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Spin(7) Instantons on Asymptotically Conical Calabi-Yau Fourfolds

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Statement of Originality

I declare that the work contained in this thesis is, to the best of my knowledge, original and my own work, unless indicated otherwise.

I declare that the work contained in this thesis has not been submitted towards any other degree or qualification at the University of Oxford or at any other university or institution.

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Abstract

The subject of this thesis is the role of Hermitian Yang-Mills (HYM) connections in Spin(7) instanton moduli spaces over asymptotically conical (AC) Calabi-Yau fourfolds. The starting point is Lewis' result [40] on the equivalence of the equations over a closed base: if a bundle over a closed CY fourfold admits HYM solutions then all Spin(7) instantons are HYM. The fundamental problem we are interested in is whether this result persists in the AC setting.

A first-order approach is to pass to the (Sasaki-Einstein) asymptotic link, which is not Ricci flat and could thus be homogeneous. The role of the Spin(7) instantons is then assumed by the G_2 instantons and that of the HYM connections by the contact instantons. It is easier to explore the relationship between these systems instead.

We provide an alternative construction of the standard octonionic instanton of Fubini and Nicolai [27], in line with the modern methods of equivariant gauge theory [43], [55]. Prior to the results of this thesis, this was the only known example of a non HYM Spin(7) instanton on a Calabi-Yau fourfold. Examining the limiting connection over the asymptotic link, we exclude the existence of HYM connections in its moduli space. We conclude that the octonionic instanton does not help in the resolution of the problem we are interested in.

We extend Lewis's theorem to the AC setting, conditioning on decay rates.

We construct the moduli space of SO(5)-invariant Spin(7) instantons with structure group SO(3) on the Stenzel space [56]. These new instantons sit exactly at the slow-rate cut-off point of our extension of Lewis's theorem. They provide a negative answer to the question we set out to answer: the moduli space is one-dimensional and contains precisely two HYM connections. One of these is the epicentre of a removable singularity/ bubbling phenomenon and the development of a corresponding Fueter section [9], [79], [81]. We compute this explicitly and verify (after suitable modifications) an infinite-energy version of Tian's energy conservation identity [9], [43], [70]. This phenomenon hints at a possible relationship between the AC Spin(7) instanton and HYM systems.

0 Context, Motivation and Overview of the Results

Instantons are special Yang-Mills connections characterized by first order PDE systems stronger than the full second order Yang-Mills equation. The prototype for such a system is the ASD equation over a closed 4-manifold (Atiyah [3], Donaldson-Kronheimer [15]). In this setting, the ASD instantons are the absolute minimizers of the free Yang-Mills action. Moduli spaces of ASD solutions are instrumental in the study of the smooth topology of simply connected, closed, oriented 4-manifolds (Donaldson [14], Freed-Uhlenbeck [25], Mariño [45]). This approach, pioneered by Donaldson, revealed the stark difference between the smooth and continuous categories.

Analogues of the ASD system are available in higher dimensions, but now depend on the choice of a closed $(n-4)$-form (here n denotes the dimension of the base space). To proceed, one would like to have a natural choice of such an object. This is often the case over base manifolds with special geometry. Notably, in the setting of exceptional holonomy, Spin(7) and G_2 structures are encoded by their Cayley calibration $\Phi \in \Lambda^4 T^\star X^8$ and associative calibration $\phi \in \Lambda^3 T^\star \Sigma^7$ respectively. We thus obtain corresponding instanton equations: the *Spin(7) instanton equations* and the *G_2 instanton equations*. The hope is to use the corresponding moduli spaces to extract geometric and topological invariants in analogy to the 4-dimensional Donaldson theory. This is broadly known as the Donaldson-Thomas [17] (DT)/ Donaldson-Segal (DS) [16] program. In some ways, these systems parallel the ASD equations: over closed manifolds they determine the absolute minima of the Yang-Mills energy. However, their analysis is generally much harder to understand due to very complicated non-compactness phenomena and the presence of heavy obstructions complicating their infinitesimal deformation theory. Consequently, the analytic definition of DT/ DS invariants still remains conjectural.

The inclusion of SU(4) in Spin(7) allows one to endow a Calabi-Yau (CY) 4-fold with a natural Spin(7) structure. On such a manifold (in fact over any Kähler manifold), one may define yet another type of instanton: the *Hermitian Yang-Mills* (HYM) connections. A natural question is to ask whether these connections are related to the Spin(7) instantons associated to the induced Spin(7) structure. One immediately observes that HYM is a stronger condition. Our aim is to study this relationship in more detail.

A natural first step is to ask whether the HYM equations are genuinely stronger, that is if *pure* (not HYM) Spin(7) instantons can exist on a complete CY fourfold. We are thus interested in constructing such an object. In the compact case, it is known that as long as an HYM connection exists, the two types of instantons coincide (Lewis [40]). Consequently, if one hopes to display a compact counterexample to equivalence, there must not be any HYM connections at all. Furthermore, we have a general existence theorem for HYM connections over stable holomorphic bundles (Uhlenbeck, Yau [73]). This restricts the choices of bundles one could look at. Finally, closed, locally irreducible manifolds of exceptional holonomy admit no continuous symmetries (Joyce, [32]). This precludes the use of symmetry techniques (dimensional reduction), forcing us to tackle the prohibitively complicated analysis head-on. We are thus motivated to look for a counterexample over a noncompact base. Since Lewis's argument is essentially an energy estimate, it does not directly carry over to the noncompact setting.

We shall narrow down our scope to asymptotically conical (AC) CY-4 geometries. These manifolds possess a single infinite end along which the geometry approaches that of a cone. In this context, it is natural to augment the gauge theoretic equations with boundary conditions reflecting the asymptotic geometry of the base. This leads to the notion of AC instantons: solutions approaching a dilation invariant limit A_∞ along the noncompact end. If one hopes to find AC Spin(7) instantons/HYM connections approaching A_∞, the latter ought to obey strong constraints. In particular, it has to satisfy gauge-theoretic equations over the 7-dimensional asymptotic link. The link of a Spin(7) cone is *nearly parallel* G_2, whereas the link of a CY-4 cone possesses a richer *Sasaki-Einstein* structure. Both of these geometries admit natural instanton equations: the G_2 *instanton equations* and the *contact instanton equations* respectively. The former is weaker than the latter so that the boundary conditions imposed by the HYM system are more demanding. This brings us to the basic object of our study. We are interested in studying moduli spaces of AC Spin(7) instantons over AC CY fourfolds, where the limiting connection is contact. Such moduli spaces could a priori contain both pure Spin(7) instantons and HYM connections. We are interested in the relationship between the two.

We note that an example of a pure AC Spin(7) instanton already exists. It is known as the *standard octonionic instanton* and it lives on the trivial Spin(7)-bundle over flat space $\mathbb{R}^8$. The problem is that its limiting connection is G_2 but not contact. The question still remains whether pure AC Spin(7) instantons can approach a limit compatible with the HYM system. Better yet, in light of Lewis' theorem, one would like to know if they can show up in moduli spaces hosting HYM connections. Furthermore, one could ask if the existence of pure Spin(7) instantons is a pathology related to the special nature of flat space. In particular, one would like to know if such objects can live over honest complete AC CY fourfolds with full holonomy SU(4).

We begin by presenting background material on the geometry of CY fourfolds, Sasaki-Einstein 7-manifolds, Spin(7) manifolds and weak G_2 manifolds. We introduce the natural instanton equations on these spaces and review Lewis' Theorem [40]. We derive their linearizations (centered at an instanton A) and specify their relationship with the corresponding spin Dirac operators (twisted by A). We finally derive the relevant Weitzenböck formulae. To the author's knowledge, the formulae for the Spin(7) case are new.

In section 2, we provide a detailed introduction to gauge theory over spaces with large isometry groups. The techniques outlined here are crucial for what is to follow: the spaces we will be working with are of cohomogeneity one. We are thus interested in the interaction between this symmetry and gauge theory. We introduce homogeneous bundles and invariant connections and prove Wang's theorem (Wang [82]) on their classification.

In section 3 we generalise Lewis' energy estimate to the AC setting.

In section 4 we give a new construction of the standard octonionic instanton. This was initially introduced by Fubini and Nicolai in the Physics article [27]. We present an alternative derivation aligned with the methods of equivariant gauge theory in the spirit of Lotay-Oliveira [43], Clarke-Oliveira [9], Oliveira [55], [54]. We hope that this elucidates the geometric context. In contrast to [27], rather than a single solution, we find a 1-parameter family degenerating at both ends. Our method guarantees that there are no other solutions enjoying the same symmetries. We thus get a complete handle on the invariant locus in the moduli space.

Section 5 is based on the author's published work (Papoulias [56]). It is concerned with the analysis of the instanton equations on the Stenzel space (Stenzel [67]). We use the cohomogeneity one SO(5) action to reduce these to tractable ODEs and proceed to study the SO(5)-invariant solutions. In the abelian case we establish local equivalence and prove a global nonexistence result. We then study the nonabelian equations corresponding to the structure group SO(3). We give an explicit one parameter family of Spin(7) instantons containing a unique HYM connection. All instantons in the family are AC. This negatively resolves the question regarding the equivalence of the two equations. Interestingly, the decay rates of these examples sit right at the slow rate cut-off point of our generalization of Lewis' estimate, demonstrating that it is sharp.

We construct the full moduli space of invariant Spin(7) instantons in this context. This involves a second family, living on another bundle. It also carries a unique HYM connection. The bundles hosting the two families are isomorphic away from a Cayley submanifold. The first family is parameterized by a half-open half-closed interval. The second family is compact. The HYM connections play a role in the compactification of the moduli space, exhibiting a novel removable singularity/ bubbling phenomenon. As we vary the parameter of the first family toward the open end of the interval, energy concentrates around a Cayley submanifold. This provides an explicit example of Tian's compactness theory (Tian [70]). Furthermore, the family bubbles off an ASD instanton in the normal directions. We explicitly compute the relevant Fueter section (Walpuski [79], [81], Oliveira-Clarke [9], Lotay-Oliveira [43]). As expected, cutting off the Cayley, we are able to obtain a limit in the C^∞_{loc} topology. Interestingly, the limit is precisely the unique HYM connection of the other family. The moduli space is then compactified by attaching it to the open end and gluing the two intervals transversally.

1 Rudiments of CY-4 Geometry and the Associated Gauge Theories

In this section, we collect background material on the geometry of CY fourfolds and the gauge theories available on them. Along the way, we introduce various related geometries: the 7-dimensional Sasaki-Einstein and nearly parallel (weak) G_2 geometries, as well as the exceptional holonomy G_2 and Spin(7) geometries. These are all intimately tied to the study of AC CY-4 spaces. They will play a central role in the sequel. We emphasize aspects related to the corresponding instanton equations: the HYM equations, the contact instanton equations, the Spin(7) instanton equations and the G_2 instanton equations.

We do not always give complete proofs and instead refer to the books (Joyce [32], Salamon [62]), the expository paper (Salamon-Walpuski [61]), as well as the PhD theses of C. Lewis (Lewis [40]) and Thomas Walpuski ([77]) among others. The foundational article (Friedrich [63]) serves as a comprehensive introduction to weak G_2 manifolds. Our aim is to provide enough material to specify and motivate the object of our study.

We begin by introducing certain linear algebraic structures special to real dimensions 7 and 8. These structures provide the pointwise flat model for the spaces we are interested in. We then pass to the global level to introduce CY fourfolds and the other geometries of interest. We introduce the corresponding instanton equations and show that they are stronger than the full Yang-Mills system. We derive the topological energy estimates exhibiting their solutions as Yang-Mills minimizers (when the base is closed). We observe that the HYM equations are stronger than the Spin(7) instanton equations and present Lewis' Theorem [40]. Finally, we linearize the Spin(7) and G_2 instanton equations and relate them to the natural (twisted) Dirac operators available on the respective spaces. We derive the relevant Weitzenböck formulae. These resemble the formulae derived by Bourguignon-Lawson [6] in 4 dimensions. We conclude the section with a precise description of the moduli spaces we are interested in.

1.1 Geometric Aspects

1.1.1 Octonionic Linear Algebra

1.1.1.1 Real Normed Division Algebras

A *normed division algebra* over $\mathbb{R}$ consists of a finite dimensional real Euclidean vector space $\left(W, \langle\cdot,\cdot\rangle\right)$ equipped with a compatible multiplication operation admitting a unital element $1 \in W$.

Concretely, W is endowed with a bilinear map:

$$\otimes^2 W \to W$$

$$u \otimes v \mapsto uv$$

satisfying

$$|uv| = |u||v|$$

and

$$1w = w1 = w \text{ for all } w \in W.$$

We define the *real* and *imaginary* parts of W as:

$$\mathfrak{Re}(W) \stackrel{\text{def}}{=} \text{Span}_{\mathbb{R}}(1), \tag{1.1}$$

$$\mathfrak{Im}(W) \stackrel{\text{def}}{=} \mathfrak{Re}(W)^{\perp}. \tag{1.2}$$

We thus obtain a natural decomposition:

$$W = \mathfrak{Re}\left(W\right) \oplus \mathfrak{Im}\left(W\right).$$

This, in turn, yields a *conjugation* operation:

$$\bar{u} \stackrel{\text{def}}{=} \begin{cases} \text{u} & \text{if } u \in \mathfrak{Re}\left(W\right), \\ \text{-u} & \text{if } u \in \mathfrak{Im}\left(W\right). \end{cases}$$

The latter can be used to write down a multiplicative inverse for every non-zero element:

$$u^{-1} \stackrel{\text{def}}{=} \frac{\overline{u}}{|u|^2}. \tag{1.3}$$

An easy computation using the axioms verifies that indeed:

$$uu^{-1} = 1,$$

as required. This confirms that W is a division algebra and thus justifies our terminology.

From here on, we will always identify the real part of a normed division algebra with the real numbers $\mathbb{R}$. Correspondingly, we will slightly abuse notation by denoting scalar multiples of the unital element $\lambda \cdot 1$ simply by λ.

It is possible to obtain a full classification of real normed division algebras. To this end, suppose that we are given a Euclidean space $\big(W, \langle \cdot, \cdot \rangle\big)$ and we are interested in upgrading it to a normed division algebra. The first step is to select a unit vector 1 to serve as the unital element. This determines the real and imaginary parts according to (1.1) and (1.2). Denote the imaginary part by:

$$V \stackrel{\text{def}}{=} \operatorname{Span}_{\mathbb{R}}(1)^{\perp}.$$

To obtain the desired binary operation on W, one has to make a further choice. It turns out that this amounts to choosing either a *triple cross product* on W or a *cross product* on V. These two structures are interchangeable and equivalent to the desired product operation on W.

A *triple cross product* on $\big(W, \langle \cdot, \cdot \rangle\big)$ is an alternating trilinear map:

$$\otimes^3 W \to W$$

$$u \otimes v \otimes w \mapsto u \times v \times w$$

satisfying:

$$\langle u \times v \times w, u \rangle = \langle u \times v \times w, v \rangle = \langle u \times v \times w, w \rangle = 0$$

$$|u \times v \times w| = |u \wedge v \wedge w|$$

A *cross product* on $\big(V, \langle\cdot,\cdot\rangle\big)$ is an alternating bilinear map:

$$\bigotimes^2 V \to V$$

$$(u,v) \mapsto u \times v$$

satisfying:

$$\langle u \times v, u\rangle = \langle u \times v, v\rangle = 0$$

$$|u \times v|^2 = |u|^2|v|^2 - \langle u, v\rangle^2$$

Suppose that an algebra structure (product) is given. The associated triple cross product on W is defined by:

$$u \times v \times w \stackrel{\text{def}}{=} \frac{1}{2}\big((u\bar{v})w - (w\bar{v})u\big)\,. \tag{1.4}$$

Similarly, the associated cross product on V is defined by:

$$u \times v \stackrel{\text{def}}{=} uv + \langle u, v\rangle. \tag{1.5}$$

These two structures are related through the equation:

$$u \times v \stackrel{\text{def}}{=} u \times 1 \times v. \tag{1.6}$$

For the converse, note that starting from a cross product on V, formula (1.5) determines how to multiply elements of V. One subsequently extends this to W by identifying the real part with $\mathbb{R}$ and having it act on W by scalar multiplication.

Correspondingly, given a triple cross product (and our choice of unit-norm identity element), one defines the normed algebra structure by:

$$uv \stackrel{\text{def}}{=} u \times 1 \times v + \langle u, 1\rangle v + \langle v, 1\rangle u - \langle u, v\rangle \cdot 1. \tag{1.7}$$

The upshot is the following. One may obtain a complete classification of cross products using techniques of elementary linear algebra. Due to the above observation, this also gives a classification of real normed division algebras (Salamon-Walpuski [61]). Essentially, the only possibilities are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

We recall the final two examples for the reader's convenience. The quaternions $\mathbb{H}$ are defined by setting:

$$\mathbb{H} = \mathrm{Span}_{\mathbb{R}}(1, i, j, k), \tag{1.8}$$

where the generators are orthonormal and anti-commuting with square -1, $ij = k$ and the product is associative. The multiplication table for $\mathbb{H}$ takes the following form:

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Note that the resulting product is not commutative.

The octonions $\mathbb{O}$ are defined by setting:

$$\mathbb{O} = \mathrm{Span}_{\mathbb{R}}(1, i, j, k, e, ei, ej, ek), \tag{1.9}$$

where the generators are orthonormal and anti-commuting with square -1, $ij = k$ and associativity can be used to determine the product of any generator with all subsequent ones in the ordered list. The multiplication table for $\mathbb{O}$ takes the following form:

$\cdot$	1	i	j	k	e	ei	ej	ek
1	1	i	j	k	e	ei	ej	ek
i	i	-1	k	$-j$	$-ei$	e	ek	$-ej$
j	j	$-k$	-1	i	$-ej$	$-ek$	e	ei
k	k	j	$-i$	-1	$-ek$	ej	$-ei$	e
e	e	ei	ej	ek	-1	$-i$	$-j$	$-k$
ei	ei	$-e$	ek	$-ej$	i	-1	k	$-j$
ej	ej	$-ek$	$-e$	ei	j	$-k$	-1	i
ek	ek	ej	$-ei$	$-e$	k	j	$-i$	-1

Note that the resulting product is neither commutative nor associative.

The triple cross product and cross product structures associated to the four real normed division algebras can be computed using (1.4) and (1.5). The cross product associated to $\mathbb{R}$ lives on a 0-dimensional space and thus vanishes. Using (1.5), we find that the cross product associated to $\mathbb{C}$ vanishes as well. The cross product on $\mathfrak{Im}(\mathbb{H})$ is the usual cross product on $\mathbb{R}^3$, familiar from vector calculus. The cross product on $\mathfrak{Im}(\mathbb{O})$ is known as the standard cross product on $\mathbb{R}^7$. It will be further explored in section 1.1.2, though not in its standard form but rather a form suitable to the context of our study.

1.1.1.2 Associative Calibrations and Cayley Calibrations

In view of our objective to globalize the above linear algebraic picture, we are interested in recasting it in a more manageable way. In this section, we will see that we can capture the data of a real normed division algebra using alternating multilinear forms.

Let $\big(W, \langle\cdot,\cdot\rangle\big)$ and $\big(V, \langle\cdot,\cdot\rangle\big)$ be as above i.e. W is a Euclidean vector space with a preferred unit vector 1 and $V = 1^\perp \subset W$.

An alternating 3-form

$$\phi \in \Lambda^3 V^\star$$

is an *associative calibration* for $(V, \langle\cdot,\cdot\rangle)$ provided that it is *nondegenerate* and *compatible with the inner product.* The first condition is that for all linearly independent $u, v \in V$

there exists some $x \in V$ such that:

$$\phi(x, u, v) \neq 0.$$

The second condition is that the map defined by

$$(u, v) \mapsto u \times v$$

where

$$\langle u \times v, x \rangle = \phi(u, v, x) \tag{1.10}$$

is a cross product on V.

It is immediate from (1.10) that associative calibrations and cross products are in one-to-one correspondence.

The Hodge dual (over V) of an associative calibration is known as a *coassociative calibration*. It is often denoted as:

$$\psi = \star\phi. \tag{1.11}$$

An alternating 4-form

$$\Phi \in \Lambda^4 W^\star$$

is called a *Cayley calibration* for $(W, \langle \cdot, \cdot \rangle)$ if it is *nondegenerate* and *compatible with the inner product*. The first condition means that for all linearly independent $u, v, w \in W$ there exists some $x \in W$ such that $\Phi(u, v, w, x) \neq 0$. The second one is that the map defined by:

$$(u, v, w) \mapsto u \times v \times w$$

$$\langle x, u \times v \times w \rangle = \Phi(x, u, v, w) \tag{1.12}$$

is a triple cross product on W.

It is immediate from (1.12) that triple cross products and Cayley calibrations are in one-to-one correspondence.

Cayley calibrations are automatically self-dual for the Hodge-star operator determined by the metric and the orientation:

$$\star\,\Phi = \Phi.$$

The correspondence between cross products on V and triple cross products on W induces a correspondence between associative calibrations ϕ over V and Cayley calibrations Φ over W. Explicitly, this takes the form:

$$\Phi = 1^{\flat} \wedge \phi + \star\phi. \tag{1.13}$$

A normed division algebra structure on W induces a triple cross product on W and a cross product on V. One may subsequently pass to the corresponding calibrations. Dimension considerations demonstrate that these vanish for $\mathbb{R}$ and $\mathbb{C}$ and that they are trivial (scalar multiples of the volume form) for $\mathbb{H}$.

From here on, we shall focus on the final normed division algebra: the octonions $\mathbb{O}$. Our interest stems from the fact that $\mathbb{O} \cong \mathbb{R}^8$ and $\mathfrak{Im}\,(\mathbb{O}) \cong \mathbb{R}^7$ provide the flat models for Spin(7) and G_2 manifolds respectively. These geometries are intimately tied to AC CY-4 geometry. Consequently, their respective flat models will play a central role in the sequel.

1.1.1.3 The groups G_2 and Spin(7)

The groups G_2 and Spin(7) arise as the stabilizers of $\mathfrak{Im}\,(\mathbb{O})$ and $\mathbb{O}$ respectively. In this sense, once we pass to the global level, they assume the role that the special orthogonal groups play in standard Riemannian geometry.

The octonionic product induces a triple cross product on $\mathbb{O}$ and a cross product on $\mathfrak{Im}\,(\mathbb{O})$. Let ϕ and Φ denote the corresponding associative and Cayley calibrations.

We define the group G_2 to be the isotropy subgroup of ϕ in $\mathrm{GL}\left(\mathfrak{Im}\left(\mathbb{O}\right)\right)$:

$$G_2 \overset{\text{def}}{=} \left\{ g \in \mathrm{GL}\left(\mathfrak{Im}\left(\mathbb{O}\right)\right) \text{ s.t. } g^\star\phi = \phi \right\}.$$

All linear automorphisms that preserve ϕ also preserve the Euclidean metric and orientation on $\mathfrak{Im}\left(\mathbb{O}\right)$. It is then the case that:

$$G_2 \subset \mathrm{SO}(\mathfrak{Im}\left(\mathbb{O}\right)) \cong \mathrm{SO}(7).$$

In fact, we have that:

$$G_2 = \left\{ g \in \mathrm{SO}(\mathfrak{Im}\left(\mathbb{O}\right)) \text{ s.t. } gu \times gv = g(u \times v) \right\}.$$

The group G_2 is a semisimple, connected, simply connected 14-dimensional Lie group [61].

We define the group Spin(7) to be the isotropy subgroup of Φ in $\mathrm{GL}(\mathbb{O})$ i.e.:

$$\mathrm{Spin}(7) \overset{\text{def}}{=} \left\{ g \in \mathrm{GL}(\mathbb{O}) \text{ s.t. } g^\star\Phi = \Phi \right\}.$$

All linear automorphisms that preserve Φ also preserve the Euclidean metric and orientation. It is therefore the case that:

$$\mathrm{Spin}(7) \subset \mathrm{SO}(\mathbb{O}) \cong \mathrm{SO}(8).$$

In fact, we have:

$$\mathrm{Spin}(7) = \left\{ g \in \mathrm{SO}(\mathbb{O}) \text{ s.t. } gu \times gv \times gw = g(u \times v \times w) \right\}.$$

The group Spin(7) is a semisimple, connected, simply connected 21-dimensional Lie group.

We now confirm that the above definition of Spin(7) agrees with the familiar one as the universal cover of SO(7). The group SO(8) acts on $\mathfrak{so}(8)$ adjointly. Since SO(8) is simple, this representation is irreducible. Restrict it to $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. The associated branching problem is easily solved. The subspace $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$ is clearly stable under the restricted action. The corresponding representation agrees with the adjoint action of Spin(7). It is irreducible since Spin(7) is simple. The homogeneous space $\frac{\mathrm{SO}(8)}{\mathrm{Spin}(7)}$ is reductive. Let $\mathfrak{m}$ be a reductive complement. Its dimension is easily computed:

$$\begin{aligned}\dim(\mathfrak{m}) &= \dim\big(\mathfrak{so}(8)\big) - \dim\big(\mathfrak{spin}(7)\big) \\ &= 28 - 21 \\ &= 7. \end{aligned} \tag{1.14}$$

Standard semisimple theory demonstrates that the lowest-dimensional non-trivial irrep of Spin(7) has dimension 7. Since Spin(7) doesn't act trivially on $\mathfrak{m}$, the latter completes the decomposition. Identifying it with $\mathbb{R}^7$, furnishes a map Spin(7) $\to$ SO(7). This is a non-trivial double covering, exhibiting Spin(7) as the universal cover of SO(7). We thus recover the usual definition from the theory of spin groups (Hamilton [29], Roe [60]).

Given that we are primarily interested in the global picture, we want our terminology to align with the theory of G-structures (Joyce [32]). For this reason, we shall often refer to Cayley calibrations as *Spin(7)-structures* and associative calibrations as G_2*-structures*. Indeed, fixing smoothly varying Cayley or associative calibrations over the tangent spaces of a base manifold X^8 or Σ^7 naturally induces a reduction of structure ([32], [37]) to Spin(7) or G_2 respectively. This is achieved by considering the frames that restore the relevant calibration to its standard form.

1.1.2 The Flat Model for CY-4 Geometry

1.1.2.1 The Flat SU(4), Spin(7) and G_2 Structures

The local model for CY-4 geometry is provided by the space $\mathbb{C}^4$ endowed with its natural SU(4) structure (J, ω, Ω). Here J denotes the standard complex structure given by scalar multiplication through i, ω denotes the standard symplectic (Kähler) form:

$$\omega = \frac{i}{2}\sum_{j=1}^{4} dz^j \wedge d\bar{z}^j$$

and Ω denotes the standard holomorphic volume form:

$$\Omega = dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4.$$

The tensors J and ω naturally induce a Riemannian metric given by:

$$g(u,v) = \omega(u, Jv).$$

Furthermore, the complex structure J induces a natural orientation. The stabilizer of the collection of these structures is the group:

$$\mathrm{SU}(4) \subset \mathrm{SO}(8).$$

The above data naturally induce a Cayley calibration on $\mathbb{C}^4$:

$$\Phi \stackrel{\text{def}}{=} \frac{\omega^2}{2} + \mathfrak{Re}\left(\Omega\right). \tag{1.15}$$

This is known as the *standard Spin(7) structure* on $\mathbb{R}^8 \cong \mathbb{C}^4$ and it constitutes the local model for Spin(7) geometry. It is clear that:

$$\mathrm{SU}(4) \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8), \tag{1.16}$$

where the intermediate group is the stabilizer of Φ. It follows that the standard SU(4) structure on $\mathbb{C}^4$ can be used to obtain a model for $\mathbb{O}$. To this end, note that $\mathbb{C}^4$ is naturally Euclidean: it possesses a natural Riemannian metric and orientation. Seeking to upgrade it to a normed division algebra, we note the availability of a natural choice of unit vector:

$$1 = (1, 0, 0, 0).$$

We use this as the unital element. The Cayley calibration Φ then determines the product as discussed in section (1.1.1.2).

In the sequel we will maintain the earlier notation W for the full 8-dimensional space and

V for the orthogonal complement of the unital element. In particular:

$$W = \mathbb{C}^4,$$
$$V = 1^\perp \subset \mathbb{C}^4.$$

Using (1.13) we obtain a natural associative calibration ϕ on $\mathbb{R}^7 \cong V$. This is known as the *standard G_2 structure* on $\mathbb{R}^7$ and it constitutes the local model for G_2 geometry.

Earlier, we remarked that the Spin(7) and G_2 geometries are intimately related with the CY-4 geometry. The above considerations make this precise—at least at the local/ flat level.

1.1.2.2 Coordinate Representations

We wish to make the previous discussion more explicit. In particular, we wish to introduce natural coordinates and obtain associated expressions for the various linear algebraic structures involved in the flat model.

To begin with, we identify:

$$\mathbb{C}^4 \cong \mathbb{R}^8$$

by writing a typical element $u \in \mathbb{R}^8$ as:

$$u = \left(x^1, y^1, x^2, y^2, x^3, y^3, x^4, y^4\right)^\intercal;$$

and associating it to $v \in \mathbb{C}^4$ where:

$$v = \left(z^1, z^2, z^3, z^4\right)^\intercal, \quad z^j = x^j + iy^j.$$

We then introduce the following notation for the frame associated to the above coordinate system on $\mathbb{R}^8$:

$$\begin{aligned} e_1 &= \partial_{x^1}, \quad e_2 = \partial_{y^1} \\ e_3 &= \partial_{x^2}, \quad e_4 = \partial_{y^2} \\ e_5 &= \partial_{x^3}, \quad e_6 = \partial_{y^3} \\ e_7 &= \partial_{x^4}, \quad e_8 = \partial_{y^4}. \end{aligned}$$

Using this notation, the unital element is given by:

$$1 = e_1.$$

The real and imaginary parts of a fixed vector are then given by orthogonally projecting to the span of e_1 and to its orthogonal complement respectively.

We denote the dual coframe by $\left(\epsilon^i\right)_{i=1\ldots 8}$ so that:

$$\epsilon^i\left(e_j\right) = \delta^i_j.$$

Transporting the CY-4 structure of $\mathbb{C}^4$ through the identification, we find that the complex structure becomes:

$$\begin{aligned} J =& \epsilon^1 \otimes e_2 + \epsilon^3 \otimes e_4 + \epsilon^5 \otimes e_6 + \epsilon^7 \otimes e_8 \\ &- \epsilon^2 \otimes e_1 - \epsilon^4 \otimes e_3 - \epsilon^6 \otimes e_5 - \epsilon^8 \otimes e_7. \end{aligned}$$

Similarly, the Kähler form ω and the holomorphic volume form Ω are given by:

$$\omega = \epsilon^{12} + \epsilon^{34} + \epsilon^{56} + \epsilon^{78}$$

and

$$\Omega = \mathfrak{Re}\left(\Omega\right) + i\mathfrak{Im}\left(\Omega\right),$$

where

$$\mathfrak{Re}\left(\Omega\right) = \epsilon^{1357} - \epsilon^{1368} - \epsilon^{1458} - \epsilon^{1467} - \epsilon^{2358} - \epsilon^{2367} - \epsilon^{2457} + \epsilon^{2468},$$

$$\mathfrak{Im}\,(\Omega) = \epsilon^{1358} + \epsilon^{1367} + \epsilon^{1457} - \epsilon^{1468} + \epsilon^{2357} - \epsilon^{2368} - \epsilon^{2458} - \epsilon^{2467}.$$

The induced Cayley calibration (1.15) takes the form:

$$\begin{aligned}\Phi =&\epsilon^{1357} - \epsilon^{1368} - \epsilon^{1458} - \epsilon^{1467} - \epsilon^{2358} - \epsilon^{2367} - \epsilon^{2457} + \epsilon^{2468}\\ &+\epsilon^{1234} + \epsilon^{1256} + \epsilon^{1278} + \epsilon^{3456} + \epsilon^{3478} + \epsilon^{5678}.\end{aligned}$$

Recalling (1.13), we immediately spot that:

$$\begin{aligned}\phi &= \epsilon^{357} - \epsilon^{368} - \epsilon^{458} - \epsilon^{467} + \epsilon^{234} + \epsilon^{256} + \epsilon^{278},\\ \psi &= -\epsilon^{2358} - \epsilon^{2367} - \epsilon^{2457} + \epsilon^{2468} + \epsilon^{3456} + \epsilon^{3478} + \epsilon^{5678}.\end{aligned}$$

It may be easily verified that Ω and Φ are self dual and:

$$\star\,\phi = \psi.$$

In this last equation, $\star$ denotes the Hodge star operator on the complement of the unital element.

Furthermore, we compute:

$$\begin{aligned}\frac{\omega^2}{2!} &= \epsilon^{1234} + \epsilon^{1256} + \epsilon^{1278} + \epsilon^{3456} + \epsilon^{3478} + \epsilon^{5678},\\ \frac{\omega^3}{3!} &= \epsilon^{123456} + \epsilon^{123478} + \epsilon^{125678} + \epsilon^{345678},\\ \frac{\omega^4}{4!} &= \epsilon^{12345678}.\end{aligned}$$

Using these, it is easy to verify the well known identities:

$$\star\,\omega = \frac{\omega^3}{3!},$$

and

$$dV_g = \frac{\omega^4}{4!}.$$

Using the formula:

$$\phi(u, v, w) = g(u \times v, w),$$

and the explicit form of ϕ in the coordinate frame, we can obtain a multiplication table for the cross product on the complement of the unital element. This takes the following form:

$\times$	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	0	e_4	$-e_3$	e_6	$-e_5$	e_8	$-e_7$
e_3	$-e_4$	0	e_2	e_7	$-e_8$	$-e_5$	e_6
e_4	e_3	$-e_2$	0	$-e_8$	$-e_7$	e_6	e_5
e_5	$-e_6$	$-e_7$	e_8	0	e_2	e_3	$-e_4$
e_6	e_5	e_8	e_7	$-e_2$	0	$-e_4$	$-e_3$
e_7	$-e_8$	e_5	$-e_6$	$-e_3$	e_4	0	e_2
e_8	e_7	$-e_6$	$-e_5$	e_4	e_3	$-e_2$	0

The octonionic product on $\mathbb{R}^8$ can now be obtained using (1.7). This yields the formula:

$$uv = \mathfrak{Re}(u)\mathfrak{Re}(v) - g\left(\mathfrak{Im}\,(u)\,, \mathfrak{Im}\,(v)\right) + \mathfrak{Re}(u)\mathfrak{Im}\,(v) + \mathfrak{Re}(v)\mathfrak{Im}\,(u) + \mathfrak{Im}\,(u) \times \mathfrak{Im}\,(v)\,.$$

The resulting multiplication table is given by:

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e_1$	e_4	$-e_3$	e_6	$-e_5$	e_8	$-e_7$
e_3	e_3	$-e_4$	$-e_1$	e_2	e_7	$-e_8$	$-e_5$	e_6
e_4	e_4	e_3	$-e_2$	$-e_1$	$-e_8$	$-e_7$	e_6	e_5
e_5	e_5	$-e_6$	$-e_7$	e_8	$-e_1$	e_2	e_3	$-e_4$
e_6	e_6	e_5	e_8	e_7	$-e_2$	$-e_1$	$-e_4$	$-e_3$
e_7	e_7	$-e_8$	e_5	$-e_6$	$-e_3$	e_4	$-e_1$	e_2
e_8	e_8	e_7	$-e_6$	$-e_5$	e_4	e_3	$-e_2$	$-e_1$

Note that the isomorphism with the standard construction of $\mathbb{O}$ is not given by directly identifying the ordered frame $(e_i)_{i=1\ldots8}$ with the generators $(1, i, j, k, e, ei, ej, ek)$. Instead, it is given by associating:

$$1 \leftrightarrow e_1, \quad i \leftrightarrow e_3, \quad j \leftrightarrow e_5, \quad k \leftrightarrow e_7$$
$$e \leftrightarrow e_2, \quad ei \leftrightarrow e_4, \quad ej \leftrightarrow e_6, \quad ek \leftrightarrow e_8.$$

1.1.2.3 Decomposition of the Space of 2-Forms

Recall that our ultimate goal is to study gauge theory on CY fourfolds. The link between CY-4 geometry and gauge theory arises from a natural splitting of the space of 2-forms. This feature permits us to correspondingly decompose gauge fields (curvature tensors) and is hence responsible for the availability of the instanton equations of interest.

The copy of Spin(7) in GL(8) arising as the stabilizer of Φ acts on $\Lambda^2 W^\star$ through the second antisymmetric power of the standard vector representation. Decomposing this into irreducibles yields the splitting:

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{21}. \tag{1.17}$$

Here we have omitted denoting the background space and the subscripts signify the dimensions of the summands. In particular:

$$\dim_{\mathbb{R}} \left(\Lambda^2_7\right) = 7,$$
$$\dim_{\mathbb{R}} \left(\Lambda^2_{21}\right) = 21.$$

We are interested in computing these summands explicitly. To this end, we introduce the Spin(7)-invariant endomorphism:

$$T_\Phi : \ \Lambda^2 \to \Lambda^2$$
$$\alpha \mapsto \star \left(\Phi \wedge \alpha\right).$$

Its eigenvalues are given by $\lambda_1 = 3$ and $\lambda_2 = -1$. Since T_Φ is Spin(7)-invariant, so are the corresponding eigenspaces E_3 and E_{-1}. In fact, we have [61]:

$$E_3 = \Lambda^2_7, \quad E_{-1} = \Lambda^2_{21}.$$

Denoting the orthogonal projectors associated to the splitting by π^2_7 and π^2_{21}, we decompose the identity as:

$$1 = \pi_7^2 + \pi_{21}^2. \tag{1.18}$$

Noting that T_Φ acts as a scalar on each piece and making use of (1.18), we compute::

$$\begin{aligned} T_\Phi &= 3\pi_7^2 - \pi_{21}^2 \\ &= 3\pi_7^2 - \left(1 - \pi_7^2\right) \\ &= 4\pi_7^2 - 1. \end{aligned}$$

Rearranging, we obtain the following explicit formula for the orthogonal projector to Λ_7^2:

$$\pi_7^2\left(\cdot\right) = \frac{1}{4}\left(\star\left(\Phi \wedge \cdot\right) + \cdot\right). \tag{1.19}$$

This can now be applied to the natural basis of Λ^2 to yield a spanning set for Λ_7^2. In hindsight, we introduce the—pairwise orthogonal—2-forms:

$$\begin{aligned} v_1 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{12} + \epsilon^{34} + \epsilon^{56} + \epsilon^{78}\right), \\ v_2 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{13} - \epsilon^{24} + \epsilon^{57} - \epsilon^{68}\right), \\ v_3 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{14} + \epsilon^{23} - \epsilon^{58} - \epsilon^{67}\right), \\ v_4 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{15} - \epsilon^{26} - \epsilon^{37} + \epsilon^{48}\right), \\ v_5 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{16} + \epsilon^{25} + \epsilon^{38} + \epsilon^{47}\right), \\ v_6 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{17} - \epsilon^{28} + \epsilon^{35} - \epsilon^{46}\right), \\ v_7 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{18} + \epsilon^{27} - \epsilon^{36} - \epsilon^{45}\right). \end{aligned}$$

A long but tedious calculation demonstrates that:

$$\Lambda_7^2 = \mathrm{Span}_{\mathbb{R}}\left(v_1, ..., v_7\right).$$

and furthermore:

$$\pi_7^2\epsilon^{12} = v_1, \quad \pi_7^2\epsilon^{13} = v_2, \quad \pi_7^2\epsilon^{14} = v_3, \quad \pi_7^2\epsilon^{15} = v_4, \quad \pi_7^2\epsilon^{16} = v_5, \quad \pi_7^2\epsilon^{17} = v_6, \quad \pi_7^2\epsilon^{18} = v_7,$$
$$\pi_7^2\epsilon^{23} = v_3, \quad \pi_7^2\epsilon^{24} = -v_2, \quad \pi_7^2\epsilon^{25} = v_5, \quad \pi_7^2\epsilon^{26} = -v_4, \quad \pi_7^2\epsilon^{27} = v_7, \quad \pi_7^2\epsilon^{28} = -v_6$$
$$\pi_7^2\epsilon^{34} = v_1, \quad \pi_7^2\epsilon^{35} = v_6, \quad \pi_7^2\epsilon^{36} = -v_7, \quad \pi_7^2\epsilon^{37} = -v_4, \quad \pi_7^2\epsilon^{38} = v_5,$$
$$\pi_7^2\epsilon^{45} = -v_7, \quad \pi_7^2\epsilon^{46} = -v_6, \quad \pi_7^2\epsilon^{47} = v_5, \quad \pi_7^2\epsilon^{48} = v_4,$$
$$\pi_7^2\epsilon^{56} = v_1, \quad \pi_7^2\epsilon^{57} = v_2, \quad \pi_7^2\epsilon^{58} = -v_3,$$
$$\pi_7^2\epsilon^{67} = -v_3, \quad \pi_7^2\epsilon^{68} = -v_2,$$
$$\pi_7^2\epsilon^{78} = v_1.$$

Note that the 2-form v_1 is in fact equal to the Kähler form ω. This establishes that ω is an eigenvector of T_Φ with eigenvalue $\lambda_1 = 3$.

Our explicit description of π_7^2 easily translates to an explicit description of Λ_{21}^2 and π_{21}^2 through (1.18). We do not list the relevant results as we will have no use for them in the sequel.

Since the original CY-4 structure (J, ω, Ω) is finer than the induced Spin(7) structure Φ, we expect it to yield a finer decomposition of Λ^2. Considering equation (1.16), we find that the pieces Λ_7^2 and Λ_{21}^2 are SU(4)-invariant. However, they are not irreducible ([40] p. 24). They each split further as follows:

$$\Lambda_7^2 = \mathrm{Span}_{\mathbb{R}}(\omega) \oplus \mathcal{C}$$
$$\Lambda_{21}^2 = \Lambda_0^{1,1} \oplus \mathcal{B}$$

Here, $\Lambda_0^{1,1}$ denotes the orthogonal complement of the Kähler form ω inside the space of real (1,1)-forms for the bi-degree decomposition induced by J. We therefore have:

$$\dim_{\mathbb{R}}\left(\Lambda_0^{1,1}\right) = 15.$$

The inclusion:

$$\Lambda_0^{1,1} \subset \Lambda_{21}^2$$

can be verified by direct computation. The space $\mathcal{B}$ is defined by taking the orthogonal complement of $\Lambda_0^{1,1}$ in Λ_{21}^2. Similarly, the space $\mathcal{C}$ is defined by taking the orthogonal complement of the Kähler form ω inside Λ_7^2. Evidently:

$$\dim_{\mathbb{R}}(\mathcal{B}) = 6,$$
$$\dim_{\mathbb{R}}(\mathcal{C}) = 6.$$

Let $\pi_{\mathcal{B}}$ and $\pi_{\mathcal{C}}$ denote the associated orthogonal projectors:

$$\pi_{\mathcal{B}} : \Lambda^2 \twoheadrightarrow \mathcal{B},$$
$$\pi_{\mathcal{C}} : \Lambda^2 \twoheadrightarrow \mathcal{C}.$$

By definition, we have:

$$\mathcal{C} = \mathrm{Span}_{\mathbb{R}}(v_2, .., v_7)$$

and

$$\pi_{\mathcal{C}} v_1 = 0, \quad \pi_{\mathcal{C}} v_i = v_i \text{ for } i = 2, ..., 7.$$

Combining these equations with the relation:

$$\pi_C = \pi_C \circ \pi_7^2, \tag{1.20}$$

we employ our knowledge of π_7^2 to find:

$$\pi_{\mathcal{C}}\epsilon^{12} = 0, \quad \pi_{\mathcal{C}}\epsilon^{13} = v_2, \quad \pi_{\mathcal{C}}\epsilon^{14} = v_3, \quad \pi_{\mathcal{C}}\epsilon^{15} = v_4, \quad \pi_{\mathcal{C}}\epsilon^{16} = v_5, \quad \pi_{\mathcal{C}}\epsilon^{17} = v_6, \quad \pi_{\mathcal{C}}\epsilon^{18} = v_7,$$
$$\pi_{\mathcal{C}}\epsilon^{23} = v_3, \quad \pi_{\mathcal{C}}\epsilon^{24} = -v_2, \quad \pi_{\mathcal{C}}\epsilon^{25} = v_5, \quad \pi_{\mathcal{C}}\epsilon^{26} = -v_4, \quad \pi_{\mathcal{C}}\epsilon^{27} = v_7, \quad \pi_{\mathcal{C}}\epsilon^{28} = -v_6$$
$$\pi_{\mathcal{C}}\epsilon^{34} = 0, \quad \pi_{\mathcal{C}}\epsilon^{35} = v_6, \quad \pi_{\mathcal{C}}\epsilon^{36} = -v_7, \quad \pi_{\mathcal{C}}\epsilon^{37} = -v_4, \quad \pi_{\mathcal{C}}\epsilon^{38} = v_5,$$
$$\pi_{\mathcal{C}}\epsilon^{45} = -v_7, \quad \pi_{\mathcal{C}}\epsilon^{46} = -v_6, \quad \pi_{\mathcal{C}}\epsilon^{47} = v_5, \quad \pi_{\mathcal{C}}\epsilon^{48} = v_4,$$
$$\pi_{\mathcal{C}}\epsilon^{56} = 0, \quad \pi_{\mathcal{C}}\epsilon^{57} = v_2, \quad \pi_{\mathcal{C}}\epsilon^{58} = -v_3,$$
$$\pi_{\mathcal{C}}\epsilon^{67} = -v_3, \quad \pi_{\mathcal{C}}\epsilon^{68} = -v_2,$$
$$\pi_{\mathcal{C}}\epsilon^{78} = 0.$$

We are now interested in an explicit description of $\mathcal{B}$ and $\pi_{\mathcal{B}}$. To this end, we work as we did for π^2_7. We recall that the splitting in question is a feature of the full SU(4) structure, rather than just the induced Spin(7) structure. This motivates us to replace Φ by $\mathfrak{Re}(\Omega)$ in the definition of T_Φ. We thus introduce the following endomorphism on Λ^2:

$$\begin{aligned} T_\Omega : \Lambda^2 &\to \Lambda^2 \\ \alpha &\mapsto \star\left(\mathfrak{Re}(\Omega) \wedge \alpha\right) \end{aligned}$$

Evidently, T_Ω is SU(4)-invariant and this property is inherited by its eigenspaces. These are given by:

$$\begin{aligned} E_0 &= \mathrm{Span}_{\mathbb{R}}(\omega) \oplus \Lambda^{1,1}_0, \\ E_2 &= \mathcal{C}, \\ E_{-2} &= \mathcal{B}. \end{aligned}$$

Here the subscripts denote the associated eigenvalues. We therefore find that:

$$T_\Omega = 2\pi_{\mathcal{C}} - 2\pi_{\mathcal{B}},$$

and consequently:

$$\pi_{\mathcal{B}} = \pi_{\mathcal{C}} - \frac{1}{2}T_\Omega. \tag{1.21}$$

A long but tedious calculation yields the values of T_Ω on the standard basis. Combining the results with our knowledge of $\pi_{\mathcal{C}}$ and equation (1.21) yields an explicit description of $\pi_{\mathcal{B}}$. In hindsight, we introduce the—pairwise orthogonal—2-forms:

$$\begin{aligned} w_1 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{13} - \epsilon^{24} - \epsilon^{57} + \epsilon^{68}\right), \\ w_2 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{14} + \epsilon^{23} + \epsilon^{58} + \epsilon^{67}\right), \\ w_3 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{15} - \epsilon^{26} + \epsilon^{37} - \epsilon^{48}\right), \\ w_4 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{16} + \epsilon^{25} - \epsilon^{38} - \epsilon^{47}\right), \\ w_5 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{17} - \epsilon^{28} - \epsilon^{35} + \epsilon^{46}\right), \\ w_6 &\overset{\text{def}}{=} \frac{1}{4}\left(\epsilon^{18} + \epsilon^{27} + \epsilon^{36} + \epsilon^{45}\right). \end{aligned}$$

We then have that:

$$\mathcal{B} = \mathrm{Span}_{\mathbb{R}}\left(w_1, ..., w_6\right),$$

and furthermore:

$$\pi_{\mathcal{B}}\epsilon^{12} = 0, \quad \pi_{\mathcal{B}}\epsilon^{13} = w_1, \quad \pi_{\mathcal{B}}\epsilon^{14} = w_2, \quad \pi_{\mathcal{B}}\epsilon^{15} = w_3, \quad \pi_{\mathcal{B}}\epsilon^{16} = w_4, \quad \pi_{\mathcal{B}}\epsilon^{17} = w_5, \quad \pi_{\mathcal{B}}\epsilon^{18} = w_6,$$

$$\pi_{\mathcal{B}}\epsilon^{23} = w_2, \quad \pi_{\mathcal{B}}\epsilon^{24} = -w_1, \quad \pi_{\mathcal{B}}\epsilon^{25} = w_4, \quad \pi_{\mathcal{B}}\epsilon^{26} = -w_3, \quad \pi_{\mathcal{B}}\epsilon^{27} = w_6, \quad \pi_{\mathcal{B}}\epsilon^{28} = -w_5$$

$$\pi_{\mathcal{B}}\epsilon^{34} = 0, \quad \pi_{\mathcal{B}}\epsilon^{35} = w_5, \quad \pi_{\mathcal{B}}\epsilon^{36} = -w_6, \quad \pi_{\mathcal{B}}\epsilon^{37} = -w_3, \quad \pi_{\mathcal{B}}\epsilon^{38} = w_4,$$

$$\pi_{\mathcal{B}}\epsilon^{45} = -w_6, \quad \pi_{\mathcal{B}}\epsilon^{46} = -w_5, \quad \pi_{\mathcal{B}}\epsilon^{47} = w_4, \quad \pi_{\mathcal{B}}\epsilon^{48} = w_3,$$

$$\pi_{\mathcal{B}}\epsilon^{56} = 0, \quad \pi_{\mathcal{B}}\epsilon^{57} = w_1, \quad \pi_{\mathcal{B}}\epsilon^{58} = -w_2,$$

$$\pi_{\mathcal{B}}\epsilon^{67} = -w_2, \quad \pi_{\mathcal{B}}\epsilon^{68} = -w_1,$$

$$\pi_{\mathcal{B}}\epsilon^{78} = 0.$$

We wish to remark that formula (1.21) simplifies when we restrict both sides to Λ^2_{21}. In particular, if $\alpha \in \Lambda^2$ is known to satisfy:

$$\pi^2_7\alpha = 0,$$

then:

$$\pi_{\mathcal{B}}\alpha = -\frac{1}{2}T_\Omega\left(\alpha\right). \tag{1.22}$$

and therefore:

$$\pi_{\mathcal{B}} = 0 \iff T_\Omega\left(\alpha\right) = 0.$$

Finally, we note that the G_2-structure ϕ on V yields an analogous decomposition:

$$\Lambda^2 V^\star = \Lambda^2_7 \oplus \Lambda^2_{14.}$$

This can be derived by proceeding along the lines of the preceding computations. For details, we refer the reader to [61].

1.1.2.4 Representations of the Clifford Algebra

Exceptional holonomy manifolds are naturally spin. At the linear level, this is captured by explicit characterizations of the spinor modules in terms of geometric data. We provide a brief overview of the theory of Clifford algebras and then recast it in terms of the octonionic linear algebra of the preceding sections.

In accordance with [61], we will opt for the "−" convention regarding the Clifford relations. In particular, given a quadratic form $\mathcal{Q}$ on a (real or complex) vector space A, we let $\mathrm{Cl}(A, \mathcal{Q})$ be the most general algebra (over $\mathbb{R}$ or $\mathbb{C}$) satisfying:

$$v^2 = -\mathcal{Q}(v), \text{ for all } v \in A.$$

Concretely, we set:

$$\mathrm{Cl}(A, \mathcal{Q}) \stackrel{\text{def}}{=} \frac{\bigoplus_{k=0}^{\infty} \bigotimes^k A}{\langle v \otimes v + \mathcal{Q}(v) \mid v \in A \rangle} \tag{1.23}$$

and check that the resulting algebra satisfies the relevant universal property.

The map:

$$\begin{aligned} \epsilon : A &\to A \\ \alpha &\mapsto -\alpha \end{aligned}$$

preserves $\mathcal{Q}$ and thus extends to an involution ϵ of the Clifford algebra. Its ± 1-eigenspaces E_1, E_{-1} furnish a natural $\mathbb{Z}_2$-grading:

$$\begin{aligned} \mathrm{Cl}(A, \mathcal{Q}) &= \mathrm{Cl}^+(A, \mathcal{Q}) \oplus \mathrm{Cl}^-(A, \mathcal{Q}) \\ &\stackrel{\text{def}}{=} E_1 \oplus E_{-1}. \end{aligned}$$

The Pin group

$$\mathrm{Pin}(A, \mathcal{Q}) \subset \mathrm{Cl}(A, \mathcal{Q})^{\times}$$

is generated by the unit sphere in A. Its positive elements form the corresponding Spin group:

$$\mathrm{Spin}(A, \mathcal{Q}) \stackrel{\text{def}}{=} \mathrm{Pin}(A, \mathcal{Q}) \cap \mathrm{Cl}^+(A, \mathcal{Q}).$$

We will typically be interested in Clifford algebras of positive-definite forms over real spaces.

Consider such a pair $(A, \mathcal{Q})$ where A has real dimension n. When $\mathcal{Q}$ is understood, we will suppress it and write:

$$\mathrm{Cl}(A) \stackrel{\mathrm{def}}{=} \mathrm{Cl}(A, \mathcal{Q}). \tag{1.24}$$

Furthermore, we define:

$$\mathrm{Cl}(n) \stackrel{\mathrm{def}}{=} \mathrm{Cl}\left(\mathbb{R}^n, g_{\mathrm{Euclidean}}\right). \tag{1.25}$$

This is identified with $\mathrm{Cl}(A, \mathcal{Q})$ upon selecting an orthonormal basis. Finally, note that we can recover the complex Clifford algebras from the positive-definite real ones. In particular:

$$\mathrm{Cl}(A \otimes \mathbb{C}, \mathcal{Q}_{\mathbb{C}}) \cong \mathrm{Cl}(A, \mathcal{Q}) \otimes \mathbb{C}, \tag{1.26}$$

where $Q_{\mathbb{C}}$ denotes the complex bi-linear extension of $\mathcal{Q}$.

A natural first step towards the classification of a given class of associative semisimple algebras is to understand their representations. The representation theory of Clifford algebras is significantly simpler over the complex numbers. Consequently, one begins by working over $\mathbb{C}\cdot$ Real representations are subsequently approached by considering the existence of equivariant real structures.

All quadratic forms on a complex vector space are isomorphic. Consequently, the complex Clifford algebras are classified by the dimension of the underlying space. The resulting classification is 2-periodic.

When $n = 2k$, $\mathrm{Cl}(2k) \otimes \mathbb{C}$ has a unique complex 2^k-dimensional irrep $\slashed{S}_{2k}$. Appealing to the structure theory of finite-dimensional complex semisimple associative algebras, we conclude that:

$$\mathrm{Cl}(2k) \otimes \mathbb{C} \cong \mathrm{End}_{\mathbb{C}}\left(\slashed{S}_{2k}\right).$$

Restricting the action to the corresponding Spin group:

$$\mathrm{Spin}(2k) \subset \mathrm{Cl}(2k) \subset \mathrm{Cl}(2k) \otimes \mathbb{C}$$

results in an irreducible splitting:

$$\slashed{S}_{2k} = \slashed{S}_{2k}^{+} \oplus \slashed{S}_{2k}^{-}. \tag{1.27}$$

The summands have equal (complex) dimension 2^{k-1}. The module $\slashed{S}_{2k}$ is known as the *spin* representation. The summands $\slashed{S}_{2k}^{+}$, $\slashed{S}_{2k}^{-}$ are known as the positive and negative *half-spin representations*. The action of $\mathbb{C}^{2k}$ interchanges them. They are not representations of $\mathrm{Cl}(2k) \otimes \mathbb{C}$.

When $n = 2k - 1$, the grading automorphism ϵ is outer. Every irrep V has a pair V^{ϵ} obtained by precomposition with ϵ. The complex Clifford algebra $\mathrm{Cl}(2k-1) \otimes \mathbb{C}$ has precisely two irreducible 2^{k-1}-dimensional complex representations: $\slashed{S}_{2k-1}$ and $\slashed{S}_{2k-1}^{\epsilon}$. By construction, their restrictions to $\mathrm{Spin}(2k-1)$ are isomorphic. The resulting module is known as the *spin representation*. There are no half-spin representations in odd dimensions. By structure theory:

$$\mathrm{Cl}(2k-1) \otimes \mathbb{C} \cong \mathrm{End}_{\mathbb{C}}\left(\slashed{S}_{2k-1}\right) \oplus \mathrm{End}_{\mathbb{C}}\left(\slashed{S}_{2k-1}^{\epsilon}\right). \tag{1.28}$$

This is best understood when related to the even-dimensional picture. Let $e_1, \ldots, e_{2k}$ denote the standard basis of $\mathbb{C}^{2k}$. The space $\mathbb{C}^{2k-1}$ is embedded as the span of $e_1, \ldots, e_{2k-1}$. Introduce the map:

$$\begin{aligned} \mathbb{C}^{2k-1} &\hookrightarrow \mathrm{Cl}(2k) \otimes \mathbb{C} \\ v &\mapsto v e_{2k}. \end{aligned} \tag{1.29}$$

Appealing to the universal property, we extend it to an embedding:

$$\mathrm{Cl}(2k-1) \otimes \mathbb{C} \hookrightarrow \mathrm{Cl}(2k) \otimes \mathbb{C}. \tag{1.30}$$

Restricting the co-domain, we obtain an identification:

$$\mathrm{Cl}(2k-1) \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{Cl}^{+}(2k) \otimes \mathbb{C}.$$

Letting Cl$(2k-1)$ act on $\not{S}_{2k}^{+}$ and $\not{S}_{2k}^{-}$ recovers $\not{S}_{2k-1}$ and $\not{S}_{2k-1}^{\epsilon}$. We thus have:

$$\mathrm{Cl}(2k-1)\otimes\mathbb{C}\cong\mathrm{End}_{\mathbb{C}}\left(\not{S}_{2k}^{+}\right)\oplus\mathrm{End}_{\mathbb{C}}\left(\not{S}_{2k}^{-}\right)\subset\mathrm{End}_{\mathbb{C}}\left(\not{S}_{2k}\right), \tag{1.31}$$

realizing Cl$(2k-1)\otimes\mathbb{C}$ as the subalgebra of $\mathrm{End}_{\mathbb{C}}\left(\not{S}_{2k}\right)$ consisting of those endomorphisms that preserve the splitting (1.27).

In the real case, quadratic forms are characterized by their signature (p,q), $p,q\in\mathbb{Z}_{\geq 0}$. The corresponding Clifford algebras are 8-periodic in $p-q$. Our interest lies in positive definite forms on spaces of dimension 7 and 8 (i.e. $p=7,\ q=0$ and $p=8,\ q=0$). Luckily, these cases are particularly simple. The spinor modules of the complex Clifford algebras admit equivariant real structures, yielding real representations of Cl(7) and Cl(8). In turn, we have:

$$\mathrm{Cl}(7)\cong\mathrm{End}_{\mathbb{R}}\left(\mathfrak{Re}(\not{S}_{2k}^{+})\right)\oplus\mathrm{End}_{\mathbb{R}}\left(\mathfrak{Re}(\not{S}_{2k}^{-})\right) \tag{1.32}$$

and

$$\mathrm{Cl}(8)\cong\mathrm{End}_{\mathbb{R}}\left(\mathfrak{Re}(\not{S}_{2k})\right). \tag{1.33}$$

In the sequel, we will only work in the real setting. As a result, we choose to suppress the operation of taking real parts and use the symbol $\not{S}$ for real—rather than complex—spinor modules.

The remainder of this section is concerned with recovering the spinor modules of $\mathrm{Cl}\left(\mathbb{O}\right)$ and $\mathrm{Cl}\left(\mathfrak{Im}\left(\mathbb{O}\right)\right)$ in terms of linear algebraic structures related to the octonionic product. In particular, we wish to understand (1.32) and (1.33) in terms of ϕ and Φ respectively.

Let V and W be the 7 and 8 dimensional vector spaces considered in the previous sections.

The spinor modules of W can be constructed by setting:

$$\not{S}_W\overset{\mathrm{def}}{=}\not{S}_W^{+}\oplus\not{S}_W^{-}. \tag{1.34}$$

where:

$$\$_W^+ \overset{\text{def}}{=} \Lambda^0 W^\star \oplus \Lambda^2_7 W^\star,$$
$$\$_W^- \overset{\text{def}}{=} \Lambda^1 W^\star.$$

We then realize each $w \in W$ as an endomorphism:

$$w : \$_W \to \$_W$$

interchanging the factors.

In particular, following [61], we view $w \in W$ as a map:

$$w : \$_W^+ \to \$_W^-$$

by declaring:

$$w \cdot (\lambda, \eta) = \lambda g\left(w, \cdot\right) + 2\eta\left(w, \cdot\right), \text{ where } w \in W, \ (\lambda, \eta) \in \$_W^+$$

and

$$w : \$_W^- \to \$_W^+$$

by declaring:

$$w \cdot \alpha = \left(-g(w, \alpha^\#), -\frac{1}{2} w^\flat \wedge \alpha - \frac{1}{2}\Phi\left(u, v, \cdot, \cdot\right)\right), \text{ where } w \in W, \ \alpha \in \$_W^-.$$

The Clifford relations follow by an explicit calculation. Note that the adjoint of:

$$w : \$_W^+ \to \$_W^-$$

is given by:

$$-w : \$_W^- \to \$_W^+.$$

We conclude that:

$$\mathrm{Cl}\left(W\right) \cong \mathrm{End}_{\mathbb{R}}\left(\$_W\right). \tag{1.35}$$

A similar construction is available for V. In this case, the spinor modules can be constructed using the G_2 structure ϕ. We set:

$$\not{S}_V = \Lambda^0 V^\star \oplus \Lambda^1 V^\star.$$

Each $v \in V$ is then realized as an endomorphism:

$$v : \not{S}_V \to \not{S}_V$$

by declaring:

$$v\,(\lambda, \alpha) = \left(-g(v, \alpha^\#), \lambda\alpha + (v \times_\phi \alpha^\#)^\flat\right), \text{ where } v \in V,\ (\lambda, \alpha) \in \not{S}_V.$$

The Clifford relations follow by an easy computation. The endomorphism corresponding to each $v \in V$ is skew-adjoint. Precomposing with the (outer) grading automorphism ϵ, we obtain the second irrep $\not{S}_V^\epsilon$. Finally, we have:

$$\mathrm{Cl}\,(V) \cong \mathrm{End}_{\mathbb{R}}\left(\not{S}_V\right) \oplus \mathrm{End}_{\mathbb{R}}\left(\not{S}_V^\epsilon\right). \tag{1.36}$$

Recall that our choice of octonionic unit induces an inclusion:

$$V = 1^\perp \subset W.$$

We wish to emphasize that 1 (a negative element of the Clifford algebra) does not agree with the Clifford unit. In turn, this gives rise to an embedding:

$$\iota : \mathrm{Cl}(V) \hookrightarrow \mathrm{Cl}(W),$$

as in (1.30).

We can thus let $\mathrm{Cl}(V)$ act on $\not{S}_W^+$ and $\not{S}_W^-$. This recovers $\not{S}_V$ and $\not{S}_V^\epsilon$. To see this, note that there is an obvious identification:

$$\Lambda^1 W^\star \cong \Lambda^0 V^* \oplus \Lambda^1 V^\star,$$

relating $\$_V$ with $\$_W^-$. To relate $\$_V$ with $\$_W^+$, we note that ([61] p. 68):

$$\Lambda^2_7 W^\star = \left\{ 1^\flat \wedge \alpha - \iota_1 \iota_{u^\sharp} \Phi \ \mid \ u \in \Lambda^1 V^\star \right\}. \tag{1.37}$$

Evidently, this provides us with an identification

$$\Lambda^2_7 W^\star \cong \Lambda^1 V^\star, \tag{1.38}$$

as required.

Finally, we obtain:

$$\mathrm{Cl}\,(V) \cong \mathrm{End}_{\mathbb{R}}\left(\$_W^+\right) \oplus \mathrm{End}_{\mathbb{R}}\left(\$_W^-\right) \subset \mathrm{End}_{\mathbb{R}}\left(\$_W\right),$$

realizing $\mathrm{Cl}\,(V)$ as those endomorphisms of $\$_W$ that preserve the grading (1.34).

1.1.3 The Global Picture: Asymptotically Conical CY Fourfolds and Related Geometries

Having completed our discussion of the flat model, we are now ready to pass to the global level by introducing manifolds locally modelled on the linear algebraic picture presented thus far. We will introduce CY fourfolds, Sasaki-Einstein manifolds, Spin(7) manifolds and nearly parallel G_2 manifolds. We are primarily interested in CY fourfolds, but—as evidenced by the flat picture—the other geometries play a central role in the theory. Indeed, as we shall see, CY fourfolds are naturally Spin(7). Furthermore, the asymptotic links of AC CY fourfolds are Sasaki-Einstein and the asymptotic links of AC Spin(7) manifolds are nearly parallel G_2.

1.1.3.1 Calabi-Yau (CY) Fourfolds

Let X^8 be an 8-dimensional smooth manifold. Suppose that it is endowed with an almost complex structure J. Let $\omega \in C^\infty\left(\Lambda^2 T^\star X^8\right)$ be a compatible symplectic form. The frames that restore J and ω to their standard version determine a smooth sub-bundle of $\mathrm{Fr}\left(TX^8\right)$ with fiber U(4), i.e. a U(4)-structure. We have that:

$$\mathrm{U}(4) \subset \mathrm{SO}(8).$$

This implies that J and ω induce a Riemannian metric and orientation. Indeed, J induces the natural complex orientation and compatibility of ω with J is equivalent to:

$$g(u,v) = \omega(u, Jv)$$

being symmetric and positive definite. The associated volume form can be recovered from ω:

$$\mathrm{Vol}_\omega = \frac{\omega^4}{4!}.$$

Suppose now that J is integrable so that X^8 is complex. If ω and J are parallel:

$$\nabla_g J = \nabla_g \omega = 0,$$

the U(4) structure is said to be *torsion-free*. This is equivalent to the Kähler condition:

$$d\omega = 0.$$

A torsion-free U(4)-structure is known as a Kähler structure. Since parallel tensors are stabilized by holonomy, Kähler metrics satisfy:

$$\mathrm{Hol}\,(g) \subseteq \mathrm{U}(4).$$

Suppose now that Ω is a trivialization of the canonical bundle $\Omega \in C^\infty\left(\Lambda^{4,0}T^\star X^8\right)$. It induces a trivialization of the top exterior power of X^8 (i.e. a volume form):

$$\mathrm{Vol}_\Omega = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega}.$$

When the volume compatibility equation is satisfied:

$$\mathrm{Vol}_\Omega = \mathrm{Vol}_\omega,$$

the frames that simultaneously restore J, ω and Ω to their standard form yield a further reduction of the frame bundle to SU(4) (i.e. an SU(4) structure). When the associated

U(4) structure is Kähler and Ω is holomorphic (i.e. $\bar{\partial}\Omega = 0$), it follows that:

$$\nabla_g \Omega = 0.$$

In this case Ω is termed a *holomorphic volume form*, the SU(4) structure is said to be *torsion-free* and we have:

$$\mathrm{Hol}\,(g) \subseteq \mathrm{SU}(4).$$

Definition 1.1. A Calabi-Yau (CY) fourfold is a smooth 8-manifold equipped with a torsion-free SU(4) structure.

Manifolds with holonomy contained in SU(4) are automatically Ricci flat i.e. they satisfy the Einstein vacuum equations. In turn, they are of special interest to theoretical physicists: compact CY manifolds play a prominent role in string compactifications.

1.1.3.2 Sasaki-Einstein 7-Manifolds and CY-4 Cones

We are primarily interested in noncompact CY fourfolds with a single end whose geometry approaches that of a cone at infinity. Evidently, a preliminary step is to introduce the asymptotic model for such objects: CY-4 cones. The latter naturally correspond to Sasaki-Einstein 7-manifolds. We will not delve deep into the interesting field of Sasaki-Einstein geometry, but rather introduce only the basics required for the sequel. For a comprehensive introduction the reader is referred to (Sparks [66]).

Let (Σ^7, g_{Σ^7}) be 7-dimensional Riemannian 7-manifold and set:

$$X^8 \overset{\mathrm{def}}{=} (0,\infty) \times \Sigma^7.$$

Equip X^8 with the *conical metric*:

$$g_{\mathcal{C}(\Sigma^7)} \overset{\mathrm{def}}{=} dr^2 + r^2 g_{\Sigma^7}, \tag{1.39}$$

where r is the natural coordinate on the first factor. It is often referred to as the *radius function* of X^8. The resulting Riemannian manifold is known as the *cone over* Σ^7:

$$\mathcal{C}(\Sigma^7) \overset{\mathrm{def}}{=} (X^8, g_{\mathcal{C}(\Sigma^7)}).$$

In this setting, the closed manifold Σ^7 is referred to as the *link* of the cone. The copy of Σ^7 sitting at radius $r > 0$ is referred to as the *slice* of the cone at r and denoted by Σ^7_r.

A contact form on (Σ^7, g) is a smooth unit-norm one-form η such that:

$$\eta \wedge d\eta \wedge d\eta \wedge d\eta \neq 0.$$

The form η equips Σ^7 with a *contact structure* i.e. a maximally non-integrable (in the sense of Frobenius), smoothly varying distribution of 6-planes:

$$C_p \stackrel{\text{def}}{=} \mathrm{K}er(\eta_{|_p}) \subset T_p\Sigma^7 \text{ for all } p \in \Sigma^7.$$

These are known as the *contact elements* of the structure. The kernel of $d\eta$ is one-dimensional and transverse to the contact distribution C. It is trivial and the unique trivialization ξ satisfying:

$$\eta(\xi) = 1 \tag{1.40}$$

is the *Reeb vector field* associated to η.

A contact structure on Σ^7 is exactly what is required to guarantee that the cone $\mathcal{C}(\Sigma^7)$ is symplectic. Contact forms on on Σ^7 correspond to symplectic forms on $\mathcal{C}(\Sigma^7)$ using the relation:

$$\omega = \frac{1}{2} d(r^2\eta). \tag{1.41}$$

Since the contact elements C_p are even-dimensional one can consider almost complex structures on them. An endomorphism field that restricts to an almost complex structure on each contact element is known as a *transverse almost complex structure*.

A *Sasakian* 7-manifold is a Riemannian, contact 7-manifold (Σ^7, g, η, ξ) equipped with a transverse almost complex structure $\mathcal{J}$ satisfying the compatibility equations:

$$\mathcal{J}^2 = -\mathrm{Id}_{TM} + \eta \otimes \xi$$
$$g(\mathcal{J}X, \mathcal{J}Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$\nabla_X \xi = -\mathcal{J}X,$$
$$(\nabla_X \mathcal{J})(Y) = g(X,Y)\xi - \eta(Y)X.$$

A Sasakian structure on Σ^7 is precisely what is required to guarantee that the cone $\mathcal{C}(\Sigma^7)$ is Kähler (and endow it with a preferred Kähler structure).

It is shown in [66] that if a Sasakian 7-manifold is Einstein then its Einstein constant must be equal to 6, so that:

$$\mathrm{Ric}_g = 6g. \tag{1.42}$$

Sasakian 7-manifolds satisfying condition (1.42) are said to be *Sasaki-Einstein*. A Sasaki-Einstein structure on Σ^7 is precisely what is required to guarantee that $\mathcal{C}(\Sigma^7)$ is Calabi-Yau. Furthermore, all the geometric data encoded in the CY-4 structure can be constructed explicitly from the tensors comprising the Sasaki-Einstein structure. It is also possible to reverse this: one can retrieve the Sasaki Einstein structure from the CY-4 structure on the cone. In light of this correspondence, Sasaki-Einstein 7-manifolds (Σ^7, g) are often defined by asking for $\mathcal{C}(\Sigma^7)$ to be Calabi-Yau.

1.1.3.3 Spin(7) Manifolds

Let X^8 be an 8-dimensional smooth manifold and consider the (nonlinear) sub-bundle of $\Lambda^4 T^\star X^8$ defined by:

$$\mathcal{A}X^8 \stackrel{\text{def}}{=} \coprod_{p \in X^8} \mathcal{A}_p X^8,$$

where:

$$\mathcal{A}_p X^8 \stackrel{\text{def}}{=} \left\{ \omega \in \Lambda^4 T_p^\star X^8 : \exists \text{ oriented linear isomorphism } T : T_p X^8 \xrightarrow{\sim} \mathbb{R}^8 \text{ taking } \omega \text{ to } \Phi_{\text{standard}} \right\}.$$

The standard fiber of this bundle is diffeomorphic to the 43 dimensional manifold $\mathrm{GL}_+(8)/\mathrm{Spin}(7)$ and is thus of codimension 27 in $\Lambda^4 T_p^\star X^8$ (Joyce [32] p.240).

Definition 1.2. A Cayley calibration on X^8 is a smooth section:

$$\Phi \in C^\infty(\mathcal{A}X^8).$$

Note that since the bundle $\mathcal{A}X^8$ is not linear, the existence of a Cayley calibration is not guaranteed. In fact, it may be obstructed topologically. An immediate constraint is that X^8 ought to be spin. In fact, the existence of a Cayley calibration is equivalent to the existence of a nowhere-vanishing spinor.

A choice of Cayley calibration determines a Spin(7) structure. Since Spin(7) $\subset$ SO(8), one automatically obtains a Riemannian metric and orientation. This is achieved pointwise. One simply writes down the standard Euclidean structures in any of the Spin(7) frames.

A Spin(7) structure is termed *torsion-free* if:

$$\nabla_g \Phi = 0.$$

In analogy to Kähler structures, a Spin(7) structure Φ is torsion-free if and only if (Joyce [32] p.240):

$$d\Phi = 0.$$

Since parallel tensors are stabilized by holonomy, the metric of a torsion-free Spin(7) structure satisfies:

$$\mathrm{Hol}\,(g) \subseteq \mathrm{Spin}(7).$$

Definition 1.3. A Spin(7) manifold is a smooth 8-manifold X^8 equipped with a torsion-free Spin(7) structure.

For constructions of Spin(7) manifolds see (Joyce [36]) in the compact case and (Bryant-Salamon [7], Foscolo [23], Lehmann [39]) in the noncompact case. Closed examples are scarce, very difficult to come by and non-explicit.

Similar to CY fourfolds, Spin(7) manifolds are automatically Ricci flat.

We have the chain of inclusions:

$$\mathrm{SU}(4) \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8).$$

It follows that SU(4)-structures naturally induce Spin(7)-structures. At the flat level, this is captured by our ability to express the standard Cayley calibration (1.15) in terms of the standard symplectic form and holomorphic volume form. It follows that CY fourfolds are—in a natural way—Spin(7) manifolds. This motivates our interest in Spin(7) geometry.

1.1.3.4 Nearly Parallel G_2 Manifolds and Spin(7) Cones

Since Spin(7) geometry is intimately tied to CY-4 geometry, we naturally seek to identify the type of geometric structure on the link Σ^7 of a Spin(7) cone $\mathcal{C}(\Sigma^7)$. The answer turns out to be a weaker version of G_2-geometry: Σ^7 needs to be *nearly parallel* G_2. We now introduce both honest (parallel) G_2-structures and their nearly parallel counterparts.

Let Σ^7 be a 7-dimensional smooth manifold and consider the (nonlinear) sub-bundle of $\Lambda^3 T^\star \Sigma^7$ defined by:

$$\mathcal{P}\Sigma^7 \stackrel{\text{def}}{=} \coprod_{p \in \Sigma^7} \mathcal{P}_p \Sigma^7,$$

where:

$$\mathcal{P}_p\Sigma \stackrel{\text{def}}{=} \left\{ \omega \in \Lambda^3 T_p^\star \Sigma : \exists \text{ oriented linear isomorphism } T : T_p\Sigma \xrightarrow{\sim} \mathbb{R}^7 \text{ taking } \omega \text{ to } \phi_{\text{standard}} \right\}.$$

The standard fiber of this bundle is diffeomorphic to the 35 dimensional manifold $\mathrm{GL}_+(7)/G_2$. It is thus an open submanifold of $\Lambda^3 T_p^\star \Sigma$ (Joyce [32] p.240).

Definition 1.4. An associative calibration on Σ^7 is a smooth section:

$$\phi \in C^\infty(\mathcal{P}\Sigma^7).$$

Note that since the bundle $\mathcal{P}\Sigma^7$ is not linear, it is not always the case that associative calibrations exist. Their existence amounts to Σ^7 admitting a nowhere-vanishing spinor. This is always the case on a 7-dimensional spin manifold. It follows that associative calibrations exist if and only if Σ^7 is spin.

A choice of associative calibration yields a G_2-structure. Since $G_2 \subset \mathrm{SO}(7)$, one obtains a Riemannian metric and orientation. This is done pointwise by writing down the standard Euclidean structures in any of the G_2-frames.

Let ϕ be an associative calibration. The corresponding coassociative calibration is defined by:

$$\psi = \star\phi.$$

A G_2-structure is called *torsion-free* if:

$$\nabla_g \phi = 0.$$

In analogy to Kähler structures, a G_2 structure is torsion-free if and only if (Joyce [32] p.229):

$$d\phi = d^\star \phi = 0.$$

Since parallel tensors are stabilized by holonomy, the metric of a torsion-free G_2-structure satisfies:

$$\mathrm{Hol}\,(g) \subseteq G_2.$$

Definition 1.5. A G_2 manifold is a smooth 7-manifold Σ^7 equipped with a torsion-free G_2-structure.

For constructions of G_2 manifolds see (Joyce [35], Joyce-Karigiannis [33]) in the compact case and (Bogoyavlenskaya [5], Bryant-Salamon [7], Foscolo-Haskins-Nordström [24]) in the noncompact case. Similar to Spin(7) manifolds, closed examples are scarce, very difficult to come by and non-explicit.

G_2 manifolds are automatically Ricci flat.

In the context of AC CY-4 geometry one naturally encounters G_2 structures with nonvanishing (but controlled) torsion.

Definition 1.6. A *nearly parallel* G_2 manifold (also known as a manifold of *weak holonomy*

G_2) is a smooth 7-manifold Σ^7 equipped with a G_2-structure satisfying:

$$d\phi = 4\psi.$$

For an introduction to nearly parallel G_2-manifolds see [63]. In contrast to honest holonomy G_2-manifolds, they are not Ricci flat, but rather positive Einstein with Einstein constant equal to 6. Furthermore, there exists a plethora of closed explicit examples many of which are homogeneous. The homogeneous ones have been completely classified. For this classification see [63] and [65].

The fundamental fact linking G_2 geometry with the 8-dimensional geometries introduced earlier is the following:

Proposition 1.7. *The cone $\mathcal{C}(\Sigma^7)$ is torsion-free $Spin(7)$ if and only if Σ^7 is nearly parallel G_2.*

This feature is partially visible at the linear level. Indeed, the standard G_2 structure lives on the imaginary octonions $\mathfrak{Im}\,(\mathbb{O})$, whereas the standard Spin(7) structure lives on the full octonions $\mathbb{O}$. The former is embedded in the latter as the orthogonal complement of the unital element. At the global level, the radial unit vector ∂_r will assume the role of the octonionic unit and its orthogonal complement—i.e. the tangent space to Σ^7—will host a natural G_2-structure.

Proof. Suppose that Σ^7 is G_2. Let ϕ be its associative calibration and let ψ be the corresponding coassociative calibration. Introduce the following 4-form on $\mathcal{C}(\Sigma^7)$:

$$\Phi \stackrel{\text{def}}{=} r^3 dr \wedge \phi + r^4\psi.$$

The scaling has been selected to guarantee that g_Φ is conical and:

$$|\Phi|_{g_{\mathcal{C}(\Sigma^7)}} = O(1) \text{ as } r \to \infty,$$

a necessary condition for Φ to turn out covariantly constant.

The form $r^3\phi$ yields a G_2 structure on every slice Σ_r. It corresponds to the rescaled metric

$r^2 g_{\Sigma^7}$. The linear algebra developed in our discussion of the flat model is sufficient to establish that Φ is a Cayley calibration on every tangent space. It thus induces a Spin(7)-structure on $\mathcal{C}(\Sigma^7)$. Since the original G_2 structure is nearly parallel, we have:

$$d\phi = 4\psi,$$
$$d\psi = 0.$$

The Leibniz rule demonstrates that:

$$\begin{aligned} d\Phi &= dr \wedge r^3(4\psi - d\phi) + r^4 d\psi \\ &= 0, \end{aligned}$$

establishing that the induced Spin(7) structure is torsion-free. It follows that $\mathcal{C}(\Sigma^7)$ is Spin(7). The converse is proved similarly. □

1.1.3.5 Sasaki-Einstein Manifolds as Nearly Parallel G_2-Manifolds

The nearly parallel G_2 condition is a relaxation of the Sasaki-Einstein condition. In fact, nearly parallel G_2 structures admit a spinorial characterization that naturally separates them into three types [63]. These are distinguished by the dimension of the space of Killing spinors. Type I nearly G_2 manifolds have a single Killing spinor. Their cones have holonomy contained in Spin(7) and are thus (incomplete) Spin(7) manifolds. Type II nearly G_2 manifolds have two linearly independent Killing spinors. These are precisely the Sasaki-Einstein 7-manifolds. As we have seen, their cones have holonomy contained in SU(4) and are thus (incomplete) Calabi-Yau fourfolds. Type III nearly G_2 manifolds have three linearly independent Killing spinors. They are known as 3-Sasakian Manifolds. Their cones have holonomy contained in Sp(2) and are therefore (incomplete) hyperkähler manifolds.

Observe that as the geometric structure of the link is refined, so is the geometric structure of the corresponding cone. This is captured by successive holonomy reductions:

$$\mathrm{Sp}(2) \subset \mathrm{SU}(4) \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8).$$

The presence of two linearly independent Killing spinors on a Sasaki-Einstein 7-manifold Σ^7 results in a U(1)-family of compatible associative calibrations. Torsion-free associative calibrations on Σ^7 correspond to Killing spinors. Scaling an associative calibration by a real number results in a corresponding rescaling of the compatible Riemannian metric. Therefore, in order to find new associative calibrations compatible with the fixed metric on Σ^7 we are forced to rotate the natural one.

There is a more explicit way to understand this. We propose a natural parameterization of the associative calibrations by the unit complex numbers. The natural holomorphic volume form Ω on the cone can be rescaled by $z \in \mathrm{U}(1)$ to yield:

$$\Omega_z \stackrel{\text{def}}{=} z\Omega.$$

This is still compatible with g, ω and J and together they form an SU(4) structure on $\mathcal{C}\left(\Sigma^7\right)$. Using Ω_z one can build the corresponding Cayley calibration:

$$\Phi_z \stackrel{\text{def}}{=} \mathfrak{Re}\left(\Omega_z\right) + \frac{\omega^2}{2}.$$

The associative calibration ϕ_z is then determined using the equation:

$$\Phi_z = dr \wedge\ r^3\phi_z + r^4 \star_{\Sigma^7} \phi_z.$$

This retrieves the U(1) family constructed above.

1.1.3.6 Asymptotically Conical (AC) Manifolds and AC CY Fourfolds

We are finally set to introduce the geometric background for the gauge theoretic problems we wish to study: asymptotically conical CY fourfolds.

Let (X^8, g_{X^8}) be a non-compact smooth 8-manifold and (Σ^7, g_{Σ^7}) a closed smooth 7-manifold. Recall that we can form the cone $\mathcal{C}(\Sigma^7)$, with metric $g_{\mathcal{C}(\Sigma^7)}$ as defined in (1.39).

Definition 1.8. The manifold X^8 is said to be *asymptotically conical* (AC) with *rate* $\mu < 2$ and *asymptotic link* (Σ, g_Σ) if there exists a compact $K \subset X^8$, a positive $R > 0$ and a diffeomorphism:

$$\Psi : (R, \infty) \times \Sigma^7 \xrightarrow{\sim} X \setminus K$$

such that for any $k \geq 0$ we have:

$$|\nabla^k_{g_{\mathcal{C}(\Sigma^7)}}(\Psi^\star g_{X^8} - g_{\mathcal{C}(\Sigma^7)})|_{g_{\mathcal{C}(\Sigma^7)}} = O(r^{\mu-2-k}). \tag{1.43}$$

Asymptotically conical metrics are conical to leading order at infinity, but their asymptotic expansions can have nonvanishing lower order terms. The decay rate captures the asymptotic behaviour of the dominant lower order term.

Note that the diffeomorphism Ψ appearing in the definition need not be unique. Composing its inverse Ψ^{-1} with the natural coordinate r on (R, ∞) we obtain a real valued map on the complement of K. This is known as an *asymptotic radius function.* We will often slightly abuse terminology by referring to r without fixing a particular choice of Ψ. All statements made in this fashion will be valid for any Ψ satisfying (1.43) and hence any choice of asymptotic radius function r.

When working on the complement of K, one often combines the coordinate vector field ∂_r (dual to dr) with the pullback of a local frame $X_1, ..., X_7$ on Σ^7. Frames of the form $(\partial_r, X_1, ..., X_7)$ are known as *asymptotic frames* on X^8. All pointwise norms $|X_i|_{g_{X^8}}$ are $O(r)$ at infinity, whereas $|\partial r|_{g_{X^8}}$ is $O(1)$. To fix this asymmetry, it will often be preferable to work with *asymptotic logarithmic radius functions* instead. Let r be an asymptotic radius function on X^8 and set:

$$t \stackrel{\text{def}}{=} \log(r).$$

Then t is said to be a *asymptotic logarithmic radius function* on X^8. In terms of such a radial coordinate, the limiting cone metric takes the form:

$$g_{\mathcal{C}(\Sigma^7)} = e^{2t}(dt^2 + g_{\Sigma^7}).$$

All elements of the asymptotic frame are then $O(e^t)$.

One naturally extends (1.43) to general tensor fields on X^8 to introduce the class of AC tensors. Each AC tensor T of rank (p,q) approaches a limiting tensor T_∞ along the asymptotic cone. The latter is obtained by homogeneously extending the pullback of a tensor on Σ^7. The degree of the extension is specified to guarantee:

$$|T_\infty|_{g_{\mathcal{C}(\Sigma^7)}} = O(1). \tag{1.44}$$

We say that T approaches T_∞ with rate $\mu < q-p$ when:

$$|\nabla^k_{g_{\mathcal{C}(\Sigma^7)}}(\Psi^\star T_{X^8} - T_\infty)|_{g_{\mathcal{C}(\Sigma^7)}} = O(r^{\mu-(q-p)-k}). \tag{1.45}$$

We wish to warn the reader that our convention deviates from the standard one. In the context of (1.43) and (1.45), it is customary to define the decay rates to be $\mu - 2$ and $\mu - (q-p)$ respectively. As a result, the rate then captures the asymptotics of the drop-off of $T - T_\infty$ in the limiting conical metric. A desirable feature of the standard convention is that all tensors encoding the geometric structure of an AC special holonomy manifold have the same rate. A disadvantage is that the rate of a fixed tensor does not only depend on the asymptotics of the components relative to an asymptotic frame, but also its rank. When analyzing concrete examples, we typically fix an asymptotic frame and work directly with the resulting components. This situation is so frequent in the sequel that we have opted for (1.43) and (1.45) in place of the standard convention. Either way, translating between the two is trivial.

The observant reader will notice that we have not restricted Σ^7 to be connected. The *number of ends* of X^8 is defined by:

$$\#\text{ of ends of } X^8 = \text{rk } H^0(\Sigma^7),$$

(i.e. the number of connected components of the asymptotic link). General AC manifolds can have multiple ends. In fact, general noncompact manifolds can have multiple ends with mismatching asymptotic geometry. We will be interested in noncompact CY fourfolds. It follows from the Cheeger-Gromoll splitting theorem that such spaces can only have a single end. Consequently, we will always take Σ^7 to be connected.

Arguing in analogy to the pure conical setting but keeping only higher order terms in the asymptotic expansions, we find that CY-4 structures on X^8 force Σ^7 to be Sasaki-Einstein. Similarly, Spin(7) structures on X^8 force Σ^7 to be nearly parallel G_2. In the sequel, Σ^7 will always be a closed, connected Sasaki-Einstein manifold (nearly parallel G_2 of type II).

Note that all the tensors $g, J, \omega, \Omega, \Phi$ encoding the SU(4) structure on X^8 are covariantly constant and thus $O(1)$ as $r \to \infty$. An alternative way to see this is to notice that the leading order terms in their asymptotic expansions are given by the corresponding conical versions. For instance, the Cayley calibration on an AC CY fourfold is given by:

$$\Phi = (r^3 + \text{ lower order terms})\; dr \wedge \phi + (r^4 + \text{ lower order terms})\; \psi$$

where ϕ and ψ are the natural associative and coassociative calibrations on Σ^7.

1.2 Gauge Theoretic Aspects

Having introduced the geometric background required for our purposes we now turn to the gauge theoretic equations of interest. For a general introduction to gauge theory see (Hamilton [29]), (Bleecker [4]) and (Naber [53]). The appendices of (Wehrheim [84]) offer a more concise exposition. For a self-contained overview of higher dimensional gauge theory see the excellent recent book (Sá Earp-Fadel [22]).

Our first task is to define the natural instanton equations available on the four geometries introduced thus far. We then present Lewis's theorem [40] relating the Spin(7) instanton and HYM systems over a closed CY fourfold. The next step is to compute the linearizations of the Spin(7) and G_2 instanton equations, relate them to the relevant twisted Dirac operators and derive the appropriate Weitzenböck formulae. We conclude this section by formally introducing the instanton moduli spaces we intend to study.

In the sequel, unless otherwise specified, G will be taken to be a real, linear, compact Lie group and P a principal G-bundle over X^8. We will always assume that $\mathrm{ad}(P)$ is endowed with an Ad-invariant fiber metric given by $-\mathrm{Tr}(\cdot)$.

1.2.1 Hermitian Yang-Mills (HYM) Connections

Let $(X^8, g, J, \omega, \Omega)$ be a Calabi-Yau fourfold. Recall from section 1 that $\Lambda^2\mathbb{O}^\star$ decomposes into SU(4)-irreducible pieces:

$$\Lambda^2\mathbb{O}^\star = \mathrm{Span}_{\mathbb{R}}(\omega_{\mathrm{standard}}) \oplus \mathcal{C} \oplus \Lambda^{1,1}_0 \oplus \mathcal{B}. \tag{1.46}$$

Fix $p \in X^8$. We identify T_pX^8 with the octonions by choosing a frame in the fiber of the SU(4)-structure over p. We thus obtain a decomposition of $\Lambda^2 T^\star_p X^8$ as above. This decomposition doesn't depend on the choice of frame: changing frame translates everything by an element of SU(4), which leaves the irreducible pieces invariant. We therefore have a vector bundle splitting:

$$\Lambda^2 T^\star X^8 = \mathrm{Span}_{\mathbb{R}}(\omega) \oplus \mathcal{C} \oplus \Lambda^{1,1}_0 \oplus \mathcal{B}.$$

where $\mathcal{B}$ and $\mathcal{C}$ are 6 dimensional real subspaces of $\mathfrak{Re}\left(\Lambda^{2,0} \oplus \Lambda^{0,2}\right)$ and $\Lambda^{1,1}_0$ is the 15-dimensional subspace of $\mathfrak{Re}\left(\Lambda^{1,1}\right)$ defined as the orthogonal complement of the Kähler form. The Hermitian Yang-Mills (HYM) connections are instantons defined using this splitting.

Definition 1.9. Let $(X^8, g, J, \omega, \Omega)$ be a Calabi-Yau 4-fold. Let P be a principal G-bundle over X^8. A connection $A \in \mathcal{A}(P)$ is HYM if:

$$F_A \in C^\infty\left(\Lambda^{1,1}_0 \otimes \mathrm{ad}(P)\right).$$

Equivalently, the HYM equation can written in the following form, reminiscent of the familiar 4-dimensional ASD equation from Donaldson theory:

$$\frac{1}{2} \star \left(\omega^2 \wedge F_A\right) = -F_A. \tag{1.47}$$

To verify this statement, we use our explicit knowledge of the linear algebra involved in the flat model. We find that:

$$\begin{aligned}\frac{1}{2}\star\left(\omega^2\wedge(\cdot)\right) &= \star\left(\Phi\wedge(\cdot)\right) - \star\left(\mathfrak{Re}(\Omega)\wedge(\cdot)\right) \\ &= 3\pi_7^2 - \pi_{21}^2 - 2\pi_{\mathcal{C}}^2 + 2\pi_{\mathcal{B}}^2 \\ &= 3\pi_{\mathrm{Span}(\omega)}^2 + \pi_{\mathcal{C}}^2 + \pi_{\mathcal{B}}^2 - \pi_{\Lambda_0^{1,1}}^2.\end{aligned} \tag{1.48}$$

Imposing (1.47) then yields:

$$4\pi_{\mathrm{Span}(\omega)}^2 F_A + 2\pi_{\mathcal{B}}^2 F_A + 2\pi_{\mathcal{C}}^2 F_A = 0.$$

This forces all but one of the components of F_A with respect to the decomposition (1.46) to vanish. The potentially nonvanishing component is the one in $\Lambda_0^{1,1}$. We thus recover the HYM condition.

Taking exterior covariant coderivatives on both sides of (1.47) proves that HYM connections are Yang-Mills, justifying our terminology. In fact—over a closed CY4 base—they are the absolute minima of the Yang-Mills action. Even though we have already set up enough background to verify this fact, we defer the proof until our discussion of Spin(7) instantons. Our claim will follow from a more general statement proved in that section.

One frequently encounters the HYM equation written in terms of the holomorphic volume form Ω. To achieve this, we first observe that it is equivalent to:

$$\begin{aligned} F_A \wedge \star\omega &= 0, \\ F_A^{2,0} = F_A^{0,2} &= 0. \end{aligned}$$

These equations are further recast as follows. Since F_A is real, if its $(0,2)$ part vanishes, so does its $(2,0)$ part. Since Ω is of bi-degree $(4,0)$, we have that:

$$F_A \wedge \Omega = F_A^{0,2} \wedge \Omega$$

and furthermore:

$$F_A^{0,2} \wedge \Omega = 0 \iff F_A^{0,2} = 0$$

From these remarks it follows that:

$$F_A^{2,0} = F_A^{0,2} = 0 \iff F_A \wedge \Omega = 0$$

We thus find that a connection is HYM if and only if:

$$F_A \wedge \star\omega = 0 \tag{1.49}$$

$$F_A \wedge \Omega = 0. \tag{1.50}$$

In light of (1.47), (1.48), (1.59) and the fact that all HYM connections are automatically Spin(7) instantons, we find that the HYM condition amounts to:

$$F_A \wedge \star\omega = 0 \tag{1.51}$$

$$F_A \wedge \mathfrak{Re}(\Omega) = 0. \tag{1.52}$$

The Spin(7) instantons are introduced in section 1.2.3.

1.2.2 Contact Instantons and Conical HYM Connections

Let $(\Sigma^7, g, \eta, \xi, \mathcal{J})$ be a Sasaki-Einstein 7-manifold. Let σ be the 3-form:

$$\sigma \stackrel{\text{def}}{=} \eta \wedge d\eta.$$

Define the following endomorphism field on Λ^2:

$$T_\sigma : \alpha \mapsto \star_\Sigma(\sigma \wedge \alpha). \tag{1.53}$$

Portilla and Earp ([44] section 2.1.4) prove that T_σ decomposes Λ^2 in smooth sub-bundles spanned by its eigenvectors :

$$\Lambda^2 = \Lambda^2_H \oplus \Lambda^2_V$$

where:

$$\Lambda_H = \text{Span}_{\mathbb{R}}(d\eta) \oplus \Lambda^2_8 \oplus \Lambda^2_6.$$

Here, Λ^2_H and Λ^2_V are the spaces of horizontal and vertical 2-forms relative to the Reeb foliation. We have:

$$\dim_{\mathbb{R}}\left(\Lambda^2_H\right) = 15,$$
$$\dim_{\mathbb{R}}\left(\Lambda^2_V\right) = 6.$$

The dimensions of the subspaces further decomposing Λ_H have been encoded in the notation as subscripts.

The *contact instantons* are special Yang-Mills connections defined using the above splitting:

Definition 1.10. Let $(\Sigma^7, g, \eta, \xi, \mathcal{J})$ be a Sasaki-Einstein 7-manifold. Let P be a principal G-bundle over Σ^7. A connection $A \in \mathcal{A}(P)$ is a contact instanton if:

$$F_A \in C^\infty\left(\Lambda^2_8 \otimes \mathrm{ad}(P)\right).$$

In our conventions, Λ^2_8 is the -2-eigenspace of T_σ. Evidently, the contact instantons are characterized by the PDE:

$$\star\left(\sigma \wedge F_A\right) = -2F_A. \tag{1.54}$$

Consider the linear map:

$$d\eta^2 \wedge (\cdot) : \Lambda^2 \to \Lambda^6$$
$$\alpha \mapsto d\eta^2 \wedge \alpha.$$

One can prove that it is surjective with kernel:

$$\mathrm{Ker}\left(d\eta^2 \wedge (\cdot)\right) = \Lambda^2_6 \oplus \Lambda^2_8.$$

It follows that the contact instantons satisfy the equation:

$$F_A \wedge d\eta^2 = 0. \tag{1.55}$$

Taking exterior covariant coderivatives on both sides of (1.54) demonstrates that the contact instantons are Yang-Mills. In fact, Portilla and Earp ([44] p.18) prove that—when Σ^7

is closed—they are precisely the absolute minima of the Yang-Mills action functional.

The contact instanton equations arise as a dimensional reduction of the HYM equations along the radial direction in $\mathcal{C}(\Sigma^7)$. To clarify what we mean by this statement, we work as follows. First, recall that the cone $\mathcal{C}(\Sigma^7)$ carries a natural SU(4) structure. The bundle P can be pulled back to $\mathcal{C}(\Sigma^7)$ through the natural map to yield:

$$P_{\mathcal{C}(\Sigma^7)} \overset{\text{def}}{=} \pi^\star P = (0,\infty) \times P. \tag{1.56}$$

This carries an obvious lift of the natural dilation $\mathbb{R}_{>0}$-action on $\mathcal{C}(\Sigma^7)$ (scaling the first factor). A connection is *dilation invariant* if it is invariant under the dilation action as a 1-form over the total space.

Proposition 1.11. *A dilation invariant connection $A \in \mathcal{A}(P)$ in temporal gauge over $\mathcal{C}(\Sigma^7)$ is HYM if and only if its value on the link Σ^7 is a contact instanton.*

Proof. A is given a 1-parameter family of connections A_r over Σ, parameterized by the values of the radius function. We compute:

$$F_A = dr \wedge \partial_r A_r + F_{A_r},$$

where the second summand is the curvature of A_r along Σ^7_r.

Dilation invariance implies that the first term vanishes and that all the A_r are equal to the same connection over Σ^7. We denote this constant value by A_Σ. We then have:

$$F_A = F_{A_\Sigma}.$$

Using (1.41), we find that:

$$\frac{\omega^2}{2} = dr \wedge \frac{r^3}{2}\sigma + \frac{r^4}{8} d\eta^2.$$

Expressing the Hodge-star on the cone with respect to the Hodge-star on its link, we find that the HYM equation (1.47) takes the form:

$$dr \wedge \frac{r^4}{8} \star_{\Sigma^7} \left(d\eta^2 \wedge F_{A_\Sigma}\right) + \frac{r^3}{2} \star_{\Sigma^7} \left(\sigma \wedge F_{A_\Sigma}\right) = -F_{A_\Sigma} \tag{1.57}$$

The link Σ^7 sits inside the cone at radius $r = 1$. Restricting to this radius and equating radial components we obtain:

$$d\eta^2 \wedge F_{A_\Sigma} = 0.$$

Equating tangential components recovers:

$$\star\left(\sigma \wedge F_{A_\Sigma}\right) = -2F_{A_\Sigma}.$$

We recognize the latter as the contact instanton equation. Recall that the former relation is a property true of any contact instanton.

We conclude that (1.57) is equivalent to the contact instanton equation (1.54) imposed on the constant value A_Σ. □

Note that the result does not hold without the temporal-gauge assumption. Even though connections over a cone can always be brought to temporal gauge [54], it is not necessarily possible to achieve this through a dilation invariant gauge transformation. The procedure can spoil the dilation invariance of the instanton under consideration.

1.2.3 Spin(7) Instantons

Let (X^8, Φ) be a Spin(7) manifold. Recall from section 1 that $\Lambda^2\mathbb{O}^\star$ decomposes into Spin(7)-irreducible pieces:

$$\Lambda^2\mathbb{O}^\star = \Lambda^2_7 \oplus \Lambda^2_{21}. \tag{1.58}$$

Reasoning as we did for the HYM equations, we obtain a vector bundle splitting:

$$\Lambda^2 T^\star X^8 = \Lambda^2_7 \oplus \Lambda^2_{21}.$$

The Spin(7) instantons are special Yang-Mills connections defined using this splitting.

Definition 1.12. Let (X^8, Φ) be a Spin(7) manifold. Let P be a principal G-bundle over X^8. A connection $A \in \mathcal{A}(P)$ is a Spin(7) instanton if:

$$F_A \in C^\infty\left(\Lambda^2_{21} \otimes \mathrm{ad}(P)\right).$$

Interest in these connections stems from the hope that integrals over their moduli spaces will produce enumerative invariants of CY fourfolds/ Spin(7) manifolds [17]. Some general moduli theory has been developed over compact bases. Lewis [40] proposes an index formula for the virtual dimension, which is corrected and clarified by Walpuski in [79]. Orientability is studied in Munoz-Shahbazi [51] and Joyce [85]. Transversality is studied in Munoz-Shahbazi [52]. The first examples of Spin(7) instantons over a compact base were constructed by Lewis [40] using gluing techniques. Further examples were constructed by Tanaka [68]. In both cases, the underlying spaces are Joyce's Spin(7) manifolds [36]. Examples over noncompact bases are easier to come by (Fubini [27], Clarke-Oliveira [9], Clarke [8]).

Recalling our study of the flat model, we find that when the Spin(7) structure on X^8 is obtained from a CY-4 structure (g, J, ω, Ω), the two associated vector bundle decompositions are related by:

$$\Lambda^2_7 = \mathrm{Span}_{\mathbb{R}}(\omega) \oplus \mathcal{C},$$
$$\Lambda^2_{21} = \Lambda^{1,1}_0 \oplus \mathcal{B}.$$

An immediate consequence is that HYM connections are automatically Spin(7): the HYM equations are stronger. We shall shortly find that Spin(7) instantons are Yang-Mills and in fact—over a compact base—precisely the minimizers of the Yang-Mills action functional. It will then follow that HYM connections are Yang-Mills minimizers as well.

The following characterization of Λ^2_{21} can be derived from our analysis of the flat model:

$$\Lambda^2_{21} = \left\{ \omega \in \Lambda^2 T^\star X^8 \text{ s.t. } \star_g (\Phi \wedge \omega) = -\omega \right\}.$$

It follows that the Spin(7) instantons are characterized by the PDE:

$$\star_g F_A = -\Phi \wedge F_A. \tag{1.59}$$

This is known as the *Spin(7) instanton equation*. We have the following immediate observation:

Proposition 1.13. *Spin(7) instantons are Yang-Mills.*

Proof. This is a simple calculation reminiscent of the corresponding calculation for the ASD case:

$$\begin{aligned} d_A^\star F_A &= - \star_g d_A \star_g F_A \\ &= - \star_g d_A \left(-\Phi \wedge F_A\right) \\ &= \star_g \left(d\Phi \wedge F_A + \Phi \wedge d_A F_A\right) \\ &= 0 \end{aligned} \tag{1.60}$$

In the final step (1.60) we used the torsion-freeness of the Spin(7) structure and the gauge-theoretic differential Bianchi identity. □

On a compact manifold without boundary, proposition (1.13) can be significantly strengthened (Lewis [40] prop. 3.1):

Theorem 1.14. *Let (M, Φ) be a compact Spin(7) manifold without boundary. Let G be a compact linear Lie group and P a principal G-bundle over X^8. Let $A \in \mathcal{A}(P)$. We have that:*

$$\mathcal{YM}(A) = \mathcal{Q}(P, [\Phi]) + 4 \int_M |\pi_7^2 F_A|^2 dV_g, \tag{1.61}$$

where $\mathcal{Q}(P, [\Phi])$ is a quantity independent of A and determined only by the Spin(7) structure Φ and the topology of the bundle. In particular:

$$\mathcal{Q}(P, [\Phi]) = 8\pi^2 \int_M p_1(P) \cup [\Phi]$$

Proof. The proof relies on the following pointwise identity:

$$- \operatorname{Tr}\left(\alpha \wedge \beta\right) \wedge \Phi = \Big(\langle \alpha, \pi_{21}^2 \beta\rangle - 3\langle \alpha, \pi_7^2 \beta\rangle\Big)\, dV_g. \tag{1.62}$$

It can be easily verified in the flat model by expressing both sides in the standard frame and using our explicit calculation of the projectors.

Using (1.62), pass to the associated quadratic form and integrate to obtain:

$$- \int_{X^8} \operatorname{Tr}(F_A^2) \wedge \Phi = \int_{X^8} |\pi_{21}^2 F_A|^2 \mathrm{dV}_g - 3 \int_{X^8} |\pi_7^2 F_A|^2 \mathrm{dV}_g.$$

Recalling that:

$$p_1(P) = -\frac{1}{8\pi^2}\mathrm{Tr}(F_A^2),$$

we rewrite this as:

$$8\pi^2 \int_{X^8} p_1(P) \cup [\Phi] = \int_{X^8} |\pi^2_{21} F_A|^2 \mathrm{dV}_g - 3\int_{X^8} |\pi^2_7 F_A|^2 \mathrm{dV}_g.$$

Recalling the definition of $\mathcal{Q}(P, [\Phi])$ appearing in the statement of the theorem, we rewrite this as:

$$\mathcal{Q}(P, [\Phi]) = \int_{X^8} |\pi^2_{21} F_A|^2 \mathrm{dV}_g - 3\int_{X^8} |\pi^2_7 F_A|^2 \mathrm{dV}_g. \tag{1.63}$$

Finally, we recall that the decomposition of Λ^2 is orthogonal so that for any $A \in \mathcal{A}(P)$ we have:

$$\mathcal{YM}(A) = \int_{X^8} |\pi^2_{21} F_A|^2 \mathrm{dV}_g + \int_{X^8} |\pi^2_7 F_A|^2 \mathrm{dV}_g. \tag{1.64}$$

Identity (1.61) follows by combining (1.63) and (1.64). □

Corollary 1.15. *Let (M, Φ) be a compact Spin(7) manifold without boundary. Let G be a compact Lie group and P a principal G-bundle over X^8. The Spin(7) instantons are precisely the absolute minimizers of the Yang-Mills action functional on $\mathcal{A}(P)$.*

Proof. The Yang-Mills energy of a connection A is given by (1.61). The Spin(7) instantons are precisely the connections for which the second term vanishes. As such, if A is any connection and $A_{\mathrm{Spin}(7)}$ is a Spin(7) instanton, we have:

$$\mathcal{YM}(A_{\mathrm{Spin}(7)}) \leq \mathcal{YM}\left(A\right).$$

In fact, the Spin(7) instantons all share the same minimal Yang-Mills energy equal to:

$$\mathcal{YM}_{\min} = \mathcal{Q}(P, [\Phi]).$$

□

1.2.4 G_2 Instantons and Conical Spin(7) Instantons

Let (Σ^7, ϕ) be a nearly parallel G_2 manifold.

Recall from section 1 that the G_2 structure induces a decomposition of $\Lambda^2\mathfrak{Im}(\mathbb{O})^\star$ into G_2-irreducible pieces:

$$\Lambda^2\mathfrak{Im}(\mathbb{O})^\star = \Lambda^2_7 \oplus \Lambda^2_{14}. \tag{1.65}$$

This globalizes to a vector bundle splitting:

$$\Lambda^2 T^\star\Sigma^7 = \Lambda^2_7 \oplus \Lambda^2_{14}.$$

The G_2 instantons are special Yang-Mills connections defined using this splitting.

Definition 1.16. Let (Σ^7, ϕ) be a nearly parallel G_2 manifold. Let P be a principal G-bundle over M. A connection $A \in \mathcal{A}(P)$ is a G_2 instanton if:

$$F_A \in C^\infty\left(\Lambda^2_{14} \otimes \mathrm{ad}(P)\right).$$

In recent years, a considerable volume of work has been completed both on the construction (Walpuski [77], [81], [19], [78], Walpuski-Sá Earp [20], [59], Lotay-Oliveira [43], Clarke [8]) and the deformation theory (Driscoll [18], Singhal [65], [64], Waldron [76], Alexandrov-Semmelmann [2]) of G_2 instantons. Interest in the subject can be primarily attributed to the Donaldson-Segal program [16].

Of course, the G_2 instanton problem can be set up on an honest torsion-free G_2 manifold. For the purposes of this thesis we restrict attention to the nearly parallel setting.

Recalling our study of the flat model, we find that when the G_2 structure on Σ^7 is obtained canonically from a Sasaki-Einstein structure $(g, \eta, \xi, \mathcal{J})$, the two associated vector bundle decompositions are related by:

$$\Lambda^2_7 = \mathrm{Span}_{\mathbb{R}}(d\eta) \oplus \Lambda^2_6,$$
$$\Lambda^2_{14} = \Lambda^2_8 \oplus \Lambda^2_V.$$

An immediate consequence is that contact instantons are automatically G_2. In fact, in the compact Sasaki-Einstein setting, a connection A is a contact instanton if and only if it is a G_2 instanton with respect to all associative calibrations in the U(1)-family of compatible nearly parallel G_2 structures on Σ^7 (Singhal [64]).

The following characterization of Λ^2_{21} can be derived from our analysis of the flat model:

$$\begin{aligned} \Lambda^2_{14} &= \left\{\omega \in \Lambda^2 T^\star \Sigma^7 \text{ s.t. } \star_g (\phi \wedge \omega) = -\omega\right\} \\ &= \left\{\omega \in \Lambda^2 T^\star \Sigma^7 \text{ s.t. } \omega \wedge \psi = 0\right\}. \end{aligned}$$

It follows that the G_2 instantons are characterized by either of the PDEs:

$$\begin{aligned} \star_g F_A &= -\phi \wedge F_A. \\ F_A \wedge \psi &= 0. \end{aligned} \tag{1.66}$$

Either of these two equivalent systems is referred to as the G_2 *instanton equation.* We have the following immediate observation:

Proposition 1.17. *G_2 instantons are Yang-Mills.*

Proof. This is a simple calculation reminiscent of the corresponding calculation for the ASD case:

$$\begin{aligned} d_A^\star F_A &= \star_g d_A \star_g F_A \\ &= \star_g d_A (-\phi \wedge F_A) \\ &= - \star_g (d\phi \wedge F_A + \phi \wedge d_A F_A) \\ &= -4 \star_g (\psi \wedge F_A) \\ &= 0 \end{aligned}$$

where we used the differential Bianchi identity, the nearly parallel condition for the G_2 structure and the G_2 instanton equations in both of their forms. □

Suppose that Σ^7 is closed. If the G_2 structure ϕ is torsion-free, the G_2 instantons are precisely the Yang-Mills minimizers. This follows by topological bounds obtained in the same fashion as the ones in the Spin(7) case. The proof does not carry over to the nearly parallel setting as the associative calibration ϕ fails to define a cohomology class. In fact, G_2 instantons over nearly parallel G_2 manifolds need not be minimizing. A counterexample is given by the canonical invariant connection on the isotropy bundle of S^7 corresponding to its exceptional homogeneous structure:

$$S^7 = \frac{\mathrm{Spin}(7)}{G_2}.$$

The bundle is trivial, implying that the minimal energy is equal to 0. The canonical connection is a non-flat G_2 instanton. See section 4 for a detailed exposition.

As an aside, note that this yields an example of a non-minimizing Yang-Mills field. A result of Bourguignon and Lawson ([6], p.216, Thm 7.7) guarantees that it is a saddle point for the Yang-Mills action. Non-minimizing Yang-Mills fields were first constructed by Parker [57] in dimension 4. Parker's examples are of cohomogeneity-one. The example on S^7 is homogeneous and therefore more elementary.

Recall that the cone $\mathcal{C}(\Sigma^7)$ has a natural Spin(7) structure. The bundle P can be pulled back to $\mathcal{C}(\Sigma^7)$ through the natural map. The G_2 instanton equations arise as a dimensional reduction of the Spin(7) instanton equations along the radial direction in $\mathcal{C}(\Sigma^7)$.

Proposition 1.18. *A dilation invariant connection $A \in \mathcal{A}(P)$ in temporal gauge over $\mathcal{C}(\Sigma^7)$ is a* Spin(7) *instanton if and only if its value on the link Σ^7 is a G_2 instanton.*

Proof. A is given by a 1-parameter family of connections A_r over Σ^7, parameterized by the values of the radius function. We compute:

$$F_A = dr \wedge \partial_r A + \bar{F}_{A_r},$$

where the second summand is the curvature of A_r along Σ^7_r.

Dilation invariance implies that the first term is annihilated, and that all the A_r are equal to the same connection over Σ^7. We denote this constant value by A_Σ. We then have:

$$F_A = F_{A_\Sigma}.$$

Recalling that:

$$\Phi = dr \wedge r^3\phi + r^4\psi,$$

and expressing the Hodge-star on the cone with respect to the Hodge-star on its link, we find that the Spin(7) instanton equation (1.59) takes the form:

$$dr \wedge r^3 \star_{\Sigma^7} F_{A_\Sigma} = -dr \wedge r^3 \phi \wedge F_{A_\Sigma} - r^4 \psi \wedge F_{A_\Sigma}. \tag{1.67}$$

The link Σ^7 sits inside the cone at radius $r = 1$. Restricting to this radius and equating radial components we obtain:

$$\star_{\Sigma^7} F_{A_\Sigma} = -\phi \wedge F_{A_\Sigma}.$$

Equating tangential components recovers:

$$\psi \wedge F_{A_\Sigma} = 0.$$

It follows that (1.67) is equivalent to the G_2 instanton equation (1.66) for the constant value A_Σ. □

As in the HYM case, the result does not hold without the temporal-gauge assumption. The full dimensional reduction of the Spin(7) instanton equation is captured by the more general G_2-monopole equation (see section 1.2.7.1).

1.2.5 Relationship of the Equations over a Compact Base: Lewis's Energy Estimate

Let X^8 be a CY fourfold. We have seen that the SU(4) structure gives rise to the HYM system. Similarly, the induced Spin(7) structure gives rise to the Spin(7) instanton system. There are thus two natural gauge-theoretic equations available on X^8. As noted earlier, it is immediate from the definitions that the HYM condition is stronger. Lewis ([40] Thm. 3.1) establishes the converse when the base is compact and HYM solutions exist (so that a certain cohomological invariant vanishes).

Recall that the curvature tensors of Spin(7) instantons take values in:

$$\Lambda^2_{21} \subset \Lambda^2.$$

This space further decomposes under the SU(4) structure:

$$\Lambda^2_7 = \Lambda^{1,1}_0 \oplus \mathcal{B}.$$

The HYM connections are characterized by having vanishing component in the $\mathcal{B}$ summand. Motivated by this, we introduce the failure of a Spin(7) instanton $A \in \mathcal{A}(P)$ to be HYM:

$$\mathcal{E}(A) \stackrel{\text{def}}{=} \int_{X^8} |\pi^2_{\mathcal{B}} F_A|^2 dV_g.$$

We then have the following theorem due to Lewis.

Theorem 1.19 (Lewis's Energy Estimate)**.** *Let $(X^8, g, J, \omega, \Omega)$ be a closed CY fourfold, G a compact Lie group and P a principal G-bundle over X^8. All smooth Spin(7) instantons on P share the same finite failure from being HYM. This quantity is a cohomological invariant, depending on the topology of the bundle and the geometry of the base.*

Proof. The proof relies on the following pointwise identity:

$$-\operatorname{Tr}(\alpha \wedge \beta) \wedge \mathfrak{Re}(\Omega) = \Big(2\langle \alpha, \pi^2_{\mathcal{B}}\beta\rangle - 2\langle \alpha, \pi^2_{\mathcal{C}}\beta\rangle\Big)\, dV_g. \tag{1.68}$$

It can be easily verified in the flat model by expressing both sides in the standard frame and using our explicit calculation of the projectors.

Using (1.68), pass to the associated quadratic form and integrate to obtain:

$$-\int_{X^8} \operatorname{Tr}(F_A^2) \wedge \mathfrak{Re}(\Omega) = 2\int_{X^8} |\pi^2_{\mathcal{B}} F_A|^2 \mathrm{dV}_g - 2\int_{X^8} |\pi^2_{\mathcal{C}} F_A|^2 \mathrm{dV}_g.$$

If A is a Spin(7) instanton, its curvature tensor has vanishing Λ^2_7 component and therefore vanishing $\mathcal{C} \subset \Lambda^2_7$ component. Consequently, Spin(7) instantons satisfy:

$$\begin{aligned}
\mathcal{E}(A) &= \int_{X^8} |\pi^2_{\mathcal{B}} F_A|^2 \mathrm{dV}_g \\
&= -\frac{1}{2}\int_{X^8} \operatorname{Tr}(F_A^2) \wedge \mathfrak{Re}(\Omega) \\
&= 4\pi^2 \int_{X^8} p_1(P) \cup [\mathfrak{Re}(\Omega)].
\end{aligned}$$

The proof is complete since the right hand side depends only on the topology of the bundle and the geometry of the base. □

We immediately obtain the following:

Corollary 1.20. *Let $(X^8, g, J, \omega, \Omega)$ be a closed CY fourfold and P a principal G-bundle over X^8. Suppose that:*

$$p_1(P) \cup [\mathfrak{Re}(\Omega)] = 0 \text{ in } H^4(X^8). \tag{1.69}$$

Then all Spin(7) instantons on P are HYM.

The following slightly weaker result will be relevant in the sequel:

Corollary 1.21. *Let $(X^8, g, J, \omega, \Omega)$ be a closed CY fourfold, G a compact Lie group and P a principal G-bundle over X^8. Suppose that P admits an HYM connection. Then all Spin(7) instantons on P are HYM.*

Proof. The failure is a topological invariant. The HYM connections have vanishing failure. Let A be the HYM connection promised by the assumptions. Use this to compute the invariant and find that it vanishes. This forces all Spin(7) instantons to be HYM. □

1.2.6 The Dirac Operator on a Spin(7) Manifold and the Linearized Spin(7) Instanton Equation

Let (X^8, Φ) be a Spin(7) manifold. Since Spin(7) is simply connected, we can lift the inclusion into SO(8) through the natural covering:

$$\begin{array}{ccc} & & \mathrm{Spin}(8) \\ & \overset{\tilde{\iota}}{\nearrow} & \downarrow \pi \\ \mathrm{Spin}(7) & \xrightarrow[\iota]{} & \mathrm{SO}(8) \end{array}$$

This allows us to associate a spin structure to the Spin(7) structure $P_\Phi \subset \mathrm{Fr}_{\mathrm{SO}}(TX^8)$ (consisting of those orthonormal frames that restore Φ to its standard form):

$$\mathrm{Spin}(TX^8) \overset{\mathrm{def}}{=} P_\Phi \times_{\tilde{\iota}} \mathrm{Spin}(8)$$

It follows that X^8 is naturally spin.

The results of section (1.1.2.4) allow us to get a more concrete handle on the corresponding spinor bundle. In particular, we set:

$$\not{S} \stackrel{\text{def}}{=} \not{S}^+ \oplus \not{S}^-.$$

where:

$$\not{S}^+ \stackrel{\text{def}}{=} \Lambda^0 \oplus \Lambda^2_7,$$
$$\not{S}^- \stackrel{\text{def}}{=} \Lambda^1.$$

We then define a Clifford action of TX^8 on $\not{S}$ interchanging the factors. Fix $p \in X^8$, we view each tangent vector $w \in T_pX^8$ as a map:

$$w : \not{S}^+_p \to \not{S}^-_p$$

by declaring:

$$w \cdot (\lambda, \eta) = \lambda g\,(w, \cdot) + 2\eta\,(w, \cdot)$$

and

$$w : \not{S}^-_p \to \not{S}^+_p$$

by declaring:

$$w \cdot \alpha = \left(-g(w, \alpha^{\#}), -\frac{1}{2} w^\flat \wedge \alpha - \frac{1}{2} \Phi\,(u, v, \cdot, \cdot) \right).$$

Note that the adjoint of:

$$w : \not{S}^+ \to \not{S}^-$$

is given by:

$$-\, w : \not{S}^- \to \not{S}^+.$$

The spin connection ∇^{spin} on $\not{S}$ is obtained by lifting the Levi-Civita connection ∇^{LC} to the spin structure $\text{Spin}(TX^8)$ and then pushing it forward to the Dirac bundle $\not{S}$. Here we view $\not{S}$ as being associated to $\text{Spin}(TX^8)$ via the spin representation. Using the fact that the constant function $f = 1$ is a harmonic spinor, ∇^{spin} can be seen to agree with ∇^{LC} on $\Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2_7$. The corresponding negative Dirac operator can be computed directly using the standard formula:

$$\not{D} = \sum_{i=1}^{8} e_i \cdot \nabla_{e_i}$$

and the explicit characterization of the Clifford action. Ultimately, one finds that it is given by:

$$\begin{aligned} \not{D}^- : \; & C^\infty(\$^-) \to C^\infty(\$^+) \\ & \alpha \mapsto (d^\star \alpha, -2\pi^2_7 d\alpha). \end{aligned} \tag{1.70}$$

Taking adjoints, we find that the positive Dirac operator is given by:

$$\begin{aligned} \not{D}^+ : \; & C^\infty(\$^+) \to C^\infty(\$^-) \\ & (f, \omega) \mapsto df - 2d^\star \omega. \end{aligned} \tag{1.71}$$

When we twist the spin structure by a vector bundle (E, ∇), the Dirac operators combine with ∇ to yield twisted Dirac operators $\not{D}^-_\nabla$ and $\not{D}^+_\nabla$. They can be written down by replacing exterior derivatives with exterior covariant derivatives in the formulae expressing $\not{D}^-$ and $\not{D}^+$.

Fix a Spin(7) instanton $A \in \mathcal{A}(P)$. The Spin(7) instanton equations (1.59) correspond to looking at the zero level-set of the nonlinear operator

$$\begin{aligned} \mathcal{F}_A : \; & C^\infty(\Lambda^1 \otimes \mathrm{ad}(P)) \to C^\infty(\Lambda^2_7 \otimes \mathrm{ad}(P)) \\ & \alpha \mapsto \pi^2_7 F_{A+\alpha} \end{aligned}$$

Using the standard formula for the curvature of the perturbation of a connection by an arbitrary $\mathrm{ad}(P)$-valued 1-form, we find that:

$$\mathcal{F}_A(\alpha) = \pi^2_7 d_A \alpha + \frac{1}{2}\pi^2_7 [\alpha, \alpha].$$

Consequently, the linearization of the Spin(7) instanton equation $\mathcal{F}_A \alpha = 0$ centered at the Spin(7) instanton $A \in \mathcal{A}(P)$ is given by:

$$d\mathcal{F}_A : \alpha \mapsto \pi^2_7 d_A \alpha.$$

We incorporate the Coulomb gauge-fixing condition:

$$d_A^\star \alpha = 0. \tag{1.72}$$

This guarantees—at least in the closed manifold setting—that any potential infinitesimal deformations are "honest" in the sense that they are L^2-orthogonal to the directions spanned by the infinitesimal gauge-action centered at A. Since the exterior covariant derivative and exterior covariant coderivative take values in different spaces, we have that:

$$d\mathcal{F}_A \alpha = 0 \text{ and } d_A^\star \alpha = 0 \iff \not{D}_A^- \alpha = 0,$$

where $\not{D}_A^-$ denotes the negative Dirac operator (1.70) twisted by A. It follows that gauge-fixed (not induced by gauge transformations) infinitesimal deformations of A to nearby Spin(7) instantons correspond to (twisted) negative harmonic spinors on X^8. In the sequel we will slightly abuse terminology by suppressing the epithet *gauge-fixed* to refer to

$$\not{D}_A^- \alpha = 0 \tag{1.73}$$

as the *linearized Spin(7) instanton equation.*

Recall that the symbol of a Dirac operator is given by the fiberwise action of cotangent vectors through Clifford multiplication. This is invertible and the Dirac operator is then elliptic. We thus obtain ellipticity for the (gauge-fixed) Spin(7) instanton equation. On an AC space, it is *uniformly elliptic* and asymptotic to the linearized G_2-monopole operator in the sense of [42], [46]. The upshot is that we do not have to worry about regularity issues.

The above can be neatly packaged in a three-term deformation complex, reminiscent of the AHS complex from Donaldson theory ([3],[15]):

$$0 \longrightarrow C^\infty(\mathrm{ad}(P)) \xrightarrow{d_A} C^\infty(\Lambda^1 \otimes \mathrm{ad}(P)) \xrightarrow{\pi^2_7 d_A} C^\infty(\Lambda^2_7 \otimes \mathrm{ad}(P)) \longrightarrow 0.$$

Folding and considering our earlier observations, we immediately find that this is elliptic.

Over a closed base, the index is easily computed using the Atiyah-Singer index theorem ([79] p.7 eq 2.24):

$$\text{Index} = \text{rk}(\text{ad}(P))(-b^0 + b^1 - b^2_7) + \frac{1}{24}\left\langle p_1(P) \cup p_1(TX^8) - 2p_1(P)^2 + 4p_2(P), [X^8]\right\rangle.$$

Over a noncompact base, there are geometric (spectral) contributions to the index. These are determined by the APS theorem (Melrose [47], Atiyah-Patodi-Singer [48], [49], [50]). Since gauge-fixed infinitesimal deformations/ obstructions are controlled by Dirac operators, Weitzenböck-type formulae become available tools for tackling the linear problem. In this setting they take the following form:

On $\not{S}^- \otimes \text{ad}(P)$ we have:

$$\not{D}^+_A \not{D}^-_A \alpha = \nabla^\star_A \nabla_A \alpha + 2\mathcal{R}^{\not{S}^-}_A[\alpha]. \tag{1.74}$$

Here, the curvature error term $\mathcal{R}^{\not{S}^-}_A$ is the order zero operator:

$$\begin{aligned} \mathcal{R}^{\not{S}^-}_A : \Lambda^1 \otimes \text{ad}(P) &\to \Lambda^1 \otimes \text{ad}(P) \\ \alpha &\mapsto \mathcal{R}^{\not{S}^-}_A[\alpha] \end{aligned}$$

defined as:

$$\begin{aligned} \mathcal{R}^{\not{S}^-}_A[\alpha](X) &\overset{\text{def}}{=} -\left[F_A \llcorner \alpha\right](X) \\ &= \sum_i \left[F_A(e_i, X), \alpha(e_i)\right]. \end{aligned}$$

The sum is taken over an orthonormal frame for the tangent space.

On $\not{S}^+ \otimes \text{ad}(P)$ we have:

$$\not{D}^-_A \not{D}^+_A (f,\omega) = \nabla^\star_A \nabla_A (f,\omega) + 2\mathcal{R}^{\not{S}^+}_A[\omega]. \tag{1.75}$$

Here, the curvature error term on $\not{S}^{+} \otimes \mathrm{ad}(P)$ is the order zero operator:

$$\mathcal{R}_A^{\not{S}^+} : \Lambda^2_7 \otimes \mathrm{ad}(P) \to \Lambda^2_7 \otimes \mathrm{ad}(P)$$
$$\omega \mapsto \mathcal{R}_A^{\not{S}^+} [\omega]$$

defined as:

$$\mathcal{R}_A^{\not{S}^+} [\omega] = \pi^2_7 R_A^{\not{S}^+} [\omega] \,,$$

where:

$$R_A^{\not{S}^+} [\omega] (X, Y) = \sum_i \left[F_A (e_i, X) \wedge \omega (e_i, Y) \right].$$

The formulae we have obtained are strongly reminiscent of the ones derived by Bourguignon and Lawson in [6].

1.2.7 The Dirac Operator on a G_2 Manifold and the Linearized G_2 Instanton Equation

We will perform a similar analysis for G_2 manifolds. We begin with the simpler torsion-free case and proceed to the nearly parallel case once this is settled.

1.2.7.1 The Torsion-Free Case

Let (Σ^7, ϕ) be a G_2 manifold. An argument analogous to the one employed in the Spin(7) setting—using the fact that G_2 is is simply connected—demonstrates that Σ^7 is naturally spin. The results of section (1.1.2.4) allow us to get a concrete handle on the corresponding spinor bundle. In particular, we set:

$$\not{S} = \Lambda^0 \oplus \Lambda^1. \tag{1.76}$$

Fix $p \in \Sigma^7$. Each $v \in V$ is realized as an endomorphism:

$$v : \not{S}_p \to \not{S}_p$$

by declaring:

$$v (\lambda, \alpha) = \left(-g(v, \alpha^\#), \lambda v^\flat + (v \times \alpha^\#)^\flat \right).$$

This defines a skew-adjoint Clifford action of $T\Sigma^7$ on $\not{S}$.

The connection on $\not{S}$ is obtained by lifting the Levi-Civita connection to a principal connection on the spin structure and pushing it forward to the Dirac bundle using the associated vector bundle construction. Using the fact that the constant function $f = 1$ is a harmonic spinor, ∇^{spin} can be seen to agree with the Levi-Civita connection on $\Lambda^0 \oplus \Lambda^1$.

The corresponding Dirac operator can be computed directly using the standard formula:

$$\not{D} = \sum_{i=1}^{7} e_i \cdot \nabla_{e_i}.$$

Carrying out the calculation yields:

$$\begin{aligned} \not{D}:\ & C^\infty(\not{S}) \to C^\infty(\not{S}) \\ & f + \alpha \mapsto d^\star \alpha + df + \star(\psi \wedge d\alpha). \end{aligned} \tag{1.77}$$

This is concisely presented in matrix form using the splitting $\not{S} = \Lambda^0 \oplus \Lambda^1$:

$$\not{D} = \begin{pmatrix} 0 & d^\star \\ d & \star_\Sigma(\psi \wedge d(\cdot)) \end{pmatrix}$$

Twisting the spin structure by a vector bundle (E, ∇) affects the Dirac operator by replacing exterior derivatives with exterior covariant derivatives:

$$\not{D}_\nabla = \begin{pmatrix} 0 & d_\nabla^\star \\ d_\nabla & \star_\Sigma(\psi \wedge d_\nabla(\cdot)) \end{pmatrix}$$

Let $A \in \mathcal{A}(P)$ be a G_2 instanton. Working as we did in the previous section, we find that the linearized (at A) G_2 instanton equation takes the form:

$$\psi \wedge d_A \alpha = 0. \tag{1.78}$$

As it stands, equation (1.78) is not elliptic. This issue persists even when we supplement it with the Coulomb gauge-fixing condition. Fortunately, the deformation theory *is* governed by an elliptic complex, and is thus well-behaved. The catch is that in contrast to the familiar ASD and Spin(7) pictures, one has to use more than three terms. Following [13], we introduce the differential complex:

$$\begin{array}{ccccc} 0 & \longrightarrow & C^{\infty}(\mathrm{ad}(P)) & \xrightarrow{d_A} & C^{\infty}(\Lambda^1 \otimes \mathrm{ad}(P)) \\ & & & & \Big\downarrow \psi \wedge d_A(\cdot) \\ 0 & \longleftarrow & C^{\infty}(\Lambda^7 \otimes \mathrm{ad}(P)) & \xleftarrow[d_A]{} & C^{\infty}(\Lambda^6 \otimes \mathrm{ad}(P)) \end{array}$$

Passing to the level of symbols, exactness can be verified by direct computation. It follows that the complex is elliptic. Over a closed base, its index is zero: when the moduli space is smooth, it is zero-dimensional. This will always be the case in the applications we have in mind: Σ^7 will be the asymptotic link of an AC CY fourfold.

If one insists on working with a single elliptic system, the G_2 instanton equations should be replaced by the G_2 *monopole equations.* The latter arise naturally as the dimensional reduction of the Spin(7) instanton equations along the radial direction of a cone.

Given a pair (A, Φ), where $A \in \mathcal{A}(P)$ and $\Phi \in C^{\infty}(\mathrm{ad}(P))$, the G_2-monopole equations read:

$$\star\,(\psi \wedge F_A) = -\nabla_A \Phi.$$

It is clear that each G_2 instanton defines a G_2-monopole by taking Φ to vanish identically. In fact, any Higgs field that is covariantly constant with respect to the instanton will work.

Suppose now that Σ^7 is closed. Applying $\nabla_A^{\star}$ to both sides of the G_2 monopole equation and integrating by parts shows that the Higgs field Φ of any G_2 monopole (A, Φ) is covariantly constant:

$$\nabla_A \Phi = 0.$$

The G_2 monopole equation then implies that A is a G_2 instanton.

$$\star\,(\psi \wedge F_A) = -\nabla_A \Phi = 0.$$

This yields the following characterization of the space of G_2 monopoles over Σ^7: they form a fibration over the space of G_2 instantons, where the fiber over A is given by:

$$\mathrm{Ker}(\nabla_A) = \mathrm{Lie}\left(\mathrm{Stab}\,(A)\right) \subset \mathrm{Lie}\left(\mathcal{G}\,(P)\right) = C^\infty\left(\mathrm{ad}(P)\right).$$

In particular, if the structure group has trivial center, then the fiber over every irreducible G_2 instanton is given by a single point. This results in a one-to-one correspondence between irreducible G_2 instantons and irreducible G_2 monopoles.

Linearizing the G_2-monopole equation at a G_2 instanton $(A, 0)$, we obtain:

$$\nabla_A f + \star(\psi \wedge d_A\alpha) = 0$$

Incorporating the Coulomb gauge-fixing condition, we get:

$$d_A^\star\alpha + \nabla_A f + \star(\psi \wedge d_A\alpha) = 0.$$

We recognize the left hand side of this equation as the (twisted) Dirac operator $\not{D}_A$. We conclude that the gauge-fixed G_2 monopole equation is elliptic. Furthermore, gauge-fixed infinitesimal deformations of a G_2 instanton A through G_2 monopoles correspond to (twisted) $\not{D}_A$-harmonic spinors (f, α). The Λ^0 component f corresponds to an infinitesimal motion along the fiber parameterizing the compatible Higgs-fields. The Λ^1 component α corresponds to an infinitesimal deformation of the underlying instanton. Over a closed base, simple integration by parts shows that:

$$\not{D}_A(f, \alpha) = 0 \Rightarrow \not{D}_A(0, \alpha) = 0. \tag{1.79}$$

In [76], Waldron derives a Weitzenböck-type formula for the square of the linearized G_2-monopole equation on nearly parallel G_2-manifolds ([76] Prop. 2.8). Restricting this to the torsion-free case, we find that the operator $\not{D}_A$ satisfies:

$$\not{D}_A^2(f, \alpha) = \nabla_A^\star\nabla_A f + +\nabla_A^\star\nabla_A\alpha - 2\left[F_A \llcorner \alpha\right].$$

1.2.7.2 The Nearly Parallel Case

Suppose now that ϕ is nearly parallel. Most of the above carry over so that we obtain a natural spin structure $\$$ and a fiberwise action of $T\Sigma^7$ defined in the exact same way. However, the constant function 1 is now not a harmonic spinor, but a $-\frac{1}{2}$-Killing spinor:

$$\begin{aligned}\nabla_X 1 &= -\frac{1}{2} X \cdot f \\ &= -\frac{1}{2} g(X, \cdot).\end{aligned} \tag{1.80}$$

It follows that when we view spinors as differential forms using our explicit construction (1.76), the lift of the Levi-Civita connection to the spin connection on $\$$ disagrees with the Levi-Civita connection on $\Lambda^0 \oplus \Lambda^1$. In fact, we see that—unlike the torsion-free case—the spin connection no longer preserves the splitting (1.76).

Equation (1.80) determines the order 0 offset of ∇^{spin} from ∇^{LC} on Λ^0:

$$\nabla_X^{\text{spin}} f = Xf - \frac{f}{2}\, g(X, \cdot).$$

To determine ∇^{spin} on Λ^1 we follow the same strategy as the one employed in the torsion-free case. This time we obtain order 0 corrections since 1 is not harmonic, but Killing. Using the explicit formula for the Clifford action $\cdot$, the fact that $\cdot$ is covariantly constant for ∇^{spin} and the fact that 1 is a $-\frac{1}{2}$-Killing spinor, we obtain:

$$\begin{aligned}\nabla_X^{\text{spin}} v = \nabla_X^{\text{spin}} (v^\# \cdot 1) &= \\ &= \nabla_X^{\text{LC}} v \cdot 1 + v^\# \cdot \nabla^{\text{spin}} 1 \\ &= \nabla_X^{\text{LC}} v - \frac{1}{2} v^\# \cdot X \cdot 1 \\ &= \nabla_X^{\text{LC}} v - \frac{1}{2} v^\# \cdot g(X, \cdot) \\ &= \nabla_X^{\text{LC}} v + \frac{1}{2} g(v^\#, X) - \frac{1}{2} (v^\# \times X)^\flat \\ &= \nabla_X^{\text{LC}} v + \frac{1}{2} g(v^\#, X) - \frac{1}{2} \phi(v^\sharp, X, \cdot) \\ &= \frac{1}{2} v(X) + \nabla_X^{\text{LC}} v + \frac{1}{2} \phi(X, v^\sharp, \cdot).\end{aligned}$$

The spin Dirac operator can then be determined by summing over an orthonormal frame:

$$\not{D} = \sum_{i=1}^{8} e_i \cdot \nabla^{\text{spin}}_{e_i}.$$

Carrying out the calculation yields:

$$\not{D} = \begin{pmatrix} 0 & d^\star \\ d & \star_\Sigma(\psi \wedge d\cdot) \end{pmatrix} + \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & -\frac{5}{2} \end{pmatrix}$$

Twisting this by a connection ∇ on a vector bundle E yields:

$$\not{D}_\nabla = \begin{pmatrix} 0 & d^\star_\nabla \\ d_\nabla & \star_\Sigma(\psi \wedge d\cdot) \end{pmatrix} + \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & -\frac{5}{2} \end{pmatrix}$$

Let A be a G_2-instanton. The linearization of the G_2-instanton equation at A takes the same form as in the torsion-free case. It follows that gauge-fixed infinitesimal deformations correspond to spinors:

$$s = (0, \alpha) \in C^\infty \left(\not{S} \otimes \text{ad}(P) \right),$$

such that:

$$\not{D}_A s = -\frac{5}{2} s.$$

In particular, we identify the gauge-fixed infinitesimal deformation space for a G_2-instanton A as the intersection of the $-\frac{5}{2}$-eigenspace of $\not{D}_A$ and the space of spinors with vanishing Λ^0 component. This recovers a result of Singhal ([65] Prop. 3.1).

Waldron establishes the following identity ([76] Prop. 2.8):

$$\begin{aligned} \begin{pmatrix} 0 & d_A^\star \\ d_A & \star_\Sigma(\psi \wedge d_A\cdot) \end{pmatrix}^2 &= \begin{pmatrix} \nabla_A^\star \nabla_A & 0 \\ 0 & \nabla_A^\star \nabla_A + 2 \star_\Sigma (\psi \wedge d\cdot) + \text{Ric}(\cdot) - 2[F_A \llcorner (\cdot)] \end{pmatrix} \\ &= \begin{pmatrix} \nabla_A^\star \nabla_A & 0 \\ 0 & \nabla_A^\star \nabla_A + 2 \star_\Sigma (\psi \wedge d\cdot) + 6 - 2[F_A \llcorner (\cdot)], \end{pmatrix} \end{aligned}$$

where we recall that the nearly parallel G_2 manifolds are Einstein with Einstein constant equal to 6. This allows us to easily compute the square of the twisted Dirac operator $\not{D}_A$:

$$\not{D}_A^2 = \begin{pmatrix} \nabla_A^\star \nabla_A + \frac{49}{4} & d_A^\star \\ d_A & \nabla_A^\star \nabla_A + \frac{49}{4} - 3 \star_\Sigma (\psi \wedge d\cdot) - 2[F_A \llcorner (\cdot)] \end{pmatrix}$$

The appearance of order 1 terms is expected as the Laplacian on the spin structure:

$$\not{S} = \Lambda^0 \oplus \Lambda^1$$

disagrees with the standard one on functions and forms. This is due to the disparity between ∇^{LC} and its lift ∇^{spin}.

1.2.8 Moduli Spaces of AC Spin(7) Instantons over AC CY Fourfolds

Our final task for this section is to formally introduce the instanton moduli spaces we wish to study. Let Σ^7 be a closed Sasaki-Einstein space. Let X^8 be an AC CY fourfold asymptotic to the cone $\mathcal{C}(\Sigma^7)$. Fix $K \subset X^8$ compact, $T > 0$ and a diffeomorphism:

$$\Psi : X^8 \setminus K \xrightarrow{\sim} (T, \infty) \times \Sigma^7,$$

such that the metric on $X^8 \setminus K$ approaches the conical metric on the target. Let π_i denote the projection to the i^{th} factor and let

$$t \stackrel{\text{def}}{=} \pi_1 \circ \Psi.$$

Extend this on the whole of X^8 by smoothly interpolating between 0 and t using a bump function. Slightly abuse notation by using the same symbol for the extension. The function t is an *asymptotic logarithmic radius function.*

Let G be a compact Lie group and P_{Σ^7} a principal G-bundle over Σ^7. Let P be a principal G-bundle over X^8 such that there is a bundle isomorphism:

$$\Xi : P_{|_{X^8 \setminus K}} \xrightarrow{\sim} \Psi^\star \pi_2^\star P_{\Sigma^7}.$$

Such bundles are termed *admissible*: they respect the asymptotic geometry of the base.

Finally, let $A_\infty \in \mathcal{A}(P_{\Sigma^7})$ be a contact instanton over Σ^7.

The *weighted Sobolev space* $W^{k,2}_\mu(\mathrm{ad}(P))$ *with* k *weak derivatives, integrability* 2 *and weight* μ is defined by looking at the space of smooth sections $s \in C^\infty(\mathrm{ad}(P))$ such that:

$$\|s\|^2_{k,2,\mu} \stackrel{\text{def}}{=} \sum_{j=0}^{k} \int_{X^8} |e^{(j-\mu)t}\nabla^j s|^2 e^{-8t}\mathrm{d}V_g < \infty \tag{1.81}$$

and subsequently taking the completion under this norm. The definition is designed so that the leading order asymptotic behaviour of a section $s \in W^{k,2}_\mu(\mathrm{ad}(P))$ is:

$$|s| = O(e^{-\alpha t}), \text{ where } a < \mu. \tag{1.82}$$

For a detailed discussion of weighted Sobolev spaces see [18], [42], [46]. The latter offers a thorough account of regularity issues for elliptic operators respecting the asymptotic geometry (i.e. *uniformly elliptic operators*).

The space of $W^{k,2}_\mu$ *connections on* P *asymptotic to* A_∞ is defined by:

$$\mathcal{A}_{k,\mu}(P, A_\infty) = A_\infty + W^{k,2}_\mu(\mathrm{ad}(P)). \tag{1.83}$$

Naturally, C^∞ connections with the appropriate asymptotics are obtained by taking:

$$\mathcal{A}_\mu(P, A_\infty) \stackrel{\text{def}}{=} \cap_{k\geq 0}\mathcal{A}_{k,\mu}(P, A_\infty). \tag{1.84}$$

The corresponding space of gauge transformations is built by exponentiating decaying sections of the adjoint bundle. The resulting gauge transformations are asymptotic to the identity. For $k > 4$ (to guarantee continuity), we define:

$$\mathcal{G}_{k,\mu}(P) \stackrel{\text{def}}{=} \left\{ s = \exp(\xi) \text{ s.t. } \xi \in W^{k,2}_\mu(\mathrm{ad}(P)) \right\}. \tag{1.85}$$

The space of smooth gauge transformations is obtained by taking:

$$\mathcal{G}_\mu(P) \stackrel{\text{def}}{=} \cap_{k\geq 4}\mathcal{G}_{k,\mu}(P). \tag{1.86}$$

The gauge action preserves $\mathcal{A}_\mu(P, A_\infty)$. This allows us to define the *moduli space of AC Spin(7) instantons on P asymptotic to A_∞*:

$$\mathcal{M}_\mu(P, A_\infty) = \frac{\left\{A \in \mathcal{A}_\mu(P, A_\infty) \ \text{ s.t. } \ \star_g (\Phi \wedge F_A) = -F_A\right\}}{\mathcal{G}_\mu(P)}.$$

Minor modifications are possible and lead to other moduli spaces of interest. For instance, one could replace the Spin(7) instanton equations by the HYM equations. Alternatively, one could expand the search for AC Spin(7) instantons by looking for ones that approach a G_2 instanton, without insisting that it be contact. To justify our choices, we recall propositions (1.11) and (1.18). Cutting out $K \subset X^8$, transforming to temporal gauge, expanding at infinity and keeping track of only the highest order terms establishes the following:

Proposition 1.22. *Let X^8 be an AC CY fourfold with Sasaki-Einstein asymptotic link Σ^7. The limit of an AC Spin(7) instanton on X^8 is a G_2 instanton on Σ^7. The limit of an AC HYM connection on X^8 is a contact instanton on Σ^7.*

Since we have taken A_∞ to be contact, $\mathcal{M}_\mu(P, A_\infty)$ contains the corresponding HYM moduli space. Our primary concern is the structure of this locus. Motivated by Lewis' estimate, the most basic question one could ask is whether it exhausts the whole space. The examples constructed in section 5 establish—for the first time—that this need not be the case.

2 Homogeneous Bundles and Invariant Connections

In this section, our aim is to provide a rapid introduction to gauge theory in the presence of symmetries. After a quick overview of general homogeneous space theory we move on to homogeneous principal bundles and invariant connections. Our discussion culminates with a complete classification of these objects in terms of representation-theoretic data (Wang [82]). The underlying principle is that the high degree of symmetry enjoyed by a homogeneous space typically reduces differential geometric questions to representation theory.

2.1 Naturally Reductive Homogeneous Spaces and the Canonical Invariant Connection

Let M be a smooth manifold and G a Lie group acting on the left by diffeomorphisms. When the action is transitive M is termed a *G-homogeneous space*. Choose a reference point $p \in M$. Since M is Hausdorff, the isotropy subgroup:

$$H \overset{\text{def}}{=} \mathrm{Stab}_G(p) = \{g \in G \text{ such that } gp = p\}$$

is closed in G. The natural right action of H on G is smooth, free and proper. It is then an application of the quotient manifold theorem (Lee [38] p.545) that the space of left cosets:

$$G/H = \{gH \text{ such that } g \in G\}$$

inherits a unique smooth structure such that the natural projection is a smooth submersion (Kobayashi, Nomizu [37] p.43). Ultimately, we find that fixing $p \in M$ yields a diffeomorphism:

$$\phi : G/H \xrightarrow{\sim} M,$$

$$gH \mapsto gp.$$

Thus, without loss of generality, when discussing homogeneous spaces we can restrict our attention to left coset manifolds.

Over a homogeneous space G/H there is a natural H-principal bundle. The total space is given by G and the projection map is the canonical projection to the quotient. This is known as the *isotropy bundle.*

For our purposes, it is sufficient to consider *naturally reductive* homogeneous spaces:

Definition 2.1. A homogeneous space G/H is naturally reductive if the Lie algebra $\mathfrak{h}$ admits a vector space complement in $\mathfrak{g}$ that is stable under the restriction of Ad_G to H.

The space $\mathfrak{m}$ is known as a reductive complement. A choice of $\mathfrak{m}$ endows the canonical H-bundle with a natural connection: the *canonical invariant connection.* In particular, one notices that the left invariant extension $\mathfrak{m}$ over G is Ad_H equivariant for the right H action. Furthermore, it gives a vector space complement for the left translate of $\mathfrak{h}$ over every point i.e. it is complementary to the distribution of vertical vectors. Consequently, it is a connection.

The isotropy group H stabilises the reference point p. Differentiating the action gives us a linear representation of H on T_pM. This is known as the *isotropy representation.* The canonical invariant connection sets up a correspondence between tangent vectors on the base and tangent vectors on G. This allows us to capture the isotropy representation purely at the level of the group. We denote the representation of H on $\mathfrak{m}$ through the adjoint action of G as $(\mathfrak{m}, \mathrm{Ad}_H)$.

Proposition 2.2. *Let G/H be a naturally reductive homogeneous space. Let $\mathfrak{m}$ be a reductive complement. The isotropy representation is isomorphic to $(\mathfrak{m}, \mathrm{Ad}_{G|_H})$. In other words, the following square commutes for all $h \in H$:*

$$\begin{array}{ccc} \mathfrak{m} & \xrightarrow{\mathrm{Ad}_h} & \mathfrak{m} \\ {\scriptstyle dp_G}\downarrow & & \downarrow{\scriptstyle dp_G} \\ T_pM & \xrightarrow[dl_h]{} & T_pM \end{array}$$

2.2 Invariant Tensor Fields

We wish to describe G-invariant tensor fields over M by tensor fields over G. We select a reference point $p \in M$. Since the action is not free, the left invariant extension of a tensor T over p is not—in general—well defined. However we have an easy characterization for when it is. Essentially, the only problem is that a particular point may be connected to

p through multiple group elements, each of which translate T differently. However, all of these elements lie in the same orbit of the H-action. If H acts trivially, the issue is resolved. Recalling the correspondence between the isotropy representation and the restriction of the adjoint action to H (proposition 2.2) we obtain:

Theorem 2.3. *Let G/H be a naturally reductive homogeneous space. Let $\mathfrak{m}$ be the reductive complement. G-invariant tensor fields over M correspond to elements of (appropriate tensor powers of) $\mathfrak{m}$ stabilized by $\mathrm{Ad}_{G|_H}$.*

2.3 Homogeneous Principal Bundles

Let $M = G/H$ be a homogeneous space. Let S be a Lie group. We are interested in studying principal S-bundles over M that are compatible with its symmetry. We begin with the following definition:

Definition 2.4. A homogeneous S-bundle over M is a principal S-bundle equipped with a left action $G \to \mathcal{G}(P)$ that lifts the G-action on M. Explicitly, for any $g \in G$ we have the following commutative square:

$$\begin{array}{ccc} P & \xrightarrow{l_g} & P \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[l_g]{} & M. \end{array}$$

Note that the action on the total space is by global gauge transformations (S-bundle automorphisms), not just diffeomorphisms. The following observation is immediate from the definition:

Proposition 2.5. *Let G and S be Lie groups. Let M be a homogeneous space for G. Let P be a homogeneous principal S-bundle over M. Then P is a homogeneous space for $G \times S$.*

Proof. We need to display a transitive left action of $G \times S$ on P. Define:

$$(g, s)p \stackrel{\mathrm{def}}{=} gps^{-1}. \tag{2.1}$$

This is clearly transitive. Since the left G-action lifts the action on M, it is fiber-transitive. Since P is a principal S-bundle, the right S-action is transitive on each fiber. □

We wish to explore the homogeneous structure exhibited in proposition (2.5). We begin by computing the stabiliser. Fix a reference point $p \in P$ and note that:

$$gps^{-1} = p \iff gp = ps.$$

Since the action of S is fiber preserving and the action of G lifts the action on M, we must have that:

$$g \in \mathrm{Stab}_G(\pi(p)).$$

The base space stabilizer is isomorphic to H. Consequently:

$$\mathrm{Stab}_{G\times S}(p) \cong H. \tag{2.2}$$

Our earlier remarks now imply that (at the level of smooth manifolds):

$$P \cong \frac{G \times S}{H}. \tag{2.3}$$

We are interested in refining this conclusion by determining the precise embedding in $G \times S$. Since H preserves the fiber pS, for each $h \in H$ there is a unique $s \in S$ such that:

$$hp = ps.$$

This yields a Lie group homomorphism $\lambda : H \to S$ uniquely determined by the equation:

$$hp = p\lambda(h). \tag{2.4}$$

The map λ is known as the *isotropy homomorphism*. It depends on the choice of reference. Varying p will conjugate λ by a fixed element of S. We now observe that:

$$\mathrm{Stab}_{G\times S}(p) = \Big\{(h, \lambda(h)) \text{ such that } h \in H\Big\} < G \times S.$$

Using λ we may define the following left action of H on S:

$$hs \stackrel{\text{def}}{=} s\lambda(h)^{-1}. \tag{2.5}$$

Using the structure of G as an H-bundle over M, the action (2.5) allows us to define the associated fiber bundle with standard fiber S:

$$G \times_{(H,\lambda)} S = (G \times S)/\sim \text{, where } (g,s) \sim (gh, h^{-1}s).$$

This quotient agrees with (2.3) and thus recovers P.

We now draw from this construction to classify homogeneous bundles up to their natural notion of isomorphism. This is as follows:

Definition 2.6. A G-homogeneous bundle isomorphism is a G-equivariant principal bundle isomorphism.

Note that the corresponding classification is finer than the usual one (i.e. a given principal bundle isomorphism type might fragment into several distinct homogeneous bundle isomorphism types). It is settled in the following theorem:

Theorem 2.7. *Let $M = G/H$ be a homogeneous space. Homogeneous principal S-bundles over M are classified by element-conjugacy classes of Lie group homomorphisms:*

$$\lambda : H \to S.$$

Here, *element-conjugacy* means conjugation by a fixed element of S. In view of this classification, we denote the homogeneous bundle corresponding to λ by P_λ.

The various agents involved in the definition of a homogeneous bundle can be packaged in a useful diagram. Let P be a homogeneous S bundle. Let $\pi : P \twoheadrightarrow M$ denote the projection map. Choosing a reference point $x \in M$ and a reference point $p \in P$ lifting x, we obtain natural maps:

$$\begin{aligned} p_G &: G \to M, \\ p_{G\times S} &: G \times S \to P, \qquad (2.6) \\ \Psi &: G \to P. \end{aligned}$$

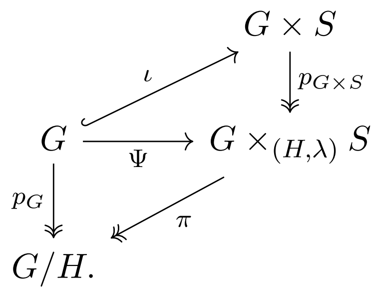
The compatibility of the actions gives:

$$p_G = \pi \circ \Psi.$$

The explicit form of the $G \times S$-action on P reveals that:

$$\Psi = p_{G\times S} \circ \iota,$$

where ι denotes the inclusion in the first factor. Ultimately, we get the following diagram:



2.4 Vector Valued ρ-Invariant Forms and Wang's Theorem

Our ultimate goal is to study invariant connections and invariant curvature forms on homogeneous principal bundles. We begin with a definition:

Definition 2.8. Let P_λ be a homogeneous S-bundle over a homogeneous space G/H. A tensorial k-form of type Ad (in the sense of [37]) is invariant if it is G-invariant as a k-form on P_λ. A connection $A \in \mathcal{A}(P_\lambda)$ is invariant if it is G-invariant as a 1-form on P_λ.

Tensorial forms and connections—regardless of whether or not they are invariant—satisfy a right equivariance property with respect to the right S-action on the bundle:

$$r_s^\star \omega = \mathrm{Ad}_{s^{-1}}(\omega). \tag{2.7}$$

We may capture property (2.7) and G-invariance simultaneously by introducing a representation ρ of $G \times S$ on $\mathfrak{s}$. Invariant tensorial forms then correspond to *ρ-invariant forms.*

In what follows, we work for a general representation of some group G on a vector space V. When we return to the setting of homogeneous bundles, the role of G will be played by $G \times S$ and the role of V will be played b $\mathfrak{s}$.

Definition 2.9. Let $M = G/H$ be a homogeneous space. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. A form $\omega \in C^\infty\left(\Lambda^k T^\star M \otimes V\right)$ is ρ-invariant if:

$$l_g^\star \omega = \rho_g(\omega). \tag{2.8}$$

We can capture ρ-invariant forms as forms over G satisfying an algebraic condition at the identity.

Proposition 2.10. *Let $M = G/H$ be a naturally reductive homogeneous space. Let $\mathfrak{m}$ be a reductive complement. Let $\rho : G \to GL(V)$ be a representation. The ρ-invariant V-valued forms on M correspond to elements $\alpha \in \Lambda^k \mathfrak{m}^\star \otimes V$ satisfying:*

$$\mathrm{Ad}_h^\star \omega = \rho_h(\omega) \text{ for all } h \in H. \tag{2.9}$$

Perturbing the canonical invariant connection by invariant tensorial forms encoded using proposition (2.10), one arrives at a complete classification of invariant connections. The following theorem is due to (Wang [82] p.8):

Theorem 2.11. *Let $M = G/H$ be a naturally reductive homogeneous space. Let $\mathfrak{m}$ be a reductive complement. Let P_λ be a homogeneous S-bundle over M. There is a one to one correspondence between invariant connections A over P_λ and linear maps:*

$$\Lambda : \mathfrak{m} \to \mathfrak{s}$$

satisfying:

$$\Lambda \circ \mathrm{Ad}_h = \mathrm{Ad}_{\lambda(h)} \circ \Lambda \text{ for any } h \in H. \tag{2.10}$$

In the above classification the canonical invariant connection corresponds to $\Lambda = 0$.

3 Asymptotically Conical Lewis Energy Estimate

Asymptotically conical manifolds admit a natural compactification obtained by adding in the asymptotic link as the *boundary at infinity.* The metric does not extend to this compactification: AC growth implies that it blows up near the boundary.

Imposing decay at infinity furnishes a hierarchy of boundary conditions. These refine the standard Dirichlet condition. Evidently, they become stronger as the rate is decreased. They are all weaker than having compact support in the interior. Analysis in the presence of the latter condition will generally resemble the closed manifold case. This is essentially because integration by parts (Stoke's theorem) works in the same way. We expect this to persist under sufficiently strong decay. In particular, we expect to be able to obtain an AC version of Lewis's theorem.

Even though ordinary characteristic classes still make sense in the AC setting, their integrals do not. Integration no longer descends to cohomology: integrals of different Chern-Weil representatives of a fixed characteristic class (Tu [72]) differ by a boundary term.

We truncate the base space at radius $T < t < \infty$ to obtain a manifold with boundary. We then integrate by parts to change representative and track the boundary term as $t \to \infty$. We are interested in determining conditions on the rate μ that will guarantee convergence to 0.

Let $(X^8, g, J, \omega, \Omega)$ be an AC CY fourfold with given asymptotic logarithmic radius function $t \geq T$. For fixed $s > T$ set:

$$X_s^8 \stackrel{\text{def}}{=} \Big\{ p \in X^8 \ \mid \ t(p) \leq s \Big\} \subset X^8$$
$$\Sigma_s^7 \stackrel{\text{def}}{=} \Big\{ p \in X^8 \ \mid \ t(p) = s \Big\} \subset X^8,$$

so that:

$$\Sigma_s^7 = \partial X_s^8.$$

It is a well known fact from the theory of characteristic classes that a primitive for the difference of two representatives of p_1 is given by the corresponding Chern-Simons form:

$$\operatorname{Tr}\left\{F_{A_1}^2\right\} - \operatorname{Tr}\left(F_{A_0}^2\right) = d\left(\operatorname{Tr}\left\{\left(F_{A_1} + F_{A_0}\right) \wedge \left(A_1 - A_0\right)\right\} - \frac{1}{3}\operatorname{Tr}\left\{\left(A_1 - A_0\right)^3\right\}\right).$$

Since Ω is closed, we obtain:

$$\operatorname{Tr}\left\{F_{A_1}^2\right\} \wedge \mathfrak{Re}\left(\Omega\right) - \operatorname{Tr}\left(F_{A_0}^2\right) \wedge \mathfrak{Re}\left(\Omega\right) = \\ d\left(\operatorname{Tr}\left\{\left(F_{A_1} + F_{A_0}\right) \wedge \left(A_1 - A_0\right)\right\} \wedge \mathfrak{Re}\left(\Omega\right) - \frac{1}{3}\operatorname{Tr}\left\{\left(A_1 - A_0\right)^3\right\} \wedge \mathfrak{Re}\left(\Omega\right)\right)$$

The following is immediate by Stokes' Theorem:

$$\int_{X_t^8} \operatorname{Tr}\left\{F_{A_1}^2\right\} \wedge \mathfrak{Re}\left(\Omega\right) - \int_{X_t^8} \operatorname{Tr}\left\{F_{A_0}^2\right\} \wedge \mathfrak{Re}\left(\Omega\right) = \int_{\Sigma_t} \operatorname{Tr}\left\{\left(F_{A_1} + F_{A_0}\right) \wedge \left(A_1 - A_0\right)\right\} \wedge \mathfrak{Re}\left(\Omega\right) \\ -\frac{1}{3}\int_{\Sigma_t} \operatorname{Tr}\left\{\left(A_1 - A_0\right)^3\right\} \wedge \mathfrak{Re}\left(\Omega\right). \quad (3.1)$$

The integral identity (3.1) provides the foundation for all subsequent results in this section. Motivated by it, we assume the existence of an HYM connection A_{HYM} on P. For each connection A over P we introduce a function:

$$\mathcal{CS}_\Omega\left[A\right] : [T, \infty) \to [0, \infty) \quad (3.2)$$

$$\mathcal{CS}_\Omega\left[A\right](t) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\Sigma_t} \operatorname{Tr}\left\{F_A \wedge \left(A - A_{\mathrm{HYM}}\right) - \frac{1}{3}\left(A - A_{\mathrm{HYM}}\right)^3\right\} \wedge \mathfrak{Re}\left(\Omega\right).$$

Our choice of notation reflects that the integrand is essentially a higher-dimensional analogue of the Chern-Simons form. In this respect, HYM connections formally adopt the role of flat connections. We thus expect $\mathcal{CS}_\Omega\left[A\right]$ to be independent of the choice of A_{HYM}. The following result guarantees that this is indeed the case.

Proposition 3.1. *Let $\left(X^8, g, J, \omega, \Omega\right)$ be an AC CY fourfold with given logarithmic radius function $t \geq T$. Let G be a real compact linear group and P an admissible principal G-bundle over X^8. Suppose that P admits an HYM connection. Let A be a connection on P. The function $\mathcal{CS}_\Omega\left[A\right]$—introduced in* (3.2)*—satisfies:*

$$\mathcal{CS}_\Omega\left[A\right](t) = \int_{X_t^8} \left|\pi_B F_A\right|^2 dV_g - \int_{X_t^8} \left|\pi_C F_A\right|^2 dV_g \ \text{ for any } t \geq T.$$

It is thus clearly independent of the choice of HYM connection involved in its definition.

Proof. For $\alpha \in \Lambda^2$, we have the following well-known formula easily following from the octonionic linear algebra discussed in section 1:

$$\star(\Omega \wedge \alpha) = 2\pi_C \alpha - 2\pi_B \alpha.$$

This allows us to analyze the pointwise symmetric bilinear form on $\Lambda^2 \otimes \mathrm{ad}(P)$ defined by:

$$(\alpha, \beta)_\Omega \overset{\text{def}}{=} \star\left(\mathrm{Tr}\{\alpha \wedge \beta\} \wedge \mathfrak{Re}(\Omega)\right).$$

In particular, we have:

$$\begin{aligned}
(\alpha, \beta)_\Omega &= \star\left(\mathrm{Tr}\{\alpha \wedge \beta\} \wedge \mathfrak{Re}(\Omega)\right) \\
&= \star\Big(\mathrm{Tr}\left\{\alpha \wedge \mathfrak{Re}(\Omega) \wedge \beta\right\}\Big) \\
&= \mathrm{Tr}\Big\{\star\left(\alpha \wedge \star(2\pi_C\beta - 2\pi_B\beta)\right)\Big\} \\
&= 2\mathrm{Tr}\left\{\star(\alpha \wedge \star\pi_C\beta)\right\} - 2\mathrm{Tr}\left\{\star(\alpha \wedge \star\pi_B\beta)\right\}
\end{aligned}$$

The pointwise fiber metric on the adjoint bundle is given by:

$$\langle \alpha, \beta \rangle = -\mathrm{Tr}\left\{\star(a \wedge \star\beta)\right\}.$$

Incorporating this in the above calculation, we find that:

$$(\alpha, \beta)_\Omega = 2\langle \alpha, \pi_B\beta \rangle - 2\langle \alpha, \pi_C\beta \rangle.$$

Consequently:

$$(F_A, F_A)_\Omega = 2|\pi_B F_A|^2 - 2|\pi_C F_A|^2.$$

This finally yields:

$$\begin{aligned}
\int_{X_t^8} \mathrm{Tr}\left\{F_A^2\right\} \wedge \mathfrak{Re}(\Omega) &= \int_{X_t^8} \star\Big(\mathrm{Tr}\left\{F_A^2\right\} \wedge \mathfrak{Re}(\Omega)\Big)\ dV_g \\
&= \int_{X_t^8} (F_A, F_A)_\Omega\ dV_g \\
&= 2\int_{X_t^8} |\pi_B F_A|^2\ dV_g - 2\int_{X_t^8} |\pi_C F_A|^2\ dV_g. \qquad (3.3)
\end{aligned}$$

We now appeal to (3.1). We take $A_1 = A$ to be the connection in question and $A_0 = A_{\mathrm{HYM}}$. The HYM equations allow us to rewrite (3.1) as:

$$\int_{X_t^8} \mathrm{Tr}\left\{F_A^2\right\} \wedge \mathfrak{Re}\left(\Omega\right) = 2\mathcal{CS}_\Omega\left[A\right](t). \tag{3.4}$$

The result follows by combining (3.3) and (3.4). □

The following corollary demonstrates that $\mathcal{CS}_\Omega\left[A\right](t)$ is a natural quantity to consider when comparing Spin(7) instantons to HYM connections over AC CY fourfolds.

Corollary 3.2. *In the context of Proposition 3.1, suppose that A is a Spin(7) instanton. We then have:*

$$\mathcal{CS}_\Omega\left[A\right](t) = \int_{X_t^8} \left|\pi_B F_A\right|^2 dV_g \ \ \textit{for any } t \geq T.$$

In particular, the function $\mathcal{CS}_\Omega\left[A\right](t)$ is non-decreasing in t.

The following result is the heart of the matter:

Proposition 3.3. ***[AC Lewis Energy Estimate]*** *Let $\left(X^8, g, J, \omega, \Omega\right)$ be an AC CY fourfold with Sasaki-Einstein asymptotic link $\left(\Sigma^7, g_{\Sigma^7}, \eta, \xi, \mathcal{J}\right)$ and given logarithmic radius function $t \geq T$. Let G be a real compact linear group and P an admissible principal G-bundle over X^8. Let A_∞ be a contact instanton over Σ^7 and fix $\mu \leq -\frac{4}{3}$. Suppose that the moduli space $\mathcal{M}_\mu\left(P, A_\infty\right)$ contains an HYM connection. Let A be an AC Spin(7) instanton in $\mathcal{M}_\mu\left(P, A_\infty\right)$ with rate α i.e.:*

$$\left|A - A_\infty\right| = O(e^{(\alpha-1)t}) \ \textit{where } \alpha < \mu. \tag{3.5}$$

Suppose further that:

$$\left|\iota_{\partial_t}\pi_B F_A\right| = O(e^{(\beta-1)t}) \ \textit{where } \beta < -4-\mu. \tag{3.6}$$

Then A is HYM.

Proof. A calculation similar to the one in the proof of Proposition 3.1 demonstrates that as $t \to \infty$:

$$\mathcal{CS}_\Omega\left[A\right](t) = \mathcal{I}_1\left[A\right](t) \cdot O(e^{-t}) - \mathcal{I}_2\left[A\right](t), \tag{3.7}$$

where:

$$\mathcal{I}_1[A](t) \stackrel{\text{def}}{=} \int_{\Sigma_t} \left\langle (A_1 - A_{\text{HYM}}), \iota_{\partial_t} \pi_B F_A \right\rangle_{|\Sigma_t} dV_{g_{|\Sigma_t}}$$

and

$$\mathcal{I}_2[A](t) \stackrel{\text{def}}{=} \frac{1}{6} \int_{X_t^8} \operatorname{Tr}\left\{ (A_1 - A_{\text{HYM}})^3 \right\} \wedge \mathfrak{Re}(\Omega).$$

The factor of e^{-t} appearing next to the first integral arises form the relation of the Hodge $\star$ on the slice at radius t to the Hodge $\star$ on X^8.

The task is to understand the leading order asymptotic behaviour of $\mathcal{I}_1$ and $\mathcal{I}_2$.

We begin by expressing the first integral as an integral over the asymptotic link—rather than the slice at radius t. Since the growth of the 7-dimensional slices is asymptotically conical, their volume forms grow like $O(e^{7t})$. Similarly, the fiber metric on 1-forms scales like $O(e^{-2t})$. Viewing 2-forms along the non-compact end of X^8 as curves of 2-forms along the asymptotic slice we obtain:

$$\mathcal{I}_1[A](t) = O(e^{5t}) \int_{\Sigma} \left\langle (A_1 - A_{\text{HYM}}), \iota_{\partial_t} \pi_B F_A \right\rangle_{|\Sigma} dV_{g_{|\Sigma}}.$$

The t-dependence of $\mathcal{I}_1$ arising from the geometry of the base X^8 has been absorbed in the coefficient $O(e^{5t})$. By (3.6), we have that the leading order term of $\iota_{\partial_t} \pi_B F_A$ at infinity is $O(e^{\beta t})$. Consequently:

$$\mathcal{I}_1[A](t) = O(e^{(5+\alpha+\beta)t}).$$

We now treat $\mathcal{I}_1[A](t)$. Since Ω is covariantly constant, its norm is $O(1)$. Consequently, the leading order term in its asymptotic expansion is $O(e^{4t})$. We therefore have:

$$\mathcal{I}_2[A](t) = O(e^{(4+3\alpha)t}).$$

Incorporating these results into (3.7) we find that:

$$\mathcal{CS}_{\Omega}[A](t) = O(e^{(4+\alpha+\beta)t}) + O(e^{(4+3\alpha)t}).$$

Condition (3.5) implies that the second summand vanishes as $t \to \infty$. Condition (3.6) implies that the first summand vanishes as well. Applying corollary 3.2 and the monotone convergence theorem we finally obtain:

$$\begin{aligned}\int_{X^8} |\pi_B F_A|^2 \, dV_g &= \lim_{t\to\infty} \int_{X^8_t} |\pi_B F_A|^2 \, dV_g \\ &= \lim_{t\to\infty} \mathcal{CS}_\Omega [A](t) \\ &= 0. \end{aligned} \tag{3.8}$$

This completes the proof. □

Proposition 3.3 applies to AC Spin(7) instanton moduli spaces $\mathcal{M}_\mu (P, A_\infty)$ that host HYM connections and have appropriately low decay parameter μ. In this setting, it essentially provides a lower bound on the asymptotic decay rate β of the failure of a pure AC Spin(7) instanton to be HYM. It asserts that unless this failure is severe (decays slowly), it doesn't exist at all. In fact, it constrains the decay of the radial part of the failure $\iota_{\partial_t}\pi_B F_A$. This is stronger than the corresponding statement involving the full tensor $\pi_B F_A$: the latter drops off only as fast as its slowest component. The lower the value of μ, the better the lower bound obtained. The crudest bound yielded is $\beta \geq -\frac{8}{3}$ and it corresponds to $\mu = -\frac{4}{3}$. It is improved to $\beta \geq -4-\mu$ as μ is decreased.

The following theorem distils the essence of the work carried out thus far:

Theorem 3.4 (Coexistence of AC Spin(7) Instantons and AC HYM Connections)**.** *Let (X^8, g, J, Ω) be an AC CY fourfold with Sasaki-Einstein asymptotic link $(\Sigma^7, g_{\Sigma^7}, \eta, \xi, \mathcal{J})$ and given logarithmic radius function $t \geq T$. Let G be a real compact linear group and P an admissible principal G-bundle over X^8. Let A_∞ be a contact instanton over Σ^7.*

- *The space $\mathcal{M}_{-2}(P, A_\infty)$ is either empty, comprised entirely of HYM connections, or comprised entirely of pure Spin(7) instantons. They all share the same finite failure.*
- *The space:*

  $$\overline{\mathcal{M}_{-2}}(P, A_\infty) = \bigcap_{\epsilon>0} \mathcal{M}_{-2+\epsilon}(P, A_\infty)$$

  is either empty, comprised entirely of HYM connections, or comprised entirely of pure Spin(7) instantons.
- *Let $-2 < \mu \leq -\frac{4}{3}$. Suppose that $\mathcal{M}_\mu (P, A_\infty)$ contains both HYM and pure Spin(7)*

solutions simultaneously. The pure Spin(7) solutions satisfy:

$$\iota_{\partial_t}\pi_B F_A = O(t^{\beta}) \text{ where } -4-\mu \leq \beta < \mu. \tag{3.9}$$

- *Let $\mu > -\frac{4}{3}$. The space $\mathcal{M}_{\mu}(P, A_{\infty})$ can contain both HYM and pure Spin(7) solutions simultaneously.*

Proof. For the first assertion, note that:

$$\pi_B F_A = \pi_B d_{A_{\mathrm{HYM}}}(A - A_{\mathrm{HYM}}) + \frac{1}{2}\pi_B [A - A_{\mathrm{HYM}}, A - A_{\mathrm{HYM}}].$$

It follows that $\pi_B F_A$ decays at least as fast as $A - A_{\mathrm{HYM}}$ i.e. strictly faster than μ. When the decay parameter is decreased to $\mu = -2$, the bound provided by proposition 3.3 yields a contradiction.

The finitude of the failure follows from the observation that -2 is the L^2 rate for AC two-forms. Its invariance follows by a slight modification of the integration by parts calculation of proposition 3.3.

For the second assertion we wish to infinitesimally strengthen the above to hold for decay rates precisely equal to $\alpha = -2$ (rather than just $\alpha < \mu \leq -2$). To this end, suppose that $\overline{\mathcal{M}}_{-2}(P, A_{\infty})$ contains both HYM connections and pure Spin(7) instantons. Let A be a pure Spin(7) instanton in $\overline{\mathcal{M}}_{-2}(P, A_{\infty})$ with failure tensor having rate β. Since $A \in \mathcal{M}_{-2+\epsilon}(P, A_{\infty})$, proposition 3.3 guarantees that:

$$-2-\epsilon \leq \beta < -2+\epsilon \text{ for all } \epsilon > 0. \tag{3.10}$$

It follows that $\beta = -2$. Since -2 is the L^2 rate for AC two-forms, $\pi_B F_A$ is not in L^2 i.e. A has infinite failure. However, we have that:

$$\lim_{t\to\infty} \mathcal{CS}_{\Omega}[A](t) < \infty. \tag{3.11}$$

Monotone convergence and corollary 3.2 furnish the requisite contradiction.

The third assertion is merely the bound provided by proposition 3.3.

The final assertion is obtained by considering the explicit examples constructed in section 5. These decay with rate precisely equal to $\alpha = -\frac{4}{3}$. Consequently, they live in $\mathcal{M}_{-\frac{4}{3}+\epsilon}(P, A_\infty)$ for any $\epsilon > 0$. $\square$

Remark 3.5. *The extension of the first assertion provided by the second one is not vacuous. The standard octonionic instanton on* $\mathbb{R}^8$ *(the subject of section* 4*) decays with rate* $\alpha = -2$ *and lives in a moduli space with no HYM connections. Nevertheless, it doesn't provide an ideal example: its limiting connection is* G_2 *but not contact. In particular, we know a priori that there are no HYM connections in any* $\mathcal{M}_\mu(P, A_\infty)$ *for* $\mu < 0$*, not just* $\overline{\mathcal{M}_{-2}}(P, A_\infty)$*.*

Remark 3.6. *To the author's knowledge, there are no known examples illustrating the third assertion. The hope would be to use* (3.9) *in order to extend the incompatibility result all the way up to* $\mu = -\frac{4}{3}$*.*

Remark 3.7. *Due to the structure of the second term* $\mathcal{I}_2(t)$ *in* (3.7)*, the methods employed in this section could never reveal information for* $\mu > -\frac{4}{3}$*. The final assertion establishes that this is no mere coincidence: the value* $\mu = -\frac{4}{3}$ *is critical.*

Remark 3.8. *One might conjecture that the only thing that can go wrong for slower rates is the unboundedness of the failure: i.e. that the failure is invariant when it is finite. Our examples on the Stenzel metric presented in section* 5 *disprove this: the failure is a non-constant, bounded, smooth function along the family.*

To conclude this section, we propose a potential application of Theorem 3.4. It might be able to exclude the existence of AC HYM connections when this is not already achieved by the asymptotic boundary condition (i.e. the choice of A_∞). The method would involve the construction of a suitable Spin(7) instanton. In particular:

- If we exhibit a pure Spin(7) instanton in $\overline{\mathcal{M}_{-2}}(P, A_\infty)$, then this moduli space contains no HYM connections at all.
- If we exhibit a pure Spin(7) instanton in $\mathcal{M}_\mu(P, A_\infty)$ with $-2 < \mu \leq -\frac{4}{3}$ such that $\beta < -4 - \mu$, then $\mathcal{M}_\mu(P, A_\infty)$ contains no HYM connections at all.

Note that in the first case, it is arguably more natural to approach the problem using Lewis' cohomological invariant. To the author's knowledge, there is currently no known example of a pure AC Spin(7) instanton satisfying either set of requirements (supercritical decay to a limit compatible with the HYM equation). If such objects do exist, this result would interestingly constitute a purely Spin(7) technique to address a classical problem in complex geometry: the existence of HYM connections over noncompact CY fourfolds (where the Donaldson-Uhlenbeck-Yau theorem [73] doesn't apply).

4 Spin(7)-Equivariant Instantons on Flat Space

We now turn to the construction of our first example of a non-trivial AC Spin(7) instanton over an AC CY fourfold: the so called *standard octonionic instanton* on $\mathbb{R}^8$. While this example is not new, we offer an alternative derivation that we hope elucidates the underlying geometry. The octonionic instanton first appeared in the Physics literature and is the subject of the article [27] by Fubini and Nicolai. Nevertheless -to the author's knowledge- it has not appeared in the mathematics literature thus far. We follow a method analogous to that in [43] to give an alternative construction, which we hope will be quicker and easier to understand by a mathematical audience. An advantage of our approach is the ability to just read off the asymptotic behaviour and other mathematical properties of interest. We find that the octonionic instanton is AC of rate -2. Its asymptotic limit is given by the canonical invariant connection on S^7 induced by the exceptional homogeneous structure:

$$S^7 = \frac{\mathrm{Spin}(7)}{G_2}.$$

In [27], the authors construct a single instanton. We find that the octonionic instanton lives in a 1-parameter family $\mathcal{M}_{\mathrm{inv}}^{\mathrm{Spin}(7)}(P_{\mathrm{Id}})$ of invariant solutions that is noncompact at both ends. The family can be obtained by pulling the Fubini-Nicolai instanton back through the natural dilation action.

The non-compactness phenomena encountered as we approach either end of $\mathcal{M}_{\mathrm{inv}}^{\mathrm{Spin}(7)}(P_{\mathrm{Id}})$ differ. This provides us with two distinct ways in which a family of AC Spin(7) instantons may degenerate.

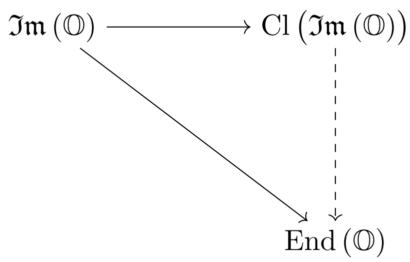
4.1 The Spin Representation of Spin(7)

Owing to the results of section 1.1.2.4, Spin(7) has a unique real irreducible 8-dimensional representation $\$$: the *spin representation*. The standard octonionic instanton lives on $\$$ and the Spin(7)-action preserves it. We are thus interested in recasting $\$$ in a way amenable to calculations. In the sequel, we follow the notation established in section 1.

The imaginary octonions $\mathfrak{Im}\,(\mathbb{O})$ act on $\mathbb{O}$ from the left by octonionic multiplication. This can be easily seen to satisfy the Clifford relations:

$$v^2 = -g(v,v) \cdot 1 \text{ in } \mathrm{End}\,(\mathbb{O})\,. \tag{4.1}$$

The universal property gives rise to a commutative diagram:



Using the basis of $\mathbb{O}$ introduced in section 1 we obtain identifications:

$$\mathbb{O} \cong \mathbb{R}^8$$
$$\mathrm{Cl}\,\big(\mathfrak{Im}\,(\mathbb{O})\big) \cong \mathrm{Cl}(7).$$

Composing with the inclusion of Spin(7) in Cl(7), we finally obtain the spin representation:

$$\mathrm{Spin}(7) \to \mathrm{Gl}(\mathbb{R}^8).$$

Explicit computation demonstrates that this is faithful and that it preserves the metric and orientation. It is thus an embedding:

$$\mathrm{Spin}(7) \hookrightarrow \mathrm{SO}(8).$$

One can also check that the spinor action preserves the standard Cayley calibration Φ. It follows that we could have alternatively defined it by composing the standard embedding of Spin(7) in SO(8) with the vector action of SO(8). While this is perhaps easier to state, our description—using octonionic multiplication—facilitates calculations.

Since $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, the spinor action preserves the 7-sphere S^7_r of radius $r > 0$. In fact, the restricted action remains transitive for any r. We thus obtain a cohomogeneity one structure on $\mathbb{R}^8$. The principal orbits are the spheres of positive radius. The origin constitutes the unique (0-dimensional) singular orbit. Note that we have effectively reduced the symmetries provided by the rotations of 8-space to a smaller group, while maintaining transitivity along the 7-spheres. There exist no Spin(7) instantons on $\mathbb{R}^8$ enjoying full rotational symmetry. As we will see however, it is possible to find instantons invariant by those rotations of 8-space comprising Spin(7). The fact that transitivity is maintained for the smaller group is crucial. It allows us to reduce the number of free variables to just one, ultimately simplifying the problem no less than dimensional reduction by full rotational symmetry.

4.2 The Exceptional Homogeneous Structure on the Round S^7

The stabilizer of:

$$p_r \stackrel{\text{def}}{=} (r,0,0,0,0,0,0,0)^\intercal \in S^7_r \tag{4.2}$$

is a copy of $G_2 \subset \mathrm{Spin}(7)$ [61] exhibiting S^7_r as a homogeneous space:

$$S^7_r = \frac{\mathrm{Spin}(7)}{G_2}.$$

In the sequel we will denote S^7_1 simply as S^7. We are interested in obtaining a reductive decomposition:

$$\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}.$$

The Lie algebra of Spin(7) naturally sits inside Cl(7) as:

$$\mathfrak{spin}(7) = \left\langle e_i e_j \right\rangle_{i<j,\, i,j=2,\dots,8} \subset \mathrm{Cl}\left(e_2, \dots, e_8\right).$$

We study the inclusion $G_2 \subset \mathrm{Spin}(7)$ at the linear level. Consider the natural homogeneous projection map identifying S^7_1 with a coset space:

$$\pi : \mathrm{Spin}(7) \twoheadrightarrow S^7$$
$$g \mapsto g p_1.$$

The relevant copy of $\mathfrak{g}_2$ in $\mathfrak{spin}(7)$ is given by:

$$\mathfrak{g}_2 = \mathrm{Ker}\left(d\pi_{|_1}\right).$$

Letting $\cdot$ denote the octonionic product, we use our construction of the spinor representation and the definition of π to explicitly characterize $d\pi$:

$$d\pi : e_i e_j \mapsto e_i \cdot e_j.$$

The relevant kernel can now be easily computed by observing the octonionic multiplication table provided in section 1. We find that $\mathfrak{g}_2$ is given by:

$$\begin{aligned} \mathfrak{g}_2 = <& e_2e_3 + e_5e_8, e_2e_3 + e_6e_7, e_2e_4 + e_5e_7, e_2e_4 - e_6e_8, \\ & e_2e_5 - e_3e_8, e_2e_5 - e_4e_7, e_2e_6 - e_3e_7, e_2e_6 + e_4e_8, \\ & e_2e_7 + e_3e_6, e_2e_7 + e_4e_5, e_2e_8 + e_3e_5, e_2e_8 - e_4e_6, \\ & e_3e_4 - e_5e_6, e_3e_4 - e_7e_8 > . \end{aligned} \tag{4.3}$$

Since the exceptional homogeneous structure on S^7 is normal, the orthogonal complement of $\mathfrak{g}_2$ under (any scalar multiple of) the Killing form is reductive. Therefore we take:

$$\mathfrak{m} = \mathfrak{g}_2^{\perp} \subset \mathfrak{spin}(7).$$

The basis vectors of $\mathfrak{spin}(7)$ given by the products e_ie_j correspond to (twice) the natural basis vectors of $\mathfrak{so}(7) \cong \mathfrak{spin}(7)$ given by the elementary antisymmetric matrices. In particular:

$$e_ie_j \leftrightarrow -2E_{ij}.$$

Here E_{ij} denotes the elementary antisymmetric matrix with (k,l) entry equal to:

$$\left(E_{ij}\right)_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

These matrices are pairwise orthogonal for the Killing form, implying the same for the basis vectors e_ie_j. Armed with the above observations, we introduce:

$$X_1 \overset{\text{def}}{=} \frac{1}{3}(e_3e_4 + e_5e_6 + e_7e_8)$$
$$X_2 \overset{\text{def}}{=} \frac{1}{3}(e_5e_7 - e_2e_4 - e_6e_8)$$
$$X_3 \overset{\text{def}}{=} \frac{1}{3}(e_2e_3 - e_5e_8 - e_6e_7)$$
$$X_4 \overset{\text{def}}{=} \frac{1}{3}(e_4e_8 - e_2e_6 - e_3e_7)$$
$$X_5 \overset{\text{def}}{=} \frac{1}{3}(e_2e_5 + e_3e_8 + e_4e_7)$$
$$X_6 \overset{\text{def}}{=} \frac{1}{3}(e_3e_5 - e_2e_8 - e_4e_6)$$
$$X_7 \overset{\text{def}}{=} \frac{1}{3}(e_2e_7 - e_3e_6 - e_4e_5)$$

Furthermore, we label the ordered basis (4.3) of $\mathfrak{g}_2$ as $X_8, ..., X_{21}$. Finally, we introduce the dual basis by $(\theta^i)_{i=1...21}$ so that:

$$\theta^i\left(X_j\right) = \delta^i_j \text{ for } i,j = 1...21.$$

This completes the explicit description of the sought reductive decomposition.

The stabilizer G_2 acts on the reductive complement through the adjoint action of Spin(7) to yield the isotropy representation. This agrees with the lowest-dimensional non-trivial irrep of G_2. Hence, the exceptional homogeneous structure on S^7 is isotropy irreducible. Note that:

$$\langle X_1, ..., X_7\rangle \text{ and } \left\langle \theta^1, ..., \theta^7\right\rangle$$

can be pushed forward by $d\pi$ to provide (co)frames for the tangent and cotangent spaces of S^7 at p_1. This process can also be carried out for $0 < r \neq 1$. In terms of the standard basis of $\mathbb{R}^8$ (introduced in section 1), the resulting frames and coframes take the form:

$$X_1 \mapsto re_2, \quad \theta^1 \mapsto \epsilon^2, \tag{4.4}$$

$$X_2 \mapsto re_3, \quad \theta^2 \mapsto \frac{1}{r}\epsilon^3, \tag{4.5}$$

$$X_3 \mapsto re_4, \quad \theta^3 \mapsto \frac{1}{r}\epsilon^4, \tag{4.6}$$

$$X_4 \mapsto re_5, \quad \theta^4 \mapsto \frac{1}{r}\epsilon^5, \tag{4.7}$$

$$X_5 \mapsto re_6, \quad \theta^5 \mapsto \frac{1}{r}\epsilon^6, \tag{4.8}$$

$$X_6 \mapsto re_7, \quad \theta^6 \mapsto \frac{1}{r}\epsilon^7, \tag{4.9}$$

$$X_7 \mapsto re_8, \quad \theta^7 \mapsto \frac{1}{r}\epsilon^8. \tag{4.10}$$

In the sequel we slightly abuse notation by omitting the application of $d\pi$. Note however that these vectors/ covectors do not admit well defined left invariant extensions. Indeed, the isotropy representation has no fixed points (it is irreducible). This demonstrates that while S^7 is parallelizable (for topological reasons) its tangent bundle doesn't admit an SO(8) invariant (not even Spin(7) invariant) trivialization. This is a consequence of the failure of associativity for the octonionic product and the corresponding insufficiency of the latter to equip S^7 with a Lie group structure.

Nevertheless, the standard Spin(7) structure (g, Φ) on $\mathbb{R}^8$ is Spin(7) invariant. Consequently, g and Φ show up as fixed points of appropriate tensor powers of the isotropy action. Note however that all other tensors involved in the flat CY-4 model are not Spin(7)-invariant.

Incorporating the radial unit vector ∂_r to the frame $\langle X_1, ..., X_7\rangle$ and its dual dr to the coframe $\left\langle \theta^1, ..., \theta^7 \right\rangle$, we obtain bases for $T_{p_r}\mathbb{R}^8$ and $T_{p_r}(\mathbb{R}^8)^\star$ along the reference ray $(p_r)_{r>0}$. Using (4.4)-(4.10) and the expressions found in section 1, we find:

$$\omega = dr \wedge r\theta^1 + r^2 \left(\theta^{23} + \theta^{45} + \theta^{67}\right),$$

$$g = dr^2 + r^2 \left(\theta^{23} + \theta^{45} + \theta^{67}\right),$$

$$J = \frac{1}{r} dr \otimes X_1 - r\theta^1 \otimes \partial_r + \theta^2 \otimes X_3 - \theta^3 \otimes X_2 + \theta^4 \otimes X_5 - \theta^5 \otimes X_4 + \theta^6 \otimes X_7 - \theta^7 \otimes X_6,$$
$$\mathfrak{Re}\,(\Omega) = dr \wedge r^3 \left(\theta^{246} - \theta^{257} - \theta^{347} - \theta^{356}\right) + r^4 \left(\theta^{1247} - \theta^{1256} - \theta^{1346} + \theta^{1357}\right),$$
$$\mathfrak{Im}\,(\Omega) = dr \wedge r^3 \left(\theta^{247} + \theta^{256} + \theta^{346} - \theta^{357}\right) + r^4 \left(\theta^{1246} - \theta^{1257} - \theta^{1347} - \theta^{1356}\right).$$

Finally, the standard Cayley calibration takes the form:

$$\Phi = dr \wedge r^3 \left(\theta^{246} - \theta^{257} - \theta^{347} - \theta^{356} + \theta^{123} + \theta^{145} + \theta^{167}\right)$$
$$+ r^4 \left(\theta^{1247} - \theta^{1256} - \theta^{1346} + \theta^{1357} + \theta^{2345} + \theta^{2367} + \theta^{4567}\right),$$

Observe that only g and Φ are stabilized by G_2—i.e. are extendable by left translations. The other tensors extend smoothly, but not Spin(7)-equivariantly.

4.3 Spin(7) Equivariant Gauge Theory on S^7: Homogeneous Bundles and Invariant Connections

Having completed our discussion of the geometry that hosts the standard octonionic instanton, we now address the gauge-theoretic setup of the construction.

We take the structure group of the gauge theory to be equal to Spin(7). As we shall see, all non-flat Spin(7) invariant Spin(7) instantons over $\mathbb{R}^8$ with structure group Spin(7) are irreducible, meaning that choosing any smaller compact Lie group that occurs as a subgroup of Spin(7) would be futile.

4.3.1 The Homogeneous Bundle

Homogeneous bundles with fiber Spin(7) over the exceptional homogeneous structure on S^7 are classified by Lie group maps:

$$\lambda : G_2 \to \text{Spin}(7).$$

We do not go through the process of classifying all such maps and instead focus on the bundle P_ι corresponding to the inclusion ι of G_2 in Spin(7) as the stabilizer of $p_1 \in S^7$. Our first task is to identify this bundle topologically.

In general, for a naturally reductive homogeneous space G/H with reductive complement $\mathfrak{m}$, the isotropy representation yields a map:

$$\lambda : H \to \mathrm{Gl}(\mathfrak{m}).$$

In the presence of a compatible metric and orientation the target becomes $\mathrm{SO}(\mathfrak{m})$. The bundle P_λ corresponding to the homomorphism λ is then the (oriented orthonormal) frame bundle of G/H. In this case, the isotropy representation is the unique 7-dimensional irrep of G_2. Identifying it with $\mathbb{R}^7$ using the X_i basis and noting that we do indeed have a compatible metric and orientation, we obtain a map:

$$\mu_1 : G_2 \to \mathrm{SO}(7).$$

This corresponds to the orthonormal oriented frame bundle of S^7. Since the latter is parallelizable, this bundle is trivial. Note however that μ_1 is not trivial (i.e. not identically equal to 1). It follows that the even though:

$$P_{\mu_1} \cong \mathrm{Fr}_{\mathrm{SO}}\left(S^7\right) \cong S^7 \times \mathrm{SO}(7)$$

holds at the topological level, the homogeneous structure on P_{μ_1} doesn't match up with the trivial one on the right-hand-side. In particular, there is no Spin(7) equivariant trivialization -an observation we arrived at earlier as well.

Composing the inclusion:

$$\iota : G_2 \hookrightarrow \mathrm{Spin}(7)$$

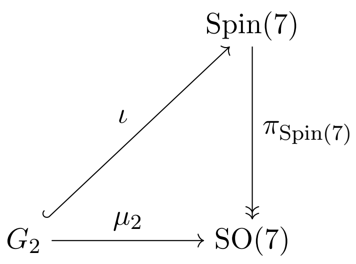
with the two sheeted covering map:

$$\pi_{\mathrm{Spin}(7)} : \mathrm{Spin}(7) \twoheadrightarrow \mathrm{SO}(7),$$

we obtain a map:

$$\mu_2 : G_2 \hookrightarrow \mathrm{Spin}(7) \twoheadrightarrow \mathrm{SO}(7).$$

This yields a 7-dimensional representation of G_2. There is precisely one non-trivial irrep of G_2 with $\dim(V) \leq 7$. Since μ_2 is non-trivial, it must be isomorphic to μ_1. In particular, the maps μ_1 and μ_2 are element-conjugate and hence define equivariantly isomorphic bundles. We conclude that after applying $\pi_{\mathrm{Spin}(7)}$, ι corresponds to P_{μ_1}: the topologically trivial (oriented orthonormal) frame bundle of S^7 with the non-trivial homogeneous structure associated to μ_1. In particular, the diagram:



exhibits P_ι as the (unique) spin structure of S^7. Representing both bundles by the standard quotient construction, the 2-sheeted covering map:

$$P_\iota \twoheadrightarrow P_{\mu_2}$$

can be constructed explicitly from the homomorphism $\pi_{\mathrm{Spin}(7)}$. Since the spin structure of S^7 is trivial, we finally conclude that (topologically):

$$P_\iota \cong \mathrm{Spin}\left(TS^7\right) \cong S^7 \times \mathrm{Spin}(7).$$

Since ι is not trivial, this identification is not equivariant.

4.3.2 The Canonical Invariant Connection

We now turn to the classification of invariant connections on P_1. Notice that in the context of our study, the symmetry group matches the structure group. This is a coincidence, and there is no relationship between the corresponding Lie algebras. To avoid confusion, we distinguish them by introducing the alternative notation:

$$\mathfrak{spin}(7) = \langle \beta_1, ..., \beta_{21} \rangle$$

for the Lie algebra of the structure group. These vectors are defined exactly like $X_1, ..., X_{21}$, but live in a different copy of the Lie algebra. This prompts us to distinguish them notationally.

We also introduce the following notation for the structure constants of $\mathfrak{spin}(7)$ in the $X_1, ..., X_{21}$ frame:

$$\left[X_i, X_j\right] = C_{ij}^k X_k.$$

The constants C_{ij}^k can be computed explicitly. The calculation is long and tedious so we choose to omit it. However, we wish to remark the following two properties that will play a central role in subsequent computations. First, we have:

$$C_{ij}^k = 0 \text{ when } k = 1, ..., 7,\ i, j = 8, ..., 21. \tag{4.11}$$

This follows by noting that the vectors $\langle X_8, ..., X_{21}\rangle$ span $\mathfrak{g}_2$, which is a subalgebra. Furthermore, we have:

$$C_{ij}^k = 0 \text{ when } i = 1, ..., 7,\ j, k = 8, ..., 21. \tag{4.12}$$

The canonical invariant connection corresponds to the differential of the classifying homomorphism. In this case, we obtain:

$$A_{\text{can}} = d\iota = \sum_{i=8}^{21} \theta^i \otimes \beta_i.$$

We wish to compute its curvature. We have:

$$F_{A_{\text{can}}} = dA_{\text{can}} + \frac{1}{2}\left[A_{\text{can}}, A_{\text{can}}\right].$$

The first term is computed as follows:

$$\begin{aligned} dA_{\text{can}} &= d\sum_{k=8}^{21} \theta^k \otimes \beta_k \\ &= \sum_{k=8}^{21} d\theta^k \otimes \beta_k \\ &= -\frac{1}{2}\sum_{k=8}^{21} C_{ij}^k \theta^{ij} \otimes \beta_k. \\ &= -\sum_{\substack{i,j=1...21 \\ i<j \\ k=8...21}} C_{ij}^k \theta^{ij} \otimes \beta_k. \end{aligned}$$

For the second term, we calculate:

$$\begin{aligned}
\frac{1}{2}\left[A_{\text{can}}, A_{\text{can}}\right] &= \frac{1}{2}\left[\sum_{i=8\ldots21}\theta^i \otimes \beta_i, \sum_{j=8\ldots21}\theta^j \otimes \beta_j\right] \\
&= \frac{1}{2}\sum_{i,j=1\ldots21}\theta^{ij}\otimes\left[\beta_i,\beta_j\right] \\
&= \sum_{\substack{i,j=1\ldots21\\ i<j}}\theta^{ij}\otimes\left[\beta_i,\beta_j\right] \\
&= \sum_{\substack{i,j=1\ldots21\\ i<j\\ k=1\ldots21}} C_{ij}^k\theta^{ij}\otimes\beta_k.
\end{aligned}$$

Summing and using properties (4.11) and (4.12), we obtain:

$$\begin{aligned}
F_{A_{\text{can}}} &= -\sum_{\substack{i,j=1\ldots21\\ i<j\\ k=8\ldots21}} C_{ij}^k\theta^{ij}\otimes\beta_k + \sum_{\substack{i,j=1\ldots21\\ i<j\\ k=1\ldots21}} C_{ij}^k\theta^{ij}\otimes\beta_k \\
&= -\sum_{\substack{i,j=1\ldots7\\ i<j\\ k=8\ldots21}} C_{ij}^k\theta^{ij}\otimes\beta_k.
\end{aligned} \tag{4.13}$$

We see that A_{can} is not flat. Inspecting (4.13) we find that its curvature tensor spans $\mathfrak{g}_2 \subset \mathfrak{spin}(7)$, implying that:

$$\text{Hol}\,(A) = G_2.$$

The reduction theorem then guarantees that A is reducible to a sub-bundle with fiber G_2. This bundle is homogeneous and corresponds to the identity map. The inclusion $\iota : G_2 \hookrightarrow \text{Spin}(7)$ extends the range of Id_{G_2} to Spin(7). Describing $P_{\text{Id}_{G_2}}$ and P_ι by the standard quotient construction, ι defines an explicit embedding of the former in the latter. We thus have at our disposal an explicit description of the G_2-bundle that A_{can} reduces to. One might wonder why we did not choose to work directly with $P_{\text{Id}_{G_2}}$. Even though A_{can} is reducible, all (non-flat) Spin(7) invariant Spin(7) instantons over P_ι are irreducible, as we shall shortly discover.

4.3.3 The Space of Invariant Connections

Invariant connections on P_ι correspond to G_2 equivariant maps:

$$(\mathfrak{m}, \mathrm{Ad} \circ \iota) \to \left(\mathfrak{spin}(7), \mathrm{Ad} \circ \iota\right).$$

Noting that the domain is just an irrep appearing in the decomposition of the target, and that Schur's lemma applies in full to odd-dimensional real representations, we find that:

$$\mathrm{Hom}_{G_2}\left(\mathfrak{m}, \mathfrak{spin}(7)\right) = \mathbb{R}.$$

Here, the scalar λ corresponds to $\lambda \iota_{\mathfrak{m}}$, where $\iota_{\mathfrak{m}}$ is the inclusion of the reductive complement $\mathfrak{m}$ in $\mathfrak{spin}(7)$.

The space of invariant connections is a 1-dimensional affine space:

$$\mathcal{A}_{\mathrm{inv}}\left(P_\iota\right) = A_{\mathrm{can}} + \mathrm{Hom}_{G_2}\left(\mathfrak{m}, \mathfrak{spin}(7)\right).$$

A general element $A \in \mathcal{A}_{\mathrm{inv}}\left(P_\iota\right)$ takes the form:

$$A = A_{\mathrm{can}} + \Lambda,$$

where:

$$\Lambda = a \sum_{k=1}^{7} \theta^k \otimes \beta_k, \quad a \in \mathbb{R}.$$

The associated curvature tensor is given by:

$$F_A = F_{A_{\mathrm{can}}} + d_{A_{\mathrm{can}}}\Lambda + \frac{1}{2}\left[\Lambda, \Lambda\right]. \tag{4.14}$$

We compute the second and third summands individually. For the second one, note that by definition:

$$d_{A_{\mathrm{can}}}\Lambda = d\Lambda + \left[A_{\mathrm{can}}, \Lambda\right].$$

The first term takes the following form:

$$
\begin{aligned}
d\Lambda &= d\sum_{k=1}^{7} a\,\theta^k \otimes \beta_k \\
&= a\sum_{k=1}^{7} d\theta^k \otimes \beta_k \\
&= -\frac{a}{2}\sum_{k=1}^{7} C^k_{ij}\theta^{ij} \otimes \beta_k \\
&= -a \sum_{\substack{i,j=1...21\\ i<j\\ k=1...7}} C^k_{ij}\theta^{ij} \otimes \beta_k \\
&= -a \sum_{\substack{i,j=1...7\\ i<j\\ k=1...7}} C^k_{ij}\theta^{ij} \otimes \beta_k - a \sum_{\substack{i=1...7\\ j=8...21\\ k=1...7}} C^k_{ij}\theta^{ij} \otimes \beta_k,
\end{aligned}
$$

where we have used the Maurer-Cartan relations and property (4.11). We perform analogous calculations for the second term:

$$
\begin{aligned}
[A_{\text{can}}, \Lambda] &= \left[\sum_{i=8}^{21} \theta^i \otimes \beta_i,\; a\sum_{j=1}^{7} \theta^j \otimes \beta_j\right] \\
&= a \sum_{\substack{i=8...21\\ j=1...7}} \theta^{ij} \otimes \left[\beta_i, \beta_j\right] \\
&= a \sum_{\substack{i=1...7\\ j=8...21}} \theta^{ij} \otimes \left[\beta_i, \beta_j\right]
\end{aligned}
$$

where in the final step we used that both θ^{ij} and $\left[\beta_i, \beta_j\right]$ are antisymmetric in i and j, so that their tensor product is symmetric. Continuing, we find:

$$
\begin{aligned}
[A_{\text{can}}, \Lambda] &= a \sum_{\substack{i=1...7\\ j=8...21\\ k=1...21}} C^k_{ij}\theta^{ij} \otimes \beta_k \\
&= a \sum_{\substack{i=1...7\\ j=8...21\\ k=1...7}} C^k_{ij}\theta^{ij} \otimes \beta_k.
\end{aligned}
$$

Summing, we finally complete our calculation of the second summand in (4.14):

$$\begin{aligned} d_{A_{\mathrm{can}}}\Lambda &= -a \sum_{\substack{i,j=1\ldots7\\ i<j\\ k=1\ldots7}} C^k_{ij}\theta^{ij}\otimes\beta_k - a \sum_{\substack{i=1\ldots7\\ j=8\ldots21\\ k=1\ldots7}} C^k_{ij}\theta^{ij}\otimes\beta_k + a \sum_{\substack{i=1\ldots7\\ j=8\ldots21\\ k=1\ldots7}} C^k_{ij}\theta^{ij}\otimes\beta_k \\ &= -a \sum_{\substack{i,j=1\ldots7\\ i<j\\ k=1\ldots7}} C^k_{ij}\theta^{ij}\otimes\beta_k. \end{aligned} \tag{4.15}$$

The third summand takes the following form:

$$\begin{aligned} \frac{1}{2}[\Lambda,\Lambda] &= \frac{1}{2}\left[\sum_{i=1}^{7} a\theta^i\otimes\beta_i, \sum_{j=1}^{7} a\theta^j\otimes\beta_j\right] \\ &= \sum_{\substack{i,j=1\ldots7\\ i<j\\ k=1\ldots21}} a^2 C^k_{ij}\theta^{ij}\otimes\beta_k \end{aligned} \tag{4.16}$$

Finally, using (4.13), (4.14), (4.15) and (4.16), we obtain:

$$F_A = \sum_{k=1}^{7}\left(\sum_{\substack{i,j=1..7\\ i<j}} C^k_{ij}\left(a^2-a\right)\theta^{ij}\right)\otimes\beta_k + \sum_{k=8}^{21}\left(\sum_{\substack{i,j=1..7\\ i<j}} C^k_{ij}\left(a^2-1\right)\theta^{ij}\right)\otimes\beta_k. \tag{4.17}$$

4.4 The Standard Octonionic Instanton

The Spin(7) action on $\mathbb{R}^8$ and the choice of reference points p_r (introduced in (4.2)) induce a polar coordinate system:

$$\mathbb{R}^8 - \{0\} \cong (0,\infty)\times S^7.$$

Projecting to the second factor (i.e. remembering only the angular coordinates), we obtain a map:

$$\pi : \mathbb{R}^8 - \{0\} \twoheadrightarrow S^7.$$

Pulling P_ι back to $\mathbb{R}^8 - \{0\}$ through π we obtain a Spin(7) bundle $\pi^\star P_\iota$. We slightly abuse notation by suppressing the pullback operation and denoting this simply by P_ι. Its smooth homogeneous extensions across the origin are parameterized by Lie group maps:

$$\lambda : \mathrm{Spin}(7) \to \mathrm{Spin}(7)$$

extending the inclusion $\iota : G_2 \hookrightarrow \mathrm{Spin}(7)$. One such map is given by the identity:

$$\mathrm{Id}_{\mathrm{Spin}(7)} : \mathrm{Spin}(7) \to \mathrm{Spin}(7).$$

The associated smooth homogeneous Spin(7) bundle $P_{\mathrm{Id}_{\mathrm{Spin}(7)}}$ over $\mathbb{R}^8$ hosts the standard octonionic instanton. Note that it is trivial, but not equivariantly so.

Invariant connections over the complement of the origin restrict to invariant connections on each homogeneous principal orbit S^7. These objects have been classified in the previous section. Since invariant connections may always be brought to temporal gauge (no dr component) through an equivariant gauge transformation [54], a general invariant connection is -up to gauge- given by:

$$A = A_{\mathrm{can}} + \Lambda,$$

where we have slightly abused notation to denote by A_{can} the pullback by π of the canonical invariant connection on P_ι. Furthermore:

$$\Lambda = a(r) \sum_{k=1}^{7} \theta^k \otimes \beta_k.$$

Using formula (4.17) and the Leibniz rule for the exterior derivative, we arrive at:

$$F_A = \sum_{k=1}^{7} \left(\frac{da}{dr} dr \wedge \theta^k + \sum_{\substack{i,j=1..7 \\ i<j}} C^k_{ij} \left(a^2 - a\right) \theta^{ij} \right) \otimes \beta_k$$
$$+ \sum_{k=8}^{21} \left(\sum_{\substack{i,j=1..7 \\ i<j}} C^k_{ij} \left(a^2 - 1\right) \theta^{ij} \right) \otimes \beta_k. \tag{4.18}$$

We now wish to impose the Spin(7) instanton equations on F_A:

$$\pi^2_7 F_A = 0. \tag{4.19}$$

Note that the projector π^2_7 has been explicitly computed in section 1—albeit in a different coordinate system. Using (4.4)-(4.10), we translate the results to the relevant frame and find that for $k = 1...7$, equation (4.19) amounts to the ODE:

$$\frac{1}{r}\frac{da}{dr} = \frac{2}{r^2}a\,(a-1)\,. \tag{4.20}$$

Furthermore, all components corresponding to $k = 8...21$ are annihilated by π^2_7 with no constraint on a. It follows that the Spin(7) instanton equation for Spin(7) invariant connections on $P_{\mathrm{Id}_{\mathrm{Spin}(7)}}$ is equivalent to the ODE (4.20). This is a nonlinear, nonautonomous singular equation with an order 1 pole at $r = 0$. Luckily it can be solved explicitly. Other than the trivial solution $a = 0$, there is a one parameter family of solutions:

$$a_\lambda(r) = \frac{1}{\lambda r^2 + 1}, \qquad \lambda \in \mathbb{R}. \tag{4.21}$$

Inspecting (4.21), we find that as $\lambda \to \pm\infty$, a_λ converges pointwise for $r > 0$ to $a = 0$. This motivates us to include the trivial solution $a = 0$ in the family α_λ for $\lambda = \infty$. Ultimately, the a_λ exhaust all possible local solutions when λ varies in the circle $\mathbb{R} \cup \{\infty\}$.
Let A_λ denote the corresponding Spin(7) instanton. We immediately observe that when $\lambda < 0$, A_λ blows up in finite time:

$$r_{\mathrm{blowup}}\,(\lambda) = |\lambda|^{-\frac{1}{2}}.$$

Furthermore when $\lambda = +\infty$ (i.e. $\alpha = 0$), we obtain the dilation invariant solution equal to A_{can} on each principal orbit S^7. We use (4.18) and the Euclidean metric to find that:

$$|F_{A_\lambda}| = O(r^{-2}) \text{ as } r \to 0.$$

In particular, the curvature tensor blows up so that A_λ has an essential singularity at $r = 0$: it does not smoothly extend across the origin on any Spin(7) bundle.

Since we are interested in global instantons we restrict to $0 \leq \lambda < \infty$. One can follow the technique of Eschenburg and Wang [21] to establish that—in this case—the asymptotics of α_λ near $r = 0$ are suitable for smooth extension across the origin. Furthermore, these solutions are evidently smooth for $r > 0$. It follows that:

$$A_\lambda = A_{\text{can}} + \frac{1}{\lambda r^2 + 1} \sum_{k=1}^{7} \theta^k \otimes \beta_k \quad \text{where } 0 \leq \lambda < \infty, \tag{4.22}$$

is a globally smooth Spin(7) instanton.

When $\lambda = 0$ one can use (4.18) to find that A_λ is flat. Since $\mathbb{R}^8$ is simply connected, $P_{\text{Id}_{\text{Spin}(7)}}$ has a unique flat connection up to gauge. It is given by the product structure. This solution is trivial and we ignore it. Note that it is asymptotic to $A_{\text{can}} + \sum_{k=1}^{7} \theta^k \otimes \beta_k$ rather than A_{can}.

When $0 < \lambda < \infty$, A_λ is known as the *standard octonionic instanton.* The Fubini-Nicolai solution [27] sits at $\lambda = 1$. The other elements of the family are obtained by dilation.

Examining (4.18) and varying the input tangent vectors, we find that F_{A_λ} spans the full Lie algebra $\mathfrak{spin}(7)$. The Ambrose-Singer holonomy theorem then implies that:

$$\text{Hol}\,(A_\lambda) = \text{Spin}(7) \text{ for } \lambda > 0.$$

Consequently, A_λ is irreducible. This finally justifies our choice of large structure group: choosing a smaller group would not yield any results at all.

It is clear from (4.22) that the standard octonionic instanton is asymptotically conical of rate -2 and with limiting connection given by A_{can}. The latter is G_2, but not contact.

The above considerations are sufficient to obtain a complete characterization of the moduli space of invariant Spin(7) instantons in this context. Equivariant gauge transformations on $P_{\text{Id}_{\text{Spin}(7)}}$ are parameterized by the center of Spin(7), which is empty. It follows that all the A_λ lie in distinct equivariant gauge equivalence classes. Since Spin(7) is semisimple, modding out invariant solutions by equivariant gauge transformations retrieves the invariant locus in the full moduli space [54]. It follows that the invariant Spin(7) instantons form—up to gauge—a copy of the positive real line $(0, \infty)$. Note that this moduli space is entirely contained in $\mathcal{M}_\mu(P_{\text{Id}_{\text{Spin}(7)}}, A_{\text{can}})$ with rate $\mu = -2 + \epsilon$ for any $\epsilon > 0$. Furthermore, it is noncompact with missing limits at both its left and right endpoints.

As $\lambda \to 0$ the noncompactness is manifested by a shift in the limiting connection. The missing limit is the product structure. It is AC (in fact dilation invariant), but—as observed earlier—it converges to $A_{\mathrm{can}} + \sum_{k=1}^{7} \theta^k \otimes \beta_k$, rather than A_{can}. This is an interesting noncompactness phenomenon special to the open manifold case: energy escapes toward infinity modifying the limiting connection, but this leads to no topological shift. Perhaps this behaviour is typical of Spin(7) instantons over AC CY fourfolds. The next chapter is concerned with a similar analysis on the Stenzel space where this sort of phenomenon will be met once more.

As $\lambda \to \infty$ the instantons A_λ attempt to converge to the dilation invariant extension of A_{can}. This solution has an *essential* point-singularity at the origin as demonstrated by the unboundedness of its curvature tensor. It thus does not smoothly extend to any bundle and convergence is only local.

5 SO(5)-Equivariant Instantons on the Stenzel Space

5.1 The Stenzel Space

The smooth manifold underlying the Stenzel space is $T^\star S^4$. We introduce the natural cohomogeneity-one SO(5)-action on it and derive general formulae expressing invariant Kähler structures coming from global invariant Kähler potentials. We finally solve the (dimensionally reduced) Monge-Ampère equation associated to the Calabi-Yau condition to obtain the Stenzel CY-4 structure. The overall technique for finding invariant objects in cohomogeneity one is essentially the same as in the paper (Lotay-Oliveira [43]). The calculations for SO(4)-invariant Kähler structures on $T^\star S^3$ have been carried out in the paper (Oliveira [55]). Our notation is the same as the one employed there.

5.1.1 Models for $T^\star S^4$ and the Cohomogeneity One SO(5) Action

We will work with two different models for the space $T^\star S^4$. It is helpful to introduce both of them, as they capture different aspects of the structures we wish to study. The first model elucidates the SO(5) symmetry; the second the complex structure.

The manifold $T^\star S^4$ admits a natural embedding into $\mathbb{R}^{10}$ as follows:

$$T^\star S^4 = \left\{ (x, y^\intercal) \;\mid\; |x| = 1,\; \langle x, y\rangle = 0 \right\} \subset T^\star \mathbb{R}^5.$$

The group SO(5) acts on the left through its natural vector action (and its linearization on forms):

$$g\,(x, y^\intercal) \overset{\text{def}}{=} \left(gx, y^\intercal g^{-1} \right). \tag{5.1}$$

We now compute the orbits and isotropy groups of (5.1):

Proposition 5.1. *The principal orbits of* (5.1) *are the positive radius sphere bundles of* $T^\star S^4$ *in the metric inherited by* $\mathbb{R}^{10}$. *They are* 7 *dimensional Stiefel manifolds with isotropy group isomorphic to* SO(3). *The singular orbit is* S^4 *sitting in its cotangent bundle as the zero section. Its isotropy subgroup is isomorphic to* SO(4).

Proof. Let $p_{R_-} = \left(x, y_{R_-}\right) \in T^\star S^4$ be the point:

$$x \overset{\text{def}}{=} (1,0,0,0,0)^\intercal\,, \quad y_{R_-} \overset{\text{def}}{=} (0, R_-, 0, 0, 0)\,. \tag{5.2}$$

An element $g \in \mathrm{SO}(5)$ stabilizes p_{R_-} if and only if:

$$gx = x \text{ and } yg^{-1} = y.$$

The first equation forces the first column of g to vanish. Since the columns are orthonormal, this forces g to lie in the lower right diagonal copy of SO(4). Repeating the same argument, we see that the second equation forces the SO(4) block to lie in the lower right diagonal copy of SO(3). Matrices lying in this subgroup definitely stabilise p_{R_-} and hence we have:

$$\mathrm{Stab}_{\mathrm{SO}(5)}(p_{R_-}) = \mathrm{SO}(3).$$

Since the action of SO(5) is transitive on S^4 and the action of SO(4) is transitive on S^3, the orbit of p_{R_-} is precisely the R_--sphere bundle in $T^\star S^4$:

$$O_{R_-} = \mathrm{SO}(5)p_{R_-} = S_{R_-}\left(T^\star S^4\right) = \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}. \tag{5.3}$$

Now work with the point:

$$p_0 = (x, 0)$$

where x is as in (5.2). Applying the same argument used in the positive radius case we see that:

$$\mathrm{Stab}_{\mathrm{SO}(5)}(p_0) = \mathrm{SO}(4).$$

Using the same reasoning as above, we immediately see that the orbit is the zero section:

$$\mathrm{SO}(5)p_0 = S^4 = \frac{\mathrm{SO}(5)}{\mathrm{SO}(4)}.$$

□

For any vector bundle E over a manifold M, we can write the following decomposition at the topological level:

$$E - M \cong (0, \infty) \times S\,(E)$$

where $S(E)$ denotes the unit sphere bundle of E. In our case, this splitting takes the form:

$$T^{\star}S^4 - S_4 \cong (0,\infty) \times S_1\left(T^{\star}S^4\right) \tag{5.4}$$

where the subscript 1 denotes the radius. This identification is explicitly given by:

$$(R_-,\omega) \mapsto R_-\omega.$$

where $\omega \in S_1\left(T^{\star}S^4\right)$ and $R_- > 0$.

Equation (5.2) provides us with a natural choice of reference point on each principal orbit O_{R_-}. This choice identifies O_{R_-} with the left coset space $\frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}$ as in (5.3). Combining the above we may write:

$$T^{\star}S^4 - S_4 \cong (0,\infty) \times \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)},$$

where:

$$\left(R_-,\ g\,\mathrm{SO}(3)\right) \mapsto gp_{R_-}. \tag{5.5}$$

Note however, that the unit sphere bundle is twisted as can be shown, for instance, by the hairyball theorem.

We may also realise $T^{\star}S^4$ as a complex submanifold of $\mathbb{C}^5$. Consider the quadratic polynomial:

$$F \stackrel{\text{def}}{=} z_1^2 + ... + z_5^2.$$

Since F is holomorphic, we may compute its derivative as:

$$dF = \partial F = \sum_{j=1}^{5} 2z_i dz^i. \tag{5.6}$$

Since every point p in $F^{-1}(1)$ must have a non-zero coordinate, $dF_{|p}$ does not vanish. It follows that 1 is a regular value for F. Since F is holomorphic, we see that:

$$X^8 \stackrel{\text{def}}{=} F^{-1}(1) \tag{5.7}$$

is a complex submanifold of $\mathbb{C}^5$ and hence Kähler. We split the complex coordinates of $\mathbb{C}^5$

into their real and imaginary parts:

$$z_j = x_j + iy_j.$$

We introduce the functions:

$$r^2 \stackrel{\text{def}}{=} |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2.$$

$$R_+^2 \stackrel{\text{def}}{=} x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2.$$

$$R_-^2 \stackrel{\text{def}}{=} y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2.$$

The following relations follow:

$$R_+^2 = \frac{r^2+1}{2}, \ \ R_-^2 = \frac{r^2-1}{2}, \ \ r^2 = R_+^2 + R_-^2.$$

Define the map:

$$\Psi : \mathbb{C}^5 \to \mathbb{R}^{10},$$

$$(z_1, ..., z_5) \mapsto \left(\frac{x}{R_+}, y^{\intercal} \right). \tag{5.8}$$

It may be easily seen that this cuts down to a diffeomorphism:

$$X^8 \xrightarrow{\sim} T^\star S^4.$$

We therefore conclude that:

Proposition 5.2. *The complex quadric X^8 is diffeomorphic to the total space $T^\star S^4$.*

This identification endows $T^\star S^4$ with a complex structure. Seeing as we are interested in studying $T^\star S^4$ as a CY 4-fold, from here on we mostly work in the complex model. The minimum value of r on X^8 is $r = 1$ and the associated level set corresponds to the singular orbit $\frac{\mathrm{SO}(5)}{\mathrm{SO}(4)}$. The latter sits inside X^8 as an embedded totally real submanifold (Patrizio [58]). When working in the complex model, we will modify the notation of the previous subsection and relabel the point p_{R_-} by p_r. We then have:

$$p_r \stackrel{\text{def}}{=} (R_+, iR_-, 0, 0, 0)^{\intercal}. \tag{5.9}$$

With this definition p_r corresponds to p_{R_-} under the identification (5.8). Furthermore, we will denote the principal orbit at radius $r > 1$ as O_r.

Our next task it to understand how to work with invariant tensors on X^8. The goal is to obtain a natural frame for $T_{p_r}X^8$ along the reference ray. To begin with, the following proposition is an immediate consequence of (5.6):

Proposition 5.3. *At a point $p \in X^8 \subset \mathbb{C}^5$, with $p_5 \neq 0$ we have:*

$$T_pX^8 = \left\{ \left(v_1, v_2, v_3, v_4, -\frac{1}{p_5}\left(p_1v_1 + p_2v_2 + p_3v_3 + p_4v_4\right)\right)^{\mathsf{T}} \ s.t.\ v_j \in \mathbb{C} \right\}. \tag{5.9}$$

We introduce a natural choice of a radial vector ∂_r on the complement of the singular orbit:

Proposition 5.4. *There exists a unique smooth vector field ∂_r on $X^8 - S^4$ characterised by the following properties:*

1. *The vector field ∂_r is tangent to $(0, \infty)$ in the splitting (5.4).*

2. *$dr\left(\partial_r\right) = 1$.*

Let $(x, y) \in X^8$. The vector field ∂_r can be expressed as follows in terms of the standard coordinate vector fields on $\mathbb{C}^5$:

$$\partial_{r_{|(x,y)}} = \frac{r}{2R_+^2}\left(\sum_{j=1}^{5} x^j \partial_{x^j_{|(x,y)}}\right) + \frac{r}{2R_-^2}\left(\sum_{j=1}^{5} y^j \partial_{y^j_{|(x,y)}}\right) \tag{5.10}$$

Evaluating the expression (5.10) at p_r we obtain:

$$\partial_r = \frac{r}{2R_+}\partial_{x^1} + \frac{r}{2R_-}\partial_{y^2}. \tag{5.11}$$

To complete the frame, we study the symmetries along the principal orbits. The Lie algebra $\mathfrak{so}(5)$ consists of all 5×5 antisymmetric matrices under the commutator bracket. It is given by:

$$\mathfrak{so}(5) = \text{Span}\left\{C_{ij} \mid 1 \leq i < j \leq 5\right\},$$

where $C_{ij} = e_{ij} - e_{ji}$ and e_{ij} is the matrix with ij entry equal to 1 and all other entries vanishing. The bracket is characterized by:

$$[C_{ij}, C_{ik}] = -C_{jk}, \tag{5.12}$$

$$[C_{ij}, C_{kl}] = 0 \text{ for } i \neq j \neq k \neq l. \tag{5.13}$$

We write:

$$X_1 = C_{12},\ X_2 = C_{13},\ X_3 = C_{14},\ X_4 = C_{15},\ X_5 = C_{23}$$

$$X_6 = C_{24},\ X_7 = C_{25},\ X_8 = C_{34},\ X_9 = C_{35},\ X_{10} = C_{45}.$$

and we denote the dual one-form of X_i by θ^i.

The adjoint representation of SO(5) on its Lie algebra can be restricted to SO(3)$<$ SO(5). An element $g \in \mathrm{SO}(3)$ then acts on $A \in \mathfrak{so}(5)$ by conjugation. It can be easily seen that this representation splits as:

$$\mathfrak{so}(5) = \langle X_1 \rangle \oplus \langle X_2, X_3, X_4 \rangle \oplus \langle X_5, X_6, X_7 \rangle \oplus \langle X_{10}, -X_9, X_8 \rangle \tag{5.14}$$

The first summand is the trivial representation. The other three are isomorphic to the vector representation of SO(3) on $\mathbb{R}^3$: the order in which the vectors appear corresponds to the standard basis $(\partial_x, \partial_y, \partial_z)$. The Lie algebra of the stabilizer is given by the final summand. We define the natural reductive complement:

$$\mathfrak{m} = \langle X_1 \rangle \oplus \langle X_2, X_3, X_4 \rangle \oplus \langle X_5, X_6, X_7 \rangle. \tag{5.15}$$

Owing to (5.14), this is closed under the action of $\mathrm{Ad}_{\mathrm{SO}(3)}$ and its left invariant extension gives the canonical invariant connection on the SO(3)-bundle:

$$\mathrm{SO}(3) \hookrightarrow \mathrm{SO}(5) \twoheadrightarrow O_r.$$

Given our choice of p_r, the right arrow is given by:

$$\pi : g \mapsto g p_r.$$

Using (5.1) and (5.8), we find that:

$$d\pi_{|\mathrm{Id}} : \mathfrak{m} \xrightarrow{\sim} T_{p_r}O_r$$

acts on a matrix A by:

$$A \mapsto \big(R_+ c_1(A), -R_- r_2(A)\big)\,, \tag{5.16}$$

where $c_1(\cdot)$ denotes the operation of taking the first column and $r_2(\cdot)$ denotes the operation of taking the second row. Using (5.16) we obtain the equations:

$$d\pi_{|\mathrm{Id}} X_1 = -R_+ \partial_{x^2_{|p_r}} + R_- \partial_{y^1_{|p_r}} \tag{5.17}$$

$$d\pi_{|\mathrm{Id}} X_2 = -R_+ \partial_{x^3_{|p_r}} \tag{5.18}$$

$$d\pi_{|\mathrm{Id}} X_3 = -R_+ \partial_{x^4_{|p_r}} \tag{5.19}$$

$$d\pi_{|\mathrm{Id}} X_4 = -R_+ \partial_{x^5_{|p_r}} \tag{5.20}$$

$$d\pi_{|\mathrm{Id}} X_5 = -R_- \partial_{y^3_{|p_r}} \tag{5.21}$$

$$d\pi_{|\mathrm{Id}} X_6 = -R_- \partial_{y^4_{|p_r}} \tag{5.22}$$

$$d\pi_{|\mathrm{Id}} X_7 = -R_- \partial_{y^5_{|p_r}} \tag{5.23}$$

Evidently, X_1, X_2, X_3, X_4 correspond to infinitesimal motions in the horizontal directions along the base S^4. The vectors X_5, X_6, X_7 correspond to infinitesimal vertical motions along the fiber of the sphere bundle.

The only vector invariant under $\mathrm{Ad}_{\mathrm{SO}(3)}$ is X_1. It extends to a globally defined, SO(5)-invariant vector field over O_r. Ultimately, it will show up as the Reeb field over the asymptotic link of the Stenzel space. Its dual is related to the canonical cotangent symplectic structure. Since the tautological (Liouville) 1-form θ_L on $T^\star S^4$ is invariant under diffeomorphisms lifted from the base, it ought to show up among the invariant 1-forms. An easy calculation demonstrates that it is given by:

$$\theta_L = -R_- \theta^1. \tag{5.24}$$

Using (5.11) and (5.17)-(5.23) we conclude that (at p_r):

$$dx^1 = \frac{r}{2R_+}dr \qquad dy^1 = R_-\theta^1 \tag{5.25}$$

$$dx^2 = -R_+\theta^1 \qquad dy^2 = \frac{r}{2R_-}dr \tag{5.26}$$

$$dx^3 = -R_+\theta^2 \qquad dy^3 = -R_-\theta^5 \tag{5.27}$$

$$dx^4 = -R_+\theta^3 \qquad dy^4 = -R_-\theta^6 \tag{5.28}$$

$$dx^5 = -R_+\theta^4 \qquad dy^5 = -R_-\theta^7 \tag{5.29}$$

Using equations (5.25)-(5.29) we obtain:

$$dz^1 = \frac{r}{2R_+}dr + iR_-\theta^1 \tag{5.30}$$

$$dz^2 = -R_+\theta^1 + i\frac{r}{2R_-}dr \tag{5.31}$$

$$dz^3 = -R_+\theta^2 - iR_-\theta^5 \tag{5.32}$$

$$dz^4 = -R_+\theta^3 - iR_-\theta^6 \tag{5.33}$$

$$dz^5 = -R_+\theta^4 - iR_-\theta^7 \tag{5.34}$$

5.1.2 SO(5)-Invariant Kähler Structures

We now turn to the problem of finding SO(5)-invariant Kähler structures on X^8. Since the second cohomology group vanishes, any Kähler structure comes from a global Kähler potential. We therefore seek a Kähler form ω on X^8 coming from a potential function $\mathcal{F}(r^2)$:

$$\omega \in \Lambda^{1,1}T^\star M,$$

$$\omega = \frac{i}{2}\partial\overline{\partial}\mathcal{F}(r^2).$$

This is automatically SO(5)-invariant. The formulae from the previous section allow us to write $\omega_{|_{p_r}}$ in terms of invariant combinations of the θ^i. We calculate:

$$\omega = \frac{i}{2}\partial\overline{\partial}\mathcal{F}(r^2) = \frac{i}{2}\partial\left(\mathcal{F}'(r^2)\overline{\partial}r^2\right)$$

$$= \frac{i}{2}\left(\mathcal{F}''(r^2)\partial r^2 \wedge \overline{\partial}r^2 + \mathcal{F}'(r^2)\partial\overline{\partial}r^2\right)$$

$$= \frac{i}{2}\mathcal{F}'(r^2)\sum_{j=1}^{5} dz^j \wedge d\overline{z}^j + \frac{i}{2}\mathcal{F}''(r^2)\sum_{j=1}^{5}\overline{z^j}dz^j \wedge \sum_{j=1}^{5} z^j d\overline{z^j}. \tag{5.35}$$

In the last step we have explicitly calculated ∂r^2, $\overline{\partial} r^2$ and $\partial\overline{\partial} r^2$ using the standard coordinates in $\mathbb{C}^5$. For this calculation, it is useful to write:

$$r^2 = \sum_{j=1}^{5} z^j \overline{z}^j$$

We now substitute (5.30)-(5.34) into (5.35) to obtain:

$$\omega = P(r)dr \wedge \theta^1 + Q(r)\left(\theta^{25} + \theta^{36} + \theta^{47}\right), \tag{5.36}$$

where we have introduced the functions:

$$P(r) \stackrel{\text{def}}{=} \frac{r}{2}\left(\frac{R_+}{R_-} + \frac{R_-}{R_+}\right)\mathcal{F}'(r^2) + 2rR_+R_-\mathcal{F}''(r^2), \tag{5.37}$$

$$Q(r) \stackrel{\text{def}}{=} R_+R_-\mathcal{F}'(r^2). \tag{5.38}$$

Direct calculation shows that the volume form associated to the Kähler structure defined by ω is given by:

$$\mathrm{Vol}_\omega = \frac{\omega^4}{4!} = -PQ^3 dr \wedge \theta^{1234567}. \tag{5.39}$$

We observe that ordering the basis vectors at p_r in increasing index and with the radial vector coming first gives a negatively oriented basis.

Having expressed the Kähler form in terms of invariant forms and the invariant potential (formula in (5.36)), we now write down the complex structure J in this language and proceed to derive an expression for the associated Kähler metric.

Recall that X^8 is a complex submanifold of $\mathbb{C}^5$. As such, the complex structure J is induced from the standard complex structure of the ambient space:

$$\partial_{x^j} \mapsto \partial_{y^j}, \quad \partial_{y^j} \mapsto -\partial_{x^j}.$$

Using equations (5.17)-(5.10), we discover:

$$JX_1 = -\frac{2R_+R_-}{r}\partial_r \qquad J\partial_r = \frac{r}{2R_+R_-}X_1 \tag{5.40}$$

$$JX_2 = \frac{R_+}{R_-}X_5 \qquad JX_5 = -\frac{R_-}{R_+}X_2 \tag{5.41}$$

$$JX_3 = \frac{R_+}{R_-}X_6 \qquad JX_6 = -\frac{R_-}{R_+}X_3 \tag{5.42}$$

$$JX_4 = \frac{R_+}{R_-}X_7 \qquad JX_7 = -\frac{R_-}{R_+}X_4. \tag{5.43}$$

The associated metric is given by:

$$g(\cdot,\cdot) = \omega(\cdot, J\cdot).$$

Using this formula in conjunction with (5.36) and (5.40)-(5.43) we find that all the off-diagonal components vanish and:

$$g(X_1, X_1) = \frac{2R_+R_-}{r}P, \quad g(\partial_r, \partial_r) = \frac{r}{2R_+R_-}P, \tag{5.44}$$

$$g(X_2, X_2) = g(X_3, X_3) = g(X_4, X_4) = \frac{R_+}{R_-}Q, \tag{5.45}$$

$$g(X_5, X_5) = g(X_6, X_6) = g(X_7, X_7) = \frac{R_-}{R_+}Q. \tag{5.46}$$

We hence obtain:

$$\begin{aligned} g = {} & \frac{rP}{2R_+R_-}dr \otimes dr + \frac{2R_+R_-P}{r}\theta^1 \otimes \theta^1 \\ & + \frac{R_+Q}{R_-}\left(\theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3 + \theta^4 \otimes \theta^4\right) \\ & + \frac{R_-Q}{R_+}\left(\theta^5 \otimes \theta^5 + \theta^6 \otimes \theta^6 + \theta^7 \otimes \theta^7\right). \end{aligned} \tag{5.47}$$

5.1.3 The Stenzel Calabi-Yau Structure

We are now interested in imposing the Calabi-Yau condition on the invariant Kähler structures discussed above. We begin by studying the canonical bundle of X^8. We first prove it is trivial by constructing an explicit holomorphic trivialization. We then derive a formula for this trivialization in terms of invariant forms.

Proposition 5.5. *The bundle K_{X^8} is holomorphically trivial.*

Proof. Let $S_i \subset \mathbb{C}^5$ be the open subset where $z_i \neq 0$. Introduce the following $(n,0)$-form on S_i:

$$\Omega_i \stackrel{\text{def}}{=} \frac{1}{z^i} dz^{i+1} \wedge dz^{i+2} \wedge \ldots \wedge dz^{i-1} \tag{5.48}$$

where the indices in 5.48 are reduced mod 5.

The forms $\iota^\star_{X^8}\Omega_i$ glue to a global holomorphic volume form on X^8. □

The first coordinate of p_r does not vanish and hence we have:

$$\Omega = \Omega_1 = \frac{1}{R_+} dz^2 \wedge \ldots \wedge dz^5$$

Using formulae (5.30)-(5.34) and performing a lengthy calculation, we discover that:

$$\begin{aligned} \mathfrak{Re}(\Omega) =& R_+^3 \theta^{1234} - R_+ R_-^2 \left(\theta^{1267} + \theta^{1537} + \theta^{1564}\right) \\ &+ \frac{r}{2} dr \wedge \left(R_+ \left(\theta^{237} + \theta^{264} + \theta^{534}\right) - \frac{R_-^2}{R_+}\theta^{567}\right) \end{aligned} \tag{5.49}$$

$$\begin{aligned} \mathfrak{Im}(\Omega) =& - R_-^3 \theta^{1567} + R_+^2 R_- \left(\theta^{1237} + \theta^{1264} + \theta^{1534}\right) \\ &+ \frac{r}{2} dr \wedge \left(R_- \left(\theta^{267} + \theta^{537} + \theta^{564}\right) - \frac{R_+^2}{R_-}\theta^{234}\right) \end{aligned} \tag{5.50}$$

Finally, we calculate the volume form associated to Ω. We first compute:

$$\Omega \wedge \overline{\Omega} = \left(\mathfrak{Re}(\Omega) + i\mathfrak{Im}(\Omega)\right) \wedge \left(\mathfrak{Re}(\Omega) - i\mathfrak{Im}(\Omega)\right) = \mathfrak{Re}(\Omega) \wedge \mathfrak{Re}(\Omega) + \mathfrak{Im}(\Omega) \wedge \mathfrak{Im}(\Omega)$$

We then use (5.49) and (5.50) to see that:

$$\Omega \wedge \overline{\Omega} = -8rR_+^2 R_-^2 dr \wedge \theta^{1234567}$$

We can now easily compute:

$$\mathrm{Vol}_\Omega = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega} = -\frac{r}{2} R_+^2 R_-^2 dr \wedge \theta^{1234567} \tag{5.51}$$

The CY-4 condition is equivalent to volume compatibility:

$$\mathrm{Vol}_\omega = \mathrm{Vol}_\Omega. \tag{5.52}$$

This boils down to a Monge-Ampère-type equation for $\mathcal{F}(r^2)$ with right hand side determined by Ω. In SO(5)-symmetry this reduces to an ODE. We derive the ODE and obtain the solution explicitly. The Calabi-Yau metric obtained through this process is known as the *Stenzel metric* (Stenzel [67], Oliveira [55]). Using (5.39) and (5.51), we see that (5.52) is equivalent to the ODE:

$$PQ^3 = \frac{r}{2} R_+^2 R_-^2. \tag{5.53}$$

Unpacking the definitions of P (5.37) and Q (5.38), translates the equation to:

$$1 = r^2 \mathcal{F}'(r^2)^4 + (r^4 - 1)\mathcal{F}'(r^2)^3 \mathcal{F}''(r^2) \tag{5.54}$$

We thus obtain a second order nonlinear ODE for $\mathcal{F}$. Observe that the metric only depends on $\mathcal{F}'$. This motivates us to introduce:

$$\mathcal{G}(r^2) \stackrel{\text{def}}{=} \mathcal{F}'(r^2)^4 \tag{5.55}$$

Writing (5.54) in terms of $\mathcal{G}$ we obtain:

$$1 = r^2 \mathcal{G}(r^2) + \frac{(r^4 - 1)}{4} \mathcal{G}'(r^2) \tag{5.56}$$

We thus reduce the equation to a first order linear ODE for $\mathcal{F}'$. This is soluble by hand using the integrating factor technique. We write $u = r^2$ and multiply the equation by

$4(u^2-1)$ to obtain:

$$\begin{aligned}
&(u^2-1)^2\frac{d\mathcal{G}}{du}+4u(u^2-1)\mathcal{G}(u)=4(u^2-1)\\
\Rightarrow&\frac{d}{du}\Big((u^2-1)^2\mathcal{G}(u)\Big)=4(u^2-1)\\
\Rightarrow&(u^2-1)^2\mathcal{G}(u)=\frac{4}{3}u^3-4u+C\\
\Rightarrow&\mathcal{G}(u)=\frac{4}{3}\frac{u^3-3u+C}{(u^2-1)^2}
\end{aligned} \tag{5.57}$$

We therefore have the solution:

$$\mathcal{F}'(r^2)=\left(\frac{4}{3}\right)^{\frac{1}{4}}\left(\frac{r^6-3r^2+C}{(r^4-1)^2}\right)^{\frac{1}{4}}.$$

We would like to select the constant so that $\mathcal{F}'$ extends continuously at $r^2=1$. This forces us to take $C=2$, so that the numerator of the fraction vanishes at $r^2=1$. We obtain:

$$\mathcal{F}'(r^2)=\left(\frac{4}{3}\right)^{\frac{1}{4}}\frac{\left(r^2+2\right)^{\frac{1}{4}}}{\left(r^2+1\right)^{\frac{1}{2}}}. \tag{5.58}$$

Our task is to write down the functions P and Q in terms of r. The relation (5.54) gives:

$$\mathcal{F}''(r^2)=\frac{1-r^2\mathcal{F}'(r^2)^4}{(r^4-1)\mathcal{F}'(r^2)^3}.$$

Combining this with the relation (5.37) we obtain:

$$P(r)=\frac{r}{2}\left(\frac{R_+}{R_-}+\frac{R_-}{R_+}\right)\mathcal{F}'(r^2)+2rR_+R_-\frac{1-r^2\mathcal{F}'(r^2)^4}{(r^4-1)\mathcal{F}'(r^2)^3}.$$

A short calculation gives:

$$P(r)=\frac{r}{2R_+R_-\mathcal{F}'(r^2)^3}.$$

Incorporating (5.58) we obtain:

$$P(r)=\left(\frac{3}{4}\right)^{\frac{3}{4}}\frac{r(r^2+1)}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{1}{2}}(r-1)^{\frac{1}{2}}}. \tag{5.59}$$

Similarly, we determine $Q(r)$. Using (5.38) we obtain:

$$Q(r) = \frac{1}{2}\left(\frac{4}{3}\right)^{\frac{1}{4}} (r^2+2)^{\frac{1}{4}}(r+1)^{\frac{1}{2}}(r-1)^{\frac{1}{2}}. \qquad (5.60)$$

Note that as $r \to 1$, we have that $P(r) \to \infty$ monotonically. This is merely a coordinate singularity: ω is constructed using a globally smooth Kähler potential.

The Stenzel metric is left SO(5) invariant but not bi-invariant: its restriction on the principal orbits does not agree with the Killing form. It does, however, enjoy a right U(1) symmetry. This is in direct analogy to the Berger metrics on the Hopf fibration (Hitchin [30]).

We now study how the norms of the basis vectors vary with r. Using (5.47) we observe the following results:

$$|X_1|^2 = \left(\frac{3}{4}\right)^{\frac{3}{4}} \frac{(r^2+1)^{\frac{3}{2}}}{(r^2+2)^{\frac{3}{4}}}, \qquad (5.61)$$

$$|\partial_r|^2 = \left(\frac{3}{4}\right)^{\frac{3}{4}} \frac{r^2(r^2+1)^{\frac{1}{2}}}{(r^2+2)^{\frac{3}{4}}(r+1)(r-1)}, \qquad (5.62)$$

$$|X_2|^2 = |X_3|^2 = |X_4|^2 = \frac{1}{2}\left(\frac{4}{3}\right)^{\frac{1}{4}} (r^2+1)^{\frac{1}{2}}(r^2+2)^{\frac{1}{4}}, \qquad (5.63)$$

$$|X_5|^2 = |X_6|^2 = |X_7|^2 = \frac{1}{2}\left(\frac{4}{3}\right)^{\frac{1}{4}} \frac{(r^2+2)^{\frac{1}{4}}(r+1)(r-1)}{(r^2+1)^{\frac{1}{2}}}. \qquad (5.64)$$

We observe that as $r \to 1$, $|\partial_r|^2$ blows up monotonically, $|X_1|^2$, $|X_2|^2$, $|X_3|^2$ and $|X_4|^2$ approach 1 and $|X_5|^2, |X_6|^2, |X_7|^2$ tend to 0. Recall that over the singular orbit, the kernel of the projection map (5.16) extends to $\mathfrak{so}(4)$ and X_5, X_6, X_7 project to 0. Consequently, the decay of their norms as $r \to 1$ is a property true of any smooth metric on $T^\star S^4$.

The SO(4) orbit of p_r is the round 3-sphere S^3_ρ of radius:

$$\rho^2 = \frac{1}{2}\left(\frac{4}{3}\right)^{\frac{1}{4}} \frac{(r^2+2)^{\frac{1}{4}}(r+1)(r-1)}{(r^2+1)^{\frac{1}{2}}}.$$

The 3-dimensional volume of S^3_ρ is given by:

$$\mathrm{Vol}\left(S^3_\rho\right) = 2\pi^2\rho^3 = \frac{2^{\frac{1}{4}}\pi^2}{3^{\frac{3}{8}}}\frac{(r^2+2)^{\frac{3}{8}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}}{(r^2+2)^{\frac{3}{4}}}.$$

The singular orbit is the round unit radius 4-sphere S^4_1 with 4-dimensional volume equal to $\frac{8\pi^2}{3}$.

As discussed in section 1, a CY 4-fold is in a natural way a Spin(7) manifold. Recall that the induced Cayley calibration can be written in terms of the Kähler form and the holomorphic volume form:

$$\Phi = \frac{\omega^2}{2} + \mathfrak{Re}(\Omega). \tag{5.65}$$

Using (5.36), we find that:

$$\omega^2 = 2PQdr \wedge \left(\theta^{125} + \theta^{136} + \theta^{147}\right) + 2Q^2\left(\theta^{2536} + \theta^{2547} + \theta^{3647}\right). \tag{5.66}$$

Combining this with (5.49) and incorporating the results into (5.65), we obtain:

$$\begin{aligned}\Phi = dr \wedge &\left[PQ\left(\theta^{125} + \theta^{136} + \theta^{147}\right) + \frac{rR_+}{2}\left(\theta^{237} + \theta^{264} + \theta^{534}\right) - \frac{rR_-^2}{2R_+}\theta^{567}\right] \\ &+ R_+^3\theta^{1234} - R_+R_-^2\left(\theta^{1267} + \theta^{1537} + \theta^{1564}\right) + Q^2\left(\theta^{2536} + \theta^{2547} + \theta^{3647}\right).\end{aligned} \tag{5.67}$$

We immediately make the following observation. When we pull back Φ to the singular S^4 by the inclusion map, only the θ^{1234} term survives. Furthermore, on S^4 we have $r = 1$. We therefore get:

$$\iota^\star_{S^4}\Phi = \theta^{1234}. \tag{5.68}$$

We conclude that the singular orbit is calibrated for Φ and is therefore a Cayley submanifold of the Spin(7) manifold (X^8, Φ). As such, it is volume minimizing in its homology class (Joyce [32]).

Modifying the radial coordinate so that the corresponding radial vector is asymptotically of unit length and expanding the Stenzel metric at infinity, one finds that it is asymptotically conical of rate $\mu = -\frac{8}{3}$. In fact, this is the *optimal rate* in the sense that it cannot be improved upon by deforming the chosen diffeomorphism (Conlon–Hein [10]):

$$\Psi : X^8 \setminus S^4 \xrightarrow{\sim} (0,\infty) \times \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}.$$

In particular, on setting:

$$s \overset{\mathrm{def}}{=} \left(\frac{4}{3}\right)^{\frac{5}{8}} r^{\frac{3}{4}}, \tag{5.69}$$

one finds that:

$$|g - h_{\mathrm{cone}}|_{h_{\mathrm{cone}}} = O(s^{-\frac{8}{3}}) \quad \text{as } s \to \infty. \tag{5.70}$$

Here, the conical metric:

$$h_{\mathrm{cone}} = ds \otimes ds + s^2 \overline{h} \tag{5.71}$$

is defined over $X^8 \setminus S^4$ and it is built on the homogeneous link $\left(\frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}, \overline{h}\right)$, where:

$$\overline{h} = \frac{9}{16}\theta^1 \otimes \theta^1 + \frac{3}{8}\sum_{j=2}^{7} \theta^j \otimes \theta^j.$$

We briefly consider uniqueness issues for the above construction. The form Ω is not the only possible choice of holomorphic trivialization for $\mathcal{K}_{X^8}$. Others may be constructed through multiplication by a nowhere-vanishing holomorphic function. Scaling by non-trivial SO(5)-invariant functions will yield new SO(5)-invariant holomorphic trivializations. Changing Ω can affect the resulting metric and Cayley calibration through (5.52) and (5.65). An obvious modification is to scale by non-zero complex numbers. Scaling by non-negative reals modifies the Calabi–Yau equation, which ultimately results in rescaling the metric. Scaling by elements of U(1) does not affect (5.52) and hence leaves the metric unaltered. It does however alter the Cayley calibration. The trivialization Ω has been chosen in hindsight for the special geometric features it entails: it makes the radius of the round S^4 equal to 1 (which serves as a natural normalization condition for the rescaling freedom) and furthermore guarantees that the latter is calibrated for the induced Spin(7) structure.

5.2 SO(5)-Invariant Instantons with Structure Group U(1)

We now have all the ingredients required to study invariant instantons on $\left(T^\star S^4, J, \omega, \Omega\right)$. We begin by studying the abelian case. The first task is to apply the results of section 2 in order to classify the relevant bundles and connections. Once this is settled, we proceed to derive the ODEs describing the evolution of invariant solutions to the gauge theoretic equations of interest. The SO(5)-invariant Spin(7) instanton equations turn out to be identical to the SO(5)-invariant HYM equations: the two problems are locally equivalent. We are able to solve the ODEs explicitly. Using the explicit solution, we observe that the corresponding local instantons blow up near the singular orbit S^4. We thus obtain a global nonexistence result.

5.2.1 Homogeneous Bundles and Invariant Connections with Structure Group U(1)

Let $r > 1$. The homogeneous U(1) bundles over the orbit $\mathcal{O}_r$ correspond to element-conjugacy (i.e. conjugation by a fixed element in the target) classes of Lie group homomorphisms:

$$\lambda : \mathrm{SO}(3) \to \mathrm{U}(1). \tag{5.72}$$

Since the target is abelian, the element-conjugacy relation is trivial: the classes are singletons. The only map of type (5.72) is $\lambda = 1$. Consequently, the only homogeneous U(1) bundle over $\mathcal{O}_r$ -up to equivariant principal bundle isomorphism- is the trivial one:

$$P_1 = \mathcal{O}_r \times \mathrm{U}(1) = \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)} \times \mathrm{U}(1). \tag{5.73}$$

SO(5)-invariant U(1)-connections on P_1 are parameterised by representation morphisms:

$$\Lambda : \left(\mathfrak{m}, \mathrm{Ad}_{\mathrm{SO}(5)_{|\mathrm{SO}(3)}}\right) \to \left(\mathfrak{u}(1), \mathrm{Ad}_{\mathrm{U}(1)} \circ \lambda\right) = (i\mathbb{R}, 1)\,. \tag{5.74}$$

Recalling the decomposition (5.15) and applying Schur's lemma, we obtain that:

$$\mathrm{Hom}_{\mathrm{SO}(3)}\left(\mathfrak{m}, \mathfrak{u}(1)\right) = i\mathbb{R}. \tag{5.75}$$

Here, the imaginary number $i\alpha$ corresponds to:

$$\Lambda_\alpha \overset{\text{def}}{=} i\alpha\theta^1. \tag{5.76}$$

The cohomogeneity one bundle over $X^8 \setminus S^4$ associated to P_1 is obtained by pulling back along the map:

$$X^8 \setminus S^4 \xrightarrow{\sim} (1, \infty) \times \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)} \twoheadrightarrow \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}.$$

We slightly abuse notation by suppressing the pullback symbol and denoting the resulting bundle by P_1. It is trivial and it admits a unique extension across the singular orbit given by $X^8 \times \mathrm{U}(1)$.

Connections over $X^8 \setminus S^4$ can be put in temporal gauge (vanishing dr component) through an equivariant gauge transformation (Lotay-Oliveira [43] p. 973, Remark 3). Consequently, each invariant connection on P_1 is equivariantly gauge equivalent to one lying in the space:

$$\mathcal{A}_{\mathrm{inv}}(P_1) = \left\{ i\alpha(r)\theta^1 \mid a \in C^\infty(1, \infty) \right\} \subset \mathcal{A}(P_1). \tag{5.77}$$

Such connections can only be related by an r-independent gauge transformation. If such a gauge transformation is equivariant, it is given by a fixed element of U(1) and it stabilises all connections. It follows that no two distinct elements of $\mathcal{A}_{\mathrm{inv}}(P_1)$ are equivariantly gauge equivalent.

We compute the curvature of $A \in \mathcal{A}(P_1)$:

$$\begin{aligned} F_A &= dA \\ &= i\frac{d\alpha}{dr} dr \wedge \theta^1 + i\alpha(r) d\theta^1. \end{aligned} \tag{5.78}$$

To simplify the second term we use the Maurer-Cartan relations (Kobayashi–Nomizu [37] p. 41). For this calculation we require the structure constants of $\mathfrak{so}(5)$. They can be computed using (5.12) and (5.13). Carrying out the calculation gives:

$$d\theta^1 = \theta^{25} + \theta^{36} + \theta^{47}.$$

Incorporating this into (5.78), we obtain:

$$F_A = i\frac{d\alpha}{dr} dr \wedge \theta^1 + i\alpha(r) \left(\theta^{25} + \theta^{36} + \theta^{47} \right). \tag{5.79}$$

The Ambrose-Singer holonomy theorem implies that any non-flat U(1) connection is irreducible. Consequently, all elements of $\mathcal{A}_{\text{inv}}(P_1)$ -excluding the trivial connection- are irreducible.

5.2.2 The Invariant ODEs on P_1: Dimensional Reduction and Explicit Solution

The Spin(7) instanton equation reads:

$$\star_g F_A = -\Phi \wedge F_A \tag{5.80}$$

We use the formulae obtained in the previous sections to express each side in terms of invariant forms. We work on $T_{p_r}X^8$ with the X_i frame. Since the metric diagonalises we have:

$$\star_g \theta^{i_0} \wedge ... \wedge \theta^{i_k} = \pm \frac{\sqrt{\det(g)}}{g_{i_0 i_0} \cdots g_{i_k i_k}} \theta^{i_{k+1}} \wedge ... \wedge \theta^{i_7}, \tag{5.81}$$

where $i_1, ..., i_n$ is an even permutation of $1, ..., n$. Using (5.81) and (5.47) we obtain the results:

$$\begin{aligned}
\star_g dr \wedge \theta^1 &= -\frac{Q^3}{P}\theta^{234567}, \\
\star_g \theta^{25} &= -PQdr \wedge \theta^{13467}, \\
\star_g \theta^{36} &= -PQdr \wedge \theta^{12356}.
\end{aligned}$$

Using these expressions we obtain:

$$\star_g F_A = -i\frac{Q^3}{P}\frac{d\alpha}{dr}\theta^{234567} - iPQ\alpha dr \wedge \left(\theta^{13467} + \theta^{12457} + \theta^{12356}\right). \tag{5.82}$$

We now use (5.67) and (5.79) to compute:

$$\Phi \wedge F_A = -3iQ^2\alpha(r)\theta^{234567} - i\left(Q^2\frac{d\alpha}{dr} + 2PQ\alpha(r)\right) dr \wedge \left(\theta^{13467} + \theta^{12457} + \theta^{12356}\right). \tag{5.83}$$

Imposing (5.80) and comparing coefficients gives two equations. These are the same and read:

$$\frac{d\alpha}{dr} = -3\frac{P}{Q}\alpha. \tag{5.84}$$

The HYM equations read:

$$F_A \wedge \star\omega = 0,$$
$$F_A \wedge \Omega = 0.$$

The latter statement holds identically. This can be seen by direct computation using (5.49), (5.50) and (5.79).

Over a Hermitian manifold of complex dimension n, we have:

$$\star_g \omega = \frac{\omega^{n-1}}{(n-1)!}, \tag{5.85}$$

where g is the Kähler metric associated to ω by the complex structure. Using (5.36) and (5.66) we compute:

$$\omega^3 = 6PQ^2 dr \wedge \left(\theta^{12536} + \theta^{12547} + \theta^{13647}\right) + 6Q^3\theta^{253647}. \tag{5.86}$$

Using (5.85), (5.86) and (5.79) we calculate:

$$\begin{aligned} F_A \wedge \star\omega &= F_A \wedge \frac{\omega^3}{3!} \\ &= -i\left(Q^3\frac{d\alpha}{dr} + 3PQ^2\alpha(r)\right) dr \wedge \theta^{1234567}. \end{aligned}$$

It follows that an SO(5)-invariant U(1)-connection is HYM if and only if:

$$\frac{d\alpha}{dr} = -3\frac{P}{Q}\alpha. \tag{5.87}$$

We observe that this equation is the same as (5.84).

Using the uniqueness part of the standard Picard theorem, we obtain:

Theorem 5.6. *An* SO(5)*-invariant* U(1)*-connection over* $X^8 - S^4$ *equipped with the Stenzel Calabi-Yau structure is a* Spin(7) *instanton if and only if it is* HYM.

We study the ODE (5.87). Using (5.59) and (5.60) we write it as:

$$\frac{da}{dr} = -\frac{9}{2}\frac{r(r^2+1)}{(r^2+2)(r+1)(r-1)}\alpha(r). \tag{5.88}$$

We integrate (5.88) directly to see that the solution takes the following form for some $K \in \mathbb{R}$:

$$\alpha(r) = \frac{K}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}} \tag{5.89}$$

An elementary calculation yields:

$$\frac{da}{dr} = -\frac{9K}{2}\frac{r(r^2+1)}{(r^2+2)^{\frac{7}{4}}(r+1)^{\frac{5}{2}}(r-1)^{\frac{5}{2}}}. \tag{5.90}$$

Recalling the formulae (5.77) and (5.79) and incorporating (5.89) and (5.90), we formulate the following theorem:

Theorem 5.7. *Let $X^8 - S^4$ be equipped with the Stenzel Calabi-Yau structure (ω, Ω, J). Let P be the unique homogeneous* U(1) *bundle over $X^8 - S^4$ (i.e. the trivial bundle). There exists a one-parameter family of smooth* SO(5)*-invariant Spin(7) instantons $A_K \in \mathcal{A}_{inv}(P)$:*

$$A_K = \frac{iK}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}}\theta^1, \quad \text{where } K \in \mathbb{R}. \tag{5.91}$$

The curvature of A_K is given by:

$$F_{A_K} = iK\left(-\frac{9}{2}\frac{r(r^2+1)}{(r^2+2)^{\frac{7}{4}}(r+1)^{\frac{5}{2}}(r-1)^{\frac{5}{2}}}dr\wedge\theta^1 + \frac{\theta^{25}+\theta^{36}+\theta^{47}}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}}\right). \tag{5.92}$$

Using (5.62), (5.61), (5.63) and (5.64), we find that:

$$|F_A|_g^2 = O\left(|r-1|^{-4}\right) \text{ as } r \to 1 \tag{5.93}$$

In particular:

$$\lim_{r\to 1}|F_A|_g^2 = +\infty$$

i.e. the pointwise norm of the invariant Spin(7) instanton A_K -measured using the Stenzel metric- blows up as $r \to 1$. Since the metric extends smoothly to the singular orbit, this behaviour is precluded for connections that are smooth over the whole space. We therefore obtain the following global nonexistence result:

Theorem 5.8. *There exist no global, abelian,* SO(5)*-invariant* Spin(7) *instantons/* HYM *connections on the Stenzel manifold* X^8 *apart from the trivial connection* $A = 0$ *(corresponding to* $K = 0$*).*

This non-existence result has to do with abelian gauge theory being too coarse to capture the behaviour we would like to see. In the following section we study the nonlinear equations associated to the structure group SO(3). The nonlinearity induced by the non-commutativity of the group *smoothes* the equations and we are able to obtain solutions that extend over the singular orbit S^4.

As a closing remark, we note that—even though the singular locus of A_K is Cayley—these instantons do not provide examples of the interesting removable singularity/ bubbling phenomena introduced in the end of section (4.4). Their singularity is *essential* rather than *removable* and this is captured by the failure of $|F_{A_K}|_g$ to be globally bounded. We have already seen that the loci of essential singularities need not be Cayley: on flat space we encountered essential point-singularities. An explicit removable singularity/ bubbling phenomenon will be encountered in the subsequent section.

5.3 SO(5)-Invariant Instantons with Structure Group SO(3)

In this section we study the nonabelian equations corresponding to the structure group SO(3). We begin by classifying the SO(3) bundles of cohomogeneity one. There are precisely two bundles of this kind on $X^8 - S^4$: P_1 and P_{Id}. They correspond to the trivial map and the identity map from SO(3) to itself respectively. We classify their smooth homogeneous extensions over S^4 and their invariant connections. We then compute expressions for the curvature fields of these connections in terms of the standard framing $\langle X^1, ..., X^7, \partial_r\rangle$. We proceed to deal with the instanton equations on the trivial bundle P_1. The situation here is simple: no new phenomena are encountered. The picture is essentially equivalent to the abelian case.

In preparation for analysis of the instanton equation on P_{Id}, we apply the technique of Eschenburg and Wang (Eschenburg-Wang [21]) to develop a necessary and sufficient condition for connection 1-forms on P_{Id} to smoothly extend over the singular orbit.

We proceed to derive the equations describing the evolution of invariant Spin(7) instantons and invariant HYM connections on the non-trivial bundle P_{Id}.

The HYM equations consist of a decoupled ODE system together with a set of constraint equations. These are easily solved to give precisely two explicit AC HYM connections on the complement of the singular orbit. We prove that they extend over S^4 and study their extensions, which we find to be topologically distinct.

The Spin(7) instanton equations yield a 2 dimensional coupled system of ODEs. It is thus already evident—at least at the local level—that Spin(7) instantons are more general than HYM connections. In fact, we can get a global result: we give an explicit formula for a 1-parameter family of AC Spin(7) instantons on X^8, only one of which is HYM. We thus (negatively) resolve the question regarding the equivalence of the two problems.

We finally analyze the full ODE system. We determine all solutions that are smooth and global in time, thus giving a construction of the moduli space $\mathcal{M}^{\mathrm{Spin}(7)}_{\mathrm{inv}}\left(X^8\right)$ of invariant Spin(7) instantons in this setting. This involves an interesting removable singularity/ bubbling phenomenon that we analyze in detail. We discover that the HYM connections play a role in its resolution and consequently in the compactification of $\mathcal{M}^{\mathrm{Spin}(7)}_{\mathrm{inv}}\left(X^8\right)$.

We note that we have not studied the asymptotics of these non-explicit solutions and have thus not established that they are AC. Nevertheless, we expect that this is the case.

5.3.1 Homogeneous Bundles and Invariant Connections with Structure Group SO(3)

5.3.1.1 Bundles and Bundle Extensions

Let $r > 1$. The homogeneous SO(3) bundles over the orbit $\mathcal{O}_r$ correspond to element-conjugacy classes of Lie group homomorphisms:

$$\lambda : \mathrm{SO}(3) \to \mathrm{SO}(3). \tag{5.94}$$

There are two such classes. They are represented by the trivial map and the identity respectively. Consequently, there are precisely two homogeneous principal SO(3) bundles over $\mathcal{O}_r$ -up to equivariant principal bundle isomorphism. We denote these by P_1 and P_{Id}. Slightly abusing notation, we also denote by P_1 and P_{Id} the pullbacks of the respective bundles along the map:

$$X^8 \setminus S^4 \xrightarrow{\sim} (1,\infty) \times \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)} \twoheadrightarrow \frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}.$$

We now classify smooth homogeneous extensions of P_1 and P_{Id} across the singular orbit S^4. These correspond to element-conjugacy classes of Lie group homomorphisms:

$$\mu : \mathrm{SO}(4) \to \mathrm{SO}(3). \tag{5.95}$$

Once such a map is chosen, one uses it to form the associated homogeneous bundle P_μ over S^4. The extension is then determined by pulling P_μ back over X^8 through the natural projection:

$$X^8 \cong T^\star S^4 \twoheadrightarrow S^4.$$

The element-conjugacy class of the restriction of μ to the lower right block copy of SO(3) determines which bundle is being extended.

We are therefore required to classify element-conjugacy classes of homomorphisms of type (5.95). Natural representatives are described by passing through the respective universal covers. We have the two-sheeted covering maps:

$$\pi_{\mathrm{Spin}(4)} : \mathrm{Sp}(1)^2 \twoheadrightarrow \mathrm{SO}(4),$$
$$\pi_{\mathrm{Spin}(3)} : \mathrm{Sp}(1) \twoheadrightarrow \mathrm{SO}(3),$$

where:

$$\pi_{\mathrm{Spin}(4)}(x,y) : \mathbb{H} \to \mathbb{H},$$
$$q \mapsto xqy^{-1}, \tag{5.96}$$

$$\pi_{\mathrm{Spin}(3)}(x) = \pi_{\mathrm{Spin}(4)}(x,x)_{|\mathfrak{Im}(\mathbb{H})}. \tag{5.97}$$

Considering (5.96) and (5.97), we obtain:

$$\mathrm{SO}(4) = \frac{\mathrm{Sp}(1)^2}{\{(1,1),(-1,-1)\}}, \quad \mathrm{SO}(3) = \frac{\mathrm{Sp}(1)}{\pm 1}.$$

There are precisely three element-conjugacy classes of homomorphisms of type (5.95), one of them being that of the trivial map. The two non-trivial classes are represented by the two projections:

$$\pi_1, \pi_2 : \mathrm{SO}(4) = \frac{\mathrm{Sp}(1)^2}{\{(1,1),(-1,-1)\}} \twoheadrightarrow \frac{\mathrm{Sp}(1)}{\pm 1} \times \frac{\mathrm{Sp}(1)}{\pm 1} \twoheadrightarrow \frac{\mathrm{Sp}(1)}{\pm 1} = \mathrm{SO}(3).$$

The resulting representations correspond to the action of SO(4) on $\Lambda^2_\pm T^\star\mathbb{R}^4$.

We conclude that there are precisely three principal SO(3)-bundles of cohomogeneity-one over X^8. We denote these as P_1, P_{π_1} and P_{π_2}. The first is the trivial bundle. It extends the trivial bundle on $X^8 \setminus S^4$. The other two bundles are non-trivial (see section 5.3.4.2). They provide distinct extensions of P_{Id}.

5.3.1.2 Invariant Connections on the Complement of the Singular Orbit

We now classify the invariant connections on the bundles P_1 and P_{Id} over $X^8 \setminus S^4$. For each of these connections we compute the associated curvature tensor in terms of the standard framing.

We introduce the following basis for $\mathfrak{so}(3)$:

$$e_1 \stackrel{\text{def}}{=} C_{12}, \;\; e_2 \stackrel{\text{def}}{=} C_{13}, \;\; e_3 \stackrel{\text{def}}{=} C_{23}. \tag{5.98}$$

Here the matrices C_{ij} are defined as in section 1.2.1 and obey the commutation relations (5.12) and (5.13). The adjoint representation of SO(3) takes the form:

$$\mathrm{Ad}_{\mathrm{SO}(3)} = \mathfrak{so}(3) = \langle e_3, -e_2, e_1 \rangle. \tag{5.99}$$

Here SO(3) acts by its natural irreducible vector representation on $\mathbb{R}^3$ and we maintain the convention that the ordering in the bracket reflects the associated identification $\mathfrak{so}(3) \cong \mathbb{R}^3$.

In general, an invariant connection $A \in \mathcal{A}_{\text{inv}}\left(P_\lambda\right)$ corresponds to a map of representations:

$$\Lambda : \left(\mathfrak{m}, \mathrm{Ad}_{\mathrm{SO}(5)_{|\mathrm{SO}(3)}}\right) \to \left(\mathfrak{so}(3), \mathrm{Ad}_{\mathrm{SO}(3)} \circ \lambda\right). \tag{5.100}$$

Given such a map, we use the canonical invariant connection $d\lambda$ as a reference and write:

$$A = d\lambda + \Lambda. \tag{5.101}$$

We first deal with P_1. In this case $\lambda = 1$ and the target representation is trivial. Recalling the splitting (5.15) and applying Schur's lemma, we see that Λ must take the form:

$$\Lambda = \theta^1 \otimes \left(a^1\, e_1 + a^2\, e_2 + a^3\, e_3\right). \tag{5.102}$$

The canonical invariant connection A_1^{can} is represented by $d\,1 = 0$. Evidently, it is flat.

Any connection over $X^8 \setminus S^4$ can be brought to temporal gauge by an equivariant gauge transformation [54]. It follows that any invariant connection on P_1 is equivariantly gauge equivalent to one lying in the space:

$$\mathcal{A}_{\text{inv}}\left(P_1\right) = \left\{\theta^1 \otimes \left(a^1(r)\, e_1 + a^2(r)\, e_2 + a^3(r)\, e_3\right) \mid a^1, a^2, a^3 \in C^\infty\left(1, \infty\right)\right\}. \tag{5.103}$$

A gauge transformation relating two elements of $\mathcal{A}_{\text{inv}}\left(P_1\right)$ must be r-independent. If it is equivariant, it is given by a fixed element of SO(3) acting on $\mathcal{A}_{\text{inv}}\left(P_1\right)$ by conjugation. It follows that the elements of $\mathcal{A}_{\text{inv}}\left(P_1\right)$ need not lie in distinct equivariant gauge equivalence classes.

A calculation analogous to the one in section 2.1 yields:

$$F_A = \left(\frac{da^i}{dr} dr \wedge \theta^1 + a^i \left(\theta^{25} + \theta^{36} + \theta^{47}\right)\right) \otimes e_i. \tag{5.104}$$

The Ambrose-Singer holonomy theorem implies that the elements of $\mathcal{A}\left(P_1\right)$ need not be irreducible. This happens -for instance- if one of the components a^i vanishes identically, in which case e_i does not lie in the holonomy algebra.

We now work on P_{Id}. In this case, the target representation is the adjoint representation of SO(3). Recalling the decomposition (5.15) and applying Schur's lemma, we see that equivariant maps of type (5.100) always vanish on the first summand and either restrict to isomorphisms or the zero map on the second and third summands. The automorphisms of $\mathrm{Ad}_{\mathrm{SO}(3)}$ are given by multiplication by fixed scalars. We conclude that for $\lambda = \mathrm{Id}$, maps of type (5.100) look like:

$$\Lambda = a\left(\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1\right) + b\left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right), \quad a, b \in \mathbb{R}.$$

Written over the symmetry group SO(5), the canonical invariant connection $A^{\mathrm{can}}_{\mathrm{Id}} = d\,\mathrm{Id}_{\mathrm{SO}(3)}$ on P_{Id} takes the form:

$$A^{\mathrm{can}}_{\mathrm{Id}} = \theta^8 \otimes e_1 + \theta^9 \otimes e_2 + \theta^{10} \otimes e_3.$$

It is not flat. Its curvature is given by:

$$\begin{aligned} F_{A^{\mathrm{can}}_{\mathrm{Id}}} &= dA^{\mathrm{can}}_{\mathrm{Id}} + \frac{1}{2}\left[A^{\mathrm{can}}_{\mathrm{Id}} \wedge A^{\mathrm{can}}_{\mathrm{Id}}\right] \\ &= \left(\theta^{23} + \theta^{56}\right) \otimes e_1 + \left(\theta^{24} + \theta^{57}\right) \otimes e_2 + \left(\theta^{34} + \theta^{67}\right) \otimes e_3. \end{aligned}$$

The radial component of an invariant tensorial 1-form is an invariant section of the adjoint bundle. In this context, these objects correspond to fixed points of $\mathrm{Ad}_{\mathrm{SO}(3)}$. This representation has no fixed points, implying that all invariant connections are already in temporal gauge. Consequently, the space of invariant connections is given by:

$$\begin{aligned} &\mathcal{A}_{\mathrm{inv}}\left(P_{\mathrm{Id}}\right) = \\ &\left\{A^{\mathrm{can}}_{\mathrm{Id}} + a(r)\left(\theta^2 e_3 - \theta^3 e_2 + \theta^4 e_1\right) + b(r)\left(\theta^5 e_3 - \theta^6 e_2 + \theta^7 e_1\right) | a, b \in C^{\infty}(1, \infty)\right\}. \end{aligned} \tag{5.105}$$

Equivariant gauge transformations correspond to central elements of SO(3). Since SO(3) is centerless, the only possibility is the identity. Consequently, each invariant connection constitutes its own equivariant gauge equivalence class.

To compute the curvature of a general element $A = A^{\mathrm{can}}_{\mathrm{Id}} + \Lambda \in \mathcal{A}_{\mathrm{inv}}(P_{\mathrm{Id}})$, we use the formula:

$$F_A = F_{A^{\mathrm{can}}_{\mathrm{Id}}} + d_{A^{\mathrm{can}}_{\mathrm{Id}}}\Lambda + \frac{1}{2}\left[\Lambda \wedge \Lambda\right].$$

Routine calculation yields:

$$\begin{aligned} d_{A^{\mathrm{can}}_{\mathrm{Id}}}\Lambda = d\Lambda + \left[A^{\mathrm{can}}_{\mathrm{Id}} \wedge \Lambda\right] = &\\ \left(b\,\theta^{14} - a\,\theta^{17}\right) \otimes e_1 + \left(a\,\theta^{16} - b\,\theta^{13}\right) \otimes e_2 + \left(b\,\theta^{12} - a\,\theta^{15}\right) \otimes e_3 &\\ + \frac{da}{dr}dr \wedge \left(\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1\right) + \frac{db}{dr}dr \wedge \left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right). & \end{aligned} \tag{5.106}$$

The final summand is also easily seen to take the form:

$$\begin{aligned} \frac{1}{2}\left[\Lambda \wedge \Lambda\right] = &\left(-a^2\,\theta^{23} - ab\,\theta^{26} + ab\,\theta^{35} - b^2\,\theta^{56}\right) \otimes e_1 \\ &\left(-a^2\,\theta^{24} - ab\,\theta^{27} + ab\,\theta^{45} - b^2\,\theta^{57}\right) \otimes e_2 \\ &\left(-a^2\,\theta^{34} - ab\,\theta^{37} + ab\,\theta^{46} - b^2\,\theta^{67}\right) \otimes e_3. \end{aligned} \tag{5.107}$$

Overall, we obtain the following expression for the curvature:

$$\begin{aligned} F_A = &\\ &\left((1-a^2)\,\theta^{23} + (1-b^2)\,\theta^{56} - ab\,\theta^{26} + ab\,\theta^{35} + b\,\theta^{14} - a\,\theta^{17} + \frac{da}{dr}dr \wedge \theta^4 + \frac{db}{dr}dr \wedge \theta^7\right) \otimes e_1 \\ + &\left((1-a^2)\,\theta^{24} + (1-b^2)\,\theta^{57} - ab\,\theta^{27} + ab\,\theta^{45} - b\,\theta^{13} + a\,\theta^{16} - \frac{da}{dr}dr \wedge \theta^3 - \frac{db}{dr}dr \wedge \theta^6\right) \otimes e_2 \\ + &\left((1-a^2)\,\theta^{34} + (1-b^2)\,\theta^{67} - ab\,\theta^{37} + ab\,\theta^{46} + b\,\theta^{12} - a\,\theta^{15} + \frac{da}{dr}dr \wedge \theta^2 + \frac{db}{dr}dr \wedge \theta^5\right) \otimes e_3. \end{aligned} \tag{5.108}$$

The Ambrose-Singer holonomy theorem implies that all elements of $\mathcal{A}_{\mathrm{inv}}\left(P_{\mathrm{Id}}\right)$ are irreducible. Since gauge equivalent, irreducible, invariant connections are equivariantly gauge equivalent (Oliveira [54] Corollary 4.5), the elements of $\mathcal{A}_{\mathrm{inv}}\left(P_{\mathrm{Id}}\right)$ all lie in distinct gauge equivalence classes.

5.3.1.3 Invariant Connections on the Extended Bundles

It remains to understand how to describe invariant connections on the extensions of P_1 and P_{Id} over S^4. For P_1 this is easy. The unique extension is given by the trivial bundle. The canonical invariant reference connection is still equal to the product structure. It follows that the ad(P_1)-valued forms (5.103) are still meaningful over the extended bundle and describe the relevant invariant connections with this choice of reference.

The situation is slightly more subtle for P_{Id}. The canonical invariant connection of P_{Id} disagrees with those of P_{π_1} and P_{π_2}. In fact, $A^{\mathrm{can}}_{\mathrm{Id}}$ does not smoothly extend on either bundle extension. To see this, we compute the canonical invariant connections of P_{π_1}, P_{π_2}. These are given by:

$$\begin{aligned} A^{\mathrm{can}}_{\pi_1} &= d\pi_1 \\ &= \left(\theta^8 + \theta^7\right) \otimes e_1 + \left(\theta^9 - \theta^6\right) \otimes e_2 + \left(\theta^{10} + \theta^5\right) \otimes e_3 \\ &= A^{\mathrm{can}}_{\mathrm{Id}} + \left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right), \end{aligned} \tag{5.109}$$

$$\begin{aligned} A^{\mathrm{can}}_{\pi_2} &= d\pi_2 \\ &= \left(\theta^8 - \theta^7\right) \otimes e_1 + \left(\theta^9 + \theta^6\right) \otimes e_2 + \left(\theta^{10} - \theta^5\right) \otimes e_3 \\ &= A^{\mathrm{can}}_{\mathrm{Id}} - \left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right). \end{aligned} \tag{5.110}$$

Now, an invariant connection $A \in \mathcal{A}_{\mathrm{inv}}(P_{\mathrm{Id}})$ over $X^8 \setminus S^4$ takes the form:

$$A = A^{\mathrm{can}}_{\mathrm{Id}} + a(r)\left(\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1\right) + b(r)\left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right). \tag{5.111}$$

We rewrite it using $A^{\mathrm{can}}_{\pi_1}$ and $A^{\mathrm{can}}_{\pi_2}$ as the reference. This yields:

$$A = A^{\mathrm{can}}_{\pi_1} + a(r)\left(\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1\right) + \left(b(r) - 1\right)\left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right), \tag{5.112}$$

$$A = A^{\mathrm{can}}_{\pi_2} + a(r)\left(\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1\right) + \left(b(r) + 1\right)\left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right). \tag{5.113}$$

The forms θ^5, θ^6 and θ^7 blow up as $r \to 1$. We conclude that a necessary condition for A to extend to P_{π_1} is:

$$\lim_{r \to 1} b(r) = 1. \tag{5.114}$$

Similarly, if A extends to P_{π_2} we have:

$$\lim_{r\to 1} b(r) = -1. \tag{5.115}$$

The connection $A_{\mathrm{Id}}^{\mathrm{can}}$ corresponds to $a = b = 0$. Both conditions (5.114) and (5.115) fail. Consequently $A_{\mathrm{Id}}^{\mathrm{can}}$ does not extend to either P_{π_1} or P_{π_2}.

5.3.2 The Invariant ODEs on P_1: Dimensional Reduction and Explicit Solution

A general invariant connection $A \in \mathcal{A}_{\mathrm{inv}}(P_1)$ defined over $X^8 - S^4$ takes the form:

$$A = \theta^1 \otimes a^i(r)\, e_i.$$

The associated curvature tensor is given by:

$$F_A = \left(\frac{da^i}{dr} dr \wedge \theta^1 + a^i \left(\theta^{25} + \theta^{36} + \theta^{47} \right) \right) \otimes e_i.$$

These expressions are manifestly similar to (5.77) and (5.79). An identical computation to the one carried out in the abelian case gives:

$$F_A \wedge \Omega = 0,$$

$$F_A \wedge \star\omega = -\left(Q^3 \frac{da^i}{dr} + 3PQ^2 a^i(r) \right) dr \wedge \theta^{1234567} \otimes e_i.$$

Consequently, the invariant Hermitian Yang-Mils equations take the form:

$$\frac{da^i}{dr} = -3\frac{P}{Q} a^i$$

Using (5.67) and computing as in the abelian case we obtain:

$$\Phi \wedge F_A = -\left[3Q^2 a^i(r) \theta^{234567} + \left(Q^2 \frac{da^i}{dr} + 2PQ\alpha(r) \right) dr \wedge \left(\theta^{13467} + \theta^{12457} + \theta^{12356} \right) \right] \otimes e_i.$$

Similarly, we compute:

$$\star_g F_A = -\left[\frac{Q^3}{P}\frac{da^i}{dr}\theta^{234567} + PQa^i dr \wedge \left(\theta^{13467} + \theta^{12457} + \theta^{12356}\right)\right] \otimes e_i$$

Consequently, the invariant Spin(7) instanton equations take the form:

$$\frac{da^i}{dr} = -3\frac{P}{Q}a^i \tag{5.116}$$

The two equations are identical. We thus obtain the following local equivalence result:

Theorem 5.9. *An* SO(5)*-invariant* SO(3)*-connection* $A \in \mathcal{A}_{\text{inv}}(P_1)$ *over* $T^\star S^4 - S^4$ *equipped with the Stenzel Calabi-Yau structure is a* Spin(7) *instanton if and only if it is* HYM.

The ODE (5.116) has already been studied in the context of the abelian equations. We thus immediately obtain a result analogous to the one we got for structure group U(1):

Theorem 5.10. *Let* $X^8 - S^4$ *be equipped with the Stenzel Calabi-Yau structure. There is a* 3*-parameter family of invariant* Spin(7) *instantons* $A_{K_1,K_2,K_3} \in \mathcal{A}_{\text{inv}}(P_1)$*:*

$$A_{K_1,K_2,K_3} = \sum_{i=1}^{3} \frac{K_i}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}}\theta^1 \otimes e_i \text{ where } K_i \in \mathbb{R}$$

The curvature of A_{K_1,K_2,K_3} *is given by:*

$$F_{A_{K_1,K_2,K_3}} = \sum_{i=1}^{3} K_i \left(-\frac{9}{2}\frac{r(r^2+1)}{(r^2+2)^{\frac{7}{4}}(r+1)^{\frac{5}{2}}(r-1)^{\frac{5}{2}}} dr \wedge \theta^1 + \frac{\theta^{25}+\theta^{36}+\theta^{47}}{(r^2+2)^{\frac{3}{4}}(r+1)^{\frac{3}{2}}(r-1)^{\frac{3}{2}}}\right) \otimes e_i$$

The Ambrose-Singer holonomy theorem implies that all the instantons of theorem 5.10 are reducible. In particular, the holonomy algebra of A_{K_1,K_2,K_3} is given by:

$$\mathfrak{hol}\left(A_{K_1,K_2,K_3}\right) = \left\langle K^i e_i \right\rangle < \mathfrak{so}(3).$$

When $A_{K_1,K_2,K_3} \neq 0$, at least one of the parameters K^i does not vanish and $\mathfrak{hol}\left(A_{K_1,K_2,K_3}\right)$ is one-dimensional. The holonomy group $\text{Hol}\left(A_{K_1,K_2,K_3}\right)$ is the associated one-parameter subgroup. Recalling the elementary fact that the one-parameter subgroups of SO(3) are embedded circles, we conclude that (when $A_{K^1,K^2,K^3} \neq 0$) the holonomy group is a copy of U(1) in SO(3). Let $Q \subset P_1$ denote the trivial U(1)-subbundle with fiber $\text{Hol}\left(A_{K_1,K_2,K_3}\right)$.

The reduction theorem implies that A_{K^1,K^2,K^3} restricts to an irreducible connection on $Q_{|_{X^8 \setminus S^4}}$. The resulting instanton is one of those promised by theorem 5.10. This makes rigorous the apparent similarities with the abelian setting.

Unless $K_1 = K_2 = K_3 = 0$, the pointwise curvature norm of A_{K_1,K_2,K_3} is unbounded as $r \to 1$. We thus obtain:

Theorem 5.11. *There are no global invariant* Spin(7) *instantons (and therefore also no HYM connections) on the trivial* SO(3)- *bundle* P_1 *over* X^8 *apart from the trivial connection* $A{=}0$.

5.3.3 Extendibility of Connections Across the Singular Orbit

In this section we develop necessary and sufficient conditions for invariant connections to extend smoothly over S^4. We begin with some general remarks setting up the framework for the problem. This preparation allows us to state the criterion of Eschenburg and Wang in the context of gauge theory. Once this is done, we analyze the extension problem for the SO(3)-bundles of the preceding sections.

5.3.3.1 Extendibility of Tensorial Forms

Let S be a Lie group and let μ be a Lie group homomorphism:

$$\mu : \mathrm{SO}(4) \to S.$$

Denote by λ the restriction of μ to the bottom right copy of SO(3).

Let P be the cohomogeneity one principal S-bundle over $T^\star S^4$ obtained by pulling $P_\mu \twoheadrightarrow S^4$ back through the cotangent bundle projection map. Its restrictions over each orbit $\mathcal{O}_r$ ($r \geq 1$) are given by:

$$P_{|\mathcal{O}_r} = \begin{cases} \mathrm{SO}(5) \times_{(\mathrm{SO}(4),\mu)} S \text{ if } r = 1 \\ \mathrm{SO}(5) \times_{(\mathrm{SO}(3),\lambda)} S \text{ if } r > 1. \end{cases}$$

Let (V, ρ) be a representation of the group S. We can form the associated vector bundle over X^8:

$$\rho\,(P) \overset{\mathrm{def}}{=} P \times_\rho V.$$

We consider the problem of extending SO(5)-invariant $\rho(P)$-valued k-forms across the singular orbit S^4.

Eschenburg and Wang give necessary and sufficient conditions for extending invariant linear tensors across the singular orbit of a cohomogeneity one space. Since we are interested in bundle-valued forms, their technique does not apply directly. We resolve this issue by passing to the total space P and working with V-valued forms instead. In this section we set up the requisite framework to implement this idea.

The manifold P is a cohomogeneity one space for the group $\mathrm{SO}(5) \times S$. Its principal orbits are isomorphic to P_λ and its singular orbit is the bundle P_μ.

Define the reference points:

$$x_r \overset{\text{def}}{=} \begin{cases} [1,1] \in P_\mu \text{ if } r = 1 \\ [1,1] \in P_\lambda \text{ if } r > 1. \end{cases}$$

With this definition, the point x_r lies in the fiber above $p_r \in \mathcal{O}_r$ for all $r \geq 1$.

Using these reference points, the isotropy subgroups corresponding to the principal and singular orbits are respectively given by:

$$\mathrm{Stab}\,(x_r) = \big\{(h, \lambda(h)) \in \mathrm{SO}(5) \times S \text{ such that } h \in \mathrm{SO}(3)\big\} \cong \mathrm{SO}(3), \tag{5.117}$$

$$\mathrm{Stab}\,(x_1) = \big\{(h, \mu(h)) \in \mathrm{SO}(5) \times S \text{ such that } h \in \mathrm{SO}(4)\big\} \cong \mathrm{SO}(4). \tag{5.118}$$

In formulae (5.117) and (5.118), SO(4) and SO(3) denote the bottom right inclusions of these groups in SO(5). In what follows, when we consider the action of SO(4) on P, it will be through its embedding in $\mathrm{SO}(5) \times S$ as the singular isotropy group (5.118).

Let ω be an invariant, tensorial form of type ρ. Its extendibility can be decided by studying the restriction $\omega_{|W}$ along a particularly simple embedded submanifold $W \subset P$. This will make the problem tractable. Let W be the union of the SO(4)-orbits of all points x_r in P:

$$W \stackrel{\text{def}}{=} \bigcup_{r \geq 1} \operatorname{Stab}(x_1)\ x_r. \tag{5.119}$$

This is a 4-dimensional linear SO(4)-representation. The SO(4)-action is obvious. The linear structure is inherited from $T^\star S^4_{p_1}$ through the projection map:

$$\pi : P \twoheadrightarrow X^8.$$

In particular, the inverse function theorem implies that π restricts to a diffeomorphism:

$$\pi : W \xrightarrow{\sim} T^\star_{p_1} S^4 \subset X^8. \tag{5.120}$$

The latter is a smoothly embedded submanifold of X^8 stable under the action of SO(4). The equivariance of π implies that W and $T^\star_{p_1} S^4$ are isomorphic SO(4)-representations. Since $T^\star_{p_1} S^4$ is a vector space, it can be naturally identified with the tangent space at its origin (e.g. by the exponential map of the underlying additive group). Endowing the latter with the isotropy action, this identification becomes equivariant. These considerations allow us to view W as the vector representation of SO(4):

$$W \cong \left\langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \right\rangle. \tag{5.121}$$

The extendibility problem for invariant tensors is addressed by examining their restrictions along W. We are thus interested in finding a useful way to describe these restrictions. Pull the bundle

$$\Lambda^k T^\star P \otimes \underline{V}$$

back to W using the inclusion map (here we denote by $\underline{V}$ the trivial vector bundle with fiber V). Since W is linear, the pullback is trivial. We will now give a particular trivialization that elucidates the action of SO(4). Using an invariant connection to decompose TP into vertical and horizontal distributions, we obtain an equivariant identification:

$$TP \cong \pi^\star TX^8 \oplus \underline{\mathfrak{s}}. \tag{5.122}$$

Furthermore, there is an obvious SO(4)-equivariant trivialization:

$$TX^8_{|\pi(W)} \cong \pi(W) \times \left(\langle X_1, ..., X_4 \rangle \oplus \left\langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \right\rangle \right). \tag{5.123}$$

Putting these together we have:

$$\left(\Lambda^k T^\star P \otimes \underline{V} \right)_{|W} \cong W \times \left(\Lambda^k \langle X_1, ..., X_4 \rangle^\star \otimes V \oplus \Lambda^k \left\langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \right\rangle^\star \otimes V \oplus \Lambda^k \mathfrak{s}^\star \otimes V \right).$$

Here, the action of SO(4) is as follows: The action on $\mathfrak{s}$ is trivial. The action on V is obtained by composing μ and ρ. Finally, the spaces $\langle X_1, ..., X_4 \rangle^\star$ and $\left\langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \right\rangle^\star$ are vector representations.

We study the restriction of ω along W_0: the vector space W punctured at its origin

$$W_0 \stackrel{\text{def}}{=} W - \{x_1\}.$$

Since tensorial forms vanish on vertical vectors, $\omega_{|W_0}$ is a section of the trivial bundle with fiber equal to:

$$E \stackrel{\text{def}}{=} \Lambda^k \langle X_1, ..., X_4 \rangle^\star \otimes V \oplus \Lambda^k \left\langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \right\rangle^\star \otimes V.$$

Due to the triviality of the bundle, the form $\omega|_{W_0}$ amounts to an SO(4)-equivariant function:

$$f : W_0 \to E.$$

The equivariance of ω implies that no information is lost in passing to f. In turn, f is completely determined by its values on the reference points x_r—forming a ray from the origin of W to infinity. This recovers our usual description of equivariant forms as curves in a group representation:

$$f \circ x_r : (1, \infty) \to E. \tag{5.124}$$

Eschenburg and Wang prove that the extendibility of ω is contingent to a representation-theoretic condition on the formal Taylor series expansion of an appropriate reparameterization of $f \circ x_r$. This series reflects the behaviour of f near x_1 and—by equivariance—of ω near the singular orbit.

The requisite reparameterization is obtained as follows. Using (5.121), the Euclidean metric on $\mathbb{R}^{10}$ induces an inner product on W. We consider the radial function of the associated norm. Concretely, we set:

$$t \overset{\text{def}}{=} R_- = \left(\frac{r^2-1}{2}\right)^{\frac{1}{2}}, \quad r(t) = \left(2t^2+1\right)^{\frac{1}{2}}. \tag{5.125}$$

We thus obtain the reparameterization:

$$\gamma \overset{\text{def}}{=} f \circ x_{r(t)}$$

The result of Eschenburg and Wang (Eschenburg-Wang [21], Lemma 1.1, p.113) asserts that ω extends smoothly over the singular orbit if and only if the following hold:

- The curve γ is smooth from the right at $t = 0$
- The formal Taylor series of γ at $t = 0$ can be written as:

  $$\gamma \sim \sum_{k \geq 0} u_k \left(x_{r(1)}\right) t^k,$$

  where $x_{r(1)} \in W \subset P$ denotes the reference point located at $t = 1$ and:

  $$u_k : W \to E$$

  is a homogeneous equivariant polynomial of degree k.

Note that we have provided explicit descriptions of the SO(4)-actions on W and E. These descriptions facilitate the computations required for applications.

5.3.3.2 Application: Extendibility of Connections

We are interested in studying the extendibility of tensorial forms ω describing connections on P (relative to the canonical invariant connection). Therefore—in the context of our application—we have:

$$S = \mathrm{SO}(3),\ V = \mathfrak{so}(3)\ , \rho = \mathrm{Ad}_{\mathrm{SO}(3)},\ k = 1.$$

Given our setup, ω will usually be available in the form (5.124). Given this data, we need to pass to the associated curve $\gamma(t)$ and express it in a basis of E coming from evaluation of homogeneous equivariant polynomials at $x_{r(1)} \in W$. To achieve this, we need to be able to find appropriate equivariant polynomials. This task can be simplified if we understand the relevant representations in terms of quaternions. To this end, we identify the spaces W and $\langle X_1, ..., X_4 \rangle$ with $\mathbb{H}$ by:

$$\left\langle X^1, X^2, X^3, X^4 \right\rangle \cong \langle 1, i, j, k \rangle \cong \left\langle \partial_{y^2}, \partial_{y^3}, \partial_{y^4}, \partial_{y^5} \right\rangle.$$

Furthermore, we lift the action of SO(4) to $\mathrm{Sp}(1)^2$ using the covering map $\pi_{\mathrm{Spin}(4)}$. Under these identifications, the SO(4)-action is captured by the usual vector representation of $\mathrm{Sp}(1)^2$ on $\mathbb{H}$.

General points $p \in W$ and $q \in \langle X_1, ..., X_4 \rangle$ can be written as:

$$\begin{aligned} p &= p^0 X_1 + p^1 X_2 + p^2 X_3 + p^3 X_4, & q &= q^0 \partial_{y^2} + q^1 \partial_{y^3} + q^2 \partial_{y^4} + q^3 \partial_{y^5} \\ &= p^0 + p^1 i + p^2 j + p^3 k & &= q^0 + q^1 i + q^2 j + q^3 k. \end{aligned}$$

With this choice of coordinates we have:

$$x_{r(1)} = 1 \in \mathrm{Sp}(1) \subset \mathbb{H}.$$

The Lie algebra $\mathfrak{so}(3)$ can be naturally identified with $\mathfrak{sp}(1) = \mathfrak{Im}\,(\mathbb{H})$ using the differential of the covering map $\pi_{\mathrm{Spin}(3)}$ (defined in (5.97)). This identification is Ad-equivariant. Explicitly, it takes the following form:

$$d\pi_{\mathrm{Spin}(3)_{|_1}} : i \mapsto -2e_3, \quad j \mapsto 2e_2, \quad k \mapsto -2e_1. \tag{5.126}$$

These considerations demonstrate that we require homogeneous $\mathrm{Sp}(1)^2$-equivariant polynomials:

$$u : \mathbb{H} \to \mathbb{H}^\star \otimes \mathfrak{Im}\,(\mathbb{H}) \oplus \mathbb{H}^\star \otimes \mathfrak{Im}\,(\mathbb{H})$$

with prescribed value at $x = 1$. Separating the components in the target, such maps take the form:

$$u(x)(p,q) = u_1(x)(p) + u_2(x)(q), \quad x,p,q \in \mathbb{H}.$$

The $\mathrm{Sp}(1)^2$-equivariance condition for $u : W \to E$ translates to the following:

$$u_1\left(ax\bar{b}\right)(p) = \mathrm{Ad}_{\mu\circ\pi_{\mathrm{Spin}(4)}(a,b)} u_1(x)(\bar{a}pb) \text{ for all } (a,b) \in \mathrm{Sp}(1)^2, \tag{5.127}$$

$$u_2\left(ax\bar{b}\right)(q) = \mathrm{Ad}_{\mu\circ\pi_{\mathrm{Spin}(4)}(a,b)} u_2(x)(\bar{a}qb) \text{ for all } (a,b) \in \mathrm{Sp}(1)^2. \tag{5.128}$$

The Case of P_{π_1}

In this case $\mu = \pi_1$. The action of $\mathrm{Sp}(1)^2$ on $\mathfrak{Im}(\mathbb{H})$ is given by projecting the group element to the first factor and conjugating by the result. Conditions (5.127), (5.128) become:

$$u_1\left(ax\bar{b}\right)(p) = a u_1(x)(\bar{a}pb)\,\bar{a} \text{ for all } (a,b) \in \mathrm{Sp}(1)^2, \tag{5.129}$$

$$u_2\left(ax\bar{b}\right)(q) = a u_2(x)(\bar{a}qb)\,\bar{a} \text{ for all } (a,b) \in \mathrm{Sp}(1)^2. \tag{5.130}$$

Using (5.112), we write a general invariant connection over $X^8 \setminus S^4$ as $A^{\mathrm{can}}_{\pi_1} + \omega$, where:

$$\omega = a(r(t))\left(\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1\right) + \left(b(r(t)) - 1\right)\left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right).$$

Using (5.26)-(5.29) and (5.126) we find that the form $\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1$ corresponds to:

$$\begin{aligned}(p,q) \mapsto & -\frac{1}{2}\left(p^1 i + p^2 j + p^3 k\right)\\ &= -\frac{1}{2}\mathfrak{Im}(p)\\ &= \frac{\langle p,1\rangle - p}{2}.\end{aligned} \tag{5.131}$$

Similarly, using (5.26)-(5.29), (5.125) and (5.126), we find that the form $\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1$ corresponds to:

$$
\begin{aligned}
(p, q) \mapsto & \frac{1}{2t}\left(q^1 i + q^2 j + q^3 k\right) \\
= & \frac{1}{2t}\mathfrak{Im}(q) \\
= & \frac{q - \langle q, 1\rangle}{2t}.
\end{aligned}
\tag{5.132}
$$

Any equivariant polynomial u satisfying:

$$
u(1)(p, q) = \frac{\langle p, 1\rangle - p}{2} \tag{5.133}
$$

has the following restriction on $S^3 \subset \mathbb{H}$:

$$
u(x)(p) = \frac{\langle x, p\rangle - p\overline{x}}{2}. \tag{5.134}
$$

Similarly, any equivariant polynomial v satisfying:

$$
v(1)(p, q) = \frac{q - \langle q, 1\rangle}{2} \tag{5.135}
$$

has the following restriction on $S^3 \subset \mathbb{H}$:

$$
v(x)(q) = \frac{q\overline{x} - \langle x, q\rangle}{2}. \tag{5.136}
$$

As soon as u and v are specified on the unit sphere, they are extended to $\mathbb{H}$ by homogeneity. The extensions depend on the degree d, which is yet unspecified. Given d, we define:

$$
u_d(x)(p) \stackrel{\text{def}}{=} \begin{cases} |x|^d u\left(\frac{x}{|x|}\right)(p) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0, \end{cases}
$$

$$
v_d(x)(p) \stackrel{\text{def}}{=} \begin{cases} |x|^d v\left(\frac{x}{|x|}\right)(p) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0. \end{cases}
$$

The admissible values of d are constrained: not all choices yield polynomial extensions. Equations (5.134) and (5.136) demonstrate that u_0 and v_0 are not constant functions. If they were polynomials they would have homogeneous degree $d = 0$ and would thus be constant. We conclude that the choice $d = 0$ is not admissible. However, it is clear that we could choose $d = 1$. This would correspond to defining the extensions by the formulae (5.134) and (5.136) on the whole of $\mathbb{H}$. Now, if two homogeneous polynomials agree on the unit sphere, they are related through multiplication by the homogeneous degree $2k$ polynomial $|x|^{2k}$. Consequently, all other admissible choices of d are obtained by adding even integers to $d = 1$. We conclude that for each odd positive integer:

$$d = 1 + 2k \tag{5.137}$$

we have precisely one homogeneous equivariant polynomial u_d of degree d satisfying (5.133) and precisely one homogeneous equivariant polynomial v_d of degree d satisfying (5.135).

We rewrite the form ω as:

$$\omega = a(r(t))\, u_d\,(1) + \frac{b(r(t)) - 1}{t}\, v_d\,(1)\,. \tag{5.138}$$

Applying the criterion of Eschenburg and Wang we obtain:

Proposition 5.12. *Let $A \in \mathcal{A}_{\text{inv}}\left(P_{\text{Id}}\right)$ be an invariant connection defined over $X^8 \setminus S^4$. Let*

$$\omega = a(r(t)) \left(\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1\right) + \left(b(r(t)) - 1\right) \left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right)$$

be the tensorial form expressing A with respect to the canonical invariant connection of P_{π_1}. Then A extends over the singular orbit on P_{π_1} if and only if the following hold:

- *The function $a(r(t))$ is smooth from the right at $t = 0$, odd and $O\,(t)$.*
- *The function $b(r(t)) - 1$ is smooth from the right at $t = 0$, even and $O\left(t^2\right)$.*

The Case of P_{π_2}

In this case $\mu = \pi_2$. The action of $\mathrm{Sp}(1)^2$ on $\mathfrak{Im}\,(\mathbb{H})$ is given by projecting the group element to the second factor and conjugating by the result. Conditions (5.127), (5.128) become:

$$u_1\left(ax\bar{b}\right)(p) = bu_1\,(x)\,(\bar{a}pb)\,\bar{b} \text{ for all } (a, b) \in \mathrm{Sp}(1)^2, \tag{5.139}$$

$$u_2\left(ax\bar{b}\right)(q) = bu_2\,(x)\,(\bar{a}qb)\,\bar{b} \text{ for all } (a, b) \in \mathrm{Sp}(1)^2. \tag{5.140}$$

Using (5.113), we write a general invariant connection over $X^8 \setminus S^4$ as $A^{\mathrm{can}}_{\pi_2} + \omega$, where:

$$\omega = a(r(t)) \left(\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1\right) + \left(b(r(t)) + 1\right) \left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right).$$

Recalling (5.131) and (5.132) we seek homogeneous equivariant polynomials u and v satisfying (5.133) and (5.135) respectively. Equivariance specifies their restrictions on $S^3 \subset \mathbb{H}$:

$$u\,(x)\,(p) = \frac{\langle x, p\rangle - \overline{x}p}{2}, \quad v\,(x)\,(q) = \frac{\overline{x}q - \langle x, q\rangle}{2}. \tag{5.141}$$

Arguing as in the case of P_{π_1} we find that there is precisely one homogeneous equivariant polynomial u_d and precisely one homogeneous equivariant polynomial v_d satisfying the requisite conditions in each odd degree $d = 1 + 2k$.

We rewrite the form ω as:

$$\omega = a(r(t))\, u_d\,(1) + \frac{b(r(t)) + 1}{t}\, v_d\,(1)\,. \tag{5.142}$$

Applying the criterion of Eschenburg and Wang we obtain:

Proposition 5.13. *Let $A \in \mathcal{A}_{\mathrm{inv}}\,(P_{\mathrm{Id}})$ be an invariant connection defined over $X^8 \setminus S^4$. Let*

$$\omega = a(r(t)) \left(\theta^2 \otimes e_3 - \theta^3 \otimes e_2 + \theta^4 \otimes e_1\right) + \left(b(r(t)) + 1\right) \left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right)$$

be the tensorial form expressing A with respect to the canonical invariant connection of P_{π_2}. Then A extends over the singular orbit on P_{π_2} if and only if the following hold:

- *The function $a(r(t))$ is smooth from the right at $t = 0$, odd and $O\,(t)$.*
- *The function $b(r(t)) + 1$ is smooth from the right at $t = 0$, even and $O\left(t^2\right)$.*

5.3.4 The Invariant HYM ODEs on P_{Id}

5.3.4.1 Dimensional Reduction and Explicit Solution We analyze the invariant HYM equation on the bundle P_{Id}. We obtain precisely two invariant HYM connections over $X^8 - S^4$. One of them extends on P_{π_1}, the other on P_{π_2}.

A general invariant connection $A \in \mathcal{A}_{\text{inv}}(P_{\text{Id}})$ takes the form:

$$A = A^{\text{can}}_{\text{Id}} + a(r)\left(\theta^2 e_3 - \theta^3 e_2 + \theta^4 e_1\right) + b(r)\left(\theta^5 e_3 - \theta^6 e_2 + \theta^7 e_1\right).$$

The curvature tensor associated to A is given by:

$$F_A = F^j_A \otimes e_j,$$

$$F^1_A \overset{\text{def}}{=} (1-a^2)\,\theta^{23} + (1-b^2)\,\theta^{56} - ab\,\theta^{26} + ab\,\theta^{35} + b\,\theta^{14} - a\,\theta^{17} + \frac{da}{dr}dr \wedge \theta^4 + \frac{db}{dr}dr \wedge \theta^7,$$

$$F^2_A \overset{\text{def}}{=} (1-a^2)\,\theta^{24} + (1-b^2)\,\theta^{57} - ab\,\theta^{27} + ab\,\theta^{45} - b\,\theta^{13} + a\,\theta^{16} - \frac{da}{dr}dr \wedge \theta^3 - \frac{db}{dr}dr \wedge \theta^6,$$

$$F^3_A \overset{\text{def}}{=} (1-a^2)\,\theta^{34} + (1-b^2)\,\theta^{67} - ab\,\theta^{37} + ab\,\theta^{46} + b\,\theta^{12} - a\,\theta^{15} + \frac{da}{dr}dr \wedge \theta^2 + \frac{db}{dr}dr \wedge \theta^5.$$

Using (5.86) we obtain:

$$\star_g\,\omega = PQ^2 dr \wedge \left(\theta^{12536} + \theta^{12547} + \theta^{13647}\right) + Q^3\theta^{253647}.$$

We then have:

$$F_A \wedge \star_g \omega = 0. \tag{5.143}$$

In particular, part of the HYM system is enforced by the symmetry ansatz. Consequently, the HYM equations reduce to:

$$F_A \wedge \mathfrak{Re}(\Omega) = 0. \tag{5.144}$$

We write:

$$F_A \wedge \mathfrak{Re}(\Omega) = F^j_A \wedge \mathfrak{Re}\,(\Omega) \otimes e_j$$

and derive the relevant equations component-wise.

Using (5.49) and (5.50) we obtain the results:

$$
\begin{aligned}
F_A^1 \wedge \mathfrak{Re}\,(\Omega) = {} & \Big(R_+^3(1-b^2) - R_+R_-^2(1-a^2)\Big)\,\theta^{123456} - 2R_+R_-^2 ab\,\theta^{123567} \\
& + \frac{r}{2}\left(R_+(1-b^2) - \frac{R_-^2}{R_+}(1-a^2)\right) dr\wedge\theta^{23567} + rR_+ab\,dr\wedge\theta^{23456} \\
& + \left(\frac{rR_+}{2}b + R_+^3\frac{db}{dr}\right) dr\wedge\theta^{12347} - \left(\frac{rR_-^2}{2R_+}b + R_+R_-^2\frac{db}{dr}\right) dr\wedge\theta^{14567} \\
& - \left(\frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{12467} + \left(\frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{13457},
\end{aligned}
\tag{5.145}
$$

$$
\begin{aligned}
F_A^2 \wedge \mathfrak{Re}\,(\Omega) = {} & \Big(R_+^3(1-b^2) - R_+R_-^2(1-a^2)\Big)\,\theta^{123457} - 2R_+R_-^2 ab\,\theta^{124567} \\
& + \frac{r}{2}\left(R_+(1-b^2) - \frac{R_-^2}{R_+}(1-a^2)\right) dr\wedge\theta^{24567} + rR_+ab\,dr\wedge\theta^{23457} \\
& - \left(\frac{rR_+}{2}b + R_+^3\frac{db}{dr}\right) dr\wedge\theta^{12346} + \left(\frac{rR_-^2}{2R_+}b + R_+R_-^2\frac{db}{dr}\right) dr\wedge\theta^{13567} \\
& + \left(\frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{12367} - \left(\frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{13456},
\end{aligned}
\tag{5.146}
$$

$$
\begin{aligned}
F_A^3 \wedge \mathfrak{Re}\,(\Omega) = {} & \Big(R_+^3(1-b^2) - R_+R_-^2(1-a^2)\Big)\,\theta^{123467} - 2R_+R_-^2 ab\,\theta^{134567} \\
& + \frac{r}{2}\left(R_+(1-b^2) - \frac{R_-^2}{R_+}(1-a^2)\right) dr\wedge\theta^{34567} + rR_+ab\,dr\wedge\theta^{23467} \\
& + \left(\frac{rR_+}{2}b + R_+^3\frac{db}{dr}\right) dr\wedge\theta^{12345} - \left(\frac{rR_-^2}{2R_+}b + R_+R_-^2\frac{db}{dr}\right) dr\wedge\theta^{12567} \\
& - \left(\frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{12357} + \left(\frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{12456},
\end{aligned}
\tag{5.147}
$$

Observe that there are similarities among the various components. In particular the vanishing of any one of them is equivalent to the full HYM system (5.144). We obtain the invariant HYM equations:

$$\frac{da}{dr} = -\frac{r}{2R_-^2}\,a, \tag{5.148}$$

$$\frac{db}{dr} = -\frac{r}{2R_+^2}\,b, \tag{5.149}$$

$$R_+^2(1-b^2) = R_-(1-a^2), \tag{5.150}$$

$$ab = 0. \tag{5.151}$$

Consequently, invariant HYM connections over P_{Id} obey the differential equations (5.148), (5.149) and satisfy the algebraic constraints (5.150), (5.151). Observe that the coefficients of the Stenzel metric do not appear anywhere in these equations.

The equations (5.148)-(5.151) can be solved explicitly. We obtain precisely two solutions:

$$a_{\mathrm{HYM}_{\pi_1}} = 0, \;\; b_{\mathrm{HYM}_{\pi_1}} = \frac{1}{R_+},$$
$$a_{\mathrm{HYM}_{\pi_2}} = 0, \;\; b_{\mathrm{HYM}_{\pi_2}} = -\frac{1}{R_+},$$

yielding two HYM connections:

$$A_{\mathrm{HYM}_{\pi_1}} \stackrel{\mathrm{def}}{=} A^{\mathrm{can}}_{\mathrm{Id}} + \frac{1}{R_+}\left(\theta^5 e_3 - \theta^6 e_2 + \theta^7 e_1\right), \tag{5.152}$$

$$A_{\mathrm{HYM}_{\pi_2}} \stackrel{\mathrm{def}}{=} A^{\mathrm{can}}_{\mathrm{Id}} - \frac{1}{R_+}\left(\theta^5 e_3 - \theta^6 e_2 + \theta^7 e_1\right). \tag{5.153}$$

The notation has been chosen in hindsight to reflect the bundle on which these connections extend. In particular, we claim that $A_{\mathrm{HYM}_{\pi_1}}$ extends to P_{π_1}, while $A_{\mathrm{HYM}_{\pi_2}}$ extends to P_{π_2}.

We treat $A_{\mathrm{HYM}_{\pi_1}}$ in detail. We shall apply proposition 5.12. In this case, the a-component vanishes and we have:

$$\begin{aligned} b_{\mathrm{HYM}_{\pi_1}}(r(t)) - 1 &= \frac{1}{R_+(r(t))} - 1 \\ &= \frac{\sqrt{2}}{\sqrt{r^2(t)+1}} - 1 \\ &= \frac{1}{\sqrt{t^2+1}} - 1, \\ &= \frac{1-\sqrt{t^2+1}}{\sqrt{t^2+1}}. \end{aligned}$$

The conditions on $a_{\mathrm{HYM}_{\pi_1}}(r(t))$ are trivially satisfied. The function $b_{\mathrm{HYM}_{\pi_1}}(r(t)) - 1$ is obviously smooth from the right at $t = 0$. It is even, since t only appears in power 2. One easily computes that both $b_{\mathrm{HYM}_{\pi_1}}(r(t)) - 1$ and its first derivative vanish at $t = 0$. Consequently, $b_{\mathrm{HYM}_{\pi_1}}(r(t)) - 1 = O(t^2)$. The criterion of Eschenburg and Wang (in the form of proposition 5.12) implies that $A_{\mathrm{HYM}_{\pi_1}}$ smoothly extends over the singular orbit to give an element:

$$A_{\mathrm{HYM}_{\pi_1}} \in \mathcal{A}_{\mathrm{inv}}\left(P_{\pi_1}\right).$$

An analogous calculation using proposition 5.13 shows that $A_{\mathrm{HYM}_{\pi_2}}$ smoothly extends over the singular orbit to give an element:

$$A_{\mathrm{HYM}_{\pi_2}} \in \mathcal{A}_{\mathrm{inv}}\left(P_{\pi_2}\right).$$

Using (5.108) we compute the curvature tensors of the solutions:

$$\begin{aligned} F_{A_{\mathrm{HYM}_{\pi_1}}} &= \left(\theta^{23} + \frac{R_-^2}{R_+^2}\,\theta^{56} + \frac{1}{R_+}\,\theta^{14} - \frac{r}{2R_+^3} dr \wedge \theta^7\right) \otimes e_1 \\ &+ \left(\theta^{24} + \frac{R_-^2}{R_+^2}\,\theta^{57} - \frac{1}{R_+}\,\theta^{13} + \frac{r}{2R_+^3} dr \wedge \theta^6\right) \otimes e_2 \\ &+ \left(\theta^{34} + \frac{R_-^2}{R_+^2}\,\theta^{67} + \frac{1}{R_+}\,\theta^{12} - \frac{r}{2R_+^3} dr \wedge \theta^5\right) \otimes e_3, \end{aligned} \tag{5.154}$$

$$\begin{aligned} F_{A_{\mathrm{HYM}_{\pi_2}}} &= \left(\theta^{23} + \frac{R_-^2}{R_+^2}\,\theta^{56} - \frac{1}{R_+}\,\theta^{14} + \frac{r}{2R_+^3} dr \wedge \theta^7\right) \otimes e_1 \\ &+ \left(\theta^{24} + \frac{R_-^2}{R_+^2}\,\theta^{57} + \frac{1}{R_+}\,\theta^{13} - \frac{r}{2R_+^3} dr \wedge \theta^6\right) \otimes e_2 \\ &+ \left(\theta^{34} + \frac{R_-^2}{R_+^2}\,\theta^{67} - \frac{1}{R_+}\,\theta^{12} + \frac{r}{2R_+^3} dr \wedge \theta^5\right) \otimes e_3. \end{aligned} \tag{5.155}$$

We study the curvature norm of the connections $A_{\mathrm{HYM}_{\pi_1}}$ and $A_{\mathrm{HYM}_{\pi_2}}$. The curvature tensors (5.154), (5.154) only differ by certain signs. Consequently, it suffices to treat $A_{\mathrm{HYM}_{\pi_1}}$.

We make the following crucial technical remark. In order to compute the norm of the curvature, we must endow the adjoint bundle of P_{π_1} with a fiber metric. To this end, it suffices to choose an Ad-invariant inner product on $\mathfrak{so}(3)$. In general, the choice of such an inner product is free. However, we shall choose $\langle\cdot,\cdot\rangle$ so that:

$$e_i \perp e_j \text{ if } i \neq j, \quad |e_i|^2 = 2. \tag{5.156}$$

This is the unique inner product on $\mathfrak{so}(3)$ such that:

$$|\xi|^2 = -\mathrm{Tr}\left(\xi^2\right). \tag{5.157}$$

This identity is required to relate the Yang-Mills energy of instantons to characteristic classes of the underlying bundle. It shall be crucial in the next section.

With this choice of inner product, we use (5.61), (5.62), (5.63), (5.64) and (5.154) to compute:

$$\begin{aligned}\left|F_{A_{\mathrm{HYM}_{\pi_1}}}\right|^2 &= 2\sum_{i=1}^{3}\left|F^i_{A_{\mathrm{HYM}_{\pi_1}}}\right|^2 \\ &= 8\sqrt{3}\frac{3r^4+10r^2+11}{\left(r^2+1\right)^3\left(r^2+2\right)^{\frac{1}{2}}}.\end{aligned}$$

In particular, we see that as $r \to \infty$:

$$|F_{A_{\mathrm{HYM}_{\pi_1}}}| = O(r^{-\frac{3}{2}}) = O(s^{-2}).$$

Here we have used the radial coordinate $s(r)$ introduced in (5.69) to exhibit the asymptotically conical growth of the Stenzel metric. This decay rate is not sufficient for the Yang-Mills energy to be finite: on an asymptotically conical n-manifold, a function is integrable when it decays faster than $O(s^{-n})$. Indeed:

$$\begin{aligned}\mathcal{YM}\left(A_{\mathrm{HYM}_{\pi_1}}\right) &= \int_{X^8}\left|F_{A_{\mathrm{HYM}_{\pi_1}}}\right|^2 dV_g \\ &= \int_{X^8}\left|F_{A_{\mathrm{HYM}_{\pi_1}}}\right|^2 PQ^3\theta^{1234567}\wedge dr \\ &= \int_0^\infty O\left(r^2\right) dr = +\infty.\end{aligned} \tag{5.158}$$

In fact, the decay rate of the curvature norm could have been inferred directly from (5.152) and (5.153). When a connection defined over an asymptotically conical manifold decays to a dilation invariant limit, it is itself termed *asymptotically conical* (Driscoll [18], p. 38). Such connections always have curvature tensors decaying like $O(s^{-2})$. Both HYM solutions are asymptotically conical seeing as they decay to the dilation invariant limit $A^{\mathrm{can}}_{\mathrm{Id}}$.

5.3.4.2 Pullbacks to the Singular Orbit S^4 Since both HYM solutions $A_{\mathrm{HYM}_{\pi_1}}$, $A_{\mathrm{HYM}_{\pi_2}}$ smoothly extend to the singular orbit, we can study their restrictions along S^4. The solution $A_{\mathrm{HYM}_{\pi_1}}$ restricts to an invariant self dual (SD) instanton on P_{π_1}, while the solution $A_{\mathrm{HYM}_{\pi_2}}$ restricts to an invariant anti-self dual (ASD) instanton on P_{π_2}. The bundles P_{π_1} and P_{π_1} have opposite charge of magnitude 1 and the two 4-dimensional instantons are essentially equivalent: they are the SD and ASD versions of the unique rotationally invariant charge one instanton on S^4.

The connection $A_{\mathrm{HYM}_{\pi_1}}$ pulls back to the canonical invariant connection of P_{π_1} over S^4:

$$\begin{aligned} A_{\mathrm{SD}} &\stackrel{\mathrm{def}}{=} A_{\mathrm{HYM}_{\pi_1}|_{S^4}} \\ &= \left(\theta^8 + \theta^7\right) \otimes e_1 + \left(\theta^9 - \theta^6\right) \otimes e_2 + \left(\theta^{10} + \theta^5\right) \otimes e_3. \end{aligned} \tag{5.159}$$

Its curvature is given by:

$$F_{A_{\mathrm{SD}}} = \left(\theta^{23} + \theta^{14}\right) \otimes e_1 + \left(\theta^{24} - \theta^{13}\right) \otimes e_2 + \left(\theta^{34} + \theta^{12}\right) \otimes e_3. \tag{5.160}$$

An explicit calculation demonstrates that A_{SD} is a self-dual instanton (Donaldson–Kronheimer [15]) on S^4. This justifies our choice of notation.

The connection $A_{\mathrm{HYM}_{\pi_2}}$ pulls back to the canonical invariant connection of P_{π_2}:

$$\begin{aligned} A_{\mathrm{ASD}} &\stackrel{\mathrm{def}}{=} A_{\mathrm{HYM}_{\pi_2}|_{S^4}} \\ &= \left(\theta^8 - \theta^7\right) \otimes e_1 + \left(\theta^9 + \theta^6\right) \otimes e_2 + \left(\theta^{10} - \theta^5\right) \otimes e_3. \end{aligned} \tag{5.161}$$

Its curvature is given by:

$$F_{A_{\mathrm{ASD}}} = \left(\theta^{23} - \theta^{14}\right) \otimes e_1 + \left(\theta^{24} + \theta^{13}\right) \otimes e_2 + \left(\theta^{34} - \theta^{12}\right) \otimes e_3. \tag{5.162}$$

An explicit calculation demonstrates that A_{ASD} is an anti-self-dual instanton (Donaldson–Kronheimer [15]) on S^4. This justifies our choice of notation.

Using the Stenzel metric and the fiber metric (5.156) we see that the curvature norms of the two connections have the same constant value on all points of S^4:

$$|F_{A_{\mathrm{SD}}}|^2 = |F_{A_{\mathrm{ASD}}}|^2 = 12.$$

Since the restriction of the Stenzel metric on the singular orbit is round of unit radius, we have:

$$\begin{aligned} \mathcal{YM}\left(A_{\mathrm{SD}}\right) &= \mathcal{YM}\left(A_{\mathrm{ASD}}\right) \\ &= \int_{S^4} |F_{A_{\mathrm{SD}}}|^2 dV_g \\ &= 12\mathrm{Vol}\left(S^4\right) \\ &= 32\pi^2. \end{aligned} \tag{5.163}$$

Owing to (5.157), any SO(3)-connection satisfies:

$$\mathrm{Tr}\left(F_A^2\right) = -\left(|F_A^+|^2 - |F_A^-|^2\right) dV_g. \tag{5.164}$$

By Chern-Weil theory, the first Pontryagin class of the underlying SO(3)-bundle is the cohomology class of the 4-form:

$$p_1 = -\frac{1}{8\pi^2}\mathrm{Tr}\left(F_A^2\right). \tag{5.165}$$

Using (5.164) and (5.165), we conclude that the self-dual SO(3)-instantons on S^4 have Yang-Mills energy equal to $8\pi^2$ times the integral of the first Pontryagin class of the underlying bundle. Similarly, the anti-self-dual instantons have Yang-Mills energy equal to $-8\pi^2$ times the integral of the first Pontryagin class. Using (5.163) we obtain:

$$p_1\left(P_{\pi_1}\right) = 4, \tag{5.166}$$

$$p_1\left(P_{\pi_2}\right) = -4. \tag{5.167}$$

We already knew that P_{π_1} and P_{π_2} are not equivariantly trivial nor equivariantly isomorphic to each other. The above calculation demonstrates that they are genuinely non-trivial and non-isomorphic (even if we drop the requirement that the identification be equivariant).

Over a manifold with vanishing second cohomology, each SO(3)-bundle is associated to a unique SU(2)-bundle through the natural two sheeted covering projection. The second Chern class of the lift is related to the first Pontryagin class of the original bundle by the equation (Donaldson–Kronheimer [15], p.41-42):

$$p_1 = -4c_2. \tag{5.168}$$

Consequently, the SU(2)-lift of A_{ASD} has topological charge equal to 1. The compactified moduli space of charge-one anti-self-dual instantons with structure group SU(2) over S^4 can be identified with the closed 5-ball B^5 (Donaldson–Kronheimer [15], p.126). The lift of A_{ASD} coincides with the center of B^5. It is the only SO(5)-invariant instanton in this moduli space. Analogous remarks hold for A_{SD}. This can be seen by reversing orientation to make it anti-self-dual.

5.3.5 The Invariant Spin(7) Instanton ODEs on P_{Id}

5.3.5.1 Dimensional Reduction A general invariant connection $A \in \mathcal{A}_{\mathrm{inv}}(P_{\mathrm{Id}})$ takes the form:

$$A = A_{\mathrm{Id}}^{\mathrm{can}} + a(r)\left(\theta^2 e_3 - \theta^3 e_2 + \theta^4 e_1\right) + b(r)\left(\theta^5 e_3 - \theta^6 e_2 + \theta^7 e_1\right).$$

The curvature tensor associated to A is given by:

$$F_A = F_A^j \otimes e_j, \text{ where}$$

$$F_A^1 \stackrel{\mathrm{def}}{=} (1-a^2)\,\theta^{23} + (1-b^2)\,\theta^{56} - ab\,\theta^{26} + ab\,\theta^{35} + b\,\theta^{14} - a\,\theta^{17} + \frac{da}{dr} dr\wedge\theta^4 + \frac{db}{dr} dr\wedge\theta^7, \tag{5.169}$$

$$F_A^2 \stackrel{\mathrm{def}}{=} (1-a^2)\,\theta^{24} + (1-b^2)\,\theta^{57} - ab\,\theta^{27} + ab\,\theta^{45} - b\,\theta^{13} + a\,\theta^{16} - \frac{da}{dr} dr\wedge\theta^3 - \frac{db}{dr} dr\wedge\theta^6, \tag{5.170}$$

$$F_A^3 \stackrel{\mathrm{def}}{=} (1-a^2)\,\theta^{34} + (1-b^2)\,\theta^{67} - ab\,\theta^{37} + ab\,\theta^{46} + b\,\theta^{12} - a\,\theta^{15} + \frac{da}{dr} dr\wedge\theta^2 + \frac{db}{dr} dr\wedge\theta^5. \tag{5.171}$$

We write:

$$\Phi \wedge F_A = \Phi \wedge F_A^i \otimes e_i.$$

Using (5.67) we compute:

$$
\begin{aligned}
\Phi \wedge F_A^1 = &\left(PQ(1-a^2) + \frac{rR_+}{2}b + R_+^3\frac{db}{dr}\right) dr\wedge\theta^{12347} \\
&+ \left(PQ(1-b^2) - \frac{rR_-^2}{2R_+}b - R_+R_-^2\frac{db}{dr}\right) dr\wedge\theta^{14567} \\
&+ \left(PQ\,ab - \frac{rR_+}{2}a - R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{12467} + \left(-PQ\,ab + \frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{13457} \\
&+ \left(\frac{rR_+}{2}(1-a^2) - \frac{rR_-^2}{2R_+}(1-a^2) - Q^2\frac{db}{dr}\right) dr\wedge\theta^{23567} + \left(rR_+ab - Q^2\frac{da}{dr}\right) dr\wedge\theta^{23456} \\
&+ \left(R_+^3(1-b^2) - R_+R_-^2(1-a^2) - Q^2 b\right)\theta^{123456} + \left(-2R_+R_-^2ab + Q^2a\right)\theta^{123567},
\end{aligned} \tag{5.172}
$$

$$
\begin{aligned}
\Phi \wedge F_A^2 = &-\left(PQ(1-a^2) + \frac{rR_+}{2}b + R_+^3\frac{db}{dr}\right) dr\wedge\theta^{12346} \\
&+ \left(-PQ(1-b^2) + \frac{rR_-^2}{2R_+}b + R_+R_-^2\frac{db}{dr}\right) dr\wedge\theta^{13567} \\
&+ \left(-PQ\,ab + \frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{12367} + \left(PQ\,ab - \frac{rR_+}{2}a - R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{13456} \\
&+ \left(\frac{rR_+}{2}(1-a^2) - \frac{rR_-^2}{2R_+}(1-a^2) - Q^2\frac{db}{dr}\right) dr\wedge\theta^{24567} + \left(rR_+ab - Q^2\frac{da}{dr}\right) dr\wedge\theta^{23457} \\
&+ \left(R_+^3(1-b^2) - R_+R_-^2(1-a^2) - Q^2 b\right)\theta^{123457} + \left(-2R_+R_-^2ab + Q^2a\right)\theta^{124567},
\end{aligned} \tag{5.173}
$$

$$
\begin{aligned}
\Phi \wedge F_A^3 = &\left(PQ(1-a^2) + \frac{rR_+}{2}b + R_+^3\frac{db}{dr}\right) dr\wedge\theta^{12345} \\
&+ \left(PQ(1-b^2) - \frac{rR_-^2}{2R_+}b - R_+R_-^2\frac{db}{dr}\right) dr\wedge\theta^{12567} \\
&+ \left(PQ\,ab - \frac{rR_+}{2}a - R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{12357} + \left(-PQ\,ab + \frac{rR_+}{2}a + R_+R_-^2\frac{da}{dr}\right) dr\wedge\theta^{12456} \\
&+ \left(\frac{rR_+}{2}(1-a^2) - \frac{rR_-^2}{2R_+}(1-a^2) - Q^2\frac{db}{dr}\right) dr\wedge\theta^{34567} + \left(rR_+ab - Q^2\frac{da}{dr}\right) dr\wedge\theta^{23467} \\
&+ \left(R_+^3(1-b^2) - R_+R_-^2(1-a^2) - Q^2 b\right)\theta^{123467} + \left(-2R_+R_-^2ab + Q^2a\right)\theta^{134567}.
\end{aligned} \tag{5.174}
$$

We now wish to compute the Hodge dual of the curvature. We will require the Hodge duals of all 2-forms θ^{ij}. These can be computed using (5.44), (5.45), (5.46) and the formula (5.81). We write:

$$\star_g F_A = \star_g F_A^i \otimes e_i$$

and compute:

$$\begin{aligned}\star_g F_A^1 =& -(1-a^2)\frac{PQR_-^2}{R_+^2}dr\wedge\theta^{14567} - (1-b^2)\frac{PQR_+^2}{R_-^2}dr\wedge\theta^{12347}\\ &+ abPQdr\wedge\theta^{12467} - abPQdr\wedge\theta^{13457}\\ &-\frac{rQ^2}{2R_+^2}b\,dr\wedge\theta^{23567} - a\frac{rQ^2}{2R_-^2}dr\wedge\theta^{23456}\\ &+\frac{da}{dr}\frac{2R_-^2Q^2}{r}\theta^{123567} - \frac{db}{dr}\frac{2R_+^2Q^2}{r}\theta^{123456},\end{aligned} \tag{5.175}$$

$$\begin{aligned}\star_g F_A^2 =&(1-a^2)\frac{PQR_-^2}{R_+^2}dr\wedge\theta^{13567} + (1-b^2)\frac{PQR_+^2}{R_-^2}dr\wedge\theta^{12346}\\ &- abPQdr\wedge\theta^{12367} + abPQdr\wedge\theta^{13456}\\ &-\frac{rQ^2}{2R_+^2}b\,dr\wedge\theta^{24567} - a\frac{rQ^2}{2R_-^2}dr\wedge\theta^{23457}\\ &+\frac{da}{dr}\frac{2R_-^2Q^2}{r}\theta^{124567} - \frac{db}{dr}\frac{2R_+^2Q^2}{r}\theta^{123457},\end{aligned} \tag{5.176}$$

$$\begin{aligned}\star_g F_A^3 =& -(1-a^2)\frac{PQR_-^2}{R_+^2}dr\wedge\theta^{12567} - (1-b^2)\frac{PQR_+^2}{R_-^2}dr\wedge\theta^{12345}\\ &+ abPQdr\wedge\theta^{12357} - abPQdr\wedge\theta^{12456}\\ &-\frac{rQ^2}{2R_+^2}b\,dr\wedge\theta^{34567} - a\frac{rQ^2}{2R_-^2}dr\wedge\theta^{23467}\\ &+\frac{da}{dr}\frac{2R_-^2Q^2}{r}\theta^{134567} - \frac{db}{dr}\frac{2R_+^2Q^2}{r}\theta^{123467}.\end{aligned} \tag{5.177}$$

The Spin(7) Instanton equations are given by:

$$\star_g F_A = -\Phi\wedge F_A.$$

Separating the components of the Lie algebra we obtain:

$$\star_g F_A^i = -\Phi\wedge F_A^i. \tag{5.178}$$

The set of equations given by imposing (5.178) is the same for each $i = 1, 2, 3$. It is as follows:

$$\frac{da}{dr} = \frac{2PQ}{R_+ R_-^2}\, ab - \frac{r}{2R_-^2} a \tag{5.179}$$

$$\frac{da}{dr} = \frac{rR_+}{Q^2} ab - \frac{r}{2R_-^2} a \tag{5.180}$$

$$\frac{db}{dr} = \frac{PQ}{R_-^2 R_+}(1-b^2) - \frac{PQ}{R_+^3}(1-a^2) - \frac{r}{2R_+^2} b, \tag{5.181}$$

$$\frac{db}{dr} = \frac{rR_+}{2Q^2}(1-b^2) - \frac{R_-^2 r}{2R_+ Q^2}(1-a^2) - \frac{r}{2R_+^2} b. \tag{5.182}$$

This system is overdetermined unless the metric coefficients satisfy the condition:

$$PQ^3 = \frac{r}{2} R_+^2 R_-^2. \tag{5.183}$$

We recognize this as the SO(5)-invariant Monge–Ampère equation (5.53) distinguishing the Stenzel metric among the Kähler metrics induced from SO(5)-invariant potentials. Using this equation, we obtain the system:

$$\begin{aligned}\frac{da}{dr} &= \sqrt{6}\frac{r(r^2+1)^{\frac{1}{2}}}{(r-1)(r+1)(r^2+2)^{\frac{1}{2}}} ab - \frac{r}{(r-1)(r+1)} a, \\ \frac{db}{dr} &= \sqrt{6}\frac{r}{(r-1)(r+1)(r^2+1)^{\frac{1}{2}}(r^2+2)^{\frac{1}{2}}} - \frac{\sqrt{6}}{2}\frac{r(r^2+1)^{\frac{1}{2}}}{(r-1)(r+1)(r^2+2)^{\frac{1}{2}}} b^2 \\ &\quad + \frac{\sqrt{6}}{2}\frac{r}{(r^2+1)^{\frac{1}{2}}(r^2+2)^{\frac{1}{2}}} a^2 - \frac{r}{r^2+1} b.\end{aligned} \tag{5.184}$$

It is useful to work in coordinates compatible with the Eschenburg-Wang analysis. We therefore switch to the variable $t = R_-$ introduced earlier. Recall that we have the relation:

$$r = \sqrt{2t^2+1}. \tag{5.185}$$

Note that this transformation is not smooth. This is not a pathology and is—in fact—precisely the reason we are interested in it. Smoothness (at r=1) of solutions written in terms of r is not related to smoothness of the associated connections. This anomaly disappears when we replace r by t.

Noting that

$$\frac{d}{dr} = \frac{\sqrt{2t^2+1}}{2t}\frac{d}{dt}$$

and performing an elementary calculation, we find that the system takes the form:

$$\begin{aligned} \frac{da}{dt} &= \frac{\mathcal{P}a}{t}\left(b - \frac{1}{\mathcal{P}}\right), \\ \frac{db}{dt} &= \frac{\mathcal{P}}{2t}\left(1-b^2\right) - \frac{\mathcal{P}\mathcal{Q}}{2}\left(1-a^2\right) - \mathcal{Q}b. \end{aligned} \tag{5.186}$$

where we have introduced the functions $\mathcal{P}, \mathcal{Q} \in C^\infty[0,\infty)$ defined by:

$$\mathcal{P}(t) \overset{\text{def}}{=} \frac{\sqrt{6}\sqrt{2t^2+2}}{\sqrt{2t^2+3}}, \tag{5.187}$$

$$\mathcal{Q}(t) \overset{\text{def}}{=} \frac{t}{t^2+1}. \tag{5.188}$$

5.3.5.2 Elementary Observations on the Dynamics We begin our analysis of the system (5.186). In this section we make a few elementary observations about the dynamics. First, we have the following:

Proposition 5.14. *The dynamics (5.186) preserve the vanishing of a and correspondingly if $a(t) \neq 0$ for some $t > 0$, then $a(t) \neq 0$ for all $t > 0$.*

Proof. The first statement is trivial. The second follows by the uniqueness part of the standard Picard theorem. □

Next we observe a symmetry in the solution space:

Proposition 5.15. *Suppose that the pair (a, b) solves the system (5.186). Then so does $(-a, b)$.*

Proof. This follows from a trivial calculation. □

We conclude the following: either $a = 0$ for all time, or a has a fixed sign throughout its lifespan. Furthermore, it suffices to study the case $a > 0$ as—owing to the above observation—all solutions (a, b) with $a < 0$ can be obtained by considering a solution where $a > 0$ and reversing its sign.

The next proposition establishes that if one solution lies above another at some instant $t^\star$, the inequality persists for all time. Here, 'lying above' is interpreted componentwise.

Proposition 5.16. *Suppose that* (a, b), $(\widetilde{a}, \widetilde{b})$ *are two solutions to the system (5.186) such that* $a(t) > 0$ *and* $\widetilde{a}(t) > 0$ *for all* $t > 0$. *Suppose further that for some time* $t^\star > 0$ *we have* $a(t^\star) > \widetilde{a}(t^\star)$ *and* $b(t^\star) > \widetilde{b}(t^\star)$. *These inequalities remain true for all* $t \geq t^\star$ *for which both solutions exist.*

Proof. Suppose not. Let t_f be the first time for which the inequality fails. There are three cases:

1. $\widetilde{a}(t_f) = a(t_f)$ and $\widetilde{b}(t_f) = b(t_f)$,
2. $\widetilde{a}(t_f) = a(t_f)$ and $\widetilde{b}(t_f) < b(t_f)$,
3. $\widetilde{a}(t_f) < a(t_f)$ and $\widetilde{b}(t_f) = b(t_f)$.

Case 1 contradicts the uniqueness part of the standard Picard theorem.

Suppose case 2 holds. Consider the evolution of $a - \widetilde{a}$:

$$\frac{d}{dt}(a - \widetilde{a}) = \frac{\mathcal{P}ab}{t} - \frac{a}{t} - \left(\frac{\mathcal{P}\widetilde{a}\widetilde{b}}{t} - \frac{\widetilde{a}}{t}\right).$$

At $t = t_f$ we have $\widetilde{a}(t_f) = a(t_f) = s > 0$ and $\widetilde{b}(t_f) < b(t_f)$. Consequently:

$$\frac{d}{dt}(a - \widetilde{a})_{|t_f} = \frac{\mathcal{P}s}{t}\left(b(t_f) - \widetilde{b}(t_f)\right) > 0.$$

It follows that $a(t) < \widetilde{a}(t)$ for some time $t < t_f$ and the intermediate value theorem contradicts the fact that t_f is the first time for which the inequalities fail.

Suppose case 3 holds. Consider the evolution of $b - \widetilde{b}$:

$$\frac{d}{dt}\left(b - \widetilde{b}\right) = \frac{\mathcal{P}}{2t}\left(1 - b^2\right) - \frac{\mathcal{P}\mathcal{Q}}{2}\left(1 - a^2\right) - \mathcal{Q}b - \left(\frac{\mathcal{P}}{2t}\left(1 - \widetilde{b}^2\right) - \frac{\mathcal{P}\mathcal{Q}}{2}\left(1 - \widetilde{a}^2\right) - \mathcal{Q}\widetilde{b}\right).$$

At $t = t_f$ we have $\widetilde{b}(t_f) = b(t_f)$ and $\widetilde{a}(t_f) < a(t_f)$. Consequently:

$$\frac{d}{dt}\left(b - \widetilde{b}\right)_{|t_f} = \frac{\mathcal{P}\mathcal{Q}}{2}\left(a(t_f) - \widetilde{a}(t_f)\right)\left(a(t_f) + \widetilde{a}(t_f)\right) > 0. \tag{5.189}$$

which leads to a contradiction as above. □

Proposition 5.17. *Suppose that* (a,b) *is a solution of (5.186) defined in a neighbourhood of* $t_0 > 0$*. Take initial data at* t_0 *satisfying* $a(t_0) > 0$*,* $b(t_0) < 0$ *and flow backwards. Either the solution* (a,b) *blows-up as* $t \to t_{\text{blowup}} > 0$ *or* $a \to +\infty$ *as* $t \to 0$*.*

Proof. We will bound a from below by a function v satisfying $v \to +\infty$ as $t \to 0$.

Consider the evolution of the product ab. Using the equations (5.186), compute:

$$\begin{aligned} \frac{d}{dt}(ab)|_t &= \dot{a}b + a\dot{b} \\ &= \frac{P}{2t}ab^2 + \frac{\mathcal{P}}{2t\left(t^2+1\right)}a + \frac{\mathcal{P}\mathcal{Q}}{2}a^3 - \left(\mathcal{Q}+\frac{1}{t}\right)ab > -\left(\mathcal{Q}+\frac{1}{t}\right)ab, \end{aligned} \tag{5.190}$$

where in the last line we used the fact that $a > 0$ for all time. By comparison, flowing backwards in time, ab stays below the solution of the I.V.P:

$$\begin{cases} \dot{u}(t) = -\left(\mathcal{Q}+\frac{1}{t}\right)u, \\ u(t_0) = a(t_0)b(t_0). \end{cases}$$

By assumption, the initial data satisfy:

$$a(t_0)b(t_0) < 0.$$

Consequently, $u < 0$ for all $0 < t < t_0$ and we conclude that the same is true of ab.

This allows us to estimate:

$$\dot{a}(t) = \frac{\mathcal{P}ab}{t} - \frac{a}{t} < -\frac{a}{t}.$$

Consequently, a lies above the solution to the following I.V.P backwards of t_0:

$$\begin{cases} \dot{v}(t) = -\frac{v}{t}, \\ v(t_0) = a(t_0). \end{cases}$$

This is easily solved explicitly and we obtain the inequality:

$$a(t) \geq \frac{a(t_0)t_0}{t} \text{ for all } 0 < t \leq t_0.$$

□

Corollary 5.18. *Let $T > 0$ and let $(a, b) \in C^1[0, T]$ be a solution of (5.186) satisfying $a \neq 0$. We have that $b(t) > 0$ for all $t \geq 0$ for which the solution exists.*

Proof. Trivially, $b(0) = \pm 1$. If not, then $\dot{b}(t)$ blows up as $t \to 0$. Hence it suffices to prove the result for $t > 0$. If we achieve this, the possibility that $b(0) = -1$ is excluded by continuity and thus we have that $b(0) = 1$.

Suppose that for some $t_0 > 0$, $b(t_0) < 0$. We have that $a(t_0) \neq 0$ by assumption. If $a(t_0) > 0$, the above proposition implies that a blows up to $+\infty$ near $t = 0$ contradicting the boundedness of the solution. If $a(t_0) < 0$, then $-a(t_0) > 0$. Since $(-a, b)$ is a solution, $-a$ blows up to $+\infty$ near $t = 0$. Hence, a blows up to $-\infty$ near $t = 0$.

Suppose that $b(t_0) = 0$ for some $t_0 > 0$. At such a point we have:

$$\dot{b}(t_0) = \frac{\mathcal{P}(t_0)}{2t_0(t_0^2 + 1)} + \frac{\mathcal{P}(t_0)\mathcal{Q}(t_0)}{2} a^2(t_0) > 0.$$

It follows that $b(t) < 0$ for some $0 < t < t_0$ and this brings us to the previous case. □

Putting the above together: if (a, b) is a global solution of (5.186), either $a = 0$ identically or the sign of a is fixed and $b > 0$.

The final proposition in this section asserts that -when $a \neq 0$- the long-term behaviour of (5.186) is essentially driven by a.

Proposition 5.19. *Let $(a, b) \in C^1[0, T]$ be a solution of (5.186). Suppose that $a \neq 0$. Either (a, b) survives for all $t > 0$, or there exists a finite blowup time $0 < t_{blowup} < \infty$ such that:*

$$|a| \to +\infty \quad as \quad t \to t_{blowup}^-.$$

Proof. By the preceding results of this section, we have that $b > 0$ and the sign of a is fixed. Proposition 5.15 permits us to assume $a > 0$ without loss of generality. By the standard escape lemma, either the solution is global, or its phase space norm blows up to $+\infty$ at some finite time $0 < t_* < \infty$.

The task is to prove that when the latter occurs, we have that $a \to +\infty$ as $t \to t_*^-$. Suppose not. Then $b \to +\infty$ as $t \to t_*^-$. Let $\epsilon_1 > 0$ be such that:

$$b(t) > \frac{1}{\mathcal{P}(t)} \text{ for } t_* - \epsilon_1 < t < t_*. \tag{5.191}$$

The equation governing a implies that it is monotonically increasing past $t_* - \epsilon_1$. Since it doesn't converge to $+\infty$, it must be bounded above. We then have that:

$$M := \sup_{t_* - \epsilon_1 < t < t_*} \left(\frac{\mathcal{P}}{2\mathcal{Q}t} + \frac{\mathcal{P}}{2} a^2 \right) < \infty.$$

Let $0 < \epsilon_2 < \epsilon_1$ be such that:

$$b(t) > M \text{ for } t_* - \epsilon_2 < t < t_*. \tag{5.192}$$

Using the equation governing b, we find that for times past $t_* - \epsilon_2$:

$$\begin{aligned}
\frac{db}{dt} &= \frac{\mathcal{P}}{2t}\left(1 - b^2\right) - \frac{\mathcal{P}\mathcal{Q}}{2}\left(1 - a^2\right) - \mathcal{Q}b. \\
&< \mathcal{Q}\left(\frac{\mathcal{P}}{2\mathcal{Q}t} + \frac{\mathcal{P}}{2}a^2 - b \right) \\
&< \mathcal{Q}\left(M - b\right) \\
&< 0.
\end{aligned}$$

It follows that b is bounded above. This yields the requisite contradiction, establishing the claim with $t_{\text{blowup}} = t_*$. □

5.3.5.3 Solutions Extending on P_{π_2}: An Explicit Family of Spin(7) Instantons Containing a Unique HYM Connection We analyze the invariant Spin(7) instanton equations on the bundle P_{π_2}. The system (5.186) reduces to a single nonlinear ODE that we can solve explicitly. We thus exhibit a 1-parameter family of Spin(7) instantons only one of which is HYM. This resolves (negatively) the question of equivalence of the two gauge theoretic problems.

Owing to proposition 5.13, solutions extending to P_{π_2} must satisfy $b(0) = -1$. Due to corollary 5.18, the a-component of such a solution must vanish identically. The system (5.186) reduces to the following ODE:

$$\frac{db}{dt} = -\frac{\mathcal{P}\mathcal{Q}}{2} + \frac{\mathcal{P}}{2t}\left(1 - b^2\right) - \mathcal{Q}b \tag{5.193}$$

This can be solved explicitly. We fix a positive reference time and parameterize solutions by their value at that time. We choose to work with $t_{\text{ref}} = \frac{\sqrt{6}}{2}$ (corresponding to $r_{\text{ref}} = 2$). This choice is arbitrary.

Writing:

$$\nu = b\left(t_{\text{ref}}\right),$$

the associated solution to (5.193) takes the form:

$$b_\nu(t) = \frac{\sqrt{2}}{2}\left(1 + \frac{\sqrt{6} - \nu\sqrt{10t^2+15}}{\sqrt{30}\nu + \sqrt{6} - \left(\sqrt{5}\nu + 2\right)\sqrt{2t^2+3}}\right)\frac{1}{\sqrt{t^2+1}}. \tag{5.194}$$

Corresponding to b_ν there is a local Spin(7) instanton on the restriction of P_{Id} over an open submanifold of the form:

$$\left(t_{\text{ref}} - \delta, t_{\text{ref}} + \delta\right) \times \frac{\text{SO}(5)}{\text{SO}(3)} \subset X^8. \tag{5.195}$$

This instanton is given by:

$$A_\nu \stackrel{\text{def}}{=} A^{\text{can}}_{\pi_2} + \left(b_\nu(t) + 1\right)\left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right). \tag{5.196}$$

An elementary calculation yields the values of ν for which there exists a time $t \geq 0$ such that:

$$\lim_{t \to t_0} b(t) = \pm\infty$$

We obtain the following:

Proposition 5.20. *Let* $\nu \in (-\infty, -\frac{2\sqrt{5}}{5}) \cup (\frac{\sqrt{10}}{5}, \infty)$*. The connection* A_ν *blows up (as witnessed—for instance—by a blowup of the pointwise curvature norm) at the time* $t_{blowup}(\nu)$

given by:

$$t_{blowup}(\nu) = \frac{\sqrt{6}}{2}\frac{\sqrt{5\nu^2-2}}{|\sqrt{5}\nu+2|}. \tag{5.197}$$

In flowing away from the reference point $t{=}t_{\mathrm{ref}}$ we have forced our solutions to be bounded at $t{=}t_{\mathrm{ref}}$ and thus neglected solutions that blow up at this time. Such solutions correspond to the limit $\nu \to \pm\infty$. We can obtain an explicit formula for them by working with a different reference time. We are not interested in this calculation since the resulting solutions cannot yield global instantons.

For ν outside of the range considered in the proposition, the solutions stay bounded for all time. These considerations lead to the following existence/classification result:

Theorem 5.21. *Let* $\nu \in [-\frac{2\sqrt{5}}{5}, \frac{\sqrt{10}}{5})$. *The connection* A_ν *is a smooth* Spin(7) *instanton on the extended bundle* P_{π_2}. *Furthermore, these are all the invariant* Spin(7) *instantons on* P_{π_2}.

Proof. For $\nu \in [-\frac{2\sqrt{5}}{5}, \frac{\sqrt{10}}{5}]$, the function b_ν is of class $C^\infty[0,\infty)$. We need to verify the extension conditions of proposition 5.13. In particular we need to prove that $b_\nu(t){+}1$ is even and $O(t^2)$ at $t=0$. We immediately exclude $\nu = \frac{\sqrt{10}}{5}$ as the associated solution satisfies $b(0) = 1$ and consequently fails the second extension condition. For $\nu \in [-\frac{2\sqrt{5}}{5}, \frac{\sqrt{10}}{5})$, the first condition is clear by looking at the formula for b_ν. The second condition is easily established by computing that:

$$b_\nu(0) + 1 = \dot{b}(0) = 0 \tag{5.198}$$

For uniqueness, we note that any invariant Spin(7) instanton on P_{π_2} obeys (5.193) and all other solutions of this equation blow up. □

In particular, we resolve (negatively) the primary question we set out to answer: the two natural gauge-theoretic systems available on a CY4 space do not coincide.

Theorem 5.22. *Let* (M^8, g, J, Ω) *be a geodesically complete Calabi-Yau 4-fold* (*i.e.* $\mathrm{Hol}(g) = \mathrm{SU}(4)$). *It is possible for* Spin(7) *instantons on* X^8 *not to satisfy the HYM equation. We shall refer to such connections as pure* Spin(7) *instantons. When* X^8 *is non-compact—*

in direct contrast to the compact case—pure Spin(7) *instantons can live on a principal G-bundle admitting HYM solutions.*

The HYM connection $A_{\mathrm{HYM}_{\pi_2}}$ (introduced in (5.153)) lies in the interior of this family and corresponds to the choice $\nu = -\frac{\sqrt{10}}{5}$. It is the only HYM connection in the family.

The boundary point $\nu_\partial = -\frac{2\sqrt{5}}{5}$ corresponds to the solution:

$$b_{\nu_\partial}(t) = -\frac{\sqrt{3}}{3}\frac{\sqrt{2t^2+3}}{\sqrt{t^2+1}}.$$

The associated Spin(7) instanton A_{ν_∂} differs from the other elements in the family in that it yields a different connection over the Stiefel manifold at infinity.

For $\nu \in \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{10}}{5}\right)$ it is easily seen that:

$$\lim_{t\to\infty} b_\nu(t) = 0. \tag{5.199}$$

Consequently, the corresponding instantons approach the canonical invariant connection of P_{Id}:

$$A_\nu^\infty = A_{P_{\mathrm{Id}}}^{\mathrm{can}}.$$

The solution corresponding to $\nu = \nu_\partial$ satisfies:

$$\lim_{t\to\infty} b_{\nu_\partial}(t) = -\frac{\sqrt{6}}{3}. \tag{5.200}$$

It follows that the associated connection at infinity is given by:

$$A_{\nu_\partial}^\infty = A_{P_{\mathrm{Id}}}^{\mathrm{can}} - \frac{\sqrt{6}}{3}\left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right) \tag{5.201}$$

In the study of gauge theoretic moduli problems over noncompact manifolds, it is common practice to fix the limiting connection at infinity. With this perspective in mind, we could view the family of Theorem 5.21 as having a missing endpoint in both directions. The noncompactness at the left endpoint is then mended by adding in the instanton A_{ν_∂} (the latter lying in another moduli space because of its different asymptotic behaviour). Perhaps, similar noncompactness phenomena (jump discontinuities of the limiting G_2 instanton over the asymptotic link) occur in general Spin(7) instanton moduli spaces.

We briefly comment on the asymptotic behaviour of the connections (A_ν). We have already computed the pointwise curvature norm of $A_{\mathrm{HYM}_{\pi_2}}$. It is of order $O(s^{-2})$ at infinity. Equations (5.199) and (5.200) imply that all elements of the family are AC. Consequently, all the associated curvature tensors decay like $O(s^{-2})$.

We conclude this section by noting that the limits $t \to 0$ and $\nu \to \frac{\sqrt{10}}{5}$ do not commute. Notably, the value $b_\nu(0)$ jumps from -1 to $+1$. Propositions 5.12 and 5.13 suggest that this signifies a topological shift—i.e. change of the underlying principal bundle. In fact, this is the first sign of the occurrence of an interesting removable singularity/ bubbling phenomenon and the development of a corresponding Fueter section. This is the subject of section 5.3.5.6.

5.3.5.4 Solutions Extending on P_{π_1} We now wish to classify solutions that smoothly extend over P_{π_1}. In the previous section we saw that the only global-in-time solution satisfying $a = 0$, $b(0) = 1$ corresponds to $\nu = \frac{\sqrt{10}}{5}$. This is given by:

$$b_\nu(t) = \frac{1}{\sqrt{t^2+1}}.$$

We recognize this as the HYM connection $A_{\mathrm{HYM}_{\pi_1}}$ (introduced in (5.152)), which—as we have already seen—extends to P_{π_1}.

Any other solution would have nonvanishing a-component. Consequently, we have to deal with the full system (5.186). The first task is to obtain short time existence and uniqueness near the pole of the ODE. This will prove that the moduli space is at most 1 dimensional. The rest of the section will deal with characterizing which of these local solutions survive for all time to yield global Spin(7) instantons.

5.3.5.4.1 Short Time Existence and Uniqueness The analysis in this section relies on the method of Eschenburg and Wang ([21], section 6). We have adapted their existence result to our equation system and refined it to include continuous dependence on initial data. This does not follow from the standard Grönwall estimate, as the I.V.P under consideration is singular. The continuity proof is based on the technique employed by McLeod, Smoller, Wasserman and Yau in ([31], p.147]).

Theorem 5.23. *Let a_0 be a fixed real number. There exists a unique solution:*

$$(a, b)_{a_0} \in C^\infty[0, t_{\max}(a_0))$$

to the system (5.186) *such that a is odd at $t = 0$, b is even at $t = 0$ and furthermore:*

$$a(0) = 0, \tag{5.202}$$

$$\dot{a}(0) = a_0, \tag{5.203}$$

$$b(0) = 1. \tag{5.204}$$

This solution satisfies the extension conditions of proposition 5.12 and thus yields a local Spin(7) *instanton on the restriction of P_{π_1} over the open submanifold defined by $0 \leq t < t_{\max}(a_0)$.*

Furthermore, we have that for any $K > 0$:

$$T_K \stackrel{def}{=} \inf \left\{ t_{\max}(a_0) \mid a_0 \in [-K, K] \right\} > 0 \tag{5.205}$$

and the following mapping is continuous:

$$\begin{aligned} [-K, K] &\to C^0 \left([0, T_K], \mathbb{R}^2 \right), \\ a_0 &\mapsto (a, b)_{a_0}. \end{aligned} \tag{5.206}$$

We will prove this result in four stages. The first step is to study the formal Taylor series of smooth solutions at $t = 0$. The second step is to derive and analyze ODEs governing perturbations of high order polynomial truncations of the series. The idea is to show that, if the order is high enough, the resulting ODEs are uniquely soluble for sufficiently short time in suitable Banach spaces. The third step is to argue that the solutions so obtained are smooth and have the correct formal series at $t = 0$. The final step is to understand how this existence/ uniqueness argument behaves under change of initial data. This involves proving that the estimates can be made to be uniform in a_0 for a_0 in compact sets and establishing the desired continuity result.

Proposition 5.24. *Fix $a_0 \in \mathbb{R}$. There exists a unique $(a,b)_{a_0} \in \mathbb{R}[[t]]^2$ solving the system (5.186) and satisfying the conditions (5.202), (5.203), (5.204). Here, differentiation is understood in the formal sense (as a derivation of the formal power series ring).*

Proof. Considering our assumptions on a and b, we introduce their formal Taylor series at $t = 0$:

$$a = \sum_{k=0}^{\infty} \frac{a_k}{(2k+1)!} t^{2k+1}, \quad b = \sum_{k=0}^{\infty} \frac{b_k}{(2k)!} t^{2k}, \text{ where } b_0 = 1. \tag{5.207}$$

Using the parity of a, b and the coefficient functions, we introduce the series:

$$a(\mathcal{P}b - 1) = \sum_{k=0}^{\infty} \frac{c_k}{(2k+1)!} t^{2k+1}, \qquad \frac{\mathcal{P}\mathcal{Q}}{2}\left(1 - a^2\right) = \sum_{k=0}^{\infty} \frac{e_k}{(2k+1)!} t^{2k+1},$$
$$\frac{\mathcal{P}}{2}\left(1 - b^2\right) = \sum_{k=0}^{\infty} \frac{d_k}{(2k)!} t^{2k} \text{ where } d_0 = 0, \qquad \mathcal{Q}b = \sum_{k=0}^{\infty} \frac{f_k}{(2k+1)!} t^{2k+1}.$$

The ODE for a translates to the condition:

$$a_k = \frac{c_k}{2k+1} \text{ for all } k \geq 0. \tag{5.208}$$

We compute c_k in terms of $a_0, ..., a_k, b_0, ..., b_k$. This yields:

$$\begin{aligned} c_k &= \frac{d^{2k+1}}{dt^{2k+1}}_{|t=0} \left(a(\mathcal{P}b - 1)\right), \\ &= a_k + \mathcal{G}(a_0, ..., a_{k-1}, b_0, ..., b_k). \end{aligned}$$

Here $\mathcal{G}$ denotes some function of coefficients of lower order. We will slightly abuse notation and maintain use of the symbol $\mathcal{G}$ in subsequent calculations -even though the particular function may not be the same. Using (5.208) we obtain:

$$\frac{2k}{2k+1} a_k = \mathcal{G}(a_0, ..., a_{k-1}, b_0, ..., b_k). \tag{5.209}$$

This determines a_k in terms of coefficients of lower order provided that $k \neq 0$. We conclude that we are allowed to choose a_0 freely.

We perform a similar calculation for b. The second equation in (5.186) translates to the relation:

$$b_{k+1} = \frac{d_{k+1}}{2k+2} - e_k - f_k \text{ for all } k \geq 0. \tag{5.210}$$

We note that e_k and f_k only involve terms depending on $a_0, ..., a_k, b_0, ..., b_k$ and it is thus unnecessary to compute them. We compute d_{k+1} in terms of $a_0, ..., a_k, b_0, ..., b_{k+1}$:

$$\begin{aligned} d_{k+1} &= \frac{d^{2k+2}}{dt^{2k+2}}_{|t=0} \left(\frac{\mathcal{P}}{2}(1-b^2) \right), \\ &= -2b_{k+1} + \mathcal{G}(b_0, ..., b_k). \end{aligned}$$

Using (5.210), we obtain:

$$\frac{k+2}{k+1} b_{k+1} = \mathcal{G}(a_0, ..., a_k, b_0, ..., b_k).$$

It follows that b_{k+1} is determined by lower order coefficients for each $k \geq 0$.

The above calculations demonstrate that the formal Taylor series at 0 is uniquely determined by induction given a choice of $a_0 \in \mathbb{R}$. □

Although the content of the preceding proposition is enough for the purposes of our existence theorem, continuity requires more refined knowledge of the formal Taylor series. In particular, we are interested in the dependence of its coefficients on a_0. We explicitly calculate the first few terms of the series associated to some fixed choice of a_0. The resulting expressions will also prove useful in our global existence analysis–the proof of proposition 5.37, in particular.

$$a(t) = a_0 t - \frac{a_0}{3} t^3 + \left(\frac{5a_0}{36} + \frac{a_0^3}{12} \right) t^5 + O(t^7), \tag{5.211}$$

$$b(t) = 1 - \frac{t^2}{2} + \left(\frac{3}{8} + \frac{a_0^2}{6} \right) t^4 + O(t^6). \tag{5.212}$$

In fact, we are able to obtain the following:

Proposition 5.25. *The coefficients of the formal Taylor series $(a, b)_{a_0}$ are polynomials (possibly of order* 0*) in a_0.*

Proof. This is certainly true for a_0, b_0 and b_1. Repeating the calculations of the preceding proposition, but keeping track of the lower order terms yields the following recurrence relations for the coefficients:

$$
\begin{aligned}
a_k =& \frac{1}{2k}\sum_{m=1}^{k}\sum_{l=0}^{m}\binom{2k+1}{2m}\binom{2m}{2l}\mathcal{P}^{\left(2(m-l)\right)}_{|t=0}a_{k-m}\,b_l,\\
b_{k+1} =& -\frac{1}{4k+8}\sum_{m=1}^{k}\sum_{l=0}^{m}\binom{2k+2}{2m}\binom{2m}{2l}\mathcal{P}^{\left(2(k-m)+2\right)}_{|t=0}\,b_{m-l}\,b_l-\frac{1}{2k+4}\sum_{l=1}^{k}\binom{2k+2}{2l}b_{k+1-l}\,b_l\\
&-\frac{k+1}{2k+4}\left(\mathcal{PQ}\right)^{(2k+1)}_{|t=0}+\frac{k+1}{2k+4}\sum_{m=1}^{k}\sum_{l=0}^{m-1}\binom{2k+1}{2m}\binom{2m}{2l+1}\left(\mathcal{PQ}\right)^{\left(2(k-m)+1\right)}_{|t=0}\,a_{m-l-1}\,a_l\\
&-\frac{k+1}{k+2}\sum_{m=0}^{k}\binom{2k+1}{2m}\mathcal{Q}^{\left(2(k-m)+1\right)}_{|t=0}b_m.
\end{aligned}
$$

The result follows by induction. □

We now discuss how to use this formal series in order to obtain an honest solution of the system (5.186). For ease of exposition, we introduce the following functions:

$$
\begin{aligned}
F_1(t,u,v) &\stackrel{\text{def}}{=} u\left(\mathcal{P}(t)\,v-1\right),\\
F_2(t,v) &\stackrel{\text{def}}{=} \frac{\mathcal{P}(t)}{2}(1-v^2),\\
F_3(t,u,v) &\stackrel{\text{def}}{=} -\frac{\mathcal{P}(t)\,\mathcal{Q}(t)}{2}(1-u^2)-\mathcal{Q}(t)\,v.
\end{aligned}
$$

We rewrite the ODE system (5.186) as:

$$
\frac{da}{dt}=\frac{F_1\left(t,a,b\right)}{t}, \tag{5.213}
$$

$$
\frac{db}{dt}=\frac{F_2\left(t,b\right)}{t}+F_3\left(t,a,b\right). \tag{5.214}
$$

Further, we let $p^a_m(t,a_0)$, $p^b_m(t,a_0)$ denote the order m Taylor polynomials corresponding to the initial data a_0. These are obtained by truncating the respective series. We also introduce the following error functions capturing the failure of the Taylor polynomials to solve (5.186):

$$
\begin{aligned}
E^a_m(t,a_0) &\stackrel{\text{def}}{=} \frac{d}{dt}p^m_a(t,a_0)-\frac{F_1\left(t,p^a_m(t,a_0),p^b_m(t,a_0)\right)}{t},\\
E^b_m(t,a_0) &\stackrel{\text{def}}{=} \frac{d}{dt}p^m_b(t,a_0)-\frac{F_2\left(t,p^b_m(t,a_0)\right)}{t}-F_3(t,p^a_m(t,a_0),p^b_m(t,a_0)).
\end{aligned}
$$

They are smooth and $O(t^m)$ at $t = 0$. To see this, recall that the full formal series was constructed by matching derivatives at the origin. Consequently, the first $m-1$ derivatives of the error functions vanish at $t = 0$.

We now introduce the Banach spaces we will be working with. For any real $T > 0$ and integer $m \geq 0$ we define:

$$\mathcal{O}_T(m) \stackrel{\text{def}}{=} \left\{ f \in C^0[0,T] \ s.t. \ \sup_{t\in[0,T]} \frac{|f(t)|}{t^m} < \infty \right\}, \tag{5.215}$$

$$\|f\|_{\mathcal{O}_T(m)} \stackrel{\text{def}}{=} \sup_{t\in[0,T]} \frac{|f(t)|}{t^m}. \tag{5.216}$$

Fix $T > 0$. We immediately observe that the error functions E^a_m, E^b_m lie in $\mathcal{O}_T(m)$ (they are $O(t^m)$ at $t = 0$). Furthermore, the functions p^a_m, $p^b_m - 1$ lie in $\mathcal{O}_T(1)$. In fact -in light of proposition 5.25- we have:

Corollary 5.26. *$E^a_m(t,\cdot)$, $E^b_m(t,\cdot)$ define continuous mappings from the space of initial data (the real numbers) into $\mathcal{O}_T(m)$. Similarly, $p^a_m(t,\cdot)$, $p^b_m(t,\cdot) - 1$ define continuous mappings from the space of initial data into $\mathcal{O}_T(1)$.*

We finally recast the problem as an integral equation for a perturbation of the polynomials (p^a_m, p^b_m). Given a pair of functions $(u,v) \in \mathcal{O}_T^{\oplus 2}(m)$ we define:

$$\Theta^1_{m,a_0}(u,v)(s) \stackrel{\text{def}}{=} \int_0^s \left(\frac{F_1\left(t, p^a_m(a_0,t) + u(t), p^b_m(a_0,t) + v(t)\right)}{t} - \dot{p}^a_m(a_0,t) \right) dt,$$

$$\Theta^2_{m,a_0}(u,v)(s) \stackrel{\text{def}}{=} \int_0^s \left(\frac{F_2\left(t, p^b_m(a_0,t) + v(t)\right)}{t} + F_3(t, p^a_m(a_0,t) + u(t), p^b_m(a_0,t) + v(t)) - \dot{p}^b_m(a_0,t) \right) dt.$$

It can be easily checked (by expanding out the integrands, counting order of vanishing and noting that integration raises this by one) that we obtain a nonlinear integral operator:

$$\Theta_{m,a_0} \stackrel{\text{def}}{=} \Theta^1_{m,a_0} \times \Theta^2_{m,a_0} : \mathcal{O}_T^{\oplus 2}(m) \to \mathcal{O}_T^{\oplus 2}(m). \tag{5.217}$$

The following proposition is the heart of the matter:

Proposition 5.27. *Let a_0 be fixed. Fix $R > 0$. For sufficiently large m (depending on F_1, F_2, F_3, R) and sufficiently small T (depending on m and a_0), the operator Θ_{m,a_0} has a* *unique fixed point* (u, v) *in* $\overline{B}_R(0) \subset \mathcal{O}_T^{\oplus 2}(m)$. *Furthermore, this fixed point is smooth on* $[0, T]$ *and the associated solution*

$$(a, b) \overset{def}{=} (p_m^a + u, p_m^b + v)$$

to the system (5.186) *satisfies (5.202), (5.203), (5.204).*

Proof. In what follows, our notation suppresses dependence on a_0. Fix $R > 0$. We will select m and T such that Θ_m is a contraction on $\overline{B}_R(0) \subset \mathcal{O}_T^{\oplus 2}(m)$.

Consider the domain:

$$D_R \overset{\text{def}}{=} [0, 1] \times \overline{B}_{2R}(0, 1) \subset \mathbb{R}^3. \tag{5.218}$$

Let $L > 0$ be a Lipschitz constant in the (u, v) variables for the restrictions of F_1, F_2, F_3 on D_R. The constant L is controlled by L^∞ bounds on the restrictions of the derivatives of the F_i on D_R. Choose:

$$m > \max\{2L, 1\}.$$

Pick T such that:

$$T < \min\left\{1, \frac{(m+1)R}{2\left(\|E_m^a\|_{\mathcal{O}_1(m)} + \left\|E_m^b\right\|_{\mathcal{O}_1(m)}\right)}, \frac{R}{\|p_m^a\|_{\mathcal{O}_1(m)}}, \frac{R}{\left\|p_m^b - 1\right\|_{\mathcal{O}_1(m)}}\right\}.$$

Clearly, for $t \in [0, T]$ we have:

$$|p_m^a(t)| \leq R, \quad |p_m^b(t) - 1| \leq R. \tag{5.219}$$

We claim that we also have:

$$\left\|\Theta_m(0, 0)\right\|_{\mathcal{O}_T(m)} \leq \frac{R}{2}. \tag{5.220}$$

To see this, we estimate as follows:

$$\begin{aligned}
\left\|\Theta_m(0,0)\right\|_{\mathcal{O}_T(m)} &= \left\|\Theta^1_m(0,0)\right\|_{\mathcal{O}_T(m)} + \left\|\Theta^2_m(0,0)\right\|_{\mathcal{O}_T(m)} \\
&= \left\|\int_0^r E^a_m(t)dt\right\|_{\mathcal{O}_T(m)} + \left\|\int_0^r E^b_m(t)dt\right\|_{\mathcal{O}_T(m)} \\
&\leq \sup_{r\in[0,T]} \frac{1}{r^m}\int_0^r |E^a_m(t)|dt + \sup_{r\in[0,T]} \frac{1}{r^m}\int_0^r |E^b_m(t)|dt \\
&\leq \sup_{r\in[0,T]} \frac{\|E^a_m\|_{\mathcal{O}_T(m)}}{r^m}\int_0^r t^m dt + \sup_{r\in[0,T]} \frac{\left\|E^b_m\right\|_{\mathcal{O}_T(m)}}{r^m}\int_0^r t^m dt \\
&\leq \sup_{r\in[0,T]} \frac{\|E^a_m\|_{\mathcal{O}_T(m)}}{m+1} r + \sup_{r\in[0,T]} \frac{\left\|E^b_m\right\|_{\mathcal{O}_T(m)}}{m+1} r \\
&\leq \frac{\|E^a_m\|_{\mathcal{O}_T(m)} + \left\|E^b_m\right\|_{\mathcal{O}_T(m)}}{(m+1)} T \leq \frac{R}{2}.
\end{aligned} \tag{5.221}$$

We now prove contraction estimates for Θ^1_m and Θ^2_m. Fix $0 \leq r \leq T$ and compute:

$$\begin{aligned}
\left|\Theta^1_m(u,v)(r) - \Theta^1_m(\widetilde{u},\widetilde{v})(r)\right| &\leq \int_0^r \frac{1}{t}\left|F_1\left(t, p^a_m + u, p^b_m + v\right) - F_1\left(t, p^a_m + \widetilde{u}, p^b_m + \widetilde{v}\right)\right| dt \\
&\leq \int_0^r \frac{L}{t}\left(|u-\widetilde{u}| + |v-\widetilde{v}|\right) dt \\
&\leq L\left(\|u-\widetilde{u}\|_{\mathcal{O}_T(m)} + \|v-\widetilde{v}\|_{\mathcal{O}_T(m)}\right)\int_0^r t^{m-1}dt \\
&\leq \frac{Lr^m}{m}\left(\|u-\widetilde{u}\|_{\mathcal{O}_T(m)} + \|v-\widetilde{v}\|_{\mathcal{O}_T(m)}\right).
\end{aligned}$$

In this calculation, the L-Lipschitz estimate is valid due to (5.219) and the fact that the uniform norm is controlled by the $\mathcal{O}_T(m)$ norm when $0 < T < 1$. We conclude that:

$$\left\|\Theta^1_m(u,v) - \Theta^1_m(\widetilde{u},\widetilde{v})\right\|_{\mathcal{O}_T(m)} \leq \frac{L}{m}\left(\|u-\widetilde{u}\|_{\mathcal{O}_T(m)} + \|v-\widetilde{v}\|_{\mathcal{O}_T(m)}\right). \tag{5.222}$$

A similar calculation yields:

$$\begin{aligned}
\left\|\Theta^2_m(u,v) - \Theta^2_m(\widetilde{u},\widetilde{v})\right\|_{\mathcal{O}_T(m)} &\leq \left(\frac{L}{m} + \frac{LT}{m+1}\right)\|v-\widetilde{v}\|_{\mathcal{O}_T(m)} + \frac{LT}{m+1}\|u-\widetilde{u}\|_{\mathcal{O}_T(m)} \\
&\leq \frac{1}{2}\left(\|u-\widetilde{u}\|_{\mathcal{O}_T(m)} + \|v-\widetilde{v}\|_{\mathcal{O}_T(m)}\right).
\end{aligned} \tag{5.223}$$

Due to (5.220), (5.222) and (5.223), the closed R-ball in $\mathcal{O}^{\oplus 2}_T(m)$ is stable under Θ. The contraction mapping theorem (CMT) yields a unique fixed point (u, v) in this ball.

This fixed point is necessarily of class $C^1[0,T]$ (by the fundamental theorem of calculus). Consequently (a,b) is C^1 and it therefore constitutes an honest solution of (5.186) on $[0,T]$. Considering the order of vanishing of u at 0 and looking at the equations, we observe that $\dot{u}(t) = O(t^{m-1})$. Conditions (5.202), (5.203), (5.204) follow.

Full regularity follows by a simple bootstrap procedure. Since flows of smooth (non-autonomous) vector fields are smooth, (u,v) is smooth on $(0,T]$. The task is to establish smoothness at 0. Smoothness on $(0,T]$ legitimizes differentiation of the equations for $t>0$. This gives an expression for the second derivatives of u and v involving terms in $\frac{u}{t^2}$, $\frac{v}{t^2}$, $\frac{\dot{u}}{t}$ and $\frac{\dot{v}}{t}$. It is thus clear that $u^{(2)}(t)$, $v^{(2)}(t) \to 0$ as $t \to 0$. Hence u, v are of class $C^2[0,T]$ with vanishing second derivative at 0. We can iterate this argument to conclude that u, v are of class $C^{m-1}[0,T]$ with vanishing derivatives at 0 up to order $m-1$. The only constraint on m required for the contraction argument to run is $m > \max\{2L, 1\}$. It follows that the operator Θ_l is a contraction for arbitrarily large $l > m$ (perhaps for shorter time T). Fixing $l > m$, we let (u_l, v_l) be the associated fixed point. Repeating the argument above, it lies in $C^{l-1}[0,T]$ with vanishing derivatives up to order $l-1$. It is thus $O(t^{m+1})$. It follows that $(u_l + p_l^a - p_m^a, v_l + p_l^b - p_m^b)$ is also $O(t^{m+1})$. Consequently -by further decreasing T as necessary- we can arrange that the latter has as small $\mathcal{O}_T^{\oplus 2}(m)$ norm as we like. In particular, we take this to be less than R. Furthermore, $(u_l + p_l^a - p_m^a, v_l + p_l^b - p_m^b)$ is a fixed point of Θ_m. But Θ_m has a unique fixed point in the closed R-ball. It follows that:

$$(u,v) = (u_l + p_l^a - p_m^a, v_l + p_l^b - p_m^b) \tag{5.224}$$

and hence that u, v lie in C^{l-1}. Since l was arbitrary, the proof is complete. □

We now have enough for the first part of theorem 5.23. The preceding proposition guarantees the existence of a smooth solution (a,b) satisfying (5.202), (5.203), (5.204). The algebraic calculation in the start of this section uniquely specifies its full formal Taylor series at $t=0$ so that it passes the extension criterion in proposition 5.12. Finally, suppose that there is another smooth solution $(\widetilde{a}, \widetilde{b})$ satisfying (5.202), (5.203), (5.204). Arguing as above, we find that the two solutions share the same formal Taylor series at 0 (the series discovered in proposition 5.24). Let m be as in proposition 5.27. We have that

$(a - p_m^a, b - p_m^b)$, $(\widetilde{a} - p_m^a, \widetilde{b} - p_m^b)$ are $O(t^{m+1})$. For short enough time T, the $\mathcal{O}_T^{\oplus 2}(m)$ norms of these functions are less than R. Since both functions are fixed points of Θ_m and lie in the closed R-ball, they are equal. Hence $(a, b) = (\widetilde{a}, \widetilde{b})$.

It remains to study the dependence of solutions on variations of the initial data a_0. We immediately obtain:

Proposition 5.28. *Fix $K > 0$. We have:*

$$T_K = \inf \left\{ t_{\max}(a_0) \mid a_0 \in [-K, K] \right\} > 0.$$

Proof. In our existence proof, once, R, L, m are fixed, T needs to be controlled from above by quantities decreasing with the $\mathcal{O}_1(m)$ norms of the error functions and the $\mathcal{O}_1(1)$ norms of p_m^a, $p_m^b - 1$. By corollary 5.26, these norms depend continuously on a_0 and are hence bounded for a_0 in a compact set. It follows that we can choose T small enough so that the contraction argument works for all $a_0 \in [-K, K]$. □

Note that the contraction constant can be taken to be the same across all $a_0 \in [-K, K]$. This is vital for the continuity proof, which we now discuss.

Proposition 5.29. *The mapping defined by:*

$$\begin{aligned} [-K, K] &\to C^0 \Big([0, T_K], \mathbb{R}^2 \Big) \\ a_0 &\mapsto (a, b)_{a_0} \end{aligned} \tag{5.225}$$

is continuous.

Proof. Consider the trivial (infinite-rank) vector bundle over $[-K, K]$:

$$E \stackrel{\text{def}}{=} [-K, K] \times \mathcal{O}_{T_K}^{\oplus 2}(m).$$

The following map is fiber-preserving and continuous:

$$\begin{aligned} S : E &\to E, \\ \big(a_0, (u, v)\big) &\mapsto \big(a_0, \Theta_{m, a_0}(u, v)\big) . \end{aligned}$$

There is a unique section s of E that is fixed by S (the one assigning to each choice of initial data the associated fixed point of $\Theta_{a_0,m}$). The task is to prove that s is continuous. To this end, we fix $x \in [-K, K]$ and prove that s is continuous at x. Fix $\epsilon > 0$ and define the following (continuous) section of E:

$$u_x(a_0) \stackrel{\text{def}}{=} (a_0, s(x)).$$

We will run the CMT iteration on each fiber with initial condition determined by u_x. Letting $0 < C < 1$ be the contraction constant of $\Theta_{a_0,m}$ and using the standard convergence rate estimate of the CMT we have:

$$\begin{aligned} \left\| \Theta^N_{m,a_0}(u_x(a_0)) - s(a_0) \right\|_{\mathcal{O}^{\oplus 2}_{T_K}(m)} &\leq \frac{\left\| \Theta_{m,a_0}(u_x(a_0)) - u_x(a_0) \right\|_{\mathcal{O}^{\oplus 2}_{T_K}(m)}}{1-C} C^N \\ &\leq \frac{2R}{1-C} C^N. \end{aligned} \tag{5.226}$$

Fix N large enough so that this quantity is controlled by $\frac{\epsilon}{2}$. Since S and u are continuous, we have:

$$\lim_{a_0 \to x} S^N u_x(a_0) = S^N u_x(x) = (x, s(x)).$$

Consequently, for a_0 sufficiently close to x, we can achieve:

$$\left\| \Theta^N_{m,a_0}\left(u_x(a_0)\right) - s(x) \right\|_{\mathcal{O}^{\oplus 2}_{T_K}(m)} < \frac{\epsilon}{2}. \tag{5.227}$$

Using (5.226), (5.227) and the triangle inequality completes the proof. □

Uniqueness implies that the solution associated to $a_0 = 0$ corresponds to $A_{\mathrm{HYM}_{\pi_1}}$. This instanton will play a central role in the analysis of the global properties of the system.

5.3.5.4.2 Global Existence for Small Initial Data The previous section yields a characterization of short-time solutions near the pole. We are now tasked with understanding which of these solutions are global. In this section we establish that:

Theorem 5.30. *There exists an $\epsilon > 0$ such that for $|a_0| < \epsilon$, the short-time solutions of theorem 5.23 are global.*

Combining this with theorem 5.33 (presented in the subsequent section), we get a complete description of the invariant instanton moduli space. Note that we have not studied the precise long-time asymptotics of the global solutions. In the notation of theorem 5.33, we expect those in $(-x, x)$ to be AC, asymptotic to a contact instanton at infinity. We expect the limiting instanton to jump as we approach the boundary points.

The heart of the small-data global existence argument lies in the following proposition. Its conditions are subsequently easily verified (for small initial data) by a continuity argument.

Proposition 5.31. *Suppose that $a_0 > 0$ and let $(a, b)_{a_0}$ be a solution to the system (5.186) such that a attains a critical point in the spacetime region:*

$$t > \frac{\sqrt{6}}{2\sqrt{1-2a^2}}, \quad 0 < a < \frac{\sqrt{2}}{2}. \tag{5.228}$$

Then $t_{max}(a_0) = +\infty$.

Proof. By proposition 5.14, $a(t) > 0$ for all $0 < t < t_{\max}(a_0)$. Looking at the ODE for a, we conclude that the critical points of a are precisely the points where $b = \mathcal{P}^{-1}$. We seek an expression for the second derivative of a at a critical point occurring at time $t = t_{\text{crit}} > 0$. Differentiating the ODE for a and setting $b = \mathcal{P}^{-1}$, we obtain:

$$\frac{d^2a}{dt^2}_{|t=t_{\text{crit}}} = \frac{3\, a\,(t_{\text{crit}})}{2t_{\text{crit}}^2\left(2t_{\text{crit}}^2+3\right)}\left[\left(4\, a\,(t_{\text{crit}})^2-2\right)t_{\text{crit}}^2+3\right]. \tag{5.229}$$

The first factor is strictly positive. Consequently, the nature of the critical point depends on the sign of:

$$F(t,a) \stackrel{\text{def}}{=} \left(4a^2-2\right)t^2+3 \tag{5.230}$$

at $\left(t_{\text{crit}}, a\,(t_{\text{crit}})\right)$. For (t, a) in the spacetime region (5.228), we have $F(t, a) < 0$. Hence, any critical point occurring in the region is a maximum.

Suppose that a maximum does occur inside the region (5.228). For a short amount of time thereafter a is decreasing. The only way that a can ever increase again is if it reaches a minimum. A minimum can only occur if $(t, a(t))$ exits the spacetime region (5.228). For this to occur, a has to increase. It follows that a decreases for as long as the solution survives. Consequently a is bounded from above. Since $a > 0$, it follows that a is also bounded from below. Since a consistently decreases after the maximum point, we have that $b(t) < \mathcal{P}(t)^{-1}$ for $t > t_{\text{crit}}$. By corollary 5.18, $b > 0$ for all time. Hence both a and b

are bounded and thus survive for all time $t \geq 0$. □

Proposition 5.31 applies provided that the initial data is small enough:

Proposition 5.32. *There exists $\epsilon > 0$ such that if $0 \leq a_0 < \epsilon$, then a_{a_0} attains a critical point in the spacetime region (5.228).*

Proof. The idea is to use a continuity argument and compare with the solution corresponding to $a_0 = 0$:

$$a_{\mathrm{HYM}}(t) = 0, \quad b_{\mathrm{HYM}}(t) = \frac{1}{\sqrt{t^2+1}}. \tag{5.231}$$

Note that $\mathcal{P}(0) = 2$ and $b(0) = 1$ (independently of the choice of a_0). Hence b always starts above $\mathcal{P}^{-1}$. For $a_0 = 0$, the solution b_{HYM} crosses $\mathcal{P}^{-1}$ at the time: $t = \frac{3\sqrt{2}}{2}$. For $a_0 > 0$, formulae (5.211) and (5.212) show that -at least for a very short time- to the right of $t = 0$ we have

$$(a, b) > (a_{\mathrm{HYM}}, b_{\mathrm{HYM}})\,, \tag{5.232}$$

where the inequality is understood componentwise. By proposition 5.16, this inequality persists for as long as the solutions exist. Consequently, if $a_0 > 0$, b_{a_0} can only cross $\mathcal{P}^{-1}$ strictly after $t = \frac{3\sqrt{2}}{2}$.

Consider only $|a_0| \leq 1$. By the second assertion of theorem 5.23, the maximal existence time of the resulting solutions is bounded below by a positive number T_1. Furthermore, these solutions depend continuously on a_0 (in the $C^0[0, T_1]$ norm). Composing with the local flow associated to taking initial conditions at $t = T_1$, we see that the maximal existence time is lower semicontinuous in a_0. Furthermore, we see that if a particular choice of a_0 yields a solution surviving past some time $t = T$, the mapping sending initial conditions to their associated solutions is continuous from an open neighbourhood of a_0 into $C^0[0, T]$.

Since $(a_{\mathrm{HYM}}, b_{\mathrm{HYM}})$ (associated to $a_0 = 0$) is global, initial data close to 0 lead to solutions that survive arbitrarily long. In particular, we can choose $\epsilon > 0$ to be small enough so that solutions associated to $0 < a_0 < \epsilon$ survive past $t = 4$. Furthermore -at the expense of taking ϵ to be even smaller- we can appeal to continuity to arrange that:

$$\sup_{t\in[0,4]} \left|a_{a_0}(t)\right| < \frac{1}{2}, \tag{5.233}$$

$$\sup_{t\in[0,4]} \left|b_{a_0}(t) - \frac{1}{\sqrt{t^2+1}}\right| < \frac{1}{2} \inf_{t\in[3,4]} \left|\frac{1}{\sqrt{t^2+1}} - \frac{1}{\mathcal{P}(t)}\right|. \tag{5.234}$$

Condition (5.233) implies that for any $\sqrt{3} < t \leq 4$ the point $(t, a(t))$ lies in the spacetime region (5.228). Condition (5.234) implies that for any $3 \leq t \leq 4$ we have:

$$b_{a_0}(t) < \frac{1}{\mathcal{P}(t)}.$$

By the intermediate value theorem, there exists a $0 < t_{\text{crit}} < 3$ where b_{a_0} crosses $\mathcal{P}^{-1}$. But we have seen that this time must be after $t = \frac{3\sqrt{2}}{2}$ and consequently after $t = \sqrt{3}$. Hence, the critical point at $t = t_{\text{crit}}$ occurs in the spacetime region (5.228). □

Theorem 5.30 easily follows from the preceding two propositions and the symmetry of the system (5.186)—as formulated in proposition 5.15.

Proof (of Theorem 5.30): Proposition 5.32 yields a threshold $\epsilon > 0$ such that for any $0 \leq a_0 < \epsilon$, the a component of the associated solution attains a critical point in the region (5.228). Proposition 5.31 then implies that $(a, b)_{a_0}$ is global. Finally, proposition 5.15 proves that solutions associated to $-\epsilon < a_0 \leq 0$ are global too. □

5.3.5.4.3 Finite Time Blowup for Large Initial Data We now wish to study the development of large initial data. We will obtain the following:

Theorem 5.33. *Suppose that:*

$$|a_0| > \frac{1}{2\operatorname{arctanh}\left(\frac{1}{2}\right)}.$$

Then $(a, b)_{a_0}$ blows up in finite time at most equal to:

$$t_{blowup}(a_0) \overset{def}{=} \frac{3\sqrt{2}}{2} \frac{\left(1 - \tanh^2\left(\frac{1}{2|a_0|}\right)\right)^{\frac{1}{2}}}{1 - 2\tanh\left(\frac{1}{2|a_0|}\right)}. \tag{5.235}$$

Furthermore, the blowup set:

$$\mathcal{S}_{blowup} \overset{def}{=} \left\{a_0 \in \mathbb{R} \ s.t. \ (a,b)_{a_0} \ \text{blows up in finite time}\right\} \tag{5.236}$$

is of the form:

$$\mathcal{S}_{blowup} = (-\infty, -x) \cup (x, \infty) \tag{5.237}$$

for some $0 < x \leq \frac{1}{2\,\mathrm{arctanh}\left(\frac{1}{2}\right)}$.

Our analysis relies on an a priori bound for $\frac{b}{a}$:

Proposition 5.34. *For any $a_0 \geq 0$, the solution $(a,b)_{a_0}$ satisfies the following inequality for all $0 \leq t < t_{max}(a_0)$:*

$$b(t) > \frac{t}{2\sqrt{t^2+1}}a(t). \tag{5.238}$$

Proof. Estimate (5.238) is clearly satisfied at $t = 0$. To show that it persists for as long as solutions survive, we let $t_\star > 0$ be any time such that:

$$b(t_\star) = \frac{t_\star}{2\sqrt{t_\star^2+1}}a(t_\star)$$

and we compute:

$$\frac{d}{dt}_{|t=t_\star}\left(b(t) - \frac{t}{2\sqrt{t^2+1}}a(t)\right) = \sqrt{6}\,\frac{b(t_\star)^2\,t_\star^2 + b(t_\star)^2 + 1}{t_\star\sqrt{2t_\star^2+2}\sqrt{2t_\star^2+3}} > 0. \tag{5.239}$$

□

Proposition 5.34 allows us to estimate:

$$\dot{a} = \frac{\mathcal{P}a}{t}\left(b - \frac{1}{\mathcal{P}}\right) > \frac{\mathcal{P}a}{t}\left(\frac{ta}{2\sqrt{t^2+1}} - \frac{1}{\mathcal{P}}\right). \tag{5.240}$$

Fix a positive reference time $t_0 > 0$. Estimate (5.240) implies that -past t_0- a is bounded below by the solution of the following I.V.P. of Riccati type:

$$\begin{cases} \dot{u}(t) = u(t)\left(\frac{\sqrt{3}}{\sqrt{2t^2+3}}u(t) - \frac{1}{t}\right), \\ u(t_0) = a(t_0). \end{cases} \tag{5.241}$$

Setting $x_0 \overset{\text{def}}{=} a(t_0) > 0$, equation (5.241) can be solved explicitly to give:

$$u_{t_0,x_0}(t) = \frac{t_0 x_0}{t\left(t_0 x_0 \operatorname{arctanh}\left(\frac{\sqrt{3}}{\sqrt{2t^2+3}}\right) - t_0 x_0 \operatorname{arctanh}\left(\frac{\sqrt{3}}{\sqrt{2t_0^2+3}}\right) + 1\right)}. \tag{5.242}$$

The task is now to determine conditions on $t_0 > 0, x_0 > 0$ such that the lower bound u_{t_0,x_0} blows up to $+\infty$ in finite time past t_0.

We introduce the *threshold function*:

$$\mathcal{R}\left(t\right) \overset{\text{def}}{=} \frac{1}{t \operatorname{arctanh}\left(\frac{\sqrt{3}}{\sqrt{2t^2+3}}\right)}. \tag{5.243}$$

An elementary calculation demonstrates that the denominator of (5.242) has a zero in the non-negative real line if and only if:

$$x_0 > \mathcal{R}(t_0). \tag{5.244}$$

Furthermore, when (5.244) is satisfied, this zero is unique and located at the time given by:

$$\mathcal{T}(t_0, x_0) \overset{\text{def}}{=} \frac{\sqrt{6}}{2}\left(1 - \frac{3}{2t_0^2+3}\right)^{\frac{1}{2}} \frac{\left(1 - \tanh^2\left(\frac{1}{t_0 x_0}\right)\right)^{\frac{1}{2}}}{\frac{\sqrt{3}}{\sqrt{2t_0^2+3}} - \tanh\left(\frac{1}{t_0 x_0}\right)}. \tag{5.245}$$

Consequently, the lower bound u_{t_0,x_0} will exhibit the requisite behaviour provided that:

$$x_0 > \mathcal{R}(t_0) \quad \text{and} \quad \mathcal{T}(t_0, x_0) > t_0. \tag{5.246}$$

As it turns out, the second condition is vacuous:

Proposition 5.35. *Fix $t_0 > 0$. We have:*

$$\mathcal{T}\left(t_0, \mathcal{R}(t_0)\right) = +\infty$$

and $\mathcal{T}(t_0, x_0)$ decreases monotonically to t_0 (as a function of x_0) for $x_0 > \mathcal{R}(t_0)$. In *particular, we have the following pointwise limit:*

$$\lim_{x_0 \to \infty} \mathcal{T}(t_0, x_0) = t_0.$$

Proof. The proof is an elementary explicit calculation which we omit. □

Incorporating the content of proposition 5.35 to our earlier discussion, we arrive at the following conclusion. For each time $t_0 > 0$, the value $\mathcal{R}(t_0)$ provides a threshold (justifying our terminology), such that if u solves (5.241) and satisfies

$$u(t_0) > \mathcal{R}(t_0),$$

then it blows up in finite time equal to $\mathcal{T}(t_0, u(t_0)) > t_0$. Fix $t_0 > 0$. For $u(t_0)$ close to (but above) the threshold, the blowup time can be arbitrarily large. As $u(t_0) \to \infty$, the blowup time approaches t_0 from above. Consequently, for very large initial data, the solution survives for arbitrarily short time past t_0.

Since u_{t_0,x_0} bounds a from below, we obtain:

Proposition 5.36. *Suppose that $(a, b)_{a_0}$ is a solution of the system (5.186) satisfying:*

$$a(t_0) > \mathcal{R}(t_0) \textit{ for some } t_0 > 0.$$

Then a blows up to $+\infty$ in finite time at most equal to $\mathcal{T}(t_0, a(t_0))$.

The task is then to verify that for large initial data a_0, the a-component of the solution eventually crosses the threshold $\mathcal{R}$, depicted below:

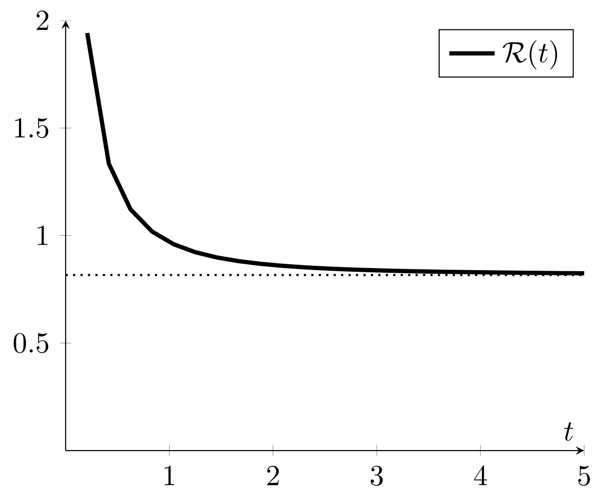


Figure 1: Graph of the threshold function $\mathcal{R}$.

We will use the reference time $t_0 = \frac{3\sqrt{2}}{2}$. This is the time where b_{HYM} -introduced in (5.231)- crosses $\mathcal{P}^{-1}$. We are able to obtain the following bound:

Proposition 5.37. *Fix $a_0 > 0$. Let $(a,b)_{a_0}$ be the development of the initial data a_0. We have:*

$$a\left(\frac{3\sqrt{2}}{2}\right) > \frac{2\sqrt{2}}{3}a_0. \tag{5.247}$$

Proof. Fix $a_0 > 0$. Formula (5.212) demonstrates that -at least for a short time-, b_{a_0} exceeds b_{HYM} to the right of $t = 0$. Proposition 5.16 establishes that $b_{a_0} > b_{\text{HYM}}$ until $t = t_{\max}(a_0)$. Incorporating this bound with the ODE governing a, we estimate (for $t > 0$):

$$\begin{aligned}\dot{a}(t) &= \frac{\mathcal{P}a}{t}\left(b - \frac{1}{\mathcal{P}}\right)\\ &> \frac{\mathcal{P}a}{t}\left(\frac{1}{\sqrt{t^2+1}} - \frac{1}{\mathcal{P}}\right)\\ &= \frac{a(t)}{t}\left(\frac{2\sqrt{3}}{\sqrt{2t^2+3}} - 1\right).\end{aligned} \tag{5.248}$$

Motivated by this computation, we introduce the following (singular) I.V.P:

$$\begin{cases} \dot{v}(t) = \frac{v(t)}{t}\left(\frac{2\sqrt{3}}{\sqrt{2t^2+3}} - 1\right), \\ v(0) = 0, \\ \dot{v}(0) = a_0. \end{cases}$$

The problem is well-posed (solutions exist and are uniquely determined by the prescribed initial data) and v takes the form:

$$v_{a_0}(t) = \frac{36\, a_0\, t}{\left(3 + \sqrt{6t^2+9}\right)^2}. \tag{5.249}$$

The function v_{a_0} has a global maximum at time $t = t_0 = \frac{3\sqrt{2}}{2}$ with value $\frac{2\sqrt{2}}{3}a_0$.
If the I.V.P. (5.249) were not singular, we could invoke estimate (5.248) to conclude that v_{a_0} bounds a_{a_0} from below for as long as the latter survives. By virtue of the above remarks, this would complete the proof. However, the standard ODE comparison argument (for nonsingular I.V.Ps) does not apply directly.

We sketch this well-known argument to identify where things go wrong. Suppose that we have two initial value problems on the real line associated to two smooth (potentially non-autonomous) vector fields F_1 and F_2 such that:

$$F_1(t, x) > F_2(t, x) \quad \text{for all } t \geq 0, \; x \in \mathbb{R}. \tag{5.250}$$

Let $s_1(t)$ and $s_2(t)$ be solutions satisfying the same initial condition:

$$x_0 = s_1(0) = s_2(0). \tag{5.251}$$

Since $F_1(0) > F_2(0)$ there is an open interval to the right of $t = 0$ where $s_1 > s_2$. Suppose, for a contradiction that this inequality does not persist for all time. At the first future crossing the difference $s_1 - s_2$ is increasing, which yields a contradiction.

Attempting to run this argument for I.V.P's with a first order singularity at $t = 0$, we observe that the initial derivatives of s_1 and s_2 match. We thus run into trouble getting s_1 to exceed s_2 in the immediate future of the initial time. The rest of the argument carries through successfully.

Fortunately, the situation can be mended by comparing higher order data at the pole. The function v_{a_0} has the following formal Taylor series at $t = 0$:

$$v_{a_0} = a_0 t - \frac{a_0}{3} t^3 + \frac{5a_0}{36} t^5 + O(t^7). \tag{5.252}$$

Comparing this with (5.211), and recalling that $a_0 > 0$ we conclude that a_{a_0} exceeds v_{a_0} in the immediate future of the initial time. The rest of the standard comparison argument carries through, ultimately establishing the requisite estimate. □

The upshot is that by choosing a_0 to be sufficiently large, we can arrange that $a(t_0)$ exceeds any number we like. In particular, we can arrange that $a(t_0)$ exceeds the threshold $\mathcal{R}(t_0)$.

We now have enough to complete the proof of theorem 5.33.

Proof (of Theorem 5.33): Evaluating (5.243) and (5.245) at the reference time $t_0 = \frac{3\sqrt{2}}{2}$ we obtain:

$$\mathcal{R}\left(t_0\right) = \frac{\sqrt{2}}{3\,\mathrm{arctanh}\left(\frac{1}{2}\right)}, \quad \mathcal{T}\left(t_0, x\right) = \frac{3\sqrt{2}}{2}\frac{\left(1-\tanh^2\left(\frac{\sqrt{2}}{3x}\right)\right)^{\frac{1}{2}}}{1-2\tanh\left(\frac{\sqrt{2}}{3x}\right)}. \tag{5.253}$$

Let (a, b) be a solution of the system (5.186) satisfying:

$$a\left(t_0\right) > \frac{\sqrt{2}}{3\,\mathrm{arctanh}\left(\frac{1}{2}\right)}. \tag{5.254}$$

Using (5.253) and proposition 5.36, we conclude that the solution blows up to $+\infty$ in finite time at most equal to $\mathcal{T}\left(t_0, a\left(t_0\right)\right)$. Proposition 5.37 guarantees that (5.254) is satisfied provided that we take:

$$a_0 > \frac{1}{2\,\mathrm{arctanh}\left(\frac{1}{2}\right)}. \tag{5.255}$$

By proposition 5.35, when $x > \mathcal{R}(t_0)$ we have that the function $\mathcal{T}(t_0, x)$ is monotonic in x. Condition (5.255) guarantees that the right hand side of (5.247) exceeds $\mathcal{R}\left(t_0\right)$ and is thus large enough for the monotonicity statement to apply. We obtain:

$$\mathcal{T}\left(t_0, a\left(t_0\right)\right) < \mathcal{T}\left(t_0, \frac{2\sqrt{2}}{3}a_0\right) = \frac{3\sqrt{2}}{2}\frac{\left(1-\tanh^2\left(\frac{1}{2|a_0|}\right)\right)^{\frac{1}{2}}}{1-2\tanh\left(\frac{1}{2|a_0|}\right)}.$$

Defining $t_{\mathrm{blowup}}(a_0)$ to be equal to the right hand side of this inequality, we have established the first assertion of theorem 5.33.

Define the positive and negative blowup sets as:

$$\mathcal{S}^{+}_{\mathrm{blowup}} \stackrel{\mathrm{def}}{=} \left\{a_0 \in \mathbb{R} \text{ s.t. } a_{a_0} \text{ blows up to } +\infty \text{ in finite time}\right\},$$
$$\mathcal{S}^{-}_{\mathrm{blowup}} \stackrel{\mathrm{def}}{=} \left\{a_0 \in \mathbb{R} \text{ s.t. } a_{a_0} \text{ blows up to } -\infty \text{ in finite time}\right\}.$$

Proposition 5.19 implies that:

$$\mathcal{S}_{\mathrm{blowup}} = \mathcal{S}^{+}_{\mathrm{blowup}} \cup \mathcal{S}^{-}_{\mathrm{blowup}}.$$

The last assertion of theorem 5.33 will follow from proposition 5.15 if we establish the existence of $x > 0$ such that:

$$\mathcal{S}^{+}_{\text{blowup}} = (x, \infty). \tag{5.256}$$

We first prove that the positive blowup set is open. Let $a_0 \in \mathcal{S}^{+}_{\text{blowup}}$ and let $t_\star$ be the blowup time of the associated solution. By definition:

$$\lim_{t \to t_\star} a(t) = +\infty.$$

Consequently, there is a time $T \in [\frac{t_\star}{2}, t_\star)$ such that:

$$a\,(T) > 2 \sup_{t \in [\frac{t_\star}{2}, t_\star]} \mathcal{R}(t).$$

By continuity with respect to variation of the initial data, we obtain a $\delta > 0$ such that for $\widetilde{a_0} \in (a_0 - \delta, a_0 + \delta)$:

$$\frac{a(T)}{2} < \widetilde{a}(T) < \frac{3}{2} a(T).$$

Consequently:

$$\widetilde{a}(T) > \sup_{t \in [\frac{t_\star}{2}, t_\star]} \mathcal{R}(t) \geq \mathcal{R}(T). \tag{5.257}$$

By proposition 5.35, the initial data $\widetilde{a_0}$ lead to finite-time blowup and $\mathcal{S}^{+}_{\text{blowup}}$ is indeed open.

Finally, by proposition 5.16, if a certain choice of $a_0 > 0$ leads to finite-time blowup, so do all $\widetilde{a_0} > a_0$. Together with openness, this property yields (5.256) for some $x \geq 0$. Theorem 5.30 implies that $x > 0$. □

5.3.5.5 The Moduli Space

The results of the preceding sections are sufficient to obtain a complete description of the moduli space of SO(5) invariant Spin(7) instantons with structure group SO(3) on the Stenzel manifold. We denote this object as $\mathcal{M}^{\text{Spin}(7)}_{\text{inv}}\left(X^8\right)$. The trivial bundle P_1 doesn't contribute to this moduli space. This is due to the nonexistence theorem 5.11. Here we ignore the trivial solution $A = 0$.

Let P be a G-homogeneous (or cohomogeneity one) principal S-bundle. There are two natural ways to set up a moduli space of G-invariant solutions to a gauge-theoretic problem on P. One is to quotient the set of invariant solutions by the group of equivariant gauge transformations. The other is carried out in two steps. Initially one quotients the set of all (not necessarily invariant) solutions by the set of all (not necessarily equivariant) gauge transformations. The action of G on the total space P induces an action on the set of all connections. This action restricts to the set of solutions and passes to the quotient. The moduli space is then defined to be the G-invariant locus. There is an obvious map from the first construction to the second construction. If the fiber S is semisimple and we restrict attention to irreducible connections, this map is an isomorphism (Oliveira [54] Corollary 4.5).

In our setting, the structure group is SO(3) (which is indeed semisimple) and furthermore, all solutions are irreducible. It follows that the two constructions coincide. We will follow the first. Recall that each invariant connection constitutes its own equivariant gauge equivalence class. Consequently, the moduli spaces on the individual bundles are:

$$\mathcal{M}\left(P_{\pi_1}\right) \stackrel{\text{def}}{=} \left\{A \in \mathcal{A}_{\text{inv}}\left(P_{\pi_1}\right) \text{ s.t. } \star_g F_A = -\Phi \wedge F_A\right\},$$
$$\mathcal{M}\left(P_{\pi_2}\right) \stackrel{\text{def}}{=} \left\{A \in \mathcal{A}_{\text{inv}}\left(P_{\pi_2}\right) \text{ s.t. } \star_g F_A = -\Phi \wedge F_A\right\}.$$

Due to the results of section 5.3.5.4, we have that $\mathcal{M}\left(P_{\pi_1}\right)$ is a compact interval. It can be parameterized by initial conditions $a_0 = \dot{a}(0)$ leading to global solutions. Using this parameterization, theorem 5.33 gives us a number $x > 0$ such that:

$$\mathcal{M}\left(P_{\pi_1}\right) \cong [-x, x]. \tag{5.258}$$

The space $\mathcal{M}\left(P_{\pi_1}\right)$ contains a unique HYM connection $A_{\text{HYM}_{\pi_1}}$ corresponding to $a_0 = 0$. It is represented by the red dot in the following diagram. The black dots represent the boundary points $\pm x$.

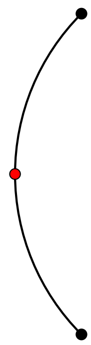


Figure 2: The Moduli Space $\mathcal{M}\left(P_{\pi_1}\right)$

Due to the results of section 5.3.5.3, we have that $\mathcal{M}\left(P_{\pi_2}\right)$ is a half-open half-closed interval. We can parameterize it by the value $\nu = b(t_0)$ at time $t_0 = \frac{\sqrt{6}}{2}$. Using this parameterization and setting:

$$\nu_\partial \stackrel{\text{def}}{=} -\frac{2\sqrt{5}}{5}, \quad \nu_* \stackrel{\text{def}}{=} \frac{\sqrt{10}}{5},$$

we have that:

$$\mathcal{M}\left(P_{\pi_2}\right) = \left\{A_\nu \text{ s.t. } \nu \in [\nu_\partial, \nu_*)\right\} \cong [\nu_\partial, \nu_*). \tag{5.259}$$

The space $\mathcal{M}\left(P_{\pi_2}\right)$ contains a unique HYM connection $A_{\mathrm{HYM}_{\pi_2}}$ corresponding to $\nu = -\frac{\sqrt{10}}{5}$. It is represented by the green dot in the following diagram. The black dot represents the boundary point ν_∂.

Figure 3: The Moduli Space $\mathcal{M}\left(P_{\pi_2}\right)$

We observe that $\mathcal{M}\left(P_{\pi_2}\right)$ is not compact. Interestingly, it admits a natural compactification. To understand the noncompactness phenomenon, we study the (missing) limit $\nu \to \frac{\sqrt{10}}{5}$. To identify what the limit should be we work on $X^8 \setminus S^4$. Using the explicit formula (5.194) with $\nu = \frac{\sqrt{10}}{5}$ yields the HYM connection $A_{\mathrm{HYM}_{\pi_1}}$. We conclude that (over $X^8 \setminus S^4$):

$$\lim_{\nu \to \nu_*} A_\nu = A_{\mathrm{HYM}_{\pi_1}}. \tag{5.260}$$

Formally, this limit should be understood in the C^∞ topology on compact sets not intersecting the (Cayley) singular orbit S^4.

We conclude that the Spin(7) instantons in $\mathcal{M}\left(P_{\pi_2}\right)$ are trying to converge to the (unique)

HYM connection of $\mathcal{M}\left(P_{\pi_1}\right)$, but fail to do so as this connection does not smoothly extend to the bundle on which they live. Notably the singularity happens around a codimension 4 Cayley submanifold (in fact a special Lagrangian). This reasoning motivates us to glue in $\mathcal{M}\left(P_{\pi_1}\right)$, by forcing the point $a_0 = 0$ to be the missing endpoint of $\mathcal{M}\left(P_{\pi_2}\right)$. This leads to the following picture of the moduli space. Crucially, it is compact.

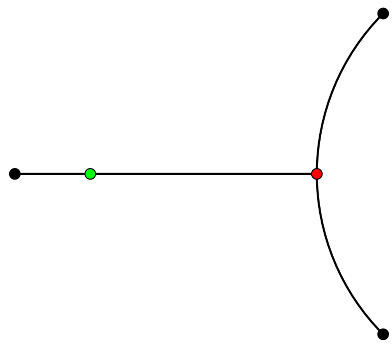


Figure 4: The Moduli Space $\mathcal{M}_{\text{inv}}^{\text{Spin}(7)}\left(X^8\right)$

This suggests a potential relationship between Spin(7) instantons and HYM connections. Indeed, they are not equivalent in general; but furthermore, the structure of $\mathcal{M}_{\text{inv}}^{\text{Spin}(7)}\left(X^8\right)$ hints that the latter might play a role in the compactification of Spin(7) instanton moduli spaces. In particular, the HYM connections might show up after resolving removable singularities developed by families of Spin(7) instantons through energy concentration (bubbling) near special Lagrangians.

5.3.5.6 Bubbling: Energy Conservation and Fueter Section We now perform a detailed analysis of the removable singularity forming as the Spin(7) instantons $(A_\nu)_{\nu_1 \leq \nu < \nu_2}$ (living on P_{π_2}) approach the HYM connection $A_{\text{HYM}_{\pi_1}}$ (living on P_{π_1}). We begin by providing a brief summary of the situation. We then proceed to formulate and prove the associated results.

As the parameter ν approaches its limiting value ν_2, the Yang-Mills energy density splits in two parts: terms approaching the energy density of $A_{\text{HYM}_{\pi_1}}$ (uniformly in all derivatives over compact sets) and terms concentrating near the singular orbit. The latter form a *bubble*: they converge to 0 uniformly over compact sets not meeting the Cayley, their L^∞ norm blows up, and while their L^2 norm is identically equal to $+\infty$, if we only integrate up to a fixed positive radius T it converges to a finite positive number independent of T. This is the energy trapped in the bubble.

Seeking a higher resolution description of the singularity formation, we zoom in near the Cayley. We find that -in the limit- ASD instantons develop along the normal directions. These instantons weave together to give a section of a fiber bundle with standard fiber given by the framed moduli space of charge-one instantons on $\mathbb{R}^4$ (Cork [11]). The latter has real dimension 8 (it is given by the trivial S^3-bundle over the closed 5-ball B^5) and carries a natural hyperkähler metric. This structure is responsible for the availability of a natural equation known as the *Fueter equation* (Walpuski [80]). The aforementioned section satisfies it: it is an example of a *Fueter section* [80], [79], [81].

This explains the energy concentration phenomenon: the difference $A_\nu - A_{\mathrm{HYM}_{\pi_1}}$ doesn't converge to 0, but rather a nontrivial limiting object living over the 4-dimensional compact Cayley. Correspondingly, the difference of the associated curvature densities doesn't converge to 0 globally but only over compact sets not meeting the Cayley. The Fueter section obtained is -in an appropriate sense- constant and equal to the instanton of charge 1 centered at 0 with scale determined by the chosen rescaling. The energy trapped in the bubble is then equal to the energy of this instanton (i.e. $32\pi^2$).

By taking the limit to be $A_{\mathrm{HYM}_{\pi_1}}$, we are effectively discarding the relevant Fueter section and thereby dispensing with the energy trapped in the bubble. When the curvature tensors of the degenerating family lie in L^2, Tian's compactness theory (Tian [70]) guarantees that the total curvature loss matches the drop in the first Pontryagin class. This doesn't quite make sense in our setting. Nevertheless, Tian's energy identity can be salvaged provided that it is reinterpreted appropriately. Lotay and Oliveira [43] produce an explicit Fueter section developing as Clarke's G_2 instantons [8] concentrate near the unique compact associative in the Bryant-Salamon space [7]. Clarke and Oliveira produce a similar example in the Spin(7) setting [9]. In both of these cases, the curvature densities are not integrable. The authors get around this issue by noticing that their difference *is*. In particular:

$$\int_M \Big| |F_{A_\nu}|^2 - |F_{A_{\mathrm{lim}}}|^2 \Big| \, dV_g < \infty. \tag{5.261}$$

They then renormalize Tian's energy identity by commuting integration and subtraction.

Our example differs in that even (5.261) fails. We resolve the issue by cutting off at a fixed positive radius $T > 0$. Energy concentration implies that the answer does not depend on the choice of $T > 0$ and Tian's identity is salvaged.

We begin by establishing the following:

Theorem 5.38. *Let $(A_\nu)_{\nu_1 \leq \nu < \nu_2}$ be the explicit family of Spin(7) instantons on P_{π_2} from theorem 5.21. As $\nu \to \nu_2$, the following hold:*

- $|F_{A_\nu}|^2 \to |F_{A_{HYM_{\pi_1}}}|^2$ *uniformly in all derivatives over compact sets not meeting S^4.*
- $|F_{A_\nu}|^2 \to |F_{A_{HYM_{\pi_1}}}|^2 + 32\pi^2 \delta_{S^4}$ *in the sense of distributions. i.e. for any compactly supported smooth function $\phi \in C_c^\infty\left(X^8\right)$:*

$$\lim_{\nu \to \nu_2} \int_{X^8} \phi \, |F_{A_\nu}|^2 dV_g = \int_{X^8} \phi \, |F_{A_{HYM_{\pi_1}}}|^2 dV_g + 32\pi^2 \int_{S^4} \phi \, dV_{g_{|_{S^4}}}.$$

Note here that the factor $32\pi^2$ in theorem 5.38 matches the Yang-Mills energy of the standard BPST instanton (Belavin, Polyakov, Schwartz, Tyupkin [1]) over $\mathbb{R}^4$ (this is four times the usual $8\pi^2$ since we are using conventions compatible with Pontryagin rather than Chern classes).

Proof. Using (5.105), we obtain:

$$F_A = F_A^i \otimes e_i,$$

where:

$$\begin{aligned}
F_A^1 &= \theta^{23} + \left(1 - b_\nu(t)^2\right)\theta^{56} + b_\nu(t)\theta^{14} + \frac{db_\nu}{dt}\frac{dt}{dr} dr \wedge \theta^7 \\
F_A^2 &= \theta^{24} + \left(1 - b_\nu(t)^2\right)\theta^{57} - b_\nu(t)\theta^{13} - \frac{db_\nu}{dt}\frac{dt}{dr} dr \wedge \theta^6 \\
F_A^3 &= \theta^{34} + \left(1 - b_\nu(t)^2\right)\theta^{67} + b_\nu(t)\theta^{12} + \frac{db_\nu}{dt}\frac{dt}{dr} dr \wedge \theta^5
\end{aligned}$$

Using (5.47), we find that all three components have equal pointwise norm. Using (5.156), we obtain:

$$|F_{A_\nu}|^2 = 6|F_A^1|^2$$

Incorporating (5.125), (5.61), (5.62), (5.63)and (5.64), we obtain:

$$|F_{A_\nu}|^2 = \frac{6\sqrt{3}(t^2+1)}{t^4\sqrt{2t^2+3}}\left(1-b_\nu^2(t)\right)^2 + 2\sqrt{3}\frac{\sqrt{2t^2+3}}{(t^2+1)^2}b_\nu^2(t) + 2\sqrt{3}\frac{\sqrt{2t^2+3}}{t^2}\left(\partial_t b_\nu(t)\right)^2$$

Using (5.194), we write:

$$b_\nu(t) = \frac{U_\nu(t)}{\sqrt{t^2+1}},$$

where:

$$U_\nu(t) \stackrel{\text{def}}{=} \frac{\sqrt{2}}{2}\left(1 + \frac{\sqrt{6}-\nu\sqrt{10t^2+15}}{\sqrt{30}\nu+\sqrt{6}-\left(\sqrt{5}\nu+2\right)\sqrt{2t^2+3}}\right).$$

Using the ODE (5.193) to simplify the derivative term, and noting that $\nu = \nu_2 = \frac{\sqrt{10}}{5}$ recovers $A_{\mathrm{HYM}_{\pi_1}}$, we finally arrive at:

$$\begin{aligned}|F_{A_\nu}|^2 - |F_{A_{\mathrm{HYM}_{\pi_1}}}|^2 &= \frac{12\sqrt{3}}{\sqrt{2t^2+3}(t^2+1)}\frac{\left(1-U_\nu^2\right)^2}{t^4} + \frac{12\sqrt{3}}{\sqrt{2t^2+3}(t^2+1)}\frac{\left(1-U_\nu^2(t)\right)}{t^2}\\ &- 2\sqrt{3}\frac{\sqrt{2t^2+3}}{(t^2+1)^3}\left(1-U_\nu^2(t)\right) - \frac{6}{(t^2+1)^3}\left(1-U_\nu^2(t)\right) - \frac{12\sqrt{3}}{\sqrt{2t^2+3}(t^2+1)^2}U_\nu(t)\frac{\left(1-U_\nu^2(t)\right)}{t^2}.\end{aligned}$$

Evidently, we are interested in studying the behaviour of the functions $1-U_\nu^2(t)$ as $\nu \to \nu_2$. We compute:

$$1-U_\nu^2(t) = \frac{(\sqrt{2}+\sqrt{5}\nu)}{2\left(\sqrt{5}\nu+2\right)^2}\left(\sqrt{2t^2+3}+\frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^2 \frac{\left(\sqrt{2}-\sqrt{5}\nu\right)t^2}{\left(t^2+\frac{3-\left(\frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^2}{2}\right)^2}. \quad (5.262)$$

It is the immediate that $U_\nu \to 1$ uniformly in all derivatives over compact sets not including $t=0$ (the corresponding level set being S^4). This establishes the first claim.

For the second claim, fix $T>0$ and test against the indicator function of the set defined by $0 \le t \le T$. The volume form of the Stenzel metric is given by:

$$\begin{aligned} dV_g &= \frac{r}{2}R_+^2R_-^2\theta^{1234567}\wedge dr\\ &= t^3(t^2+1)\theta^{1234567}\wedge dt.\end{aligned}$$

Weighing it by the difference of the curvature densities we obtain:

$$\left(|F_{A_\nu}|^2 - |F_{A_{\mathrm{HYM}_{\pi_1}}}|^2\right) dV_g = I_\nu(t)\theta^{1234567} \wedge dt,$$

where:

$$\begin{aligned} I_\nu(t) \stackrel{\text{def}}{=} & \frac{12\sqrt{3}}{\sqrt{2t^2+3}} \frac{\left(1-U_\nu^2\right)^2}{t} + \frac{12\sqrt{3}}{\sqrt{2t^2+3}} t\left(1-U_\nu^2(t)\right) - 2\sqrt{3}\frac{\sqrt{2t^2+3}}{(t^2+1)^2} t^3\left(1-U_\nu^2(t)\right) \\ & - \frac{6}{(t^2+1)^2} t^3\left(1-U_\nu^2(t)\right) - \frac{12\sqrt{3}\,U_\nu(t)}{\sqrt{2t^2+3}(t^2+1)} t\left(1-U_\nu^2(t)\right). \end{aligned} \tag{5.263}$$

Consequently, we have:

$$\begin{aligned} \lim_{\nu\to\nu_2} \frac{1}{\mathrm{Vol}(S^4)} \int_{X^8_{\leq T}} \left(|F_{A_\nu}|^2 - |F_{A_{\mathrm{HYM}_{\pi_1}}}|^2\right) dV_g &= \frac{\mathrm{Vol}\left(\frac{\mathrm{SO}(5)}{\mathrm{SO}(3)}\right)}{\mathrm{Vol}\left(S^4\right)} \lim_{\nu\to\nu_2} \int_0^T I_\nu(t)dt \\ &= \mathrm{Vol}\left(S^3\right) \lim_{\nu\to\nu_2} \int_0^T I_\nu(t)dt \\ &= 2\pi^2 \lim_{\nu\to\nu_2} \int_0^T I_\nu(t)dt \end{aligned} \tag{5.264}$$

Recall that $1 - U_\nu^2 \to 0$ pointwise for a.e. $t \geq 0$. The following is the heart of the matter: certain terms in (5.263) are not uniformly bounded for $0 < \nu < \nu_2$, $0 \leq t \leq T$. Intuitively, as the bubble gets squished near S^4, the L^∞ norm of these terms blows up. They may therefore yield a non-zero contribution in the limit. All other terms may be ignored as their respective contributions vanish by dominated convergence (DCT). Furthermore, since all terms are uniformly bounded away from $t = 0$, the DCT guarantees that the answer is independent of the choice of $T > 0$.

We now wish to understand which terms in $I_\nu(t)$ are uniformly bounded for $0 < \nu < \nu_2$, $0 \leq t \leq T$. We claim that this holds for $1 - U_\nu^2(t)$. Using (5.262), we write:

$$1 - U_\nu^2(t) = \frac{(\sqrt{2}+\sqrt{5}\nu)}{2\left(\sqrt{5}\nu+2\right)^2} \left(\sqrt{2t^2+3} + \frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^2 V_\nu(t), \tag{5.265}$$

where all factors other than V_ν are evidently bounded and:

$$V_\nu(t) \stackrel{\text{def}}{=} \frac{\left(\sqrt{2}-\sqrt{5}\nu\right)t^2}{\left(t^2+\frac{3-\left(\frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^2}{2}\right)^2}.$$

We claim that V_ν is uniformly bounded as well. To see this, we note that $V_\nu(t)$ is non-negative -hence uniformly bounded below- and we estimate:

$$\begin{aligned}
V_\nu(t) &= \frac{4\left(2+\sqrt{5}\nu\right)^4\left(\sqrt{2}-\sqrt{5}\nu\right)t^2}{4\left(\sqrt{5}\nu+2\right)^4t^4+12\left(\sqrt{5}\nu+2\right)^2\left(\sqrt{5}\nu+\sqrt{2}\right)\left(\sqrt{5}\nu-\sqrt{2}\right)t^2+9\left(\sqrt{2}+\sqrt{5}\nu\right)^2\left(\sqrt{2}-\sqrt{5}\nu\right)^2} \\
&\leq \frac{4\left(2+\sqrt{5}\nu\right)^4\left(\sqrt{2}-\sqrt{5}\nu\right)t^2}{12\left(\sqrt{5}\nu+2\right)^2\left(\sqrt{5}\nu+\sqrt{2}\right)\left(\sqrt{5}\nu-\sqrt{2}\right)t^2} = \frac{\left(\sqrt{5}\nu+2\right)^2}{3\left(\sqrt{2}+\sqrt{5}\nu\right)}.
\end{aligned}$$

The final expression is evidently bounded above. This establishes the claim.

Applying the reasoning outlined above and using (5.263) and (5.264) we find that:

$$\begin{aligned}
&\lim_{\nu\to\nu_2}\frac{1}{\mathrm{Vol}(S^4)}\int_{X^8_{\leq T}}\left(|F_{A_\nu}|^2-|F_{A_{\mathrm{HYM}_{\pi_1}}}|^2\right)dV_g \\
&= 2\pi^2\lim_{\nu\to\nu_2}\int_0^T\frac{12\sqrt{3}}{\sqrt{2t^2+3}}\frac{\left(1-U_\nu^2\right)^2}{t}dt \\
&= 2\pi^2\lim_{\nu\to\nu_2}\int_0^T\frac{12\sqrt{3}(\sqrt{2}+\sqrt{5}\nu)^2}{4\left(\sqrt{5}\nu+2\right)^4\sqrt{2t^2+3}}\left(\sqrt{2t^2+3}+\frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^4\frac{V_\nu^2(t)}{t}dt \\
&= \frac{48\pi^2}{\left(2+\sqrt{2}\right)^4}\lim_{\nu\to\nu_2}\int_0^T\eta_\nu(t)\,\frac{V_\nu^2(t)}{t}dt, \qquad (5.266)
\end{aligned}$$

where,

$$\eta_\nu(t) \stackrel{\text{def}}{=} \frac{\sqrt{3}}{\sqrt{2t^2+3}}\left(\sqrt{2t^2+3}+\frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^4.$$

The functions $\eta_\nu(t)$ are all C^∞ with ν-independent bounds on their derivatives at 0. Hence, for t smaller than a ν-independent threshold $T^\star>0$, Taylor's theorem gives:

$$\eta_\nu(t) = \left(\sqrt{3}+\frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^4+O(t). \qquad (5.267)$$

Here, $O(t)$ denotes a function bounded above by Ct, where $C > 0$ is a constant independent of ν.

Recall that (5.266) does not depend on T, and we are thus justified to assume that $T < T^\star$ so that the bound (5.267) holds. Otherwise we integrate up to $T^\star$ instead. Incorporating this bound in (5.266) we obtain:

$$\begin{aligned}
&\lim_{\nu\to\nu_2} \frac{1}{\mathrm{Vol}(S^4)} \int_{X^8_{\leq T}} \left(|F_{A_\nu}|^2 - |F_{A_{\mathrm{HYM}_{\pi_1}}}|^2 \right) dV_g \\
&= \frac{48\pi^2}{\left(2+\sqrt{2}\right)^4} \lim_{\nu\to\nu_2} \int_0^T \left(\sqrt{3} + \frac{\sqrt{30}\nu + \sqrt{6}}{\sqrt{5}\nu + 2} \right)^4 \frac{V_\nu^2(t)}{t} dt + \frac{48\pi^2}{\left(2+\sqrt{2}\right)^4} \lim_{\nu\to\nu_2} \int_0^T O(t) \frac{V_\nu^2(t)}{t} dt \\
&= \frac{48\pi^2}{\left(2+\sqrt{2}\right)^4} \lim_{\nu\to\nu_2} \int_0^T \left(\sqrt{3} + \frac{\sqrt{30}\nu + \sqrt{6}}{\sqrt{5}\nu + 2} \right)^4 \frac{V_\nu^2(t)}{t} dt \\
&= \frac{6912\pi^2}{\left(2+\sqrt{2}\right)^4} \lim_{\nu\to\nu_2} \int_0^T \frac{V_\nu^2(t)}{t} dt
\end{aligned} \tag{5.268}$$

In the third line we have dropped the second summand as its integrand is uniformly bounded. We are finally left with the task of computing the integral of $t^{-1}V_\nu^2(t)$. Note that these functions may not be uniformly bounded. As we shall see, this is indeed the case: they yield a non-zero contribution in the limit.

Integration by parts reveals that:

$$\int_0^\epsilon \frac{t^3}{\left(t^2+A\right)^4} dt = \frac{3A\epsilon^4 + \epsilon^6}{12A^2\left(A+\epsilon^2\right)^3} \quad \epsilon > 0,\ A \in \mathbb{R}.$$

Consequently, we obtain:

$$\begin{aligned}
\lim_{\nu\to\nu_2}\int_0^T \frac{V_\nu^2(t)}{t}dt &= \lim_{\nu\to\nu_2}\left(\sqrt{2}-\sqrt{5}\nu\right)^2\int_0^T \frac{t^3}{\left(t^2+\frac{3-\left(\frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^2}{2}\right)^4}dt \\
&= \lim_{\nu\to\nu_2}\left(\sqrt{2}-\sqrt{5}\nu\right)^2\frac{\frac{3}{2}\left(3-\left(\frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^2\right)T^4+T^6}{12\left(\frac{3-\left(\frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^2}{2}\right)^2\left(\frac{3-\left(\frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^2}{2}+T^2\right)^3} \\
&= \lim_{\nu\to\nu_2}\frac{4\left(\sqrt{5}\nu+2\right)^8\left(\left(10\nu^2+8\sqrt{5}\nu+8\right)T^2+18-45\nu^2\right)T^4}{27\left(\left(10\nu^2+8\sqrt{5}\nu+8\right)T^2+6-15\nu^2\right)^3\left(\sqrt{2}+\sqrt{5}\nu\right)^2} \\
&= \frac{\left(2+\sqrt{2}\right)^4}{216}.
\end{aligned}$$

We note here that the limit is independent of the choice of cutoff $T > 0$, as expected. Incorporating this into (5.268), we finally obtain:

$$\lim_{\nu\to\nu_2}\frac{1}{\mathrm{Vol}(S^4)}\int_{X^8_{\leq T}}\left(|F_{A_\nu}|^2-|F_{A_{\mathrm{HYM}_{\pi_1}}}|^2\right)dV_g = 32\pi^2. \tag{5.269}$$

Distributional convergence follows from L^1 approximation of compactly supported smooth functions by indicators of compact sets. □

We now zoom in near the concentration locus to understand the development of the relevant Fueter section. We find that there exists a suitable rescaling rate such that when we pull A_ν back to N_pS^4 we recover the unique SO(5)-invariant, charge one, scale λ ASD instanton A_λ^{ASD} on $N_pS^4 \cong \mathbb{R}^4$.

Theorem 5.39. *Let A_ν be the explicit family of instantons on P_{π_2}. Fix a scale $\lambda > 0$ and a point p on the compact Cayley S^4. Write $\exp_p^{\mathrm{NS}^4}$ for the composition of the normal exponential map with the $\flat$ map of the round unit-radius metric and:*

$$\begin{aligned}
s_\mu :& N_pS^4 \to T_p^\star S^4 \subset X^8 \\
& x \mapsto \exp_p^{\mathrm{NS}^4}(\mu x)
\end{aligned}$$

for its μ-rescaling. There exists a monotonically decreasing sequence of positive real num-

bers $\delta_\nu(\lambda)$ satisfying $\delta_\nu(\lambda) \to 0$ as $\nu \to \infty$ and encoding the appropriate rescaling speed in the sense that:

$$\lim_{\nu \to \nu_2} s^\star_{\delta_\nu} A_{\nu_{|p}} = A^{\mathrm{ASD}}_\lambda.$$

Proof. By SO(5) invariance, it suffices to take p to be the standard reference point $p_0 \in S^4$.

The normal exponential map induces a natural identification between $N_{p_0}S^4$ and $T_{p_0}S^4$. Left composition with the isomorphism $TS^4 \cong T^\star S^4$ induced by the round unit-radius metric yields:

$$\exp^{\mathrm{NS}^4}_{p_0} : N_{p_0}S^4 \xrightarrow{\sim} T^\star_{p_0}S^4 \subset X^8. \tag{5.270}$$

Our first task is to make (5.270) explicit. The vector space $N_{p_0}S^4$ carries an inner product given by the value of the Stenzel metric at the point p_0. This allows us to identify:

$$N_{p_0}S^4 - \{0\} = (0, \infty) \times S^3, \tag{5.271}$$

where S^3 is the unit sphere and we have normalised the radial direction so that $(1, 0)$ has unit length.

Even though $T^\star_{p_0}S^4$ possesses the structure of a vector space, in this context we are viewing it as an embedded submanifold of X^8. As such, the Stenzel metric restricts to a Riemannian metric on $T^\star_{p_0}S^4$, rather than a fixed inner product. Recall that we have a ray of reference points $(p_t)_{0<t<\infty}$ embedded in $T^\star_{p_0}S^4 \subset X^8$ and orthogonal to the three-spheres arising as the orbits of the SO(4)-action. This results to an identification:

$$T^\star_{p_0}S^4 - \{0\} = (0, \infty) \times S^3. \tag{5.272}$$

Here, the first factor encodes the parameter t and the second factor corresponds to the orbit at that value of t. Recall that we have natural frames for the tangent spaces to X^8 along the reference ray $(p_t)_{0<t<\infty}$:

$$T_{p_t}X^8 = \mathrm{Span}(\partial_t, X_1, ..., X_8).$$

Forgetting X_1, X_2, X_3, X_4 yields frames for the tangent spaces to $T^\star_{p_0}S^4$ along $(p_t)_{0<t<\infty}$:

$$T_{p_t}\left(T^\star_{p_0}S^4\right) = \mathrm{Span}(\partial_t, X_5, X_6, X_7).$$

The metric takes the form:

$$g_{|p_t} = \frac{tP(t)}{2\sqrt{t^2+1}\sqrt{2t^2+1}}dt \otimes dt + \frac{tQ(t)}{\sqrt{t^2+1}}\left(\theta^5 \otimes \theta^5 + \theta^6 \otimes \theta^6 + \theta^7 \otimes \theta^7\right).$$

By solving an ODE, one may introduce a new radial coordinate $s = s(t)$ defined by imposing:

$$g(\partial_s, \partial_s) = 1. \tag{5.273}$$

This results in a modification of the identification (5.272) by a diffeomorphism on the first factor.

Fix a non-zero $v \in N_{p_0}S^4$. Using the splitting (5.271) write this as:

$$v = \left(|v|_{g_{p_0}}, \frac{v}{|v|_{g_{p_0}}}\right). \tag{5.274}$$

Using the modified splitting (5.272), introduce the curve:

$$\begin{aligned} \gamma_v :&(0,\infty) \to T^\star_{p_0}S^4 \\ x \mapsto &\left(|v|x, \frac{v^\flat}{|v|}\right). \end{aligned} \tag{5.275}$$

A moment's thought reveals that γ_v is the unique geodesic passing through p_0 with velocity v.

Using the modification of (5.272) on the RHS and (5.271) on the LHS, we have that:

$$\exp^{NS^4}_{p_0}\left(|v|_{g_{p_0}}, \frac{v}{|v|_{g_{p_0}}}\right) = \left(|v|_{g_{p_0}}x, \frac{v^\flat}{|v|_{g_{p_0}}}\right).$$

In our choice of coordinates the normal exponential map becomes the identity function. From here on, we will not distinguish between the two vector spaces. All structures present on one vector space can be passed over to the other using $\exp^{NS^4}_{p_0}$. In particular, we obtain a reference ray in $N_{p_0}S^4$ and a linear frame along it.

A trivial computation allows us to determine the δ-rescaled exponential map:

$$s_\delta(v) = \delta v^\flat.$$

Before proceeding we make the following crucial observation. The ODE characterizing $s = s(t)$ is not easy to solve explicitly. However, one can easily establish that s and t agree to first order at $t = 0$:

$$s(t) = t - \frac{1}{120}t^5 + \frac{5}{756}t^7 + O(t^9).$$

The bubbling pheonomenon we wish to study involves blowing up near $p_0 \in S^4$. The rescaling we perform annihilates all higher order terms in the limit. It therefore suffices to work with t rather than s and the radial modification in the identification (5.271) is unnecessary.

Having set up the geometric framework for the proof, we need to determine the appropriate rescaling rate and compute the relevant limit.

Using (5.108), (5.110) and (5.265) we find that the pullback of the canonical reference connection $A^{\mathrm{can}}_{\pi_2}$ to $T_{p_0}S^4$ is flat. In fact, it is the product structure associated to the obvious SO(4)-equivariant trivialization of P_{π_2} along $T_{p_0}S^4 \cong \mathbb{R}^4$. Using it as a reference, restricting the instantons $(A_\nu)_{\nu_1 \le \nu < \nu_2}$ along $T_{p_0}S^4$ and zooming-in near S^4 (pulling back by s_δ), we obtain:

$$s_\delta^\star A_{\nu_{|p_t}} = \left(\frac{U_\nu(\delta t)}{\sqrt{\delta^2 t^2 + 1}} + 1\right)\left(\theta^5 \otimes e_3 - \theta^6 \otimes e_2 + \theta^7 \otimes e_1\right). \tag{5.276}$$

By (5.262) we have:

$$U_\nu^2(\delta t) = 1 - \frac{(\sqrt{2}+\sqrt{5}\nu)}{2\left(\sqrt{5}\nu+2\right)^2}\left(\sqrt{2\delta^2t^2+3} + \frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^2 \frac{\left(\sqrt{2}-\sqrt{5}\nu\right)\delta^2t^2}{\left(\delta^2t^2 + \frac{3-\left(\frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^2}{2}\right)^2}$$

$$= 1 - \left(\sqrt{2\delta^2t^2+3} + \frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^2 \frac{2C(\nu)\frac{(\sqrt{2}-\sqrt{5}\nu)}{\delta^2}t^2}{\left(2t^2 + 3C(\nu)\frac{(\sqrt{2}-\sqrt{5}\nu)}{\delta^2}\right)^2}$$

where:

$$C\left(\nu\right) \overset{\text{def}}{=} \frac{\left(\sqrt{2}+\sqrt{5}\nu\right)}{\left(\sqrt{5}\nu+2\right)}.$$

Setting:

$$\delta_{\nu}(\lambda) \overset{\text{def}}{=} \frac{\sqrt{2}}{2}\left(3C\left(\nu\right)\left(\sqrt{2}-\sqrt{5}\nu\right)\lambda\right)^{\frac{1}{2}},$$

we find that:

$$U_{\nu}^{2}(\delta_{\nu}t) = 1 - \left(\sqrt{2\delta_{\nu}^{2}t^{2}+3} + \frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^{2}\frac{\frac{4}{3\lambda}t^{2}}{\left(2t^{2}+\frac{2}{\lambda}\right)^{2}}$$

As $\nu \to \nu_2$, we have:

$$\begin{aligned}\lim_{\nu\to\nu_2} U_{\nu}^{2}(\delta_{\nu}t) &= 1 - \lim_{\nu\to\nu_2}\left(\sqrt{2\delta_{\nu}^{2}t^{2}+3} + \frac{\sqrt{30}\nu+\sqrt{6}}{\sqrt{5}\nu+2}\right)^{2}\frac{\frac{4}{3\lambda}t^{2}}{\left(2t^{2}+\frac{2}{\lambda}\right)^{2}}\\ &= \frac{\left(t^{2}-\frac{1}{\lambda}\right)^{2}}{\left(t^{2}+\frac{1}{\lambda}\right)^{2}}\end{aligned}$$

Consequently, (5.276), yields:

$$\begin{aligned}\lim_{\nu\to\nu_2} s_{\delta_{\nu}}^{\star}A_{\nu_{|p_t}} &= \lim_{\nu\to\nu_2}\left(\frac{t^{2}-\frac{1}{\lambda}}{t^{2}+\frac{1}{\lambda}}\frac{1}{\sqrt{\delta_{\nu}^{2}t^{2}+1}}+1\right)\left(\theta^{5}\otimes e_{3}-\theta^{6}\otimes e_{2}+\theta^{7}\otimes e_{1}\right)\\ &= \frac{2\lambda t^{2}}{\lambda t^{2}+1}\left(\theta^{5}\otimes e_{3}-\theta^{6}\otimes e_{2}+\theta^{7}\otimes e_{1}\right)\end{aligned}$$

Finally, using (5.126) to translate the result from the (e_1, e_2, e_3) basis of $\mathfrak{so}(3)$ to its basis provided by the unit quaternions, we find:

$$\lim_{\nu\to\nu_2} s_{\delta_{\nu}}^{\star}A_{\nu_{|p_t}} = -\frac{\lambda t^{2}}{\lambda t^{2}+1}\left(\theta^{5}\otimes i+\theta^{6}\otimes j+\theta^{7}\otimes k\right).$$

We recognize this expression as the gauge potential of the standard BPST instanton [1] on $\mathbb{R}^4$ centered at 0 with scale $\lambda > 0$. This completes the proof. □

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