb{CP}^n$ and complete it to an orthonormal basis (of size $2n$), the Ricci curvature $\mathrm{Ric}(u, u)$ is given by

$$
\begin{aligned}
\mathrm{Ric}_p(u,u) &= K(Ju,u) + \sum_{i=3}^{2n} K(v_i,u) \\
&= 4 + 1 \cdot (2n-2) \\
&= 2(n+1).
\end{aligned}
$$

The scalar curvature is then given by

$$
\begin{aligned}
\mathrm{Scal}_p(\mathbb{CP}^n) &= \sum_{i=1}^{2n} \mathrm{Ric}(e_i,e_i) \\
&= 2n\mathrm{Ric}_p(e_1,e_1) \\
&= 4n(n+1).
\end{aligned}
$$

This corresponds to an average curvature of:

$$
\frac{\mathrm{Scal}_p(\mathbb{CP}^n)}{2n \cdot (2n-1)} = 1 + \frac{3}{2n-1},
$$

which for $n = 1$ gives a value of 4. In fact, $\mathbb{CP}^1 \cong S^2$, and this quotient construction in particular produces a copy of S^2 with radius $\frac{1}{2}$ - matching this result.

2 Collapse with Bounded Curvature: First Examples

Now that the relevant background in Riemannian geometry - curvature and submersions - has been set up, we come to the part of the text concerned with the *collapse* of Riemannian manifolds. We shall explain precisely what this means below.

2.1 Introduction: Volume, Diameter, Curvature and Injectivity Radius

Given any Riemannian manifold (M^n, g) and any $\delta > 0$, one may scale lengths in M by δ, by taking the metric $g_\delta = \delta^2 g$. Under this procedure, the diameter and injectivity radius of M scale by a factor of δ, the volume by δ^n and the sectional curvature by $\frac{1}{\delta^2}$. Therefore, as $\delta \to 0$, assuming that these these four quantities are finite (which holds if M is compact), the diameter D, volume V, and injectivity radius I tend to zero, while the curvature K tends to $\pm\infty$ where the manifold is not flat. Conversely, of course, if $\delta \to \infty$, then one sees $V, D, I \to \infty$ while $K \to 0$ as M becomes "bigger and flatter".

This rather crude procedure illustrates the expected relationship between these four quantities. If one shrinks to zero, another may become unbounded. For example, under a family of changing metrics, it is natural to expect $V \to 0$ as $D \to 0$, but this can be avoided if $K \to -\infty$. Similarly, if $I \to 0$, we see separate geodesics out of each $p \in M$ intersecting arbitrarily close to p, and one might picture the case of the end of an ellipsoid with eccentricity approaching 1, in which case $K \to \infty$. If this is happening at all points, one might picture the case of ever smaller spheres.

It is interesting, therefore, to ask how it might happen that $I \to 0$ at all points, without K becoming unbounded (apart from the trivial case where M is flat).

Definition 2.1. *If a family of metrics g_δ on M achieves $I \to 0$ uniformly, while K is uniformly bounded (both on compact subsets), as $\delta \to 0$, then M is said to* ***collapse with bounded curvature.***

In this section, we shall study a first few basic examples of collapse with bounded curvature, and see how one often obtains a "limit space", in a suitable sense, at the end of a collapsing family of metrics. We shall introduce the Gromov-Hausdorff distance in order to make this precise, and in the next section, develop the theory of F-structures, a step towards classifying how collapse with bounded curvature may occur in general (it turns out that the only way is to shrink along "almost flat" fibres in a manifold).

Much of the material found in these two sections is based on that of [29] and [4].

2.2 First Example: the Berger Spheres

View S^3 as sitting inside $\mathbb{R}^4 = \mathbb{C}^2 = \mathbb{H}$. There is an action of $S^1 \subset \mathbb{C}$ on $S^3 \subset \mathbb{C}^2$ simply by

$$e^{i\theta}.(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2),$$

and as in Section 1.3.3, the complex projective line is simply the quotient of S^3 by this isometric action (in the sense of Section 1.3). The metric that makes the quotient map a Riemannian submersion is called the **Fubini-Study metric**, and this metric makes $\mathbb{CP}^1$ isometric to a sphere of radius $\frac{1}{2}$; therefore, the complex projective line has constant curvature $\frac{1}{4}$. The submersion $S^3 \to S^2$ is called the **Hopf fibration.**

One may equally well view S^3 as the set of unit quaternions by the bijection $\mathbb{C}^2 \to \mathbb{H}$, $f : (z_1, z_2) \mapsto z_1 + jz_2$. Explicitly,

$$f(a + bi, c + di) = a + bi + cj - dk.$$

The $-k$ may appear incongruous. We shall see its purpose presently: under the bijection f, the infinitesimal action of S^1 on S^3 given by the componentwise action on $\mathbb{C}^2$, is

$$\partial_\theta|_{f(z_1,z_2)} = \left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_0 e^{i\theta} z_1 + j \left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_0 e^{i\theta} z_2 = iz_1 - kz_2.$$

On the other hand, extending the vectors $i, j, k \in \mathrm{Im}\mathbb{H} = T_1 S^3$ to left-invariant vector fields e_1, e_2, e_3, we have

$$\begin{aligned} e_1|_{z_1 + jz_2} &= (z_1 + jz_2).i \\ &= z_1 i + jz_2 i = iz_1 + jiz_2 \\ &= iz_1 - kz_2. \end{aligned}$$

The point of this is that under the bijection f, the left-invariant vector field e_1 on S^3 gives the direction of the S^1 action at each point; that is, the fibre direction in the Hopf fibration. Therefore, contracting lengths in the e_1 direction at each point may be referred to as "collapsing along the fibres of the Hopf fibration".

Let us, therefore, collapse along these fibres. Let g_δ be the left-invariant metric

$$g_\delta = \delta^2 e_1^* \otimes e_1^* + e_2^* \otimes e_2^* + e_3^* \otimes e_3^*,$$

for each $\delta > 0$. The collection (S^3, g_δ) for $\delta \in (0, \infty)$ are known as the **Berger spheres.** They are all left-invariant, meaning that left-translation in the group S^3 is isometric; but only g_1 is right-invariant. Let us compute $\nabla_{e_a} e_b$ and the sectional curvatures through

$\mathrm{Span}(e_a, e_b)$, to obtain a picture of what is happening, say, as $\delta \to 0$. We use the Koszul formula (see Theorem 1.2),

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle.$$

In the case $\{X, Y, Z\} \subset \{e_1, e_2, e_3\}$, the formula simplifies; specifically, the first three terms vanish. Recalling that $[e_a, e_b] = 2\epsilon_{abc}e_c$ by the Hamilton relations for quaternions, one calculates the values

$$\begin{bmatrix} \nabla_{e_1}e_1 & \nabla_{e_1}e_2 & \nabla_{e_1}e_3 \\ \nabla_{e_2}e_1 & \nabla_{e_2}e_2 & \nabla_{e_2}e_3 \\ \nabla_{e_3}e_1 & \nabla_{e_3}e_2 & \nabla_{e_3}e_3 \end{bmatrix} = \begin{bmatrix} 0 & (2-\delta^2)e_3 & (\delta^2-2)e_2 \\ -\delta^2 e_3 & 0 & e_1 \\ \delta^2 e_2 & -e_1 & 0 \end{bmatrix}.$$

(In particular, the flows $\dot{\gamma}_a(t) = e_a|_{\gamma_a(t)}$ are geodesics for each a and every δ.) Use the formula for sectional curvature,

$$K(X, Y) = \frac{R(X, Y, X, Y)}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2} = \frac{\langle \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]}Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2},$$

to calculate (again, most of the terms in the calculation vanish) that

$$K(e_1, e_2) = \delta^2, K(e_2, e_3) = 4 - 3\delta^2,\ K(e_3, e_1) = \delta^2.$$

For all δ, the flow of e_1 through any point is a geodesic path, as mentioned above; it is a loop of length tending to zero as $\delta \to 0$. Therefore, the injectivity radius of (S^3, g_δ) is tending to zero at all points as $\delta \to 0$.

If one takes orthonormal vectors $u, v \in \mathrm{Span}(e_1, e_2, e_3) = T_1S^3$, then the bottom half of the expression for $K(u, v)$ equals 1 and the top half a linear combination of $R(e_i, e_j, e_k, e_l)$ values. This tells us that *all* sectional curvatures remain bounded as $\delta \to 0$, and any sectional curvature through e_1, the fibre direction, tends to zero, while $K(e_2, e_3)$ tends to 4, the sectional curvature of the quotient S^3/S^1. Therefore, the Berger spheres S_δ for $\delta \to 0$ may be viewed as S^3 collapsing with bounded curvature, and it appears that S^3 is "converging in curvature" to the quotient $S^2(\frac{1}{2}) = \mathbb{CP}^1$.

An aside: the curvature values above tell us that

$$(\mathrm{Ric}(e_i, e_j)) = \begin{bmatrix} 2\delta^2 & 0 & 0 \\ 0 & 4-2\delta^2 & 0 \\ 0 & 0 & 4-2\delta^2 \end{bmatrix}.$$

Then, the scalar curvature is the trace of the Ricci (2,0) tensor in this orthonormal basis:

$$\mathrm{Scal}(p) = 8 - 2\delta^2$$

for $p = (1,0,0,0) \in S^3$, and consequently at all points since the metrics are left-invariant. Therefore, if one makes δ *large,* then one has a metric of constant, arbitrarily large, *negative* scalar curvature on S^3. (This may sound surprising; after all, S^3 is a sphere. In fact, *every compact manifold* of dimension $\geqslant 3$ admits a metric with constant negative scalar curvature. See [19].)

However, note that one does not obtain negative Ricci curvature with any of these g_δ. As a further aside, this is consistent with the fact that e_1 is a notrivial Killing field for all g_δ; consistent, because of the following result, proven in chapter 7 of Petersen [25]:

Theorem 2.2 (Bochner). *If M is compact, oriented and $Ric < 0$, then there are no nontrivial Killing fields.*

Conclusion: by scaling lengths along the fibres of the Hopf fibration, S^3 has a sequence of metrics with sectional curvatures remaining bounded and injectivity radius tending to zero uniformly.

2.3 Second Example: the Heisenberg Group

We shall now witness a similar phenomenon with another 3-dimensional manifold that has the structure of a Lie group. The **Heisenberg group** is defined to be the Lie group

$$H = \begin{bmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{bmatrix}.$$

Under the bijection

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},$$

a left-invariant metric on H is a metric on $\mathbb{R}^3$ invariant under

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x+a \\ y+b \\ z+ay+c \end{bmatrix}.$$

That is, invariant under translation in the y and z directions and under

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x+a \\ y \\ z+ay \end{bmatrix},$$

whose orbits one may draw to get an idea of what this metric looks like.

Let X, Y, Z be the left-invariant vector fields that equal

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

respectively at the identity. Similarly to Section 2.2, consider the family of metrics

$$g_\delta = X^* \otimes X^* + Y^* \otimes Y^* + \delta^2 Z^* \otimes Z^*.$$

In Cartesian coordinates,

$$X = \partial_x, \quad Y = \partial_y + x\partial_z, \quad Z = \partial_z,$$

and

$$ds_\delta^2 = dx^2 + (1 + \delta^2 x^2)dy^2 - 2\delta^2 x dy dz + \delta^2 dz^2.$$

As before, one uses the Koszul formula to obtain

$$\begin{bmatrix} \nabla_X X & \nabla_X Y & \nabla_X Z \\ \nabla_Y X & \nabla_Y Y & \nabla_Y Z \\ \nabla_Z X & \nabla_Z Y & \nabla_Z Z \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}Z & -\frac{\delta^2}{2}Y \\ -\frac{1}{2}Z & 0 & \frac{\delta^2}{2}X \\ -\frac{\delta^2}{2}Y & \frac{\delta^2}{2}X & 0 \end{bmatrix},$$

a strange mixture of symmetry and anti-symmetry.

One calculates the sectional curvatures as

$$K(X, Y) = R(X, Y, X, Y) = -\frac{3}{4}\delta^2,$$

$$K(X, Z) = \frac{R(X, Z, X, Z)}{\delta^2} = \frac{1}{4}\delta^2,$$

$$K(Y, Z) = \frac{R(Y, Z, Y, Z)}{\delta^2} = \frac{1}{4}\delta^2.$$

Once again, all sectional curvatures are expressible as a linear combination of the above three, so that $K(\Pi) \to 0$ for any tangent plane Π as $\delta \to 0$.

Similarly to the case with Berger spheres, it appears as if (H, g_δ) is "converging" to the Riemannian quotient (in this case, the flat plane) as $\delta \to 0$, with some regularity. (Indeed, convergence is occuring in the *pointed Hausdorff topology*, for any basepoint in H. See [8], Chapter 6 for details; we shall not mention this concept again.) However, since $\nabla_Z Z = 0$, the vertical Euclidean line is a geodesic for all g_δ, of infinite length; so that unlike in the example of the Berger spheres, it does not follow that the injectivity radius of (H, g_δ)

tends to zero at all points. Nonetheless, one may take the quotient $\Lambda\backslash H$ of H by the (isometric) left action of the discrete subgroup

$$\Lambda = \begin{bmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{bmatrix}.$$

That is, under the bijection between H and $\mathbb{R}^3$, one makes the identifications

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \sim \begin{bmatrix} x \\ y+1 \\ z \end{bmatrix} \sim \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \sim \begin{bmatrix} x+1 \\ y \\ z+y \end{bmatrix}.$$

Then, the image of $t \mapsto (x, y, t)$, for fixed $0 \leqslant x, y \leqslant 1$, is a geodesic circle of length $O(\delta)$, meaning that the injectivity radius is tending to 0 uniformly at all points of $\Lambda\backslash H$, which therefore collapses with bounded curvature along the fibres of an S^1 bundle over $\mathbb{T}^2$. Because $\mathbb{T}^2$ is flat, it may then be collapsed with bounded curvature to a point, by simply shrinking the metric.

As a final remark, one may cross with S^1_θ to obtain $S^1 \times \Lambda\backslash H$, the *Kodaira-Thurston manifold*. This manifold is of interest in symplectic topology, as a simple example of a symplectic manifold which is not Kähler. One may then consider the metrics $h_\delta = \delta^2 d\theta^2 + g_\delta$, for g_δ as defined above, and see that the Kodaira-Thurston manifold collapses with bounded curvature along the fibres of a $\mathbb{T}^2$-bundle over $\mathbb{T}^2$.

2.4 Gromov-Hausdorff Distance and Convergence

In the two examples of the previous section, we were saying that it *appeared* as though the total space was converging towards the quotient space. It is time to make this precise with a notion of *distance* between Riemannian manifolds.

This notion leads one to consider metric spaces in general. Naïvely, one may wish to define the distance between two metric spaces M, M' as the infimal distance obtainable between them by embedding both within some larger metric space. So first, one desires a notion of distance between two subsets of a fixed metric space M. The *Hausdorff distance* provides this.

Definition 2.3. *For $p \in M$ and $A \subset M$, denote dist$(p, A) = \inf_{q\in A} d(p, q)$. Then, if $A^r := \{p \in M \mid d(p, A) < r\}$, (the r-"thickening" of A) the **Hausdorff distance** between A and B is*

$$d_H(A, B) := \inf\{r < 0 \mid A \subset B^r \text{ and } B \subset A^r\}.$$

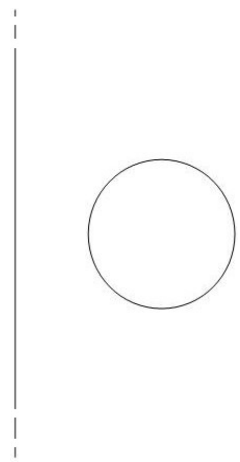
Therefore, $d_H(A, B)$ is the infimal r such that every point of A is within r of B, and every *point of B is within r of A. An equivalent characterisation is*

$$d_H(A,B) = \max(\sup_{p\in A} dist(p,B), \sup_{q\in B} dist(q,A)).$$

Note that $d_H(A,\overline{A}) = 0$. For this reason, one often either restricts to closed subsets or identifies sets which have the same closure.

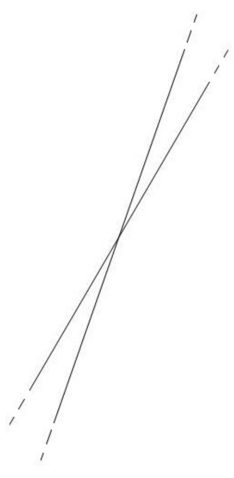
Example 2.4. [Basic Examples of Hausdorff Distance] For concreteness, we set $M = \mathbb{R}^2$ with the Euclidean metric.

- Say A is bounded and B is not. Then, $\sup_{p\in A} \mathrm{dist}(p,B)$ is finite. However, B must contain points arbitrarily far from A, so that $\sup_{p\in B} \mathrm{dist}(p,A) = \infty$. As the maximum of the two, $d(A,B) = \infty$.

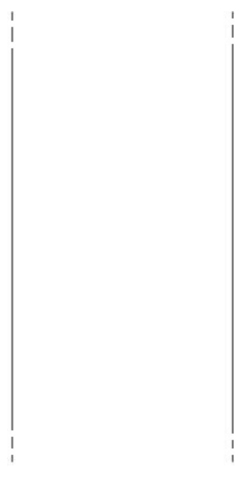


- Take two lines l_1, l_2 in the plane. The shortest distance from $p \in l_1$ to l_2 is given by the line segment pq, for $q \in l_2$, with $pq \perp l_2$. If l_1 and l_2 are parallel, then the length of this line is independent of p, and equals $d_H(l_1,l_2)$. If l_1 and l_2 are not parallel, then as $p \to \infty$, so $\mathrm{dist}(p,l_2) \to \infty$. Hence, there are points on each line arbitrarily far from the other line, so that $d_H(l_1,l_2) = \infty$.

Infinite distance:



Finite distance:



- Take two circles c, C of radii $r \leqslant R$. Let D be the distance between their centres; assume for simplicity $D > r + R$, so that the circles do not intersect, nor does C enclose c. We have

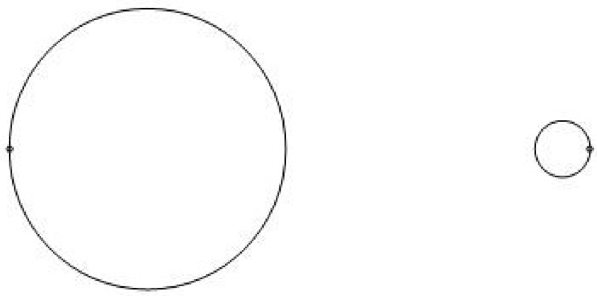
$$\sup_{p \in c} \mathrm{dist}(p, C) = D + r - R,$$

while

$$\sup_{p \in C} \mathrm{dist}(p, c) = D + R - r.$$

These suprema are attained at the points marked in the image below. (Indeed, restricting to closed sets, the suprema in the definition of the Hausdorff distance are always achieved, so that "max" could be written instead.) As the maximum of the two,

$$d_H(c, C) = D + R - r.$$



- The last example applies to closed discs as well as their boundary circles.

In order to demonstrate that the Berger spheres are genuinely converging to S^2, and the squashed Heisenberg groups to the flat plane, one must take the short conceptual step from Hausdorff to **Gromov-Hausdorff distance.** The Gromov-Hausdorff distance between two metric spaces is

$$d_{GH}(M, M') := \inf\{d_H(i(M), i'(M'))\},$$

where the infimum is taken over all pairs of isometric injections $i : M \hookrightarrow X, i' : M' \hookrightarrow X$, as i, i' and X vary.

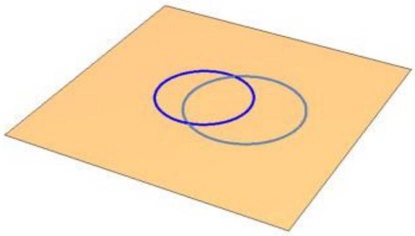
For simplicity, we shall restrict our attention to *compact* spaces for the time being: this is more of a restriction than is strictly necessary, but it covers our two earlier examples and ensures firstly that every distance we are dealing with is finite, and secondly that two non-isometric spaces are at positive Gromov-Hausdorff distance. (See [29], Proposition 3.2.)

Remark 2.5. One can in fact achieve the Gromov-Hausdorff distance between two spaces M, M' by exclusively considering metrics on the disjoint union $M \sqcup M'$, which restrict to the given metrics on M and M' (we shall call such metrics on $M \sqcup M'$ **admissible**). This is by the following procedure: for any embeddings $i : M \hookrightarrow X$, $i' : M' \hookrightarrow X$, set

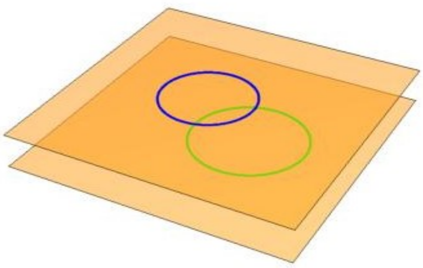
$X_1, X_2 = X$, and for $p \in X$, write p_i when considering p as an element of X_i. Then, define a metric on $X_1 \sqcup X_2$ by

$$d(p_1, q_2) = d(p, q) + \epsilon,$$

for any fixed $\epsilon > 0$. This has the effect of taking two copies of X and separating them by distance ϵ:



$M, M' =$ circles, $X =$ plane.



The circles are separated.

The metric on $X \sqcup X$ restricts to the subspace $i_1(M) \sqcup i_2'(M') \simeq M \sqcup M'$, with

$$d_H(M, M') = \epsilon + d_H(i(M), i'(M')).$$

Letting $\epsilon \to 0$, one sees that the infimal Hausdorff distance between copies of M, M' in an ambient space equals the infimal Hausdorff distance between them as one varies admissable metrics on $M \sqcup M'$.

The most important tool in demonstrating Gromov-Hausdorff convergence of compact spaces (manifolds, for the purposes of this section) is that of *Gromov-Hausdorff approximations:*

Definition 2.6. *An ϵ-**Gromov-Hausdorff approximation** between two metric spaces M and M', also referred to simply as an "ϵ-approximation", is a relation on $M \times M'$ which is onto both factors, and such that $|d_M(p,q) - d_{M'}(p',q')| < \epsilon$ if pRp' and qRq'.*

One does not require that an approximation be a well-defined function. This relaxation makes things easier: for example, for M an ϵ-dense subset of M', one can define pRp' if and only if $d(p,p') < \epsilon$, to obtain a 2ϵ-approximation.

Note that a function $f : M \to M'$ may be considered as a 3ϵ-approximation if $|d(f(p), f(q)) - d(p,q)| < \epsilon$ for all $p, q \in M$, and if $f(M)$ is ϵ-dense in M': one simply sets $pRf(p)$ for $p \in M$, and also $p \sim p' \in M'$ if $d(f(p), p') < \epsilon$.

The reason that Gromov-Hausdorff approximations are useful is the following result:

Lemma 2.7. *The infimal ϵ such that there is an ϵ-approximation between two metric spaces M and M', is proportional to $d_{GH}(M, M')$. Hence, one may demonstrate Gromov-Hausdorff convergence of a sequence of manifolds M_n by exhibiting ϵ_n-approximations between M_n and M_∞, for $\epsilon_n \to 0$.*

2.5 The Berger Spheres and Heisenberg Group Revisited

Now we are ready to view the examples in Sections 2.2 and 2.3 anew. Denote $\tilde{M}$ for S^3 or $\Lambda\backslash H$ (as in Section 2.2), and M for the quotient $S^2(\frac{1}{2})$ or $\mathbb{R}^2/\mathbb{Z}^2$, respectively. For each $\delta > 0$, the map $\pi : (\tilde{M}, g_\delta) \to M$ is a Riemannian submersion, and in fact, as the fibres are shrinking, π provides an $O(\delta)$-approximation. This is by the following argument: for $\tilde{p}, \tilde{q} \in \tilde{M}$ with images $p, q \in M$, the minimising geodesic pq lifts to a horizontal geodesic from $\tilde{p}$ to q', where q' is a point in the same fibre F as $\tilde{q}$. Hence, $d_\delta(\tilde{p}, \tilde{q}) \leqslant d(p,q) + \mathrm{diam}_\delta(F)$, for each δ. However, any path γ from $\tilde{p}$ to $\tilde{q}$ descends to a path from p to q, with length no greater than $l(\gamma)$, since $(\tilde{M}, g_\delta) \to M$ is a Riemannian submersion. Therefore, $d(p,q) \leqslant d_\delta(\tilde{p}, \tilde{q})$. Combining these two inequalities with the fact that $\mathrm{diam}_\delta F \to 0$ as $\delta \to 0$ proves that π is an $O(\delta)$-approximation.

Indeed, in this way one observes Gromov-Hausdorff convergence for any Riemannian submersion of compact spaces: if $\tilde{M}$ is the total space, then $T\tilde{M}$ splits as $P \oplus F$, where F is the fibre direction $\ker \pi_*$ and $P = F^\perp$. Then, one may write the metric as $g = g_F + g_P$, and let $g_\delta = \delta^2 g_F + g_P$. By the same reasoning as above, $(\tilde{M}, g_\delta)$ Gromov-Hausdorff converges to the quotient M.

However, collapse with bounded curvature fails to hold shrinking along the fibres of a general Riemannian submersions, even for quotients by compact groups. Firstly, if fibres are noncompact, there is no reason why the injectivity radius of $(\tilde{M}, g_\delta)$ should approach zero, as shown by the trivial example of $(\mathbb{R}^2, ds_\delta^2 = \delta^2 dx^2 + dy^2) \to (\mathbb{R}_y, ds^2 = dy^2)$, $(x, y) \mapsto y$. Hence, there may not be collapse. Secondly, the O'Neill formula, Theorem 1.11, guarantees that the sectional curvature $K_\delta(\Pi)$ through any *horizontal* plane remains bounded, and converges to $K(\pi_*\Pi)$, the curvature of its image in the quotient, as $\delta \to 0$; however, the sectional curvature of a *vertical* or a *mixed* plane (that is, $\mathrm{Span}(v, w)$ for v vertical and w horizontal) may become unbounded. Therefore, in the case of compact fibres, (M, g_δ) converges to M, but "without regularity". For a trivial example, take $S^3 \circlearrowright S^3$ by left multiplication, where the quotient is a point. Collapsing along the fibres of this action means shrinking S^3 in all directions, and the sectional curvature of g_δ equals $\frac{1}{\delta^2}$, which tends to infinity as $\delta \to 0$. We shall shortly see that the correct way to generalise the examples of the Berger spheres and the Heisenberg group is to find a more general way to collapse along *flat submanifolds*.

3 Collapse with Bounded Curvature: F-Structures and General Theory

In this section, we shall explore in more detail the idea of achieving collapse with bounded curvature by shrinking circles. Say a manifold M^{k+l} admits k commuting circle actions, so that $S^1 \times ... \times S^1 = \mathbb{T}^k \circlearrowright M$, and we know that we may collapse with bounded curvature by shrinking along the orbits of the first factor. The manifold collapses to the quotient $\overline{M} = M/S^1$, and the actions of the other S^1 factors pass to this quotient, meaning one may collapse along the next S^1 orbit in $\overline{M}$ to achieve $\overline{M}/S^1 = M/\mathbb{T}^2$, and so on until $M' = M/\mathbb{T}^k$. It appears that one may as well shrink along the fibres of all of the S^1 actions, that is, along the $\mathbb{T}^k$ action, simultaneously. Let us make this precise.

Lemma 3.1. *Let $\mathbb{T}^k \circlearrowright M$ be an isometric action on a Riemannian manifold (M, g), with all orbits of dimension k. Write $TM = T \oplus P$, for the orbital and perpendicular distributions, and write the metric on M as $g = g_T + g_P$. Setting*

$$g_\delta = \delta^2 g_T + g_P,$$

then as $\delta \to 0$, (M, g_δ) collapses with curvature uniformly bounded on each compact subset. Furthermore, the family (M, g_δ) has $M/\mathbb{T}^k$ as its Gromov-Hausdorff limit.

Proof. We exhibit the metric g_δ in coordinates that make it clear that sectional curvature remains bounded. Fix a point and a local submanifold $S^l \ni p$ transversal to the orbits, where $\dim M = n = k + l$. By Frobenius's theorem, T is integrable, with maximal leaves equal to orbits of $\mathbb{T}^k$. (P need not be integrable. This means that S cannot necessarily be chosen with $TS = P$, and so S may fail to be perpendicular to orbits at nearby points.) Use coordinates $y_1, ..., y_l$ for S, and Euclidean coordinates $x_1, ..., x_k$ for $\mathbb{T}^k$. Then, for a nearby point q to p, one may express q as $t.p'$, for $t = (x_1, ..., x_k) \in \mathbb{T}^k$ and $p' = (y_1, ..., y_l) \in S$; then, assign coordinates $(x_1, ..., x_k, y_1, ..., y_l)$ to q. It is worth mentioning that since $\mathbb{T}^k$

acts isometrically, the components of g_P at (x, y) are independent of x, so that M is locally expressed as a warped product of S and $\mathbb{T}^k$. That is, the metric is "constant" on each orbit, meaning that the orbits are *flat* compact submanifolds.

As $\delta \to 0$, so $\left\|\frac{\partial}{\partial x_i}\right\|_\delta \to 0$. Therefore, change to adapted coordinates $u_i = u_i^\delta = \delta x_i$, so that

$$\left\|\frac{\partial}{\partial u_i}\right\|_\delta = \left\|\frac{1}{\delta}\frac{\partial}{\partial x_i}\right\|_\delta$$

is a number depending only on y, not on δ and u_i. Writing

$$\frac{\partial}{\partial y_i} = X_i + V_i \in T \oplus P,$$

the metric g_δ is expressed in the coordinates u, y as

$$\begin{bmatrix} \left(\langle \partial_{u_i}, \partial_{u_j}\rangle_\delta\right)_{i,j} & \left(\langle \partial_{u_i}, \partial_{y_j}\rangle_\delta\right)_{i,j} \\ \left(\langle \partial_{y_i}, \partial_{u_j}\rangle_\delta\right)_{i,j} & \left(\langle \partial_{y_i}, \partial_{y_j}\rangle_\delta\right)_{i,j} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\langle \partial_{x_i}, \partial_{x_j}\rangle_1\right)_{i,j} & \delta\left(\langle \partial_{x_i}, X_j\rangle_1\right)_{i,j} \\ \delta\left(\langle X_i, \partial_{x_j}\rangle_1\right)_{i,j} & \delta^2\left(\langle X_i, X_j\rangle_1\right)_{i,j} + \left(\langle V_i, V_j\rangle_1\right)_{i,j} \end{bmatrix}$$

$$=: \begin{bmatrix} A(y) & \delta B(y) \\ \delta B(y)^T & \delta^2 C(y) + D(y) \end{bmatrix},$$

where A, B, C, D depend on $y \in N$ alone, as mentioned above, because the $\mathbb{T}^k$ action is isometric. As $\delta \to 0$, this matrix converges to

$$g_0 := \begin{bmatrix} A(y) & 0 \\ 0 & D(y) \end{bmatrix}.$$

If the transversal submanifold S_y is defined for $y \in (-r, r)^l \subset \mathbb{R}^l$, then each g_δ is defined for the coordinates $(u, y) \in (-\pi, \pi)^k \times (-r, r)^l$. However, since the components of g_δ in u, y are independent of u, in fact they define a metric on $\mathbb{R}^k_u \times (-r, r)^l_y$, simply by allowing u_i to take all real values. Therefore, the g_δ may be viewed as metrics on $\mathbb{R}^k \times (-r, r)^l$, which converge at each point to the limit metric g_0 as $\delta \to 0$. Since sectional curvature varies continuously with the metric, and since the portion of (M, g_δ) with $|x| < \pi, |y| < r$ is isometric to the part of $\mathbb{R}^k_u \times (-r, r)^l_y$ with $|u| < \delta\pi$, the sectional curvature of $(-\pi, \pi)^k_x \times (-r, r)^l$, i.e. that of $\mathbb{T}^k \times S$, is seen to be bounded as $\delta \to 0$. Hence, the sectional curvature of M is bounded as $\delta \to 0$, and uniformly on compact subsets.
Finally, the collapse induces Gromov-Hausdorff convergence to the quotient $M/\mathbb{T}^k$ simply by taking as Gromov-Hausdorff approximations the relation $p \sim [p]$, for each $p \in M$ and $[p]$ in the orbit space $M/\mathbb{T}^k$. □

A few observations concerning the above proof:

- The fact that the action is isometric, with the orbits flat manifolds, was instrumental in demonstrating bounded curvature.

- It is not necessary that the action be globally defined for this argument to work. Indeed, Cheeger and Gromov ([4], Theorem 2.1) use precisely the same argument for collapsing along the fibres of a *pure, polarised F-structure*: see below for the relevant definitions.

- Furthermore, as pointed out by Cheeger and Gromov, the torus action could be replaced by that of a *nilpotent group* N. This is because (compact quotients of) nilpotent groups, like circles, may be collapsed with bounded curvature to a point, by the metrics g_q as defined in [29], Section 4.1. (It is a fact that every nilpotent group embeds as a group of upper-triangular matrices with diagonal entries equal to 1.) The δ-scaling in the above proof is then replaced by the sequence of metrics $h_\delta = g_\delta + h_P$.

Example 3.2. A curved torus M embedded in $\mathbb{R}^3$ with radii $r < R$, may be endowed with angular coordinates θ (in the (x, y)-plane) and ϕ (cross-sectional), so that the metric induced by the embedding is

$$ds^2 = r^2 d\phi^2 + (R + r\cos\phi)^2 d\theta^2.$$

Abstractly, M is simply $S^1_\theta \times S^1_\phi$, and as such there are many circle actions on M. However, it may be calculated that any action other than rotation in the ϕ direction is not isometric, and that collapsing along the orbits blows up the curvature. But if one collapses along the fibres of ϕ-rotation, then one has

$$ds^2_\delta = r^2 d\phi^2 + \delta^2 (R + r\cos\phi)^2 d\theta^2,$$

and one calculates that

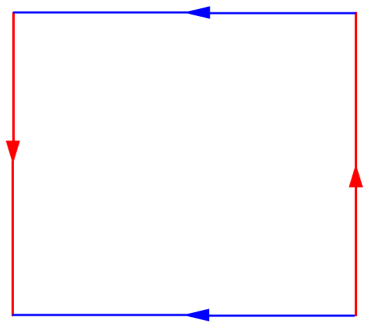
$$K_\delta(\theta, \phi) = \frac{\cos\phi}{r(R + r\cos\phi)},$$

unchanging with δ. This is consistent with the fact that in this case (using the notation of Lemma 3.1), the perpendicular distribution P *is* integrable, so that $B(\theta), C(\theta) = 0$ and $g_\delta = g_0$ on $\mathbb{R}_\theta \times (-\pi, \pi)_\phi$.

3.1 F-Structures: Purity and Polarisation

In this subsection, we introduce F-structures.

Example 3.3. [A Local Circle Action] Consider the Klein bottle $\mathbb{K}$ as a circle bundle over the circle S^1_ϕ. Here the base circle is shown horizontal:



If a "strip" is defined as the union of fibres over a proper open sub-interval, then on each strip, there is an S^1_θ action, given merely by rotation of the fibres. However, if this action is continuously carried around the base circle, one observes holonomy given by $\theta \mapsto -\theta$. Visibly, this is not a global circle action. However, if $\mathbb{K}$ is given a metric which makes the fibre rotations into Killing fields, one may still scale along the fibres of the circle "action", to obtain collapse with bounded curvature and Gromov-Hausdorff limit the base circle.

The essence of this example was a bundle over S^1 with circle actions on the fibres, but holonomy around the base precluding a global action. One may manufacture many similar examples by taking the mapping torus of any group automorphism $f : \mathbb{T}^k \to \mathbb{T}^k$; that is, the space

$$\mathbb{T}^k \times \mathbb{R}/(t, x) \sim (f(t), x + 1).$$

In some cases, collapse with bounded curvature may still be performed.

The appropriate language for dealing with sort of phenomenon is that of sheaves of groups and partial actions. We introduce the terminology:

Definition 3.4. *A* ***sheaf action*** $\mathscr{F} \circlearrowright M$ *on a manifold* M*, consists of the following data:*

- *A sheaf of Lie groups* $\mathscr{F}$ *on* M*.*
- *For each open* $U \subset M$*, a partial action of* $\mathscr{F}(U)$ *on* U*: that is, a smooth map defined on an subset* $\mathscr{F}(U) \times U \supset D \to U$*, denoted* $(g, p) \mapsto gp$*, such that* $id.p = p$ *is defined for all* $p \in U$*, and* $g(g'p)$ *is defined whenever* $(gg')p$ *is, and the two are equal.*

Definition 3.5. *An* ***F-structure*** *on a space* M *(a manifold, for our purposes) is a sheaf action* $\mathscr{F} \circlearrowright M$*, such that all stalks are isomorphic to tori, and for each* p*, there is a neighbourhood* $U = U_p \ni p$ *such that*

- $\mathscr{F}(U') \to \mathscr{F}_q$ *is an isomorphism for any connected* $q \in U' \subset U$*, and any* q *in the closure of the orbit of* p*.*
- *The action of* $\mathscr{F}(U)$ *on* U *lifts to a global action on a finite, normal cover* $\tilde{U}$*.*

- *U is a union of $\mathscr{F}$-orbits.*

In other words, $\mathscr{F}$ is a sheaf which locally lifts to complete actions on covers of saturated neighbourhoods, and which is constant when restricted to the closure of an orbit. We have seen these features in Example 3.3.

Remark 3.6. Any global action $\mathbb{T}^k \circlearrowright M$ becomes an F-structure by setting $\mathscr{F}$ to be the constant sheaf with stalks equal to $\mathbb{T}^k$.

The above is a rather involved definition, but the right one: for example, being able to lift to a complete action on a finite cover implies that by averaging inner products on tangent spaces over the orbit of a point in the cover, one may manufacture an invariant metric for any F-structure. Note that the first bullet point does *not* imply that $\mathscr{F}$ is locally constant: see Example 3.8. If one strengthens the second bullet point to demand that $\tilde{U} = U$, then one has what is called a **T-structure.** Example 3.3 is of course a T-structure, choosing the U to be strips.

Remark 3.7. The T-structure on $\mathbb{K}$ in Example 3.3 lifts to a global action on the double cover $\mathbb{T}^2$, given by rotation around the cross-sections. Such an F-structure, where one may take $U_p = M$, is called **elementary.** Therefore, Example 3.3 is an elementary F-structure, and a T-structure, but not an elementary T-structure, since such a thing would be a global torus action.

Example 3.8. Consider the standard rotation action $S^1 \circlearrowright S^2$, with the poles fixed. Take polar coordinates θ, ϕ, so that ∂_ϕ is the infinitesimal action of S^1. Then, the round metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ may be scaled along the S^1 action for $ds_\delta^2 = d\theta^2 + \delta^2 \sin^2\theta d\phi^2$. Away from the poles, the conditions of Lemma 3.1 are satisfied; and the orthogonal distribution $\mathbb{R}\partial_\phi$ is integrable, so that the Gaussian curvature remains constant ($K_\delta = 1$), similarly to Example 3.2. Then, the sphere collapses along the fibres with Gromov-Hausdorff limit the closed interval $[0, \pi]$. However, for $\delta \neq 1$, conical singularities of g_δ immediately form at the poles. A similar phenomenon will occur whenever one attempts to shrink along a circle action which has fixed points: conical singularities form at the fixed points. In this case, the process does not deserve to be called “collapse with bounded curvature”: there is no sensible notion at all of curvature at the conical points. Rather than probe the nature of these singularities, in this section we shall henceforth restrict to the case of actions with *no fixed points.*

We introduce a few more relevant definitions.

Definition 3.9. *An F-structure is* ***pure*** *if the sheaf $\mathscr{F}$ is locally constant. It is* ***polarised*** *if the dimension of each stalk $\mathscr{F}_p$ equals the dimension of the orbit $\mathscr{F}p$. The* ***rank*** *of $\mathscr{F}$ at p is the dimension of the orbit $\mathscr{F}p$. Therefore, if $\mathscr{F}$ is both pure and polarised, it has constant rank. In this case, the proof of Lemma 3.1 generalises immediately, with $\mathbb{T}^k$ orbits* *replaced by $\mathscr{F}$ orbits. (The statement, contained in Lemma 3.1, that orbits are flat, holds for a general F-structure; indeed, "F" stands for "flat".) A polarised substructure of $\mathscr{F}$ is called a* ***polarisation*** *of $\mathscr{F}$. If $\mathscr{F}$ is either* ***mixed*** *(that is, not pure) or non-polarised, the theory of bounded curvature collapse is more complicated.*

Example 3.10. Consider the T-structure associated to the action

$$\mathbb{T}^n \circlearrowright S^{2n-1} \subset \mathbb{C}^n, \quad (\theta_1, ..., \theta_n) : (z_1, ..., z_n) \mapsto (e^{i\theta_1} z_1, ..., e^{i\theta_n} z_n).$$

This is not polarised: indeed, $\dim(\mathrm{orb}(z)) = n - \#(z_k = 0)$, while all stalks equal $\mathbb{T}^n$. However, one may manufacture a polarised substructure: for a connected open set $U \subset S^{2n-1}$, set

$$\mathscr{F}'(U) = G_1 \times ... \times G_n,$$

where $G_k = 1$ if U intersects the sub-sphere $S_k : z_k = 0$, and S^1 if it does not. As such, $\mathscr{F}'$ is not pure, as the dimension of $\mathscr{F}'_p$ changes from point to point.

Therefore, $\mathscr{F}$ is non-polarised but pure (by Remark 3.6), while $\mathscr{F}' \subset \mathscr{F}$ is polarised but mixed. As shown in [4], Theorem 3.1, mixed collapse with bounded curvature blows up the diameter of a manifold, so that while S^{2n-1} may be collapsed along $\mathscr{F}'$ with bounded curvature, one does not see the quotient (a closed interval) emerging as a Gromov-Hausdorff limit.

Example 3.11. [Polarisations of the Previous Example] Set $n = 2$ in the previous example. Then, a *pure* polarisation of $\mathscr{F}$ is given by any one-parameter subgroup $t \mapsto (e^{ipt}, e^{iqt})$ acting coordinate-wise on $S^3 \subset \mathbb{C}^2$, provided that $p, q \neq 0$. These polarisations generalise Section 2.2. They do not recover the stalks of $\mathscr{F}$, and so they will yield different quotient spaces. If $\frac{p}{q}$ is not rational, then the polarisation is not an $\mathscr{F}$-structure: it is an $\mathbb{R}$-action, whose orbits are not closed, and Lemma 3.1 does not apply. (To see how incompatible non-closed structures can be with taking quotients, consider an irrational subgroup of $\mathbb{T}^2$ acting canonically on $\mathbb{T}^2$. Every orbit is dense, and so the quotient has the indiscrete topology.) In this case, one must consider the *closure* of the one-parameter group, which is the whole torus $S^1 \times S^1$, and as mentioned above, the quotient is a closed interval.

We examine the case where $0, 1 \neq \frac{p}{q}$ is rational. Pure polarised collapse occurs as in Lemma 3.1, and the Gromov-Hausdorff limit is the quotient space $M = S^3/S^1_t$. We shall describe M. First, choose appropriate coordinates on S^3: take

$$(\theta, \phi_1, \phi_2) \mapsto (\cos\theta e^{i\phi_1}, \sin\theta e^{i\phi_2}),$$

so that

$$ds^2 = |dz_1|^2 + |dz_2|^2 = d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2.$$

The coordinates (θ, ϕ_1, ϕ_2) may be forced to lie in the box $[0, \frac{\pi}{2}] \times [0, 2\pi] \times [0, 2\pi]$. Upon quotienting by the S^1_t action, the sets $\theta = 0$ and $\theta = \pi$ become two single points. Away from these, a point of S^3 may be viewed as lying in

$$\left(0, \frac{\pi}{2}\right)_\theta \times \frac{[0, 2\pi]_{\phi_1}}{0 \sim 2\pi} \times \frac{[0, 2\pi]_{\phi_2}}{0 \sim 2\pi}.$$

Then an orbit of the S^1 polarisation comprises the points $(\theta, \phi, \frac{q}{p}\phi)$, for a fixed θ, as ϕ varies. Each orbit intersects $\phi_2 = 0$, and the orbit containing $(\theta, \phi, 0)$ also intersects $\phi_2 = 0$ at the points $(\theta, \phi + \frac{2kp}{q}\pi, 0)$, for $k \in \mathbb{Z}$. Here use that $\frac{p}{q}$ is rational, so that the quotient M is given by coordinates $\theta \in [0, \pi]$ and $\phi \in [0, \frac{2p}{q}\pi]$, with $\theta = 0$ and $\theta = \pi$ collapsed to two points, and (θ, ϕ) identified with $(\theta, \phi + \frac{2q}{p}\pi)$.

Now, we identify the submersion metric on M by pulling back along the local section $(\theta, \phi) \mapsto (\theta, \phi, 0)$: the metric on M is given by the inner products between the *horizontal lifts* of ∂_θ and ∂_ϕ (see Section 1.3 for details). To find these horizontal lifts, one must remove the orbital components: since $\partial_\theta \perp \partial_t = p\partial_{\phi_1} + q\partial_{\phi_2}$, it is the case that

$$\partial_\theta^{\text{horiz}} = \partial_\theta.$$

Then,

$$\begin{aligned}(\partial_\phi)^{\text{horiz}} &= \partial_{\phi_1} - \frac{\langle \partial_{\phi_1}, p\partial_{\phi_1} + q\partial_{\phi_2}\rangle}{\langle p\partial_{\phi_1} + q\partial_{\phi_2}, p\partial_{\phi_1} + q\partial_{\phi_2}\rangle}(p\partial_{\phi_1} + q\partial_{\phi_2}) \\ &= \frac{q^2 \sin^2\theta \partial_{\phi_1} - pq\cos^2\theta \partial_{\phi_2}}{p^2\cos^2\theta + q^2\sin^2\theta},\end{aligned}$$

so that

$$\langle \partial_\theta, \partial_\theta\rangle_M = \langle \partial_\theta^{\text{horiz}}, \partial_\theta^{\text{horiz}}\rangle_{S^3} = 1,$$

$$\langle \partial_\theta, \partial_\phi\rangle_M = \langle \partial_\theta^{\text{horiz}}, \partial_\phi^{\text{horiz}}\rangle_{S^3} = 0,$$

and

$$\langle \partial_\phi, \partial_\phi\rangle_M = \langle \partial_\phi^{\text{horiz}}, \partial_\phi^{\text{horiz}}\rangle_{S^3} = \frac{q^2\cos^2\theta\sin^2\theta}{p^2\cos^2\theta + q^2\sin^2\theta} =: y(\theta)^2.$$

It may be calculated (by software) that $|y'(\theta)| < 1$ for all values $0 < \theta < \pi$, so that by [25], pages 10-11, M may be realised as the revolution of a curve $(x(\theta), y(\theta))$ about the x-axis, for $x'(\theta)^2 + y'(\theta)^2 = 1$. Thus, M is a pear-shaped S^2 which is singular at the two intersections with the x-axis, which correspond to the two orbits $z_k = 0$ in S^3.

As we shall now see, it is theoretically preferable from the point of view of collapse with bounded curvature to be polarised but not pure, than pure but not polarised.

3.2 Further Results: Polarised and Unpolarised Collapse. The Existence of an F-Structure on Sufficiently Collapsed Manifolds.

Given a polarised F-structure, there is a standard procedure for covering M by saturated sets U_a, with substructures $\mathscr{F}_a \subset \mathscr{F}|_{U_a}$ such that each $\mathscr{F}_a$ is *pure* on U_a, and for all p, there is some a with $p \in U_a$ and $\mathscr{F}_{a,p} = \mathscr{F}_p$. We shall call this taking a **good atlas**. The $\mathscr{F}_a$ may have different ranks (indeed, if they do not, then $\mathscr{F}$ is pure and one proceeds as in Lemma 3.1).

For example, we describe a good atlas for $\mathscr{F}'$ as defined in Example 3.10. A multi-index

$$a = \{a_1, ..., a_l\} \subset \{1, ..., n\}$$

corresponds to the sub-sphere S_a where S^{2n-1} intersects the axes $z_{a_1} = 0, ..., z_{a_l} = 0$. Take U_a to be a saturated neighbourhood of this sub-sphere, say where each $|z_{a_k}| < \epsilon$ for a small ϵ. Then, take $\mathscr{F}_a$ to be $G_1 \times ... \times G_n$, where $G_k = 1$ if $k \in a$ and $G_k = S^1$ otherwise. $\mathscr{F}_a$ acts by rotating the complex coordinates that do not vanish on a; that is, the coordinates of S_a. Finally, set $U = \{z : z_k \neq 0 \; \forall \; k\}$ (the generic points that lie on no complex axis), and $\mathscr{F}_U = \mathbb{T}^k$.

Theorem 3.1 in [4] states that volume and injectivity radius collapse along a polarised F-structure may be performed with bounded curvature, with the sole caveat that the diameter of the manifold blows up. For completeness, we sketch the argument. One seeks metrics g_δ for $\delta \to 0$ which are collapsing along the orbits of $\mathscr{F}$. If one simply defines g_δ to scale by δ in orbital directions, then similarly to Example 3.11, singularities form along the orbits of smaller dimension, and this may not be called "collapse with bounded curvature". Instead, the procedure in [4], Theorem 3.1, is to take a good atlas $U_1, U_2, ...$, which is ordered in such a way that the rank of $\mathscr{F}_i$ is no greater than the rank of $\mathscr{F}_{i+1}$, and then for a fixed δ, to shrink the metric along the orbits of the $\mathscr{F}_i$ one at a time. The argument of [4] relies on the U_i being locally finite and precompact (hence, it only applies for polarised F-structures on *compact* manifolds), so that the metric at any given point is only changed finitely many times as one proceeds through $U_1, U_2, ...$ If one changed the metric infintely many times at a point, then there would not necessarily be any control on the sectional curvature.

More generally, there exist manifolds that carry F-structures which admit no polarisations. Example 1.7 in [4] is one such. However, collapse may still be performed, and the theory of non-polarised collapse is developed by Cheeger and Gromov in the latter half of [4]. This means that collapse with bounded curvature can be achieved along *any F-structure of positive rank.*

Then, in the second part [5], it is shown that in each dimension n there is a "critical value" $c = c(n)$ such that if M^n is more than c-collapsed, in the sense that for all $p \in M$,

$$\sqrt{|K(\Pi)|} < \frac{\mathrm{inj}(p)}{c}$$

whenever $\Pi^2 \leqslant T_qM$ and $d(p,q) < \frac{c}{\text{inj}(p)}$, then M must admit an F-structure of positive rank. Hence, if M collapses, there is an F-structure. In particular, by Proposition 1.5 of [4], if M collapses with bounded curvature at all points, then the Euler characteristic $\chi(M)$ equals zero. Therefore, if $\chi(M) \neq 0$, one may only collapse *parts* of M, or else allow the curvature to blow up at some points.

However, the theory of F-structures is not all there is to collapse with bounded curvature. A sufficiently-collapsed manifold must *admit* an F-structure, but this does not imply that this F-structure is the only way it may collapse, as we shall now discuss. (For a start, as for $M = S^2$, a manifold may admit several different non-commuting circle-actions, so it is not the case that there is “one maximal” F-structure.)

The first study of collapse was carried out by Gromov [15] and Ruh [27], who were concerned with **almost flat** manifolds: those admitting complete metrics such that

$$\text{diam}(M)^2 \max_{\Pi \in Gr(2,TM)} |K(\Pi)|$$

becomes arbitrarily small. (In particular, they must be compact.) Hence, their interest was in *diameter* rather than injectivity radius. It was discovered that an almost flat manifold M must have a finite cover diffeomorphic to $\Lambda\backslash N$, for N a nilpotent group and Λ a lattice (such a manifold is called **infranil**: for example, M may be a quotient by the *left* action of a particular lattice in N, and the *right* action of a finite subgroup). As such, since any nilpotent group embeds in the group of upper-triangular unipotent matrices, an almost flat manifold collapses with bounded curvature to a point, by the metrics given in [29], Section 4.1. If N is the group of *all* upper-triangular unipotent matrices, and Λ the integral elements of N, then the quotient $M = \Lambda\backslash N$ does admit an F-structure, in accordance with [5]; however, this is merely S^1 acting on the upper-right-most matrix entry, and the quotient is $\left(\frac{n(n-1)}{2} - 1\right)$-dimensional: F-structures do not allow M to collapse to a point.

Then, in [13] and [14], Fukaya proved that if a sequence M_k of compact Riemannian manifolds Gromov-Hausdorff converges to a manifold L of lower dimension, while the sectional curvatures remain bounded (which is impossible, for example, in bounded curvature collapse by a mixed F-structure, since the diameter blows up in that case), then for large enough k, M_k is realised as a fibre bundle over L with infranil fibres. Therefore, convergence from M_k to L may also be realised by collapse along these infranil fibres: the dimension of the fibres cannot vary from point to point as in the theory of F-structures.

Finally, Cheeger, Fukaya and Gromov [18] combined the theory of F-structures and infranil fibrations to prove the existence of *nilpotent Killing structures* - sheaves of actions by nilpotent Lie algebras, with extra regularity - in great generality on sufficiently collapsed manifolds, and without compactness assumptions.

3.3 Volume Comparison, Precompactness and Diffeofiniteness

As a coda to this section, we include statements of the following theorems, which are clearly of interest in the theory of collapse with bounded curvature.

Theorem 3.12 (Bishop-Gromov Volume Comparison). *Let M^{n+1} be a complete manifold with $Ric_M \geqslant nK$. Let $\tilde{M}$ equal the simply-connected manifold of constant curvature K and dimension $n+1$ (that is, a suitably scaled sphere, Euclidean or hyperbolic space, according to the sign of K). Denote $v(r)$ for the volume of a ball of radius r in $\tilde{M}$. Then, for any $p \in M$ and $r > 0$, we have*

$$vol(B_M(p,r)) \leqslant v(r),$$

and if there is equality for any $r > 0$, then M is locally isometric to $\tilde{M}$ (and therefore covered by it). Furthermore, the value

$$\frac{vol(B_M(p,r))}{v(r)}$$

is non-increasing as $r \uparrow$.

Example 3.13. In Section 1.3.3, the Ricci curvature of $\mathbb{CP}^n$ was calculated to be Ric $=(2n+2)g = \frac{2n+2}{2n-1}(2n-1)g$. Therefore, set $K = \frac{2n+2}{2n-1}$. As shown in [28], pages 167-168, the volume of a ball of radius $r \leqslant \frac{\pi}{\sqrt{K}}$ in $S^{2n}\left(\frac{1}{\sqrt{K}}\right)$ is

$$\frac{\mathrm{vol}(S^{2n-1})}{K^n}\int_0^r \sin^{2n-1}(t)dt$$

It is also shown that a ball of radius $r \leqslant \frac{\pi}{2}$ in $\mathbb{CP}^n$ has volume equal to

$$\mathrm{vol}(S^{2n-1})\int_0^r \sin^{2n-1}(t)\cos(t)dt.$$

So in this case, the volume comparison theorem says that

$$\int_0^r \sin^{2n-1} dt \geqslant K^n \int_0^r \sin^{2n-1}(t)\cos(t)dt,$$

that is

$$(2n-1)^n \int_0^r \sin^{2n-1} dt \geqslant (2n+2)^n \int_0^r \sin^{2n-1}(t)\cos(t)dt.$$

This inequality could also be worked out by hand.

Theorem 3.14 (Gromov Precompactness). *Let M_k be a sequence of compact Riemannian manifolds of dimension $n+1$ such that*

$$Ric \geqslant nK, \quad diam(M_k) \leqslant D$$

holds for all k, for some fixed $K \in \mathbb{R}, D > 0$. Then, there is a subsequence that Gromov-Hausdorff converges to a (possibly singular) limit space.

Theorem 3.15 (Cheeger Diffeofiniteness). *Given a family $\mathscr{E}$ of Riemannian manifolds satisfying a uniform lower bound on volume, upper bound on diameter, and both bounds on sectional curvature, there is a finite collection of manifolds M_k such that every member of $\mathscr{E}$ is diffeomorphic to some M_k.*

Cheeger diffeofiniteness tells one that given any sequence of Riemannian manifolds with these bounds, after passing to a subsequence, one is simply considering different metrics on the same manifold. Then, Gromov precompactness tells one that after passing to a further subsequence, there is a Gromov-Hausdorff limit. Finally, volume comparison is instrumental in proving Gromov precompactness, as one may read in [17]. A proof of the volume comparison result, minus the rigidity (that equality implies isometry), may be found in [25].

4 Solving Elliptic PDEs on a Compact Riemannian Manifold

In this chapter, we take a break from pure geometry to develop some theory for finding solutions to elliptic partial differential equations, in the Riemannian manifold setting. In section 4.1, we introduce the setting in which we will be looking for solutions - a larger class of functions than C^∞ - which is where we'd like to eventually find them. In sections 4.2 and 4.3, we study existence of 'weak' solutions to PDEs - first in the specific case of Poisson's equation and then for the more general second order elliptic case, and we also tackle the problem of demonstrating that the solutions we found are in fact smooth. Throughout, we work on an n-dimensional Riemannian manifold, M.

4.1 Hölder Spaces and Sobolev Spaces

A common strategy for solving PDEs of any type is to split the problem into two parts - *existence* and *regularity*. The idea is that often, looking for a solution in the function space we would like to find it in is hard. For example, we'd like to find a smooth solution to Poisson's equation (6), but the space of smooth functions is not nice analytically - it isn't complete, for example, so isn't a Banach space. We therefore conduct our search in a larger space, with nicer properties. Once we've found our "weak" solutions, we may find that they must be smooth after all! Locating a solution is then the *existence* part, and checking smoothness is the *regularity* part.

We tackle existence first, and so begin by introducing larger spaces in which to look for 'weak' solutions.

4.1.1 Hölder Spaces

Before talking about Sobolev spaces, which will be the setting for our search for a solution, we quickly introduce the simpler *Hölder spaces*, which we will also need. Both of these spaces take up the challenge of finding a concept of differentiability that is weaker than the usual one, allowing in more functions and making a larger space.

The first idea comes from the concept of Lipschitz continuity; a function $u : M \to \mathbb{R}$ is *Lipschitz continuous* if

$$\sup_{x,y\in M, x\neq y} \frac{|u(x) - u(y)|}{dist(x, y)} = C(u) < \infty,$$

where $\mathrm{dist}(x, y)$ is the geodesic distance on M. This implies that u is continuous, but is stronger since it provides a constant quantifying the 'strength' of the continuity. It is however weaker than differentiability, for example the real function $f(x) = |x|$ is not differentiable at the origin, despite being Lipschitz continuous. We can weaken the concept of differentiability even further, by demanding that, for given $\gamma \in (0,1]$,

$$\sup_{x,y \in M, x \neq y} \frac{|u(x) - u(y)|}{dist(x,y)^\gamma} = C(u) < \infty.$$

If a function u satisfies this condition, we say it is *γ-Hölder continuous.* The constant $C(u)$ is the *γ-Hölder seminorm* of the function u, and denoted $[u]_{C^{0,\gamma}}(M)$.

We want our function space, the γ-Hölder space $C^{k,\gamma}$, to contain k times differentiable functions, with those derivatives γ-Hölder continuous. However, the derivatives of the function u are tensors, and imitating the above definition with a general tensor T runs into a problem - $T(x)$ and $T(y)$ belong to different spaces, so cannot be subtracted! To solve this problem we use the Levi-Civita connection. The connection provides us with an isomorphism

$$\tau_{x,y} : T_x M \to T_y M$$

given by parallel transport along a unique geodesic between x and y. $\tau_{x,y}$ induces a map on the dual spaces

$$\tau^*_{x,y} : T^*_y M \to T^*_x M$$

and any tensor powers of these, and then the γ-Hölder seminorm of a tensor $T \in \Gamma(T^{0,k}(M))$ can be defined as:

$$[T]_{C^{0,\gamma}(M)} := \sup_{0 < dist(x,y) < inj(M)} \frac{\|T(x) - \tau^*_{x,y} T(y)\|_g}{\text{dist}(x,y)^\gamma}.$$

This covers the Hölder continuity. To also ensure k-times continuous differentiability, we also include as part of the definition the usual norm on the space $C^k(M)$:

$$\|u\|_{C^k(M)} := \sum_{i=0}^{k} \sup_{x \in M} \|\nabla^i u(x)\|_g.$$

Definition 4.1. *The* γ-Hölder norm *is defined on functions $u : M \to \mathbb{R}$ as:*

$$\|u\|_{C^{k,\gamma}(M)} := \|u\|_{C^k(M)} + [\nabla^k u]_{C^{0,\gamma}(M)},$$

and the γ-Hölder Space *is defined to be the space of all functions for which this norm is finite.*

4.1.2 Sobolev Spaces

The other important generalised function space is the *Sobolev space* $W^{k,p}(M) \subset L^p(M)$, which is the space of functions with k "weak" derivatives. To best introduce this concept, we look first at the Euclidean case.

As an example, consider the function $f : (0, 2) \to \mathbb{R}$ defined by:

$$f(x) = \begin{Bmatrix} x & \text{for } x \in (0,1) \\ 2 - x & \text{for } x \in [1,2) \end{Bmatrix}.$$

Though the derivative does not exist, the function

$$g(x) = \begin{Bmatrix} 1 & \text{for } x \in (0,1) \\ -1 & \text{for } x \in [1,2) \end{Bmatrix}$$

acts like one, in the sense that for all $\phi \in C_c^\infty([0,2])$,

$$\int_{[0,2]} f(x)\partial_x \phi dx = -\int_{[0,2]} g(x)\phi dx,$$

one can see this by splitting the integral into the regions $[0,1]$ and $[1,2]$ and integrating by parts. We can use this 'integration by parts' technique to define a *weak* αth derivative by demanding that an analogous identity holds:

Definition 4.2. *Suppose $u, v \in L^1(U)$ for a subset $U \subset \mathbb{R}^n$, and let α be a multi-index. Then v is the* weak αth partial derivative of u *so long as the following identity holds for all test functions $\phi \in C_c^\infty(U)$:*

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v\phi dx.$$

If the above holds, we write

$$D^\alpha(u) := v.$$

It is the case that weak derivatives are unique (see for example [11], Chapter 5). To extend the above definition to a function u on a compact manifold, one can simply demand that, given any finite open cover $\{U_i\}$, if $\{\phi_i\}$ is a partition of unity subordinate to that cover, then the local representations of the functions $\{\phi_i u\}$ have a weak derivative. We can then define

$$D^\alpha u := \sum_i D^\alpha(\phi_i u).$$

It is the case that these spaces have a norm, and with respect to these norms they are complete (Banach) spaces. In fact they are (and therefore can be defined as) the completion of the smooth functions on M, with respect to these norms:

Definition 4.3. *The* Sobolev space $W^{k,p}(M)$ *on a compact manifold M is defined to be the completion of the space of smooth functions on M, $C^\infty(M)$, with respect to the norm:*

$$||u||^p_{W^{k,p}(M)} := \sum_{j=0}^k \int_M \|\nabla^j u\|_g^p \text{dvol}.$$

In particular, the space $W^{k,2}(M)$ is a Hilbert space, with associated inner product:

$$\langle u, v\rangle_{W^{k,2}(M)} := \sum_{m=0}^{k} \int_M \langle \nabla^m u, \nabla^m v\rangle_g \mathrm{dvol}.$$

Note that in these spaces, the jth covariant derivative may not exist, but we can define the norm and inner product nevertheless by taking the limit of Cauchy sequences in $C^\infty(M)$. As noted, $W^{k,2}(M)$ is a Hilbert space - we will denote it $H^k(M)$ from now on.

It is worth recording a few important theorems about Sobolev spaces here, for later use. The first one bounds the size of integral zero functions in terms of the derivative:

Theorem 4.4 (The Poincaré Inequality for Manifolds)**.** *Let $u \in W^{1,p}_{int0}(M)$ be an integral zero Sobolev function, where $1 \leqslant p < n$. Then:*

$$\|u\|_{L^q(M)} \leqslant C\|Du\|_{L^p(M)},$$

where $1 \leqslant q \leqslant \frac{np}{n-p}$.

Finally, these two theorems are examples of *Sobolev inequalities*; a good reference for proofs (in the Euclidean setting) is [11]. Intuitively, they express the fact that existence of sufficiently many weak derivatives results in nice properties - compact containment in an L^q space in the first case and some continuous derivatives in the second.

Theorem 4.5 (Rellich-Kondrachov Compactness Theorem)**.** *For $1 \leqslant p < n$ and $1 \leqslant q < \frac{np}{n-p}$,*

$$W^{1,p}(M) \Subset L^q(M),$$

where $\Subset$ denotes a compact containment, and in particular

$$W^{1,p}_0(M) \Subset L^q(M).$$

Theorem 4.6 (Continuous Sobolev Embedding Theorem)**.** *If*

$$\frac{(k-r-\alpha)}{n} = \frac{1}{p},$$

for $\alpha \in (0, 1]$, then

$$W^{k,p}(M) \subset C^{r,\alpha}(M).$$

Note that this final theorem implies that

$$\bigcap_{k=0}^{\infty} W^{k,p}(M) = C^\infty(M).$$

4.2 The Poisson Equation on a Manifold

As a key example of a second order partial differential equation, consider the classical Poisson equation, which is intended to be solved over our compact Riemannian manifold M:

$$\Delta u = f. \tag{6}$$

Before we try to solve this, however, we must discuss what is meant by the symbol Δ. In flat space this is of course the standard Laplacian differential operator:

$$\Delta u := \delta^{ij} \partial_i \partial_j u = (\text{div} \circ D) u.$$

We want to imitate the latter definition, by generalising the operators D (here denoting the *gradient vector*) and div to work in this geometric setting.

4.2.1 The Laplace-Beltrami Operator

We start by generalising the gradient vector Du to M. In $\mathbb{R}^n$, the gradient vector encapsulates all directional derivatives of a function u - the dot product of Du with a vector X gives the directional derivative of u in the direction X. The equivalent formulation for a Riemannian manifold is

$$\langle \nabla u, X \rangle = \partial_X u,$$

where we use ∇ (for now) to denote the gradient vector field - to distinguish it from the pushforward du. In local coordinates therefore, writing

$$\nabla u = Y^i \partial_i,$$

we solve for Y^i:

$$\begin{aligned} \langle \nabla u, \partial_k \rangle_g &= g_{ij} Y^i \delta^{jk} \\ \iff \quad \partial_k u &= g_{ij} Y^i \delta^{jk} \\ \iff \quad g^{ki} \partial_k u &= g^{ki} g_{ik} Y^i \\ \iff \quad g^{ki} \partial_k u &= Y^i. \end{aligned}$$

Therefore, we see that the gradient must be given by $\nabla u = g^{ij}\, \partial_i u \, \partial_j$.

Our second operator that must be generalised is the divergence operator, div. For a vector field $X^i \partial_i$ in $\mathbb{R}^n$, div is defined as:

$$div(X^i \partial_i) = \partial_i X^i.$$

It can also, equivalently, be defined as the *adjoint operator* of $-\nabla$:

Lemma 4.7.

$$\langle \text{div} X, f \rangle_{L^2} = \langle X, -\nabla f \rangle_{L^2}.$$

Proof.

$$\begin{aligned} \langle \text{div} X, f \rangle_{L^2} &= \int_{\mathbb{R}^n} (\partial_i X^i) f \text{ dvol} \\ &= -\int_{\mathbb{R}^n} X^i (\partial_i f) \text{ dvol} \qquad \text{(by parts)} \\ &= \langle X, -\nabla f \rangle_{L^2}. \end{aligned}$$

□

Since we now have a working definition of the gradient vector, we imitate this formulation of the divergence, using the L^2 inner product defined over the manifold M. In the following, recall that the volume element on an n-dimensional Riemannian manifold is given by $\sqrt{|g|} dx^1 \wedge \cdots \wedge dx$.

$$\begin{aligned} \langle X, \nabla f \rangle_{L^2} &= \int_M \langle X, \nabla f \rangle_g \text{dvol} \\ &= \int_M \langle X^i \partial_i, g^{kj} \partial_k f \partial_j \rangle_g \text{dvol} \\ &= \int_M g^{kj} g_{ij} X^i \partial_k f \text{dvol} \\ &= \int_M X^k \partial_k f \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n \\ &= -\int_M \partial_k (X^k \sqrt{|g|}) f dx^1 \wedge \cdots \wedge dx^n \\ &= \int_M -\frac{1}{\sqrt{|g|}} \partial_k (X^k \sqrt{|g|}) f \text{dvol}, \end{aligned}$$

and so it makes sense to define

$$\text{div} X = \frac{1}{\sqrt{|g|}} \partial_k (X^k \sqrt{|g|}).$$

The Riemannian Laplacian, or the *Laplace-Beltrami operator*, is therefore defined as

$$\Delta := -\text{div} \circ \nabla.$$

The minus sign is merely convention - here the Laplace-Beltrami operator has been chosen to correspond to the negative of the Euclidean Laplacian.

There is another useful formulation of the Laplacian which uses the exterior derivative on differential forms:

Definition 4.8 (Laplace-deRham operator). *The Laplace-deRham operator is defined to be*

$$\Delta_d R := dd^* + d^* d,$$

where d is the exterior derivative and d^* is the codifferential, the adjoint of the exterior derivative. On functions, the Laplace-deRham and Laplace-Beltrami operators are the same.

As an application of this alternate definition, we characterise the harmonic functions on a compact manifold.

Theorem 4.9 (Harmonic functions on a compact manifold). *The only solutions $u \in C^\infty(M)$ to*

$$\Delta u = 0$$

are the constants, $u \in \mathbb{R}$.

Proof. Certainly the constants *are* solutions, so we now show they are the only ones.

$$\begin{aligned}
\Delta u = 0 &\Leftrightarrow \forall v \in C^\infty(M), \ \ \langle \Delta u, v\rangle_{L^2(M)} \\
&\Leftrightarrow \forall v \in C^\infty(M), \ \ \langle (dd^* + d^* d)u, v\rangle_{L^2(M)} \\
&\Rightarrow \forall v \in C^\infty(M), \ \ \langle d^* du, v\rangle_{L^2(M)} \quad \text{(since d* is 0 on functions)} \\
&\Rightarrow \forall v \in C^\infty(M), \ \ \langle du, dv\rangle_{L^2(M)} \\
&\Rightarrow \|du\|_{L^2(M)} = 0 \\
&\Rightarrow u \in \mathbb{R}.
\end{aligned}$$

□

4.2.2 Existence of Weak Solutions to the Poisson Equation

We now aim to solve the Poisson equation, (6). The first thing to notice, in the case of this particular problem, is that is is unsolvable unless

$$\int_M f \mathrm{dvol} = 0,$$

this can be seen by integrating the Poisson equation above and using Stokes' theorem. Certainly then, this is a necessary condition for solutions to exist. We aim to show in this section that it is a *sufficient* condition also. We will work in the Sobolev space $H(M) = H^1_{int0}(M)$, which is the space of functions in $H^1(M)$ but with integral 0. This is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_H := \int_M \langle \nabla f, \nabla g\rangle_g \mathrm{dvol},$$

(this corresponds to the Sobolev norm $\|u\|_H$ shown to be equivalent to the usual norm on $W^{1,2}(M) = H^1(M)$ in the last section, via the Poincaré inequality). Crucially, it is also the completion of the space $C^\infty_{int0}(M)$ of integral 0 smooth functions on M.

A problem arises immediately when we start to look for solutions in this space: given $u \in H(M)$, u may not even have a well-defined derivative, so in what sense can $\Delta u = f$? To answer this, note that $u \in C^\infty(M)$ solves the Poisson equation if and only if for all test functions $g \in C^\infty(M)$,

$$
\begin{aligned}
\langle g, \Delta u\rangle_{L^2(M)} &= \langle g, f\rangle_{L^2(M)} \\
\Leftrightarrow \langle \nabla g, \nabla u\rangle_{L^2(M)} &= \langle g, f\rangle_{L^2(M)}. \\
\Leftrightarrow \langle g, u\rangle_H &= \langle g, f\rangle_{L^2(M)},
\end{aligned}
$$

and since this final line also makes sense for elements of $H^1(M)$, this is the problem we aim to solve. We will call solutions to this modified problem 'weak solutions'.

The key tool in solving this is the Riesz Representation Theorem:

Theorem 4.10 (Riesz Representation). *Given a Hilbert space H, there is a one-to-one correspondence between bounded linear maps $\alpha : H \to \mathbb{R}$ and elements $a \in H$, such that*

$$\alpha(g) = \langle a, g\rangle_H.$$

To use this, notice

$$\alpha_f(g) = \langle g, f\rangle_{L^2(M)}$$

is a linear map $C^\infty_{int0} \to \mathbb{R}$. If it were the case that this linear map extended to a linear map $\overline{\alpha_f} : H(M) \to \mathbb{R}$, then using Riesz again (after extending), there would exist an element $u \in H(M)$ such that for all test functions g,

$$
\begin{aligned}
\langle u, g\rangle_{H(M)} &= \overline{\alpha_f}(g) \\
&= \langle f, g\rangle_{L^2(M)},
\end{aligned}
$$

completing the existence part of the problem. So, our only remaining question is, can we extend α to a bounded linear operator on H? It is not as simple as 'a bounded linear operator on a normed vector space extends uniquely to a bounded linear operator on the completion', since we are using a completely different norm on $H(M)$ to the L^2 norm of $C^\infty(M)$.

Let's work through it. For $\alpha_f = \langle f, \cdot\rangle_{L^2(M)}$ to lift to a function on H, we make the definition:

$$\overline{\alpha}_f(h) = \lim_{i\to\infty} \alpha_f(h_i) = \lim_{i\to\infty} \langle f, h_i \rangle_{L^2(M)},$$

where h_i is a Cauchy sequence in C^∞_{int0} tending to h via the H norm. For this to be well-defined, the limit must exist, which may be a problem as the inner product in this definition is not the one used to complete $C^\infty_0(M)$. We are saved by the Poincaré inequality, which tells us that

$$\|u\|_{L^2(M)} \leqslant C\|u\|_{H(M)},$$

so the H-Cauchy sequence h_i is also L^2-Cauchy. Thus the limit exists, and the bounded linear operator extends to an operator on H. Note that the Poincaré inequality requires the integral of our functions to be 0, so it was important that we worked in the space of integral 0 functions. This is where the condition that f integrates to 0 made its appearance.

We have proven the following theorem:

Theorem 4.11. *The Poisson equation* (6) *is solvable for* $u \in H^1(M)$ *if and only if*

$$\int_M f \mathrm{dvol} = 0.$$

4.3 Generalising the Existence Problem to Elliptic Operators

Showing that solutions to the Poisson equation are smooth is tricky (though the details in this specific case are in [21]), so we choose instead to move on and demonstrate existence and smoothness of solutions to the equation, 0

$$Lu = f \text{ on } M \tag{7}$$

for a more general differential operator L. Once we have characterised the situations where smooth solutions can be found, we will return to the Poisson equation.

The study of *elliptic* differential operators is particularly nice - ellipticity makes both existence and regularity proofs simpler, so we will restrict to these. We therefore first define what is meant by an elliptic operator, and then move on to generalise the idea of 'weak solution' introduced in the previous section.

4.3.1 Differential Operators

The first thing to define is a differential operator. The easiest way to understand them is to work locally, since we know what we'd like differential operators to look like over $\mathbb{R}^n$.

Definition 4.12. *Let E, F be smooth vector bundles over the manifold M, and denote the space of sections of these bundles by $C^\infty(E), C^\infty(F)$. Then a linear map*

$$L : C^\infty(E) \to C^\infty(F)$$

is a differential operator *of order k, if*

1. *$\forall u \in C^\infty(E)$, $supp(Lu) \subset supp(u)$,*
2. *For all $p \in M$, there is a smooth chart (U, ϕ) around p over which the bundles are trivial, such that in this chart,*

$$L := \sum_{|\alpha| \leqslant k} A_\alpha \partial^\alpha,$$

where α denotes a multi-index, $\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x^\alpha}$, and A_α are matrices of size $\text{rank}(F) \times \text{rank}(E)$. The Laplacian, which we extensively looked at in the previous section, is therefore a differential operator,

$$\Delta : C^\infty(M \times \mathbb{R}) \to C^\infty(M \times \mathbb{R}),$$

in this notation. Another example is the exterior derivative on forms: for example, taking the example of the derivative taking 2-forms of $\mathbb{R}^3$ to 3-forms of $\mathbb{R}^3$:

$$d : C^\infty\left(\bigwedge\nolimits^2 T^*\mathbb{R}^3\right) \to C^\infty\left(\bigwedge\nolimits^3 T^*\mathbb{R}^3\right),$$

we find that

$$d(a_1 dx^1 \wedge dx^2 + a_2 dx^2 \wedge dx^3 + a_3 dx^3 \wedge dx^1) = \partial_1 a_1 + \partial_2 a_2 + \partial_3 a_3,$$

and so

$$d \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \sum_{i=1}^{3} \left(\delta_{1,i}, \delta_{2,i}, \delta_{3,i}\right) \cdot \partial_i \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Therefore d is a partial differential operator by the above definition.

A very important notion is that of the *formal adjoint* to an operator:

Definition 4.13. *An operator Q is the* formal adjoint *of the operator P if, for all $u \in C_0^\infty(E), v \in C_0^\infty(F)$, we have:*

$$\int_M \langle Pu, v \rangle_F \text{ dvol} = \int_M \langle u, Qv \rangle_E \text{ dvol}.$$

Formal adjoints are important because every differential operator has a unique one! As an example, the Laplace-Beltrami operator as defined in the last section is *self-adjoint*, since:

$$\int_M \langle \Delta u, v\rangle_g \text{ dvol} = \int_M \langle \nabla u, \nabla v\rangle_g \text{ dvol} = \int_M \langle u, \Delta v\rangle_g \text{ dvol}.$$

We will be most interested in the case where L is a second-order differential operator, and our bundles are trivial of rank 1, so we are working just with functions from M to $\mathbb{R}$. In this case, restricting to a co-ordinate patch $U \subset \mathbb{R}^n$, L will look like:

$$Lu = -\sum_{i,j=1}^{n} \partial_j(a^{ij}(x)\partial_i u) \;+\; \sum_{i=1}^{n} b^i(x)\partial_i u \;+\; c(x)u$$

(this is called *divergence form*). Since we are working locally, on $\mathbb{R}^n$, we may assume *symmetry* of the matrix $(A(x))_{ij} = a^{ij}(x)$.

4.3.2 Elliptic Operators

A second-order differential operator as defined above is said to be *elliptic* if there exists a $\Theta > 0$ such that in any chart, the matrix $A(x)$ is positive definite, with smallest eigenvalue larger than Θ. In particular, this ensures that $A(x)$ is invertible.

Importantly, if the operator L is elliptic, then its adjoint, L^* is as well. To show this, if we expand the L^2 inner product $\langle Lu, v\rangle_{L^2}$ locally, we get:

$$\begin{aligned}
\langle Lu, v\rangle_{L^2} &= -\sum_{i,j=1}^{n} \int_U \partial_j(a^{ij}\partial_i u)v \text{ dx} + \sum_{i=1}^{n} \int_U b^i \partial_i uv \text{ dx} + \int_U cuv \text{ dx} \quad \text{(locally)} \\
&= \sum_{i,j=1}^{n} \int_U a^{ij}\partial_i u \partial_j v \text{ dx} - \sum_{i=1}^{n} \partial_i(b^i v) \text{ dx} + \int_U cuv \text{ dx} \\
&= -\sum_{i,j=1}^{n} \int_U \partial_i(a^{ij}\partial_j v)u \text{ dx} - \sum_{i=1}^{n} b^i \partial_i vu \text{ dx} + \int_U \left(c - \partial_i b^i\right) vu \text{ dx} \\
&= \langle u, L^* v\rangle_{L^2},
\end{aligned}$$

showing that the adjoint is locally defined by

$$L^* v = -\sum_{i,j=1}^{n} \partial_i(a^{ij}\partial_j v) - \sum_{i=1}^{n} b^i \partial_i v + \left(c - \sum_{i=1}^{n} \partial_i b^i\right)v.$$

Then the matrix A is the same for L^* as for L, so L is elliptic if and only if L^* is.

As an example of an elliptic operator, the Laplace-Beltrami operator is given locally by:

$$\begin{aligned}
\Delta &= -div \circ \nabla \\
&= -div(g^{ij}\partial_i u \partial_j) \\
&= -\frac{1}{\sqrt{|g|}}\partial_k(g^{ik}\partial_i u\sqrt{|g|}) \\
&= -g^{ik}\partial_k\partial_i u - \frac{1}{\sqrt{|g|}}(\partial_i u \partial_k(g^{ik}\sqrt{|g|}) \\
&= -\partial_k(g^{ik}\partial_i u) + b^i\partial_i u,
\end{aligned}$$

where we have reorganised the expression into divergence form. This is elliptic by positive-definiteness of the matrix g^{ij}.

In the case of general operators on bundles, to define ellipticity we require the notion of the *symbol* of a differential operator:

Definition 4.14. *Let L be a differential operator, $L = \sum_{|\alpha|\leqslant k} A_\alpha \partial^\alpha$, and let ω be a 1-form (covector field), $\omega = \omega_i dx^i$. The* ***total symbol*** *$\sigma_L(\omega)$ of the operator L in the direction of ω is the bundle homomorphism:*

$$\begin{aligned}
\sigma_L(\omega) : E &\to F \\
e &\mapsto \sum_{|\alpha|\leqslant k} \omega^\alpha A_\alpha.
\end{aligned}$$

The ***principal symbol*** *$\hat{\sigma}_L(\omega)$ is the highest order part of the total symbol:*

$$\begin{aligned}
\hat{\sigma}_L(\omega) : E &\to F \\
e &\mapsto \sum_{|\alpha|= k} \omega^\alpha A_\alpha.
\end{aligned}$$

A couple of important properties of the principal symbol are that it is a *homomorphism*, i.e

$$\hat{\sigma}_{L_2\circ L_1}(\omega) = \hat{\sigma}_{L_2}(\omega) \circ \hat{\sigma}_{L_1}(\omega),$$

and it respects formal adjoints, in that

$$\hat{\sigma}_L(\omega)^* = \hat{\sigma}_{L^*}(\omega).$$

For proofs of these facts and further information about the symbol, see [1].

Ellipticity then has a nice characterisation in terms of this symbol: an operator L is **elliptic** if, for all $m \in M$, $\hat{\sigma}_L(\omega)|_m$ is invertible at all $\omega|_m \neq 0$, i.e. the principal symbol is a linear isomorphism for nonzero covectors. As an example, the Laplace-Beltrami operator's principal symbol is:

$$\begin{aligned}\hat{\sigma}_\Delta(\omega) &= -\sum_{i,j=1}^{n} g^{ij}\omega_i\omega_j \\ &= -\langle \omega, \omega\rangle_g \\ &= -\|\omega\|_g^2 \neq 0,\end{aligned}$$

which shows that the Laplace-Beltrami operator is elliptic. Generally speaking, if we define a **Laplace-type operator** to be one with principal symbol a multiple of $\|\omega\|_g^2$, then this must also be elliptic - there are many such operators including the *Hodge Laplacian* defined earlier.

As another example, if we define a **Dirac-type operator** to be an operator D such that DD^* and D^*D are Laplace-type, then by the homomorphism property of the principal symbol, D is also elliptic. These operators also turn up frequently in geometric analysis, an example is $d+\delta$ (on the graded bundle of all differential forms), where d is the exterior derivative and δ is the codifferential. We will use a Dirac-type operator in Section 5.2.

4.3.3 The Search for Solutions

As before, we would like to search first for *weak* solutions to the equation (7); we first explain what is meant by this. Using our definition of Sobolev spaces given above, we can extend the domain of L to the space $W^{k,p}(M)$, simply treating each ∂_i in the definition as a weak derivative. In particular, we define

$$L_k : H^k(M) \to H^{k-2}(M)$$

as the unique extension of the linear operator L to the space $H^k(M)$ (We are using Hilbert Sobolev spaces from now on to make use of adjoints and orthogonal decompositions in our arguments). Note that, importantly, it is a *bounded* linear operator on $H^k(M)$, because locally, ($u = 0$ on ∂U):

$$\begin{aligned}\|Lu\|_{H^{k-2}(U)} &= \Big\| -\sum_{i,j=1}^{n} a^{ij}(x)\partial_j\partial_i u \;+\; \sum_{i=1}^{n}\tilde{b}^i(x)\partial_i u \;+\; c(x)u\Big\|_{H^{k-2}(U)} \\ &\leqslant \sup_{i,j,x}(a^{i,j}(x))\sum_{i,j=1}^{n}\|\partial_j\partial_i u\|_{H^{k-2}(U)} + \sup_{i,x}(b^i(x))\sum_{i=1}^{n}\|\partial_i u\|_{H^{k-2}(U)} + \sup_x \|u\|_{H^{k-2}(U)} \\ &\leqslant C\|u\|_{H^k(U)}.\end{aligned}$$

We then define a *weak solution* of (7) as a solution of

$$L_k u = \rho.$$

We now tackle the problem of existence of smooth solutions to (7). The key ingredient needed for both existence of weak solutions and regularity, special to elliptic operators, is the following pair of "elliptic estimates":

Theorem 4.15 (Elliptic Estimates). *If L is an elliptic operator of order r over a compact manifold, then as long as $1 < p < \infty$ and $\alpha \in (0,1)$, then we have the following two regularity estimates for solutions u to the equation (7):*

- $\|u\|_{W^{k+r,p}} \leqslant C(\|Lu\|_{W^{k,p}} + \|u\|_{L^p})$
- $\|u\|_{C^{k+r,\alpha}} \leqslant C(\|Lu\|_{C^{k,\alpha}} + \|u\|_{C^{0,\alpha}})$.

This immediately gives us the regularity part of the argument:

Theorem 4.16. *Let L be a second order elliptic partial differential operator on a compact manifold M, and let L_k be the extension of this operator to the Hilbert space $H^k(M)$. A solution to the weak differential equation*

$$L_k u = f$$

for $f \in C^\infty(M)$ is in fact a smooth function, and therefore a solution of the smooth problem,

$$Lu = f.$$

This implies that $ker(L_k) = ker(L)$.

Proof. If u weakly solves (7) for smooth f, the estimate tells us that u has bounded Sobolev norm for *any* k. Since the intersection of all Sobolev spaces is just the smooth functions, u must be smooth. □

Once we have this theorem, the rest is quite straightforward:

Theorem 4.17 (Fredholmness of Elliptic Operators). *Let L be a second order elliptic partial differential operator on a compact manifold M, and let L_k be the extension of this operator to the Hilbert space $H^k(M)$. Then:*

1. *$dim(ker(L_k)) < \infty$.*
2. *$Im(L_k)$ is closed.*
3. *The dual space of coker (L_k) is $ker(L^*)$, where L^* is the formal adjoint of L.*
4. *There is a decomposition $H^{k-2}(M) = kerL^* \oplus im(L_k)$, where the decomposition is orthogonal with respect to the L^2 inner product.*

Along with elliptic regularity, this tells us that the original problem (7) is solvable with $u \in C^\infty(M)$ if and only if

$$\forall \rho \in \ker L^*,\ \langle f, \rho \rangle_{L^2} = 0.$$

For the proof, we follow the techniques in [7].

Proof. Take L, L_k as in the statement of the theorem.

1. Since $\ker(L_k) \subset H^k(M)$ is closed, it follows that it is a Banach space under the $H^k(M)$ norm. To show that $\ker(L_k)$ is finite-dimensional, it therefore suffices to show that the unit ball B is compact.
   Now by Theorem 4.5, B is precompact w.r.t the L^2 norm. Then by Theorem 4.15,

$$\begin{aligned}\|u\|_{H^k(M)} &\leqslant C(\|L_k u\|_{H^{k-2}(M)} + \|u\|_{L^2(M)}) \\ &= C\|u\|_{L^2(M)} \text{ for } u \in \ker(L_k),\end{aligned}$$

   so precompactness in $L^2(M)$ implies precompactness in $H^k(M)$, as required.

2. Theorem 4.15 implies that, for u orthogonal to $\ker(L_k)$,

$$\|u\|_{H^k(M)} \leqslant \|L_k u\|_{H^{k-2}(M)}.$$

   We use this to show that $\mathrm{im}(L_k)$ is closed. Take a sequence $L_k u_i$ converging to v in $H^{k-2}(M)$ - we may assume $u_i \in \ker(L_k)^\perp$. Then $L_k u_i$ is Cauchy in $H^{k-2}(M)$, implying by the above inequality that u_i is Cauchy in $H^k(M)$, converging to some u such that $L_k(u) = v$.

3/4. Denote the dual space of X by X^*. Then,

$$(\mathrm{coker}(L_k))^* = \Big(\frac{H^{k-2}(M)}{\mathrm{im}(L_k(M))}\Big)^* = \{\phi \in (H^{k-2}(M))^* : \phi(\mathrm{im}(L_k)) = \{0\}\},$$

giving an embedding

$$\begin{aligned}\iota : \ker(L^*) &\hookrightarrow (\mathrm{coker}(L_k))^* \\ u &\mapsto \langle u, \cdot \rangle_{L^2(M)}\end{aligned}$$

(note that $u \in \ker(L^*)$ implies that $\langle u, L\phi\rangle_{L^2(M)} = 0$ for $\phi \in C^\infty(M)$, and then by continuity $\langle u, L_k\phi\rangle_{L^2(M)} = 0$ for $\phi \in H^k(M)$). We must show that ι is a surjection. For $k = 2$,

$$\begin{aligned}(\mathrm{coker}(L_2))^* &= \Big(\frac{L^2(M)}{\mathrm{im}(L_2)}\Big)^* \\ &= \{\phi \in L^2 : \ \phi \text{ vanishes on } \mathrm{im}(L_2)\} \\ &= \{\phi \in L^2 : \ \forall u \in H^2(M),\ \langle \phi, L_2 u\rangle_{L^2(M)} = 0\} \\ &= \{\phi \in L^2 : \ \forall u \in C^\infty(M),\ \langle \phi, L u\rangle_{L^2(M)} = 0\} \\ &= \{\phi \in L^2 : L^*\phi = 0 \text{ weakly}\} \\ &= \{\phi \in C^\infty : L^*\phi = 0\} \quad \text{(by elliptic regularity)} \\ &= \ker(L^*).\end{aligned}$$

We therefore just demonstrated that $\text{im}(L_2)^{\perp} = (coker(L_2))^*$, so we have shown (4.) for the case $k = 2$:

$$L^2(M) = \ker(L^*) \oplus \text{im}(L_2).$$

Now let $k > 2$, and take a $v \in H^{k-2}(M)$. Then $v \in L^2(M)$ as well, so we can decompose it:

$$v = \phi + L_2 u,$$

where $\phi \in \ker(L*)$, $u \in H^2(M)$. Then:

$$\begin{aligned} L_2 u &= v - \phi \\ \implies u &\in H^k(M), \end{aligned}$$

by Theorem 4.15. Therefore,

$$H^{k-2}(M) = \ker(L^*) \oplus \text{im}(L_k).$$

This also shows, by dimension counting, that ι is onto.

□

As an example of this in action, return one final time to the example of the Poisson equation on a manifold, (6). The Laplace-Beltrami operator is a self-adjoint operator, so by Theorem 4.17 there is a smooth solution if and only if

$$\forall v \in \ker(\Delta), \;\; \langle v, f \rangle_{L^2(M)} = \int_M v f dx = 0.$$

However we know that $\ker(\Delta) = \mathbb{R}$ by Theorem 4.9, so there is a solution if and only if f integrates to 0 - which is the result we already proved in Theorem 4.11.

5 A Collapsing Sequence of Hyperkähler Metrics on the $K3$ Surface

In this chapter we will give a summary of Foscolo's recent construction [12] of a sequence of hyperkähler metrics on the $K3$ surface which collapse to the 3-dimensional orbifold $T^3/\mathbb{Z}_2$ with the flat metric. This is part of a larger program in which also the collapse of 7-dimensional manifolds with holonomy G_2 to 6-dimensional Calabi-Yau manifolds is studied, which makes precise relations between different string theories predicted by physicists.

It is also of considerable interest from the point of view of Riemannian geometry. By Gromov's precompactness theorem, a sequence of compact Riemannian 4-manifolds with Ricci-curvature bounded from below and diameter bounded from above has a subsequence which converges to some limit space in the Gromov-Hausdorff topology. If it is a sequence of Einstein 4-manifolds with additional volume bound from below, i.e. the sequence doesn't collapse, and a uniform bound on the Euler characteristic the regularity of the limit space is well understood. Nakajima [24] and Anderson [2] show that the limit space is an Einstein 4-orbifold, outside of finitely many points the convergence is in $\mathcal{C}^{k,\alpha}$ and around the singularities the convergence is modelled by rescaled Ricci-flat ALE spaces. This bubbling phenomenon also occurs in the theory of harmonic maps and Yang-Mills instantons. In the case of a collapsing sequence of Ricci-flat metrics on a compact 4-manifold it is known that the collapse occurs with bounded curvature outside of finitely many points [6]. However, almost nothing is known about the strucure of the collapse around the singularities. So far there are two constructions of collapsing sequences of Ricci-flat metrics on the $K3$ surface. In Foscolo's example of a collapse to a three-dimensional limit space the structure of the collapse around the singularities is analogous to the non-collapsing case. It is modelled on rescaled gravitational instantons. But in this case with volume growth as 3-dimensional euclidean space rather than 4-dimensional euclidean space. These gravitational instantons are called ALF spaces. However, in the collapsing case it is not always true that that curvature concentrates around singular points as bubbling of rescaled gravitational instantons. Gross and Wilson [16] constructed a family of hyperkähler metrics on a K3 elliptic fibration over $\mathbb{C}P^1$. In this example the singular fibres are modelled on the Ooguri-Vafa metric.

The fundamental difficulty in constructing a collapsing family of Ricci-flat metrics with bounded curvature on a compact simply-connected manifold is the following: Cheeger-Gromov theory of collapse with bounded curvature suggests that if the limit space has one dimension less then the collapse occurs as in the example of Berger spheres (see sections 3.2 and 2.2): the total space is a principal circle bundle and the collapse is generated by shrinking of the fibres. However, a compact simply-connected manifold with vanishing

Ricci curvature cannot have an isometric circle action: By Bochner's formula

$$|\nabla\xi|^2 = \frac{1}{2}\Delta|\xi|^2 + \mathrm{Ric}(\xi,\xi)$$

we see that the generating Killing vector field ξ is parallel [25][Chapter 7.1, p. 191, Theorem 36]. With the Weitzenböck identity

$$\Delta = \nabla^*\nabla + \mathrm{Ric}$$

we see that the 1-form $\xi^\flat = g(\xi,\cdot)$ is harmonic. By the Hodge theorem there are no non-trivial harmonic 1-forms on a compact, simply-connected manifold.

There are three main problems to solve: Firstly, the idea is to produce collapse with bounded curvature outside of finitely many points by constructing a hyperkähler metric on a circle fibration over the limit space $T^3/\mathbb{Z}_2$ outside of finitely many points and then shrink the fibres. We need a tool to construct this kind of metric. Secondly, to make this a collapse of hyperkähler metrics on the K3 surface we need to glue in appropriate "building blocks" to resolve the singularities. For this we need to compare the asymptotic geometry of the building blocks with the geometry of the fibration around the singularities. Thirdly, since gluing is involved we need to set up a good deformation theory in this situation to correct the gluing errors. The first two problems are solved by using the Gibbons-Hawking ansatz which is described in detail in section 5.3 and the third problem is solved by working in the framework of definite triples, which is described in section 5.2.

5.1 Hyperkähler Manifolds

A hyperkähler manifold (M, g, I_1, I_2, I_3) is a smooth Riemannian manifold (M, g) with three (integrable) complex structures I_1, I_2, I_3 which satisfy the quaternionic relation $I_1I_2I_3 = -1$ and (M, g) is Kähler with respect to each of them. This has strong consequences. For $a_1, a_2, a_3 \in \mathbb{R}$ with $a_1^2 + a_2^2 + a_3^2 = 1$ the metric is Kähler with respect to $a_1I_1 + a_2I_2 + a_3I_3$. Therefore, g is Kähler with respect to a whole 2-sphere of complex structures. Furthermore, the dimension $\dim \mathrm{M} = 4m$ is divisible by 4 and the holonomy $\mathrm{Hol}(g)$ is contained is the symplectic group $\mathrm{Sp}(m)$. Because of the inclusion $\mathrm{Sp}(m) \subset \mathrm{SU}(2m)$ all hyperkähler manifolds are Calabi-Yau manifolds and Ricci-flat. In dimension 4 we have equality $\mathrm{Sp}(1) = \mathrm{SU}(2)$ and all Calabi-Yau manifolds of real dimension 4 are hyperkähler. We will now restrict the discussion to dimension 4. The three complex structures I_1, I_2, I_3 yield three Kähler forms $\omega_1, \omega_2, \omega_3$. It is often convenient to work in the framework of definite triples.

5.2 Definite Triples

The following discussion of definite triples is based on [12][section 2].

Let (M, μ_0) be an oriented 4-manifold with orientation form μ_0. In this situation μ_0 allows us to identify $\bigwedge^4 T^*M$ with the trivial line bundle $M \times \mathbb{R}$. Therefore the wedge product induces a symmetric bilinear form on $\bigwedge^2 T_x^*M$ at each point $x \in M$. A triple of 2-forms $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ is called a *definite triple* if it spans a 3-dimensional positive definite subspace at each point. Every triple of 2-forms $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ gives rise to the matrix-valued function

$$Q_{ij} = \frac{\frac{1}{2}\omega_i \wedge \omega_j}{\mu_0}, \quad i, j = 1, 2, 3.$$

It is a definite triple if and only if Q is a positive definite matrix at each point. If we set

$$\mu_{\underline{\omega}} = (\det Q)^{\frac{1}{3}}\mu_0, \quad Q_{\underline{\omega}} = (\det Q)^{-\frac{1}{3}}Q,$$

then $\mu_{\underline{\omega}}$ and $Q_{\underline{\omega}}$ are independent of the original orientation form μ_0.

At each point $\bigwedge^2 T_x^*M$ is a 6-dimensional vector space. Note that a 3-dimensional positive definite subspace cannot be chosen in a unique way. However, a Riemannian metric gives a natural choice of such a subspace. The Hodge-star operator $* : \bigwedge^2 T_x^*M \to \bigwedge^2 T_x^*M$ associated to the metric is an isomorphism with eigenvalues ± 1. $\bigwedge^2 T_x^*M$ splits into eigenspaces $\bigwedge^2_+ T_x^*M$ and $\bigwedge^2_- T_x^*M$. A section of $\bigwedge^2_+(M)$ is called a *self-dual* 2-form and a section of $\bigwedge^2_-(M)$ is called an *anti-self-dual* 2-form. $\bigwedge^2_+ T_x^*M$ is positive definite. It turns out that the choice of this subspace only depends on the conformal class [23][section 6.4]. Vice versa the choice of a 3-dimesnional positive definite subspace gives a conformal structure [10][Section 1.1.5, Self-duality and special isomorphisms]. Hence, a definite triple induces a Riemannian metric in two steps: First, it gives rise to a conformal structure by specifying a maximal positive definite subspace at each point and then we get a metric by requiring that the volume form is given by $\mu_{\underline{\omega}}$.

If the triple is closed, i.e. all three forms are closed, and $Q_{\underline{\omega}} = \mathrm{id}$, then the triple defines a hyperkähler structure.

One of the key ingredients in Foscolo's construction is the fact that definite triples have a good perturbation theory. If $\underline{\omega}$ is closed and close to a hyperkähler triple, i.e. the error $\|Q_{\underline{\omega}} - \mathrm{id}\|_{C^0}$ is sufficiently small, then the linearisation of the deformation which perturbs $\underline{\omega}$ into a hyperkähler triple is given by

$$(D \oplus \mathrm{id}) \otimes \mathbb{R}^3 : (\Omega^1(M) \oplus \mathcal{H}^+_{\underline{\omega}}) \otimes \mathbb{R}^3 \to (\Omega^0(M) \oplus \Omega^+(M)) \otimes \mathbb{R}^3,$$

where D is the Dirac operator

$$D = d^* + d^+ : \Omega^1(M) \to \Omega^0(M) \oplus \Omega^+(M).$$

Here $\mathcal{H}^+_{\underline{\omega}}$ is the 3-dimensional vector space of harmonic self-dual 2-forms with respect to the metric given by $\underline{\omega}$ and d^+ is the self-dual part of d. Foscolo shows that the linearisation is invertible as an operator on weighted Hölder spaces. An introduction to analysis on manifolds is given in part 4 of this text.

5.3 Gibbons-Hawking Ansatz

In this section we describe the Gibbons-Hawking ansatz which is an important tool in Foscolo's construction. We follow [12][section 3.1] and [20].

Let $U \subset \mathbb{R}^3$ be an open subset and $\pi : \mathcal{P} \to U$ a principle U(1)-bundle. Suppose h is a positive function on U and θ a connection on $\mathcal{P}$. We identify $\mathfrak{u}(1) \cong \mathbb{R}$. Then θ can be understood as a left-invariant 1-form on $\mathcal{P}$ with the normalisation $\theta(\xi) = 1$, where the vector field ξ is the infinitesimal generator of the U(1)-action. Suppose the pair (θ, h) solves the abelian *monopole equation*

$$d\theta = \pi^*(*dh), \tag{8}$$

where $*$ is the Hodge star operator on U with the euclidean metric. This gives an easy way to construct a hyperkähler structure on $\mathcal{P}$. Set

$$g = h\pi^* g_{\mathbb{R}^3} + h^{-1}\theta^2. \tag{9}$$

Denote by $\hat{\partial}_i := \partial_i - \theta(\partial_i)\xi$ the horizontal lift of ∂_i for $i = 1, 2, 3$. Then $\{h^{-\frac{1}{2}}\hat{\partial}_1, h^{-\frac{1}{2}}\hat{\partial}_2, h^{-\frac{1}{2}}\hat{\partial}_3, h^{\frac{1}{2}}\xi\}$ is an orthonormal frame. Each unit vector in $\mathbb{R}^3$ gives rise to an integrable complex structure on $\mathcal{P}$. Considered as a constant vector field on U, its horizontal lift is mapped to a multiple of the Killing vector field ξ such that the metric is preserved. We will explain this in more detail for the unit vector ∂_1. The almost complex structure

$$I_1(\hat{\partial}_1) = h\xi, \quad I_1(\hat{\partial}_2) = \hat{\partial}_3 \tag{10}$$

clearly preserves g. $T^{1,0}\mathcal{P}$ at each point is spanned by $\{\hat{\partial}_1 - ih\xi, \hat{\partial}_2 - i\hat{\partial}_3\}$. Therefore the annihilator of $T^{1,0}\mathcal{P}$ is generated by $\{hdx_1 - i\theta, dx_2 - idx_3\}$. By (8) this ideal is closed under exterior differentiation:

$$\begin{aligned}
d(hdx_1 - i\theta) =& dh \wedge dx_1 - id\theta \\
=& dh \wedge dx_1 - i * dh \\
=& \partial_2 h\ dx_2 \wedge dx_1 + \partial_3 h\ dx_3 \wedge dx_1 \\
& - i(\partial_1 h\ dx_2 \wedge dx_3 + \partial_2 h\ dx_3 \wedge dx_1 + \partial_3 h\ dx_1 \wedge dx_2) \\
=& (-\partial_2 h - i\partial_3 h)dx_1 \wedge dx_2 + (-\partial_3 h + i\partial_2 h)dx_1 \wedge dx_3 - i\partial_1 h\ dx_2 \wedge dx_3 \\
=& ((-\partial_2 h - i\partial_3 h)dx_1 + i\partial_1 h\ dx_3) \wedge (dx_2 - idx_3).
\end{aligned}$$

Hence the almost complex structure I_1 is integrable by Frobenius' theorem. Because of

$$g(I_1(\hat{\partial}_1), \xi) = hg(\xi, \xi) = 1,$$
$$g(I_1(\hat{\partial}_2), \hat{\partial}_3) = g(\hat{\partial}_3, \hat{\partial}_3) = h,$$

the Kähler form with respect to I_1 is given by

$$\omega_1 = dx_1 \wedge \theta + h\ dx_2 \wedge dx_3. \tag{11}$$

The fact that (θ, h) solves equation (8) makes ω_1 closed and g Kähler with respect to I_1:

$$\begin{aligned} d\omega_1 &= -dx_1 \wedge d\theta + dh \wedge dx_2 \wedge dx_3 \\ &= -dx_1 \wedge (*dh) + dh \wedge dx_2 \wedge dx_3 \\ &= -dx_1 \wedge ((\partial_1 h) dx_2 \wedge dx_3) + (\partial_1 h)\ dx_1 \wedge dx_2 \wedge dx_3 \\ &= 0. \end{aligned}$$

Analogously, ∂_2 and ∂_3 give rise to the Kähler forms

$$\omega_2 = dx_2 \wedge \theta + h\ dx_3 \wedge dx_1, \tag{12}$$
$$\omega_3 = dx_3 \wedge \theta + h\ dx_1 \wedge dx_2 \tag{13}$$

respectively. Furthermore, we have

$$\begin{aligned} \omega_i \wedge \omega_j &= \Big(dx_i \wedge \theta + \sum_{ab} h\ \epsilon_{iab}\ dx_a \otimes dx_b\Big) \wedge \Big(dx_j \wedge \theta + \sum_{cd} h\ \epsilon_{jcd}\ dx_c \otimes dx_d\Big) \\ &= \sum_{ab} h\ \epsilon_{iab}\ dx_a \otimes dx_b \wedge (dx_j \wedge \theta) + \sum_{cd} h\ \epsilon_{jcd}\ dx_c \otimes dx_d \wedge (dx_i \wedge \theta) \\ &= 2\delta_{ij}\ h\ dx_1 \wedge dx_2 \wedge dx_3 \wedge \theta. \end{aligned}$$

Therefore in the terminology of section 5.2 we have $Q = \mathrm{id}$ with respect to the orientation $\mu_0 = h\ dx_1 \wedge dx_2 \wedge dx_3 \wedge \theta$. This means that $\{\omega_1, \omega_2, \omega_3\}$ is a hyperkähler triple.

The projection map $\pi : \mathcal{P} \to U \subset \mathbb{R}^3$ has an important interpretation. Inserting the Killing field ξ of the U(1)-action into the Kähler forms gives

$$\omega_i(\cdot, \xi) = dx_i, \quad i = 1, 2, 3.$$

This means that $\pi = (x_1, x_2, x_3)$ is the hyperkähler moment map.

The Gibbons-Hawking ansatz is very useful because it reduces the usually very difficult problem of constructing Ricci-flat metrics to finding a harmonic function and check the topological condition that $*dh$ is the curvature of a connection on a line bundle over the space. It is natural to use the rotationally symmetric harmonic function $\frac{1}{|x|}$ on $\mathbb{R}^3$. By choosing a finite number of singularities $p_1, \ldots, p_n \in \mathbb{R}^3$ and a constant $\lambda \geqslant 0$, by superposition we get the harmonic function

$$h = \lambda + \sum_{j=1}^{n} \frac{k_j}{|x - p_j|}. \tag{14}$$

To check that $*dh$ is the curvature of a line bundle over $\mathbb{R}^3 - \{p_1, \ldots, p_n\}$, we need to check that $\frac{1}{2\pi} * dh$ induces an integral cohomology class as line bundles are classified by the second integral cohomology group via the first Chern class. Since $H_2(\mathbb{R}^3 - \{p_1, \ldots, p_n\}, \mathbb{Z})$ is generated by spheres around the punctures, it is enough to check that the integral of $\frac{1}{2\pi} * dh$ over some sphere around each puncture is an integer. This is the case if all k_j are integers. Solutions to the monopole equation of this kind are called **Dirac monopoles**.

5.4 Gravitational Instantons

The following discussion follows [12][section 3].

A **gravitational instanton** is a complete, noncompact hyperkähler 4-manifold with a curvature decay condition at infinity. It is most common to require **finite energy** $\|Rm\|_{L^2} < \infty$ of the Riemann curvature tensor. For classification results it is helpful to require **faster than quadratic curvature decay** $|Rm| = O(r^{-2-\varepsilon})$ for some $\varepsilon > 0$.

A gravitational instanton with more than one end is disconnected at infinity, i.e. disconnested after removing a compact subset, and hence contains a line. Since it is also Ricci-flat, it must be a cylinder by the Cheeger-Gromoll splitting theorem. The curvature decay condition then forces the cross section to be flat. Therefore each non-flat gravitational instanton has only one end. The Ricci-flatness gives another restriction to the geometry: By the Bishop-Gromov volume comparison theorem the volume growth of a geodesic ball of radius r is at most r^4 (see Theorem 3.12).

Gravitational instantons with maximal volume growth are called **asymptotically locally euclidean**, or **ALE**. Examples are $\mathbb{C}^2$ with its standard hyperkähler structure and the Eguchi-Hanson metric on T^*S^2. ALE spaces are local models for how to desingularize hyperkähler 4-orbifolds. In the Kummer construction of Calabi-Yau metrics on the K3 surface, one starts with a 4-torus $T^4 = \mathbb{C}^2/\Lambda$ and takes the quotient $T^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by multiplication of -1. The 16 singularities are modelled on $\mathbb{C}^2/\mathbb{Z}_2$ and are resolved by gluing in an Eguchi-Hanson space. With a standard perturbation argument one gets a Ricci-flat metric on the K3 surface. This was carried out rigorously among others by Donaldson[9].

Gravitational instantons with cubic volume growth are called **asymptotically locally**

flat, or **ALF**. Minerbe [22] shows that under the assumption of faster than quadratic curvature decay there are no gravitational instantions with volume growth $\mathrm{Vol}(B(p,r) = O(t^a)$ for some number $a \in (3,4)$. Furthermore, using the theory of collapse with bounded curvature due to Cheeger and Gromov, he shows that the unique end $M - K$ of an ALF space (M,g) is a circle fibration $\pi : M - K \to (\mathbb{R}^3 - B_R)/\Gamma$, where $\Gamma = \{\mathrm{id}\}$ or $\mathbb{Z}_2$, and the metric is asymptotically a submersion

$$g = \pi^* g_{\mathbb{R}^3/\Gamma} + \theta^2 + O(r^{-\mu})$$

for a connection θ on π and some $\mu > 0$. If $\Gamma = \{\mathrm{id}\}$ the ALF space is called **cyclic** and if $\Gamma = \mathbb{Z}_2$ the ALF space is called **dihedral**.

We can describe the asymptotic geometry of ALF spaces even better by using the Gibbons-Hawking ansatz. Consider $h_k := 1 + \frac{k}{2\rho}$, where ρ is a radial function of $\mathbb{R}^3$. $*dh_k$ is the curvature of some connection θ_k on H^k, the principle U(1)-bundle associated with the line bundle $\mathcal{O}(k)$ over $\mathbb{C}P^1 \cong S^2$. Then (θ_k, h_k) solves the monopole equation (8) and the Gibbons-Hawking ansatz gives the hyperkähler metric

$$g_k = \left(1 + \frac{k}{2\rho}\right)(d\rho^2 + \rho^2 g_{\mathbb{S}^2}) + \left(1 + \frac{k}{2\rho}\right)^{-1} \theta_k^2. \tag{15}$$

We could also use $\lambda + \frac{k}{2\rho}$, but by scaling we can always reduce to the case $\lambda = 1$.

Definition 5.1. *[12][p. 9, definition 3.6] Let (M^4, g) be an ALF gravitational instanton.*

(i) If M is cyclic we say it is of type A_k for some $k \geqslant -1$ if there exists a compact set $K \subset M$, $R > 0$ and a diffeomorphism $\phi : H^{k+1} \to M - K$ such that

$$|\nabla^l_{g_{k+1}}(g_{k+1} - \phi^* g)|_{g_{k+1}} = O(r^{-3-l}) \tag{16}$$

for every $l \geqslant 0$.

(ii) If M is dihedral we say it is of type D_m for some $m \geqslant 0$ if there exists a compact set $K \subset M$, $R > 0$ and a double cover $\phi : H^{2m-4} \to M - K$ such that the group $\mathbb{Z}_2$ of deck transformations is generated by the standard involution on H^{2m-4} and

$$|\nabla^l_{g_{2m-4}}(g_{2m-4} - \phi^* g)|_{g_{2m-4}} = O(r^{-3-l}) \tag{17}$$

for every $l \geqslant 0$.

To conclude this section, let's return to Dirac monopoles explained at the end of the previous section. The fundamental difference between ALE and ALF spaces can be seen by their description with the Gibbons-Hawking ansatz. If we choose the constant λ in (14) to be 1, then the length of the circles given by the metric (9) is $h^{-\frac{1}{2}}$ if the Killing field has period one. If $|x|$ is very large, $\frac{1}{|x-p_1|}, \ldots, \frac{1}{|x-p_n|}$ become small and h^{-1} is roughly 1 and the length of the circles doesn't grow "at infinity". Since only the three-dimensional base space "grows", the volume growth is cubic. In this case we get an ALF space. However, if we choose λ to be zero, then for example in the case of one singularity we have up to a constant $h^{-1} = |x|$ and the circles become larger and larger. In this case we get an ALE space. In section 5.6 we will give some explicit examples of Dirac monopoles. If we choose one singularity, in the ALE case we just recover the euclidean metric on $\mathbb{C}^2$. The ALF "cousin" of the euclidean space is the Taub-NUT space. If we take two singularities, then in the ALE case we get the Eguchi-Hanson space on T^*S^2. The two singularities correspond to the two fixed points of the standard circle action on S^2. Taking multiple singularities gives the multi-Eguchi-Hanson spaces in the ALE case and the multi-Taub-NUT spaces in the ALF case. Dihedral ALF spaces can not be constructed by the Gibbons-Hawking ansatz, only their asymptotic geometry can be described by it.

5.5 Foscolo's Construction

The easiest way to construct a family of hyperkähler metrics on the $K3$ surface is to carry out a family of Kummer constructions described in section 5.4. We can consider the 4-torus $T^4 = T^3 \times S^1$ as a trivial circle bundle over the 3-torus. By scaling of the metric on the circle we get a family of 4-tori $T^4_l = T^3 \times S^1_l$, where $l > 0$ denotes the length of the circles. By again considering the action $\mathbb{Z}_2$ induced by multiplication of -1 on $\mathbb{C}^2$, we get a family of orbifolds $T^4_l/\mathbb{Z}_2$. Each orbifold has 16 singularities and they come in pairs modelled on $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$, which can be resolved by gluing in two copies of Eguchi-Hanson spaces to get a dihedral ALF space of type D_2. Gluing them into the orbifold gives a family of hyperkähler metrics collapsing to $T^3/\mathbb{Z}_2$ as $l \to 0$. Foscolo generalizes this idea by considering a non-trivial circle bundle over a punctured 3-torus. The crux of the matter is to find out which ALF spaces one can glue in.

Let τ denote the involution on T^3 and $\{q_1, \ldots, q_8\}$ the eight fixed points. Let $\{p_1, \tau(p_1), \ldots, p_n, \tau(p_n)\}$ be another set of $2n$ $\mathbb{Z}_2$-invariant points. Write $T^* := T^3 - \{q_1, \ldots, q_8, p_1, \tau(p_1), \ldots, p_n, \tau(p_n)\}$ for the punctured 3-torus. Assume we can find a principle U(1)-bundle $\mathcal{P} \to T^*$, such that the involution τ lifts to $\mathcal{P}$, and a Dirac monopole (h, θ) on $\mathcal{P}$ with asymptotics

$$h \sim \frac{2m_j - 4}{2\rho_j} \quad \text{as } \rho_j \to 0, \tag{18}$$

$$h \sim \frac{k_i}{2\rho_i} \quad \text{as } \rho_i \to 0, \tag{19}$$

where $\rho_j = \mathrm{dist}(\cdot, q_j)$ and $\rho_i = \mathrm{dist}(\cdot, p_i)$ respectively, for some non-negative integers m_j, $j = 1, \ldots, 8$, and some positive integers k_i, $i = 1, \ldots, n$. The Gibbons-Hawking ansatz then gives a hyperkähler metric on $\mathcal{P}/\mathbb{Z}_2$. Comparing the asymptotics (18) and (19) with the asymptotics of ALF spaces described in definition 5.1 suggests after adding a constant to h we can resolve the singularities of the orbifold $\mathcal{P}/\mathbb{Z}_2$ by gluing in a dihedral ALF space of type D_{m_j} at the singularity at q_j and a cyclic ALF space of type A_{k_i-1} at the singularity at p_i. Foscolo shows that all of this is possible if and only if the topological balancing condition

$$\sum_{j=1}^{8} m_j + \sum_{i=1}^{n} k_i = 16. \tag{20}$$

is satisfied.

The condition (20) has a remarkable consequence. If $m_j \geqslant 2$ for $j = 1, \ldots, 8$, then this forces equality $m_j = 2$ for all j and $n = 0$. This means that we can only glue in a dihedral ALF space of type D_2 at each of the fixed points and we can't have any further singularities. This is the case of the classical Kummer construction described above on a trivial circle bundle over the 3-torus. If it wasn't for the D_0 ALF space, the Atiyah-Hitchin space, and the D_1 ALF space, the double cover of the Atiyah-Hitchin space, Foscolo's construction wouldn't give new examples. That the Atiyah-Hitchin space and its double cover are exceptional from the other dihedral ALF spaces can be seen by looking at the description of the asymptotic geometry via the Gibbons-Hawking ansatz. For D_0 and D_1 we use the harmonic functions $1 - 4\frac{1}{2\rho}$ and $1 - 2\frac{1}{2\rho}$ respectively. This means that for $\rho \to 0$ the harmonic function goes to $-\infty$. The coefficient of $\frac{1}{2\rho}$ in the expansion of the harmonic function is called **mass**. In contrast to this D_2 has zero mass and D_m for $m > 2$ has positive mass.

By gluing in the corresponding ALF spaces we get a family of smooth 4-manifolds M_ε. ε corresponds to the size of the ALF spaces which are glued in. The framework of definite triples is very useful in this situation. In contrast to the Kummer construction [9] it is not a holomorphic gluing which means that the complex structures on the building blocks don't match together. Instead the definite triples on the building blocks are glued together to get a family of definite triples $\underline{\omega}_\varepsilon$ with a control on the error $\|Q_{\underline{\omega}_\varepsilon} - \mathrm{id}\|_{C^0}$. As mentioned in section 5.2 the deformation is unobstructed and via the implit function theorem Foscolo deforms them into a genuine hyperkähler triple. The balancing condition (20) forces the underlying manifold to be the K3 surface.

5.6 Easiest Examples for the Gibbons-Hawking Ansatz

In this section we compute the metric (9) given by the Gibbons-Hawking ansatz for a Dirac monopole with one singularity. We will see that in the ALE case we recover the euclidean metric on $\mathbb{C}^2$ and in the ALF case we get the Taub-NUT metric.

Let us review the Hopf fibration (see also section 2.2). The three dimensional sphere considered as a submanifold of $\mathbb{C}^2$ is given by $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1|^2 + |z_2|^2 = 1\}$. Therefore each $(z_1, z_2) \in S^3$ can be written as $(z_1, z_2) = (r_1 e^{i\xi_1}, r_2 e^{i\xi_2})$ for some non-negative r_1 and r_2 which satisfy $r_1^2 + r_2^2 = 1$. This means that we can find $\phi \in [0, \pi]$

with $(z_1, z_2) = (\cos\left(\frac{\phi}{2}\right)e^{i\xi_1}, \sin\left(\frac{\phi}{2}\right)e^{i\xi_2})$. The projection $\mathcal{P} : S^3 \to S^2$ can be described as $\mathcal{P}(z_1, z_2) = \varphi_S^{-1}(\frac{z_1}{z_2})$ where $\varphi_S^{-1} : \mathbb{C} \cup \{\infty\} \to S^2$ is the inverse of the stereographic projection at the north pole for the sphere in $S^2 \subset \mathbb{R}^3$ centred at the origin and is explicitly given by

$$\varphi_S^{-1}(z) = \left(\frac{z+\bar{z}}{z\bar{z}+1}, \frac{z-\bar{z}}{i(z\bar{z}+1)}, \frac{z\bar{z}-1}{z\bar{z}+1}\right).$$

On S^2 we will use the spherical coordinates $(\sin(\phi)\cos(\psi), \sin(\phi)\sin(\psi), \cos(\phi))$ with $\phi \in (0, \pi)$ and $\psi \in (0, 2\pi)$. In these coordinates

$$\mathcal{P}\Big(\cos\left(\frac{\phi}{2}\right)e^{i\xi_1}, \sin\left(\frac{\phi}{2}\right)e^{i\xi_2}\Big) = (\sin(\phi)\cos(\psi), \sin(\phi)\sin(\psi), \cos(\phi)) \tag{21}$$

with $\psi = \xi_1 - \xi_2$ [23][p.16, (0.3.5)]. Note that the Lie group $U(1) = S^1$ acts on S^3 by multiplication (on the right) and by the definition of the projection the fibres of $\mathcal{P}$ are exactly the orbits of this action. By describing trivialisations we will show that S^3 is a principal $U(1)$-bundle over S^2. If $U_S := S^2 - \{(0, 0, 1\}$ and $U_N := S^2 - \{(0, 0, -1)\}$ we can describe the bundle structure by

$$\Psi_S : \{z_2 \neq 0\} \subset S^3 \to U_S \times U(1), \tag{22}$$

$$(z_1, z_2) \mapsto \left(\mathcal{P}(z_1, z_2), \frac{z_2}{|z_2|}\right), \tag{23}$$

with inverse

$$\Psi_S^{-1}(x, g) = (z_1, z_2)\Big(g\frac{|z_2|}{z_2}\Big), \tag{24}$$

where (z_1, z_2) is any point in the fibre $\mathcal{P}^{-1}(x)$ and

$$\Psi_N : \{z_1 \neq 0\} \subset S^3 \to U_N \times U(1), \tag{25}$$

$$(z_1, z_2) \mapsto \left(\mathcal{P}(z_1, z_2), \frac{z_1}{|z_1|}\right), \tag{26}$$

with inverse

$$\Psi_N^{-1}(x, g) = (z_1, z_2)\Big(g\frac{|z_1|}{z_1}\Big), \tag{27}$$

where (z_1, z_2) is any point in the fibre $\mathcal{P}^{-1}(x)$. In spherical coordinates (ϕ, ψ) on S^2 the transition function from Ψ_S to Ψ_N is given by $g_{NS}(\phi, \psi) = e^{i\psi}$ [23][p.19, (0.3.9)]. Because the winding number of the transition function is 1 we see that $c_1 = 1$ and the Hopf fibration is the principal $U(1)$-bundle over $S^2 = \mathbb{C}P^1$ associated to the line bundle $\mathcal{O}(1)$.

With these trivialisations, the coordinate $e^{i\theta}$ on $U(1)$ and the spherical coordinates on S^2

we get two coordinate charts which are adapted to the action of $U(1)$:
If $x = (\sin(\phi)\cos(\psi), \sin(\phi)\sin(\psi), \cos(\phi))$ then $(z_1, z_2) = (\cos\left(\frac{\phi}{2}\right)e^{i\psi}, \sin\left(\frac{\phi}{2}\right)) \in \mathcal{P}^{-1}(x)$ by (21) and hence by (24) we get the coordinate map

$$\Phi_S^{-1}(\phi, \psi, \theta) = \Big(\cos\left(\frac{\phi}{2}\right)e^{i(\psi+\theta)}, \sin\left(\frac{\phi}{2}\right)e^{i\theta}\Big)$$

and analogously

$$\Phi_N^{-1}(\phi, \psi, \theta) = \Big(\cos\left(\frac{\phi}{2}\right)e^{i\theta}, \sin\left(\frac{\phi}{2}\right)e^{i(\theta-\psi)}\Big).$$

By radial extension these coordinates on S^3 will give us the coordinates

$$\Phi_S^{-1}(r, \phi, \psi, \theta) = \Big(r\cos\left(\frac{\phi}{2}\right)e^{i(\psi+\theta)}, r\sin\left(\frac{\phi}{2}\right)e^{i\theta}\Big)$$

on $\mathbb{R}^4 - \{0\}$. We will describe the Gibbons-Hawking ansatz and the euclidean metric on $\mathbb{R}^4$ in these coordinates. Because Φ_S^{-1} covers a dense subset of $\mathbb{R}^4$ this is enough to compare the two metrics. We will see that up to a constant they coincide. For this we first compute the coordinate frame with respect to these coordinates:

$$\begin{aligned}
\partial_r &= \Big(\cos\Big(\tfrac{\phi}{2}\Big)\cos(\psi+\theta), \cos\Big(\tfrac{\phi}{2}\Big)\sin(\psi+\theta), \sin\Big(\tfrac{\phi}{2}\Big)\cos(\theta), \sin\Big(\tfrac{\phi}{2}\Big)\sin(\theta)\Big),\\
\partial_\phi &= \Big(-\frac{r}{2}\sin\Big(\tfrac{\phi}{2}\Big)\cos(\psi+\theta), -\frac{r}{2}\sin\Big(\tfrac{\phi}{2}\Big)\sin(\psi+\theta), \frac{r}{2}\cos\Big(\tfrac{\phi}{2}\Big)\cos(\theta), \frac{r}{2}\cos\Big(\tfrac{\phi}{2}\Big)\sin(\theta)\Big),\\
\partial_\psi &= \Big(-r\cos\Big(\tfrac{\phi}{2}\Big)\sin(\psi+\theta), r\cos\Big(\tfrac{\phi}{2}\Big)\cos(\psi+\theta), 0, 0\Big),\\
\partial_\theta &= \Big(-r\cos\Big(\tfrac{\phi}{2}\Big)\sin(\psi+\theta), r\cos\Big(\tfrac{\phi}{2}\Big)\cos(\psi+\theta), -r\sin\Big(\tfrac{\phi}{2}\Big)\sin(\theta), r\sin\Big(\tfrac{\phi}{2}\Big)\cos(\theta)\Big).
\end{aligned}$$

This gives

$$g_{\mathbb{R}^4} = dr^2 + r^2\left(\frac{1}{4}d\phi^2 + \cos^2\left(\frac{\phi}{2}\right)d\psi^2 + d\theta^2 + \cos^2\left(\frac{\phi}{2}\right)(d\psi\otimes d\theta + d\theta\otimes d\psi)\right).$$

For the Gibbons-Hawking ansatz we will use the harmonic function $h = \frac{1}{2r}$ on $\mathbb{R}^3 - \{0\}$. $\omega = \operatorname{Im}(\bar{z}_1 dz_1 + \bar{z}_2 dz_2)$ is a connection on the principle bundle $\mathcal{P} : S^3 \to S^2$ [23][p. 331, 6.6.1]. In our coordinates it is given by

$$\begin{aligned}
&\bar{z}_1 dz_1 + \bar{z}_2 dz_2\\
=&\cos\left(\frac{\phi}{2}\right)e^{-i(\psi+\theta)}d\Big(\cos\left(\frac{\phi}{2}\right)e^{i(\psi+\theta)}\Big) + \sin\left(\frac{\phi}{2}\right)e^{-i\theta}d\Big(\sin\left(\frac{\phi}{2}\right)e^{i\theta}\Big)\\
=&i\Big(\cos^2\left(\frac{\phi}{2}\right)d\psi + d\theta\Big),
\end{aligned}$$

i.e.

$$\omega = \cos^2\left(\frac{\phi}{2}\right)d\psi + d\theta = \frac{1}{2}(1+\cos(\phi))d\psi + d\theta.$$

and radial extension of ω gives a connection on the radial extension of the Hopf fibration $\mathcal{P} : \mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$ with $*dh = d\omega$. Indeed we have $d\omega = -\frac{1}{2}\sin(\phi)d\phi \wedge d\psi$ and $*dh = *\Big(-\frac{1}{2r^2}dr\Big) = -\frac{1}{2}\sin(\phi)d\phi \wedge d\psi$. In the coordinates (r, ϕ, ψ) on $\mathbb{R}^3 - \{0\}$ the euclidean metric is given by

$$g_{\mathbb{R}^3} = dr^2 + r^2(d\phi^2 + \sin^2(\phi)d\psi^2).$$

The Gibbons-Hawking ansatz gives

$$\begin{aligned}
g_{gh} &= h\mathcal{P}^* g_{\mathbb{R}^3} + h^{-1}\omega^2 \\
&= \frac{1}{2r^2}(4r^2dr^2 + r^4(d\phi^2 + \sin^2(\phi)d\psi^2)) + 2r^2\Big(\cos^2\left(\frac{\phi}{2}\right)d\psi + d\theta\Big)^2 \\
&= 2\Big(dr^2 + r^2\Big(\frac{1}{4}d\phi^2 + \frac{1}{4}\sin^2(\phi)d\psi^2\Big)\Big) \\
&+ 2r^2\Big(\cos^4\left(\frac{\phi}{2}\right)d\psi^2 + d\theta^2 + \cos^2\left(\frac{\phi}{2}\right)(d\psi\otimes d\theta + d\theta\otimes d\psi)\Big) \\
&= 2\Big(dr^2 + r^2\Big(\frac{1}{4}d\phi^2 + \Big(\frac{1}{4}\sin^2(\phi) + \cos^4\left(\frac{\phi}{2}\right)\Big)d\psi^2 + d\theta^2 + \cos^2\left(\frac{\phi}{2}\right)(d\psi\otimes d\theta + d\theta\otimes d\psi)\Big)\Big) \\
&= 2\Big(dr^2 + r^2\left(\frac{1}{4}d\phi^2 + \cos^2\left(\frac{\phi}{2}\right)d\psi^2 + d\theta^2 + \cos^2\left(\frac{\phi}{2}\right)(d\psi\otimes d\theta + d\theta\otimes d\psi)\right)\Big) \\
&= 2g_{\mathbb{R}^4}
\end{aligned}$$

as

$$\begin{aligned}
& 2\cos^2\left(\frac{\phi}{2}\right) - 1 = \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) = \cos(\phi) \\
\Leftrightarrow \quad & \cos^2\left(\frac{\phi}{2}\right) = \frac{1}{2}(1 + \cos(\phi))
\end{aligned}$$

and

$$\frac{1}{4}\sin^2(\phi) + \cos^4\left(\frac{\phi}{2}\right) = \frac{1}{4}\sin^2(\phi) + \frac{1}{4}(1 + 2\cos(\phi) + \cos^2(\phi)) = \frac{1}{2}(1 + \cos(\phi)) = \cos^2\left(\frac{\phi}{2}\right).$$

Let's do the same computation in Euler angles

$$(z_1, z_2) = \Big(\cos\left(\frac{\phi}{2}\right)e^{\frac{i}{2}(\theta+\psi)}, \sin\left(\frac{\phi}{2}\right)e^{\frac{i}{2}(\theta-\psi)}\Big).$$

Note that the projection in Euler angles is the same as in the previous coordinates by (21).

The coordinate tangent frame is given by:

$$\begin{aligned}
\partial_r &= \Big(\cos\big(\tfrac{\phi}{2}\big)\cos\big(\tfrac{\theta+\psi}{2}\big), \cos\big(\tfrac{\phi}{2}\big)\sin\big(\tfrac{\theta+\psi}{2}\big), \sin\big(\tfrac{\phi}{2}\big)\cos\big(\tfrac{\theta-\psi}{2}\big), \sin\big(\tfrac{\phi}{2}\big)\sin\big(\tfrac{\theta-\psi}{2}\big)\Big),\\
\partial_\phi &= \Big(-\frac{r}{2}\sin\big(\tfrac{\phi}{2}\big)\cos\big(\tfrac{\theta+\psi}{2}\big), -\frac{r}{2}\sin\big(\tfrac{\phi}{2}\big)\sin\big(\tfrac{\theta+\psi}{2}\big), \frac{r}{2}\cos\big(\tfrac{\phi}{2}\big)\cos\big(\tfrac{\theta-\psi}{2}\big), \frac{r}{2}\cos\big(\tfrac{\phi}{2}\big)\sin\big(\tfrac{\theta-\psi}{2}\big)\Big),\\
\partial_\psi &= \Big(-\frac{r}{2}\cos\big(\tfrac{\phi}{2}\big)\sin\big(\tfrac{\theta+\psi}{2}\big), \frac{r}{2}\cos\big(\tfrac{\phi}{2}\big)\cos\big(\tfrac{\theta+\psi}{2}\big), \frac{r}{2}\sin\big(\tfrac{\phi}{2}\big)\sin\big(\tfrac{\theta-\psi}{2}\big), -\frac{r}{2}\sin\big(\tfrac{\phi}{2}\big)\cos\big(\tfrac{\theta-\psi}{2}\big)\Big),\\
\partial_\theta &= \Big(-\frac{r}{2}\cos\big(\tfrac{\phi}{2}\big)\sin\big(\tfrac{\theta+\psi}{2}\big), \frac{r}{2}\cos\big(\tfrac{\phi}{2}\big)\cos\big(\tfrac{\theta+\psi}{2}\big), -\frac{r}{2}\sin\big(\tfrac{\phi}{2}\big)\sin\big(\tfrac{\theta-\psi}{2}\big), \frac{r}{2}\sin\big(\tfrac{\phi}{2}\big)\cos\big(\tfrac{\theta-\psi}{2}\big)\Big).
\end{aligned}$$

This gives

$$g_{\mathbb{R}^4} = dr^2 + \frac{1}{4}r^2(d\phi^2 + d\psi^2 + d\theta^2 + \cos(\phi)(d\psi\otimes d\theta + d\theta\otimes d\psi))$$

Below (30) we show that the Dirac monopole in these coordinates is given by $(\frac{1}{2r}, \frac{1}{2}(d\theta + \cos(\phi)d\psi))$.
The Gibbons-Hawking ansatz gives

$$\begin{aligned}
g_{gh} &= h\mathcal{P}^* g_{\mathbb{R}^3} + h^{-1}\omega^2\\
&= \frac{1}{2r^2}(4r^2dr^2 + r^4(d\phi^2 + \sin^2(\phi)d\psi^2)) + 2r^2\frac{1}{4}(\cos(\phi)d\psi + d\theta)^2\\
&= 2(dr^2 + \frac{r^2}{4}(d\phi^2 + \sin^2(\phi)d\psi^2) + 2\frac{r^2}{4}(\cos^2(\phi)d\psi^2 + d\theta^2 + \cos(\phi)(d\psi\otimes d\theta + d\theta\otimes d\psi))\\
&= 2(dr^2 + \frac{1}{4}r^2(d\phi^2 + d\psi^2 + d\theta^2 + \cos(\phi)(d\psi\otimes d\theta + d\theta\otimes d\psi)))\\
&= 2g_{\mathbb{R}^4}.
\end{aligned}$$

Let's turn our attention to the ALF case, i.e. we will compute the metric given by the Dirac monopole $(1 + \frac{1}{2r}, \omega)$. It will turn out that this is the Taub-NUT space. This metric is given in [20][p. 297, (1)] with respect to a left-invariant orthonormal frame on $S^3 \cong \mathrm{SU}(2)$. So we will first compute this frame in Euler angles.

We identify S^3 and $SU(2)$ via

$$(z_1, z_2) \mapsto \begin{bmatrix} z_1 & -\bar{z_2} \\ z_2 & \bar{z_1} \end{bmatrix}.$$

A basis of the Lie algebra $\mathfrak{su}(2)$ is given by the Pauli matrices

$$X_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \qquad X_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

The Maurer-Cartan form of $SU(2)$ is given by

$$\Theta = g^{-1}dg = \begin{bmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{bmatrix} \begin{bmatrix} dz_{11} & dz_{12} \\ dz_{21} & dz_{22} \end{bmatrix} = \begin{bmatrix} z_{22}dz_{11} - z_{12}dz_{21} & z_{22}dz_{12} - z_{12}dz_{22} \\ -z_{21}dz_{11} + z_{11}dz_{21} & -z_{21}dz_{12} + z_{11}dz_{22} \end{bmatrix}$$
$$= \begin{bmatrix} \bar{z}_1 dz_1 + \bar{z}_2 dz_2 & -\bar{z}_1 d\bar{z}_2 + \bar{z}_2 d\bar{z}_1 \\ -z_2 dz_1 + z_1 dz_2 & z_2 d\bar{z}_2 + z_1 d\bar{z}_1 \end{bmatrix}.$$

The Maurer-Cartan form gives a left-invariant coframe $\{\sigma_1, \sigma_2, \sigma_3\}$ which is orthonormal with respect to the standard metric [23][p.291, Lemma 5.9.1] via

$$\Theta = \sigma_1 X_1 + \sigma_2 X_2 + \sigma_3 X_3.$$

We can write each $X \in \mathfrak{su}(2)$ as

$$\begin{bmatrix} i\alpha & \beta + i\gamma \\ -\beta + i\gamma & -i\alpha \end{bmatrix}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$ [25][Chapter 1, Example 11] . Therefore,

$$\begin{aligned} \sigma_1 &= \mathrm{Re}(-\bar{z}_1 d\bar{z}_2 + \bar{z}_2 d\bar{z}_1) \\ \sigma_2 &= \mathrm{Im}(-\bar{z}_1 d\bar{z}_2 + \bar{z}_2 d\bar{z}_1) \\ \sigma_3 &= \mathrm{Im}(\bar{z}_1 dz_1 + \bar{z}_2 dz_2). \end{aligned}$$

Let's compute them in Euler angles. We have

$$d\bar{z}_2 = d\Big(\sin\left(\frac{\phi}{2}\right) e^{-\frac{i}{2}(\theta-\psi)}\Big) = -\frac{i}{2}\sin\left(\frac{\phi}{2}\right) e^{-\frac{i}{2}(\theta-\psi)}(d\theta - d\psi) + \frac{1}{2}\cos\left(\frac{\phi}{2}\right) e^{-\frac{i}{2}(\theta-\psi)} d\phi,$$

$$-\bar{z}_1 d\bar{z}_2 = \frac{i}{2}\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) e^{-i\theta}(d\theta - d\psi) - \frac{1}{2}\cos^2\left(\frac{\phi}{2}\right) e^{-i\theta} d\phi,$$

$$d\bar{z}_1 = d\Big(\cos\left(\frac{\phi}{2}\right) e^{-\frac{i}{2}(\theta+\psi)}\Big) = -\frac{i}{2}\cos\left(\frac{\phi}{2}\right) e^{-\frac{i}{2}(\theta+\psi)}(d\theta + d\psi) - \frac{1}{2}\sin\left(\frac{\phi}{2}\right) e^{-\frac{i}{2}(\theta+\psi)} d\phi,$$

$$\bar{z}_2 d\bar{z}_1 = -\bar{z}_1 d\bar{z}_2 = -\frac{i}{2}\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) e^{-i\theta}(d\theta + d\psi) - \frac{1}{2}\sin^2\left(\frac{\phi}{2}\right) e^{-i\theta} d\phi,$$

$$\begin{aligned} -\bar{z}_1 d\bar{z}_2 + \bar{z}_2 d\bar{z}_1 &= -\frac{i}{2}\sin(\phi) e^{-i\theta} d\psi - \frac{1}{2} e^{-i\theta} d\phi, \\ &= -\frac{i}{2}\sin(\phi)\cos(\theta) d\psi - \frac{1}{2}\sin(\phi)\sin(\theta) d\psi - \frac{1}{2}\cos(\theta) d\phi + \frac{i}{2}\sin(\theta) d\phi. \end{aligned}$$

Hence

$$\sigma_1 = \mathrm{Re}(-\bar{z_1} d\bar{z_2} + \bar{z_2} d\bar{z_1}) = -\frac{1}{2}(\sin(\phi)\sin(\theta)d\psi + \cos(\theta)d\phi),$$
$$\sigma_2 = \mathrm{Im}(-\bar{z_1} d\bar{z_2} + \bar{z_2} d\bar{z_1}) = \frac{1}{2}(\sin(\theta)d\phi - \sin(\phi)\cos(\theta)d\psi).$$

Furthermore,

$$dz_1 = d\Big(\cos\Big(\frac{\phi}{2}\Big)e^{\frac{i}{2}(\theta+\psi)}\Big) = \frac{i}{2}\cos\Big(\frac{\phi}{2}\Big)e^{\frac{i}{2}(\theta+\psi)}(d\theta + d\psi) - \frac{1}{2}\sin\Big(\frac{\phi}{2}\Big)e^{\frac{i}{2}(\theta+\psi)}d\phi,$$

$$\bar{z_1} dz_1 = \frac{i}{2}\cos^2\Big(\frac{\phi}{2}\Big)(d\theta + d\psi) - \frac{1}{2}\sin\Big(\frac{\phi}{2}\Big)\cos\Big(\frac{\phi}{2}\Big)d\phi,$$

$$dz_2 = d\Big(\sin\Big(\frac{\phi}{2}\Big)e^{\frac{i}{2}(\theta-\psi)}\Big) = \frac{i}{2}\sin\Big(\frac{\phi}{2}\Big)e^{\frac{i}{2}(\theta-\psi)}(d\theta - d\psi) + \frac{1}{2}\cos\Big(\frac{\phi}{2}\Big)e^{\frac{i}{2}(\theta-\psi)}d\phi,$$

$$\bar{z_2} dz_2 = \frac{i}{2}\sin^2\Big(\frac{\phi}{2}\Big)(d\theta - d\psi) + \frac{1}{2}\sin\Big(\frac{\phi}{2}\Big)\cos\Big(\frac{\phi}{2}\Big)d\phi.$$

Hence,

$$\sigma_3 = \mathrm{Im}(\bar{z_1} dz_1 + \bar{z_2} dz_2) = \frac{1}{2}(d\theta + \cos(\phi)d\psi).$$

To sum up, on SU(2) we have the left-invariant orthonormal frame

$$\sigma_1 = -\frac{1}{2}(\sin(\phi)\sin(\theta)d\psi + \cos(\theta)d\phi), \tag{28}$$
$$\sigma_2 = \frac{1}{2}(\sin(\theta)d\phi - \sin(\phi)\cos(\theta)d\psi), \tag{29}$$
$$\sigma_3 = \frac{1}{2}(d\theta + \cos(\phi)d\psi). \tag{30}$$

Let's now construct the Taub-NUT metric with the Gibbons-Hawking ansatz. This time we will use the projection $\mathcal{P} : \mathbb{R}^4 \to \mathbb{R}^3$ which sends the 3-sphere of radius r to the 2-sphere of radius r via the Hopf-fibration. We get

$$\begin{aligned} g_{\mathrm{NUT}} &= V\mathcal{P}^* g_{\mathbb{R}^3} + V^{-1}\omega_3^2 \\ &= \Big(1 + \frac{1}{2r}\Big)(dr^2 + r^2((d\phi^2 + \sin^2(\phi)d\psi^2)) + \Big(1 + \frac{1}{2r}\Big)^{-1}\sigma_3^2 \\ &= \Big(1 + \frac{1}{2r}\Big)dr^2 + 4r\Big(r + \frac{1}{2}\Big)(\sigma_1^2 + \sigma_2^2) + \frac{2r}{1+2r}\sigma_3^2 \\ &= \frac{\rho+1}{4\rho}d\rho^2 + \rho(\rho+1)(\sigma_1^2 + \sigma_2^2) + \frac{\rho}{\rho+1}\sigma_3^2 \end{aligned}$$

with $\rho = 2r$. This coincides with the formula given in [20][p. 297, (1)].

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