Canonical Objects in Complex Geometry and Physics

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Rapport de stage

Ce mémoire rend compte d'un stage effectué au *Mathematical Institute* de l'université d'Oxford sous la direction de Jason Lotay, pour le second semestre de l'année de M1 à l'ENS de Paris, entre mars et juillet 2024. Pendant ces quelques mois, je travaillais en autonomie, et j'exposais mes résultats et mes interrogations auprès de mon encadrant environ une fois par semaine. J'ai aussi pu profiter de mon séjour pour suivre le séminaire hebdomadaire de géométrie d'Oxford, ainsi que quelques cours de physique théorique.

Le thème initialement convenu pour ce stage était l'étude des équations de Yang-Mills hermitiennes déformées en théorie de jauge. Ce sujet m'a permis de me familiariser avec quelques outils de géometrie algébrique et d'analyse géométrique, et m'a naturellement amené à étudier le cadre plus général des équations Z-critiques. Quelques semaines furent également dédiées à l'étude du système de Hull-Strominger, ce qui m'a encouragé à apprendre les fondements de le géométrie généralisée de Hitchin. En outre, parmi les nombreuses impasses que j'ai pu visiter pendant ce stage, je suis heureux d'avoir découvert quelques rudiments sur les super-variétés, la cohomologie BRST classique, ou encore la géométrie conforme de Weyl.

Ces diverses questions ont abouti à la rédaction de notes sur différentes approches originales des équations Z-critiques [Ser24], pour lesquelles j'ai profité d'échanges avec Ruadhaí Dervan, de l'université de Glasgow. Tout au long du stage, j'ai également eu le plaisir de partager des discussions très enrichissantes avec Thibault Langlais, ainsi qu'avec les étudiants et étudiantes du groupe de physique mathématique de l'institut.

Ce texte est donc à la fois un tour d'horizon d'une partie des sujets abordés, et un exercice de création *a posteriori* d'une certaine cohérence dans un stage plutôt décousu ; les contributions originales exposées dans [Ser24] sont seulement évoquées. La difficulté principale a été de rester clair en respectant les limites imposées sur le nombre de pages – j'ai essayé d'être aussi généreux que possible dans les références bibliographiques afin de combler les raccourcis et petits mensonges du corps de l'exposé.

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Introduction

It could be argued that the introduction of Riemann surfaces was among the first major developments in modern differential geometry. At any rate, the works of Riemann certainly set the stage for the study of complex manifolds, and provided close ties to topology, complex analysis, and algebraic geometry. Among other results, the classical **uniformization theorem** can be seen as the inaugural theorem of complex geometry [Yau05]. One way to state it is to say that any Riemann surface admits a metric of constant scalar curvature, which can be taken to be equal to -1, 0, or $+1$ depending on the topology of the surface. These metrics may be considered in some sense as *canonical*, and allow us to construct moduli spaces – that is spaces parametrizing the complex structures on a given surface – by working with those natural representatives.

On higher-dimensional complex manifolds that admit a Kähler structure – see section 1.3.1, attempts to find canonical metrics have led to the introduction of **extremal Kähler metrics**. However, unlike for Riemann surfaces – i.e. one-dimensional complex manifolds – it was quickly realized that the existence of such metrics cannot be guaranteed in general. For instance, the **Futaki invariant** is a non-trivial obstruction to the existence of a class of extremal metrics known as **Kähler-Einstein metrics**. More generally, it was understood that some kind of **stability condition**, in the sense of geometric invariant theory, was necessary for a manifold to admit an extremal metric. This led to the very important – and recently established – **Yau-Tian-Donaldson conjecture**, see [Szé14]. On complex manifolds that do not admit a Kähler structure, it is hoped that solutions to the **Hull-Strominger system** (3.6) may provide suitable generalizations of canonical metrics.

A similar story can be told about connections on vector bundles over a complex manifold. The methods of geometric invariant theory that motivated the Yau-Tian-Donaldson conjecture suggest an analogous statement about **Hermitian Yang-Mills connections** (1.10), which may be seen as canonical connections on a vector bundle. This statement is the **Kobayashi-Hitchin correspondence**, of which we will say more in the following. The proof of the Kobayashi-Hitchin correspondence by Donaldson, Uhlenbeck, and Yau [Don85; UY86] was a tremendous achievement, and it is now hoped that similar results may be obtained for a more difficult system known as the **deformed Hermitian Yang-Mills equations** (2.6). Moreover, these considerations have recently motivated a generalization of the deformed Hermitian Yang-Mills equations, the so-called **Z-critical equations**, which are defined in relation to specific stability conditions that appeared naturally in algebraic geometry.

Throughout the text, we have tried to emphasize the interaction between theoretical physics and geometry: symplectic geometry was first developed as a description of classical mechanics, and symplectic reduction has also been used for the quantization of gauge theories – see [Fig06]. Furthermore, the equations that we study here have all been motivated by physical problems: the Hermitian Yang-Mills equations (1.10) are equations for instantons over complex manifolds, their deformed version (2.6) are equations of motion for the B-model of string theory, and the Hull-Strominger system (3.6) first appeared in the study of heterotic supergravity.

In section 1, we introduce the Hermitian Yang-Mills equations, and give an overview of the Kobayashi-Hitchin correspondence after a quick review of complex geometry and geometric invariant theory. Section 2 is about the deformed Hermitian Yang-Mills equations and mirror symmetry – it is somewhat shorter as it only reproduces parts of the accompanying set of notes [Ser24]. Section 3 discusses the Hull-Strominger system along with more sophisticated tools from generalized geometry.

Notations and conventions We assume some familiarity with differential geometry, and we refer to [Lee12] for a good general introduction. If M is a smooth manifold, the tangent and cotangent bundles of M are denoted by $\mathrm{T}M$ and $\mathrm{T}^\vee M$ respectively. The space of sections of $\mathrm{T}M$ – that is vector fields on M – is denoted by $\mathfrak{X}(M)$, and has a natural Lie algebra structure. We let $\mathcal{F}(M)$ denote the algebra of real smooth functions on M, and $\mathcal{A}^\bullet(M)$ its graded complex of differential forms. This means that $\mathcal{A}^k(M)$ is the space of sections of the bundle $\bigwedge^k \mathrm{T}^\vee M$. If E is a vector bundle on M, we also define spaces $\mathcal{A}^\bullet(E) = \mathcal{A}^\bullet$ of smooth E-valued forms – in particular, $\mathcal{A}^0(E)$ coincides with the space $\Gamma(E)$ of sections of E. We have an exterior derivative denoted by d on forms, and the cohomology of the complex $(\mathcal{A}^\bullet(M), \mathrm{d})$ is the familiar de Rham cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$.

We will use basic notions of Riemannian geometry – see [GHL04], and it will also be helpful to have some knowledge of the theory of connections on vector bundles – see e.g. [DK90; Tel12]. If E is a vector bundle on M, and $A \in \mathcal{A}^1(\operatorname{End} E)$ is a local potential for a connection on E, we write $\mathrm{D}_A = \mathrm{d} + A$ for the corresponding differential operator $\mathrm{D}_A : \mathcal{A}^\bullet(E) \longrightarrow \mathcal{A}^{\bullet+1}(E)$, and $\mathrm{F}_A = \mathrm{D}_A \circ \mathrm{D}_A \in \mathcal{A}^2(\operatorname{End} E)$ for its curvature. Given a connection on E, Chern-Weil theory yields representatives of characteristic classes of E – i.e. *topological* invariants of E – in terms of the curvature form of a connection. All this material is covered, with physical applications in mind, in the excellent [Nak03].

Acknowledgements I wish to thank Jason Lotay for his supervision throughout my visit, and for suggesting this subject in the first place. I also thank Nicolas Tholozan for getting me in touch with Jason back in November. I am grateful to everyone in the Mathematical Institute for a very nice welcome, and I am indebted to Thibault Langlais and Ruadhaí Dervan for very helpful and enjoyable discussions.

1 Geometric quotients and Kobayashi-Hitchin correspondence

The goal of this section is to construct quotients of geometric objects that preserve some geometric structure. We give two approaches – one from symplectic geometry, the other from algebraic geometry – and show that they are closely related. This provides a motivation for the Kobayashi-Hitchin correspondence.

1.1 Moment maps and symplectic reduction

Our first approach to geometric quotients is very differential in spirit. More specifically, it relies on symplectic geometry; the goal here is to explain the reduction theorem of Marsden and Weinstein. We refer to [Can08] for a very good introduction with a lot of further material.

1.1.1 Symplectic geometry

We begin by giving a short review of symplectic geometry, i.e. the study of symplectic manifolds.

Definition 1.1 (Symplectic manifold)**.** *Let M be a smooth manifold. A* ***symplectic structure*** *on M is the data of a closed non-degenerate two-form $\omega \in \mathcal{A}^2(M)$ on M. This means that $\mathrm{d}\omega = 0$, and that the induced natural map $\omega : \mathrm{T}M \to \mathrm{T}^\vee M$ is an isomorphism. The pair (M, ω) defines a* ***symplectic manifold****.*

In definition 1.1, the natural map $\omega : \mathrm{T}M \to \mathrm{T}^\vee M$ is given by the **interior product**, which we also denote for any $X \in \mathrm{T}M$ by:

$$\omega(X) := \iota_X\omega := \omega(X, \cdot) \in \mathrm{T}^\vee M.$$

Symplectic geometry is naturally motivated by classical mechanics, as a symplectic manifold can be considered as an adequate model for the phase space of a mechanical system – see [Arn89].

The non-degeneracy hypothesis in definition 1.1 allows us to use some intuition from Riemannian geometry, where we also have an identification between the tangent and cotangent bundles via a two-tensor, in this case the metric. There are however some crucial differences: whereas any smooth manifold M admits a Riemannian metric [GHL04, §2.2], we have immediate obstructions to the existence of a symplectic form. For example, it is easy to show that M must be even-dimensional, say $2n$. Furthermore, unlike in the Riemannian setting, where a metric can always be pulled-back to a submanifold, it is not always the case that a symplectic form restricts to a non-degenerate form. As an extreme case of this, a submanifold $N \subset M$ of dimension n is said to be **Lagrangian** if the symplectic form restricts to zero on it, that is if $\omega|_N = 0$. These manifolds will be of central importance in section 2. The interaction between symplectic and Riemannian structures will be explored further when we deal with Kähler geometry in section 1.3.1.

Upon introducing a new kind of geometric structure, it is always fruitful to ask about its symmetries. If (M, ω) is a symplectic manifold, we will say that a diffeomorphism φ of M is a **symplectomorphism** if it preserves the symplectic structure, that is if:

$$\varphi^*\omega = \omega, \tag{1.1}$$

where the star denotes the **pull-back** of a differential form via φ – see [Lee12, p. 360]. Symplectomorphisms form a subgroup $\mathrm{Symp}(M) \subset \mathrm{Diff}(M)$ of the group of diffeomorphisms of M. Given a group G, a **symplectic action** of G on M is the data of a morphism $G \to \mathrm{Symp}(M)$.

We may also express the compatibility condition (1.1) in terms of infinitesimal symmetries of M. The Lie algebra of the infinite-dimensional Lie group $\mathrm{Diff}(M)$ of diffeomorphisms of M may be identified – somewhat formally – to the algebra $\mathfrak{X}(M)$ of vector fields on M: this is essentially the correspondence between vector fields and flows on M [Lee12, Theorem 9.12]. Under this identification, an infinitesimal diffeomorphism $X \in \mathfrak{X}(M)$ is compatible with the symplectic structure if and only if:

$$\mathcal{L}_X\omega = \mathrm{d}\iota_X\omega + \iota_X\mathrm{d}\omega = \mathrm{d}\iota_X\omega = 0,$$

where we have used Cartan's magic formula [Lee12, Theorem 14.35] and the closedness of ω. A vector field $X \in \mathfrak{X}(M)$ is therefore said to be a **symplectic vector field** if the form $\iota_X\omega$ is closed. Letting $\mathfrak{S}(M) \subset \mathfrak{X}(M)$ denote the space of symplectic vector fields – which may be thought of as the Lie algebra of $\mathrm{Symp}(M)$ – an **infinitesimal symplectic action** is the data of a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{S}(M)$.

Since ω is non-degenerate, the interior product $X \mapsto \iota_X\omega$ yields an isomorphism $\mathfrak{X}(M) \xrightarrow{\sim} \mathcal{A}^1(M)$ between vector fields and one-forms. Under this identification, symplectic vector fields correspond, by definition, to *closed* one-forms. A vector field $X \in \mathfrak{X}(M)$ is said to be **Hamiltonian** if it corresponds to an *exact* one-form, i.e. if we can write $\iota_X\omega = \mathrm{d}h$ for some $h \in \mathcal{F}(M)$, that is uniquely defined up to a constant provided M is connected. Letting $\mathfrak{H}(M)$ denote the space of Hamiltonian vector fields, this discussion is summed up in a commutative diagram, which may be seen as an isomorphism of exact sequences induced by ω:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}^0_{\mathrm{closed}}(M) \arrow[r] & \mathcal{A}^0(M) \arrow[r, "\mathrm{d}"] & \mathcal{A}^1_{\mathrm{closed}}(M) \arrow[r] & \mathrm{H}^1_{\mathrm{dR}}(M) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[u, equal] & \mathcal{F}(M) \arrow[r] \arrow[u, equal] & \mathfrak{S}(M) \arrow[r] \arrow[u, hook, two heads, "\omega"] & \mathfrak{S}(M)/\mathfrak{H}(M) \arrow[r] \arrow[u, hook, two heads] & 0
\end{tikzcd} \tag{1.2}$$

1.1.2 Moment maps

Moment maps were first introduced as a generalization of the concept of linear and angular momentum in classical mechanics. Because of their central rôle in the study of symplectic reduction – see section 1.1.3 – they have since had a far-reaching influence in geometry, in particular through the work of Donaldson [Don02].

Let (M, ω) be a symplectic manifold, and suppose we are given a symplectic action of a group G on M. This means that the corresponding infinitesimal action $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ takes its values in the algebra $\mathfrak{S}(M)$ of symplectic vector fields. Assume now that the infinitesimal action is via Hamiltonian vector fields, i.e. that ρ projects to zero in the quotient $\mathfrak{S}(M)/\mathfrak{H}(M)$ – notice that this is always the case if $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$. From the bottom exact sequence in (1.2), we can then ask for ρ to be lifted to a map $\tilde{\rho}$:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r] & \mathcal{F}(M) \arrow[r] & \mathfrak{S}(M) \arrow[r] & \mathfrak{S}(M)/\mathfrak{H}(M) \arrow[r] & 0 \\ & & & \mathfrak{g} \arrow[u, "\rho"] \arrow[ul, dashed, "\tilde{\rho}"] & & \end{tikzcd}$$

More explicitly, reading through the top sequence in (1.2), the function $\tilde{\rho}$ is such that for $\xi \in \mathfrak{g}$:

$$\omega(\rho(\xi), \cdot) = \mathrm{d}(\tilde{\rho}(\xi)).$$

Now if $\langle \cdot, \cdot \rangle$ denotes the canonical bracket between $\mathfrak{g}$ and its dual $\mathfrak{g}^\vee$, the data of a lift $\tilde{\rho}$ may equivalently be presented as a function $\mu : M \to \mathfrak{g}^\vee$, defined for $x \in M$ and $\xi \in \mathfrak{g}$ by:

$$\langle \mu(x), \xi \rangle := (\tilde{\rho}(\xi))(x),$$

i.e. the component $\mu_\xi := \langle \mu, \xi \rangle$ of μ in the ξ-direction is the function $\tilde{\rho}(\xi)$. This defines the moment map.

Definition (Moment map)**.** *Let G be a Lie group with a symplectic action on M, and let $\rho : \mathfrak{g} \to \mathfrak{S}(M)$ be the induced infinitesimal action. A* ***moment map*** *is a map $\mu : M \to \mathfrak{g}^\vee$ such that for $\xi \in \mathfrak{g}$, $\iota_{\rho(\xi)}\omega = \mathrm{d}\mu_\xi$.*

Since G also acts naturally on $\mathfrak{g}^\vee$ via the coadjoint action [Can08, Section 21.5], it is natural to ask for μ to be **equivariant** with respect to the actions on M and $\mathfrak{g}^\vee$. With this extra condition, we can show that the moment map is defined uniquely up to the addition of a central constant.

Example 1.2 (Moment map on projective spaces)**.** *Recall that the complex projective space $\mathbb{P}^N$ is defined as the space of lines in the complex vector space $\mathbb{C}^{N+1}$. It admits a natural symplectic structure defined by the* ***Fubini-Study metric****, see [Voi02, pp. 77-79]. If a group G acts by isometries of $\mathbb{C}^{N+1}$, that is $\rho : G \to \mathrm{U}(N+1)$, then ρ descends to an action on $\mathbb{P}^N$ that preserves the Fubini-Study form. This action is in fact Hamiltonian with equivariant moment map:*

$$\mu_\xi(x) = \frac{\tilde{x}^{\mathrm{T}} \rho_*(\xi) \tilde{x}}{2\pi \mathrm{i} \, \|\tilde{x}\|^2},$$

for all $\xi \in \mathfrak{g}$, and any lift $\tilde{x} \in \mathbb{C}^{N+1}$ of $x \in \mathbb{P}^N$. See [Kir84] for more detail.

1.1.3 Symplectic reduction

Our problem is simple: given a symplectic manifold M with a symplectic action of a Lie group G, can we construct a quotient with a natural symplectic structure? This naive question is not as simple as it sounds; for example, we know that a symplectic manifold must be even-dimensional, so that even if the quotient M/G has a smooth structure, it is possible that it does not admit a symplectic form. This problem naturally leads us to the theory of symplectic reduction. For more detail, we refer again to [Can08], or, for a more algebraic approach, to the excellent lecture notes [Fig06].

Suppose that the G-action is Hamiltonian with equivariant moment map μ. If G is connected, one can show easily that the zero-locus $M_0 := \mu^{-1}(0)$ of the moment map is stable under the action of G. We may therefore form the quotient $\tilde{M} = M_0/G$. The situation is now described in the following diagram:

$$\begin{tikzcd} & M_0 \arrow[dl, hook', "i"'] \arrow[dr, two heads, "\pi"] & \\ M & & \tilde{M} \end{tikzcd} \tag{1.3}$$

This is the setting first proposed by Marsden and Weinstein [MW74] to form quotients of symplectic manifolds.

Theorem 1.3 (Symplectic reduction)**.** *Let (M, ω) be a compact symplectic manifold, and consider an Hamiltonian action of a connected Lie group G on M with equivariant moment map μ for which $0 \in \mathfrak{g}^\vee$ is a regular value, so that M_0 is a submanifold of M. Suppose the induced action of G on M_0 is free and proper, then the* ***symplectic reduction*** *of M, defined as $M /\!\!/ G := \tilde{M}$, is a smooth manifold which admits a canonical symplectic form $\tilde{\omega}$. Referring to the notations of the diagram (1.3), this symplectic form satisfies $i^*\omega = \pi^*\tilde{\omega}$.*

From a physical point of view, if (M, ω) is thought of as the phase space of some mechanical system, then the symplectic reduction $(\tilde{M}, \tilde{\omega})$ is the right framework to describe this system without the redundancy introduced by the symmetry.

1.2 Geometric invariant theory

In this section, we give a rough introduction to geometric invariant theory, and refer to [Tho06] or [Dol03] for more detail. See also the first sections of the thesis [McC23] for an introduction with an eye towards Z-critical connections. Even though we will mostly focus on conceptual aspects of the theory, and use the material of this section merely as formal guidelines, we point out that a rigorous approach to infinite-dimensional problems in geometric invariant theory was recently proposed in [DFR24], and should apply to our later discussion of the Kobayashi-Hitchin correspondence and deformed Hermitian Yang-Mills equations.

1.2.1 Quotients in algebraic geometry

The basic idea of algebraic geometry is that *geometric* spaces should be studied though the *algebraic* properties of their spaces of functions. This simple mantra has had a long and successful history, that culminated with the invention of schemes – see e.g. [Har77]. On a less exalted level, classical algebraic geometry yields a functorial correspondence between polarized varieties and finitely generated graded rings without zero-divisors.

A **polarized variety** is the data (M, L) of a projective algebraic variety M, with an ample line bundle L. For our purposes, this means that we assume that M is embedded in a complex projective space $\mathbb{P}^N$ as the zero-locus of a finite number of polynomial equations, and that the bundle L is the pullback of the **tautological bundle** on $\mathbb{P}^N$, that is:

$$L = \{(x, u) \in M \times \mathbb{C}^{N+1} \,|\, u \in x\}.$$

This is essentially Kodaira's embedding theorem – see [Voi02, Théorème 7.11]. In that setting, M also inherits from $\mathbb{P}^N$ the Fubini-Study metric of example 1.2, making it into a symplectic manifold. The sections of the tensor powers of L yield a graded ring, the **structure ring** of M

$$\mathcal{O}_M := \bigoplus_{k\geq 0} \Gamma\left(L^{\otimes k}\right),$$

whose elements are identified to homogeneous polynomials over M. More precisely, if we let $\tilde{M} := \pi^{-1}(M)$, where π is the natural projection from $\mathbb{C}^{N+1} \setminus \{0\}$ to $\mathbb{P}^N$, then $\mathcal{O}_M$ is identified to the space of $\mathbb{C}^{\times}$-invariant algebraic functions on $\tilde{M}$. Algebraic geometry in the language of Serre gives us a correspondence between graded rings and polarized varieties. Namely, if $R = R^{\bullet}$ is a finitely generated graded ring without zero-divisors, then we may define an algebraic variety $M = \mathrm{Projm}(R)$ together with a line bundle L on M whose graded ring of sections is $\mathcal{O}_M = R$. The points of M are the homogeneous ideals of R that are maximal *among those not containing the* ***irrelevant ideal*** $R^+ := \oplus_{k\geq 1} R^k$. Geometrically, a point $x \in M$ corresponds to the ideal $\mathfrak{m}_x \subset R$ of homogeneous polynomials vanishing on the line x – the condition that $R_+ \not\subset \mathfrak{m}_x$ ensures that the zero-locus of $\mathfrak{m}_x$ is not reduced to the origin, but is indeed a line in $\tilde{M}$.

Now suppose that we have a **linearised group action** on a polarized variety (M, L), i.e. an action through bundle automorphisms of L. Another way to say this is that the group acts on M via a morphism to $\mathrm{GL}_{N+1}(\mathbb{C})$ acting on $\tilde{M}$. We also assume that the group is **reductive**, i.e. that it is the complexification $G^{\mathbb{C}}$ of a compact Lie group G – this is a technical hypothesis that will always be fulfilled in the following. Say that we want to construct the quotient of (M, L) under this action. The above discussion allows us to transform a difficult problem – defining the quotient of an algebraic variety by a group – into the much simpler task of defining what the structure ring of the quotient should be: we simply decide that the functions on the quotient correspond to those sections on M that are invariant under the $G^{\mathbb{C}}$-action. We therefore define the **geometric invariant theory (GIT) quotient** to be:

$$M /\!\!/ G^{\mathbb{C}} := \mathrm{Projm}\left(\mathcal{O}_M^{G^{\mathbb{C}}}\right),$$

where $\mathcal{O}_M^{G^{\mathbb{C}}} \subset \mathcal{O}_M$ is the subring of $G^{\mathbb{C}}$-invariant sections. The definition above makes sense since the graded ring $\mathcal{O}_M^{G^{\mathbb{C}}}$ is always finitely generated by **Nagata's theorem** – see [Dol03, Section 3.4].

Though it is very natural, considering only the sections on M that are $G^{\mathbb{C}}$-invariant is quite a drastic choice. Indeed, a point in the quotient corresponds to an ideal of the ring of invariant that is maximal *among those ideals that do not contain the irrelevant part.* This means that a point x is represented in the quotient if and only if there exists a *non-constant* invariant section s such that $s(x)$ is non-zero.

Definition (Semitability)**.** *A point $x \in M$ is* ***semistable*** *if there exists, for some $k > 0$, an invariant section $s \in \Gamma(L^{\otimes k})^{G^{\mathbb{C}}}$ such that $s(x) \neq 0$. We let M^{ss} denote the locus of semi-stable points.*

By the above discussion, the GIT quotient can be identified to an actual set-theoretic quotient:

$$M /\!\!/ G^{\mathbb{C}} = M^{\mathrm{ss}}/G^{\mathbb{C}}. \tag{1.4}$$

One may show that x is semi-stable if and only if the topological closure of the orbit $G^{\mathbb{C}} \cdot \tilde{x}$ of any non-zero lift $\tilde{x} \in \tilde{M}$ of x does not contain zero. A point is said to be **polystable** if the orbit $G^{\mathbb{C}} \cdot \tilde{x}$ is closed, which of course implies semi-stability. The GIT quotient is such that non-closed orbits are identified together, so that if we let M^{ps} denote the locus of polystable points of M, we can strengthen equation (1.4) to $M /\!\!/ G^{\mathbb{C}} = M^{\mathrm{ps}}/G^{\mathbb{C}}$.

Remark 1.4 (Choice of linearisation)**.** *Note that in the previous discussion, we assume that the group acts not only on M, but on the whole bundle L. This means that we choose a* ***linearisation*** *of the action on M. This choice hides a number of subtleties, since the GIT quotient and the notion of stability will depend on the choice of linearisation. This is discussed at length in [Tho06], see also [DH98].*

1.2.2 The Kempf-Ness theorem

Let (M, L) be a polarized projective variety with induced Fubini-Study form ω, together with a linearised projective action of a group $G^{\mathbb{C}}$ such that the compact subgroup G acts by isometries of ω. The picture to have in mind here is:

$$\begin{tikzcd}
G^{\mathbb{C}} \arrow[r] & \mathrm{GL}_{N+1}(\mathbb{C}) \arrow[r, squiggly, "\text{linearised action}"] & L \arrow[d] \\
G \arrow[u, hook] \arrow[r] & \mathrm{U}(N+1) \arrow[u, hook] \arrow[r, squiggly, "\text{symplectic action}"] & (M, \omega)
\end{tikzcd}$$

We thus have a linearised action of $G^{\mathbb{C}}$ on a polarized variety on the one hand, and a symplectic action of G on a symplectic manifold on the other hand: the two approaches of sections 1.1.3 and 1.2.1 give us two ways to construct a quotient space. The Kempf-Ness theorem relates these two constructions:

Theorem 1.5 (Kempf-Ness)**.** *Let a reductive group $G^{\mathbb{C}}$ act on a polarized variety (M, L) in such a way that the induced action of the subgroup G is Hamiltonian with equivariant moment map $\mu : M \to \mathfrak{g}^\vee$. A $G^{\mathbb{C}}$-orbit in M contains a zero of the moment map if and only if it is polystable. In that case, the zeroes of the moment map in a given $G^{\mathbb{C}}$-orbit form a G-orbit – see also figure 1. In other words, we have a set-theoretic identification between the symplectic reduction and the GIT quotient:*

$$\underbrace{M /\!\!/ G = M_0/G}_{\textit{symplectic quotient}} \simeq \underbrace{M^{\mathrm{ps}}/G^{\mathbb{C}} = M /\!\!/ G^{\mathbb{C}}}_{\textit{GIT quotient}}. \tag{1.5}$$

The basic idea of the Kempf-Ness theorem is nicely illustrated by the following picture, adapted from [Tho06]:

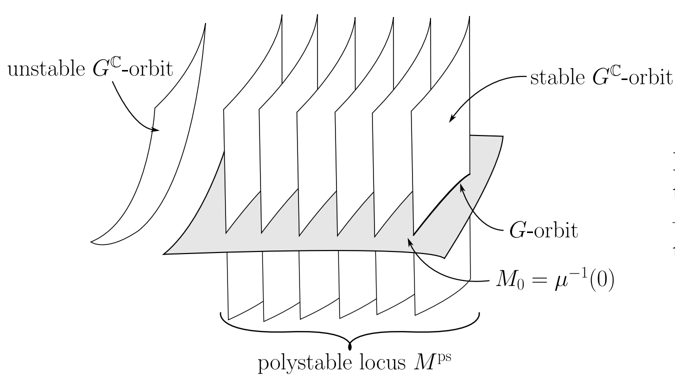

Figure 1: Pictorial representation of the Kempf-Ness theorem: the space of *polystable* $G^{\mathbb{C}}$-orbits in M – i.e. the GIT quotient – is shown to be the same as the space of G-orbits in M_0 – i.e. the symplectic quotient.

We may in fact go beyond mere equality as sets in theorem 1.5 – see [Kir84] – but one must then be mindful of the subtleties that arise when mixing algebraic and analytical structures in geometry. Another subtle point is that the symplectic reduction in the left-hand-side of equation (1.5) seemingly depends only on the action of G on M, whereas we have pointed out in remark 1.4 that the GIT quotient in the right-hand-side depends on a choice of linearisation of the action of the complexified group $G^{\mathbb{C}}$. The reason for this apparent discrepancy is that the linearisation is actually hidden in the choice of the moment map – this is explained in [Tho06].

Example 1.6 (A cute proof of the spectral theorem)**.** *Using theorem 1.5, we can give a nice proof of the fact that normal matrices, i.e. matrices that commute with their adjoint, are diagonalizable. Let $\mathrm{SL}_n(\mathbb{C})$ act on $\mathrm{M}_n(\mathbb{C})$ by conjugation – notice by the way that $\mathrm{SL}_n(\mathbb{C})$ is reductive as since it is the complexification $\mathrm{SU}(n)^{\mathbb{C}}$ of a compact Lie group. It is known that the closed orbits under this action, i.e. the closed similarity classes, correspond to diagonalizable matrices, so that a matrix is polystable if and only if it is diagonalizable.*

On the other hand, $\mathrm{M}_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$ has a natural symplectic structure ω. It is clear that the $\mathrm{SU}(n)$-action preserves ω, and it turns out that we have a moment map given by the commutator $\mu(M) = \frac{1}{2}[M, M^*] \in \mathfrak{su}_n^\vee$. The zero-locus of μ is of course the space of normal matrices.

Using the Kempf-Ness theorem, we conclude that a matrix is diagonalizable if and only if it is similar to a normal matrix, and that a normal matrix can be diagonalized in a unitary basis.

Example 1.6 serves as a nice illustration of the GIT approach to constructing quotient. Say that you want to construct a space of complex matrices up to similarity; GIT essentially tells us to forget about some pathological points, and simply take the quotient of the dense subset of *diagonalizable* matrices. The same philosophy applies when we construct moduli spaces of bundles over a variety: we only consider those bundles that are *stable*, as we explain in the next section.

1.3 The Kobayashi-Hitchin correspondence

In this section, we give a heuristic approach to the Kobayashi-Hitchin correspondence as an infinite dimensional instance of the Kempf-Ness theorem. An excellent reference on the Kobayashi-Hitchin correspondence is [LT95], and we also recommend the discussion in [Gar16]. We start by recalling some basic concepts of complex geometry – a good reference here is [Voi02].

1.3.1 A quick review of complex geometry

Complex structures A **complex manifold** of dimension n is defined as a manifold whose local coordinates are complex numbers $z^1, \cdots, z^n$ such that the transition maps are holomorphic. Another equivalent approach to complex geometry is to introduce the notion of complex structure.

Definition 1.7 (Complex structure)**.** *Let M be a smooth manifold, a **complex structure** on M is an endomorphism J of the tangent bundle $\mathrm{T}M$ such that $J^2 = -\mathrm{id}_{\mathrm{T}M}$, and which satisfies some integrability condition.*

On a complex manifold, the operator J is induced by multiplication by i in the complex charts. Conversely, given a complex structure, the integrability condition above ensures that we can construct a holomorphic atlas on M by the Newlander-Nirenberg theorem [Voi02, Théorème 2.24].

The data of a complex structure on M splits the complexified tangent bundle $\mathrm{T}M \otimes \mathbb{C}$ into $\pm$i-eigenbundles denoted by $\mathrm{T}^{1,0}M$ and $\mathrm{T}^{0,1}M$. In particular, $\mathrm{T}^{1,0}M$ is identified with the **holomorphic tangent bundle** of M, which we also denote by $\mathcal{T}_M$. The integrability condition of definition 1.7 amounts to saying that $\mathcal{T}_M$ is closed under the Lie bracket. The space of complex exterior forms $\mathcal{A}^{\bullet,\bullet}(M,\mathbb{C})$ has an induced bigrading whereby $\mathcal{A}^{p,q}(M,\mathbb{C})$ is locally spanned by terms of the form $\mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\overline{z}^{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}^{j_q}$, for indices $1 \leq i_1, \cdots, i_p, j_1, \cdots, j_q \leq n$. In particular, in a complex chart, we define a **holomorphic volume-form**:

$$\Omega := \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n, \tag{1.6}$$

which is a local holomorphic section of the **canonical bundle** $\mathcal{K}_M := \bigwedge^n \mathcal{T}_M^\vee$ of M. We may find a global non-vanishing holomorphic section of $\mathcal{K}_M$ if and only if it is trivial, which implies in particular the vanishing of the first Chern class $c_1(M)$ of M. If this section can also be chosen to be parallel with respect to the connection induced by a Kähler metric – see definition 1.9 – then M is a **Calabi-Yau manifold**. This means that the existence of a Calabi-Yau structure depends on the **holonomy** of the Levi-Civita connection of (M,ω). We refer to [GJH03; Yau09] for more detail, and a survey of various other conventions in the literature.

Integrability of the complex structure allows one to split the exterior derivative on forms as

$$\begin{aligned} \mathrm{d} \,:\, \mathcal{A}^{p,q}(M,\mathbb{C}) &\longrightarrow \mathcal{A}^{p+1,q}(M,\mathbb{C}) \oplus \mathcal{A}^{p,q+1}(M,\mathbb{C}) \\ \sigma &\longmapsto \partial\sigma + \overline{\partial}\sigma. \end{aligned}$$

The anti-holomorphic part $\overline{\partial}$ is the **Dolbeault operator**, and serves as a prototype for the notion of semi-connection in definition 1.8. The usual property $\mathrm{d}^2 = 0$ implies that $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$.

Holomorphic bundles One must be careful to distinguish between *complex* and *holomorphic* vector bundles. While a complex vector bundle is simply a vector bundle whose fibres are complex vector spaces, a holomorphic vector bundle is also required to have holomorphic transition maps between local trivializations. We will use calligraphic letters $\mathcal{E}$, $\mathcal{F}$, etc. to denote complex vector bundles with a holomorphic structure.

Definition 1.8 (Semi-connection)**.** *Let M be a complex manifold, and E a complex vector bundle on M. A **semi-connection** on E is a first order differential operator $\overline{\delta} : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^{0,1}(E)$ that satisfies the following Leibniz-type identity: for all $f \in \mathcal{F}(M) \otimes \mathbb{C}$ and $s \in \mathcal{A}^0(E)$,*

$$\overline{\delta}(f\, s) = (\overline{\partial} f)\, s + f\, \overline{\delta} s.$$

A semi-connection $\overline{\delta}$ extends naturally to a map $\overline{\delta} : \mathcal{A}^{\bullet,\bullet}(E) \to \mathcal{A}^{\bullet,\bullet+1}(E)$ via the Leibniz rule. The space of semi-connections on E is denoted by $\overline{\mathscr{A}}(E)$. By a standard calculation, it is an affine space directed by the space $\mathcal{A}^{0,1}(\operatorname{End} E)$ of $\operatorname{End} E$-valued forms of degree $(0,1)$. If $\mathcal{E}$ is a holomorphic vector bundle, then we have a natural semi-connection $\overline{\partial}_{\mathcal{E}}$ that is induced by using the Dolbeault operator on local coordinates. There is also a converse result that gives a convenient description of the space of holomorphic structures on a complex bundle:

Proposition. *A semi-connection $\overline{\delta}$ on a complex bundle E induces a unique holomorphic structure $\mathcal{E}$ such that $\overline{\partial}_{\mathcal{E}} = \overline{\delta}$ if and only if it is **integrable**, i.e. if $\overline{\delta} \circ \overline{\delta} = 0$.*

Hermitian and Kähler geometry Hermitian geometry is the complex analogue of Riemannian geometry in the same way that a Hermitian product generalizes a Euclidean product.

Definition (Hermitian metric)**.** *Let E be a complex vector bundle on M. A **Hermitian metric** h is the data of a Hermitian product on each fibre of E that varies smoothly over M. If M has a complex structure, and h is a Hermitian metric on the holomorphic bundle $\mathcal{T}_M$, then the pair (M,h) defines a **Hermitian manifold**.*

Let (M, h) be a Hermitian manifold. Splitting $h = g - \mathrm{i}\omega$ into real and imaginary part, we see that the data of a Hermitian metric is equivalent to the data of a Riemannian metric g and a real two-form ω that are compatible with the complex structure J in the sense that:

$$g(J\cdot, J\cdot) = g \qquad \text{and} \qquad \omega \in \mathcal{A}^{1,1}(M, \mathbb{R}),$$

The form ω is called the **fundamental form** of h. For a fixed complex structure, the data of either h, g, or ω is equivalent, and we will refer to any of these as a "Hermitian metric". We define the **Lefshetz operator** as the degree-two map defined on exterior forms by $\mathrm{L}_\omega(\sigma) := \sigma \wedge \omega$. If M is compact, the adjoint operator for the metric induced by h on $\mathcal{A}^\bullet(M)$ is denoted by Λ_ω. For example, let us point out for future reference that if $\sigma \in \mathcal{A}^2(M)$, we have:

$$(\Lambda_\omega \sigma)\, \omega^n = n\, \sigma \wedge \omega^{n-1}. \tag{1.7}$$

It is often useful to assume stronger hypotheses on the fundamental form of a Hermitian metric.

Definition 1.9 (Special Hermitian metrics)**.** *Let M be an n-dimensional complex manifold, a Hermitian metric h with fundamental form ω on M is said to be **Kähler** if ω is closed, i.e. $\mathrm{d}\omega = 0$. In particular, ω is a symplectic form. If ω is co-closed, or equivalently if $\mathrm{d}\omega^{n-1} = 0$, then the metric is said to be **balanced**.*

Kähler geometry is the most convenient framework to bring together complex, Riemannian, and symplectic geometry, while the balanced condition is a weaker hypothesis that is also well studied in Hermitian geometry. These extra hypotheses allow us to work with cohomology classes instead of differential forms: if ω is balanced, we define the **balanced class** of the metric as $\tau := [\omega^{n-1}] \in \mathrm{H}^{n-1,n-1}(M, \mathbb{R})$. If the metric is also Kähler, we let $\kappa := [\omega] \in \mathrm{H}^{1,1}(M, \mathbb{R})$ define the **Kähler class**, so that $\tau = \kappa^{n-1}$.

As we know from Riemannian geometry, it is sometimes necessary to impose compatibility conditions between a connection and the metric structure. A connection D_A on a Hermitian budle (E, h) is said to be a **Hermitian connection** if

$$\mathrm{d}(h(u, v)) = h(\mathrm{D}_A u, v) + h(u, \mathrm{D}_A v),$$

for all sections u, v of E. The space of Hermitian connections – denoted by $\mathscr{A}(E, h)$ – is an infinite dimensional affine space directed by $\mathcal{A}^1(\mathrm{End}(E, h))$, $\mathrm{End}(E, h)$ being the bundle of isometric endomorphisms of (E, h).

Given a Hermitian vector bundle (E, h) over M, we have a natural map:

$$\mathrm{Dol} \,:\, \mathscr{A}(E, h) \longrightarrow \overline{\mathscr{A}}(E), \tag{1.8}$$

that sends an Hermitian connection to its $(0,1)$-part. One can show [Tel12] that it is an isomorphism, called the **Dolbeault isomorphism**. Given a holomorphic structure $\mathcal{E}$ on E, the Hermitian connection corresponding to the semi-connection $\overline{\partial}_{\mathcal{E}}$ is the familiar **Chern connection**. The integrability condition for semi-connections amounts to asking that $\mathrm{F}_A^{0,2} = 0$ for a Hermitian connection A.

Finally, let us recall from Chern-Weil theory that for any connection on a Hermitian bundle, the form

$$\alpha := \frac{\mathrm{i}}{2\pi} \mathrm{tr}\,(\mathrm{F}_A) \in \mathcal{A}^2(M) \tag{1.9}$$

is a representative of the first Chern class $c_1(E)$ of E in degree-two de Rham cohomology – see [MS74] or [Nak03].

1.3.2 The Hermitian Yang-Mills equations

The Hermitian Yang-Mills equations appear naturally in the study of Yang-Mills theory over complex manifolds – see [Tel12]. Let (M, ω) be a compact Hermitian manifold, and E a complex vector bundle on M. The **Hermitian Yang-Mills equations** for a connexion D_A on E are:

$$\begin{aligned} \mathrm{F}_A^{0,2} &= 0 \\ \mathrm{i}\Lambda_\omega \mathrm{F}_A &= \lambda\, \mathrm{id}_E, \end{aligned} \tag{1.10}$$

where λ is a real constant. The first equation is the condition that D_A induces a holomorphic structure on E. If D_A is chosen to be the Chern connection on a holomorphic bundle with a given Hermitian metric, the second equation is a proportionality requirement that is reminiscent of the Einstein condition in Riemannian geometry. In fact, some authors – e.g. [Kob87; LT95] – refer to (1.10) as the **Hermite-Einstein equations**.

Taking the trace of (1.10), and using equations (1.9) and (1.7), the second line implies:

$$\lambda\, \mathrm{rk}(E)\, \omega^n = 2\pi n\, \alpha \wedge \omega^{n-1},$$

where $\mathrm{rk}(E)$ is of course the rank of E. This is an equality of (n, n)-forms on M. However, by de Rham's theorem, integrating over M yields an equation in the cohomology of M, and hence a *topological* obstruction:

$$\lambda = \frac{2\pi}{(n-1)!\, \mathrm{vol}_\omega(M)\, \mathrm{rk}(E)} \int_M \alpha \wedge \omega^{n-1}. \tag{1.11}$$

We will come back to this expression in section 1.3.3.

In the seminal article [AB83], Atiyah and Bott introduce a symplectic form on the affine space $\mathscr{A}(E,h)$ of Hermitian connections on (E,h):

$$\omega_{\mathrm{AB}}(a,b) := -\int_M \mathrm{tr}(a \wedge b) \wedge \omega^{n-1}, \tag{1.12}$$

where $a, b \in \mathrm{T}_{\mathrm{D}_A}\mathscr{A}(E,h) \simeq \mathcal{A}^1(\mathrm{End}(E,h))$. Now the **gauge group** $\mathcal{G}$ is defined as the group of unitary automorphisms of the Hermitian bundle (E,h). We have an action of $\mathcal{G}$ on the space of connections given by:

$$g \cdot \mathrm{D}_A = \mathrm{D}_{g\cdot A} = g \circ \mathrm{D}_A \circ g^{-1} \quad \text{i.e.} \quad g \cdot A = gAg^{-1} + g\mathrm{d}g^{-1}.$$

Using the assumption that ω^{n-1} is closed, we may show that the action of the gauge group preserves the symplectic form ω_{AB}. In fact, identifying the Lie algebra $\mathrm{Lie}(\mathcal{G})$ of the gauge group to the space of sections $\mathcal{A}^0(\mathrm{End}(E,h))$, we have the following – see [Gar16, Proposition 4.8]:

Proposition 1.10 (Moment map for the gauge group action)**.** *The action of the gauge group $\mathcal{G}$ on $\mathscr{A}(E,h)$ is Hamiltonian, and we have an equivariant moment map given for any Hermitian connection $\mathrm{D}_A \in \mathscr{A}(E,h)$ by:*

$$\mu(\mathrm{D}_A) = \mathrm{F}_A \wedge \omega^{n-1} + \frac{\mathrm{i}\lambda}{n}\,\omega^n\,\mathrm{id}_E \in \mathcal{A}^{2n}(\mathrm{End}(E,h)) = (\mathrm{Lie}(\mathcal{G}))^\vee,$$

where the identification with the dual space of sections is explicitly given for $b \in \mathrm{Lie}(\mathcal{G})$ by:

$$\langle \mu(\mathrm{D}_A), b\rangle = \int_M \mathrm{tr}\left(\mu(\mathrm{D}_A) \circ b\right) = \frac{1}{n}\int_M \mathrm{tr}\left(\left(\Lambda_\omega \mathrm{F}_A + \mathrm{i}\lambda\,\mathrm{id}_E\right) \circ b\right)\omega^n. \tag{1.13}$$

We easily recognize the Hermite-Einstein operator of equation (1.10) in the expression (1.13) of the moment map, so that the zero-locus of μ is the space of Hermitian Yang-Mills connections. In particular, this means that the symplectic reduction of $\mathscr{A}(E,h)$ under the Hamiltonian action of $\mathcal{G}$ is given by the quotient of the space of Hermitian Yang-Mills connections by the unitary gauge group. Now the complexified gauge group $\mathcal{G}^{\mathbb{C}}$ is the group of complex automorphisms of E, which also acts on the space of Hermitian connections. By analogy with theorem 1.5, and keeping in mind figure 1, we can expect the following statement:

Theorem 1.11 (Kobayashi-Hitchin correspondence *or* Donaldson-Uhlenbeck-Yau theorem)**.** *A complex vector bundle E on M admits a Hermitian Yang-Mills connection if and only if it is polystable. In this case, the Hermitian Yang-Mills connection is unique up to the action of $\mathcal{G}$.*

For this theorem to even make sense, we of course need to define a notion of stability for bundles over M. According to section 1.2.1, this means that we should choose an ample line bundle on the space of holomorphic vector bundles – the so-called **Quot scheme** – and a linearisation of the action of the gauge group, see [Tho06]. For our purposes, we will only state the resulting stability condition, known as slope stability, in the next section.

Remark. *In fact, as is shown in [LT95], theorem 1.11 goes beyond a mere characterization of the existence of solutions to (1.10). It may actually be shown that the Dolbeault isomorphism of equation (1.8) induces an isomorphism between the moduli spaces of Hermitian Yang-Mills connections on the one hand, and stable holomorphic structures on the other hand.*

1.3.3 Slope stability

The notion of stability that is relevant for the Kobayashi-Hitchin correspondence is slope stability. It was first introduced in algebraic geometry by Mumford and Takemoto as a suitable stability condition to construct moduli spaces of vector bundles [Tho06; HL10]. If E is a complex vector bundle on M, the **degree** of E is defined by

$$\deg_\omega(E) := \int_M \alpha \wedge \omega^{n-1},$$

where α is defined from the curvature of a connection on E by equation (1.9). If ω is a balanced metric, the degree of E is a topological quantity – it can be written in terms of the balanced class as $\deg_\tau(E) = (c_1(E) \smile \tau) \frown [M]$. Notice that, on a Riemann surface, we recover the classical notion of degree that appears for example in the Riemann-Roch theorem [Har77]. We now define the **slope** of E as:

$$\mu_\tau(E) := \frac{\deg_\tau(E)}{\mathrm{rk}(E)}.$$

We point out that this expression already shows up in the topological obstruction (1.11).

Definition 1.12 (Slope stability)**.** *Let $\mathcal{E}$ be a holomorphic vector bundle on a balanced manifold (M,ω). Then $\mathcal{E}$ is said to be* ***slope stable****, or* ***τ-stable****, if for every holomorphic subbundle $\mathcal{F} < \mathcal{E}$ – actually for every proper coherent subsheaf, see [LT95] – one has $\mu_\tau(\mathcal{F}) < \mu_\tau(\mathcal{E})$. The bundle is said to be* ***slope polystable*** *if it can be expressed as a direct sum of stable bundles with the same slope.*

With definition 1.12 in mind, theorem 1.11 says that *a complex vector bundle admits a Hermitian Yang-Mills connection if and only if it is slope polystable.* It is not extremely difficult to prove that a Hermitian Yang-Mills connection induces a stable holomorphic structure. The hard part of the proof goes the other way, and was achieved by Donaldson Uhlenbeck and Yau for Kähler manifolds [Don85; UY86]. For a proof in the case of a general Hermitian metric, see [LT95] and references therein.

2 The deformed Hermitian Yang-Mills equations

The deformed Hermitian Yang-Mills equations were first introduced in [Mar+00] as instanton equations in string theory. Since then, they have been intensely studied by mathematicians and physicists alike – we refer to [CXY18] for a review. This short section is a condensed version of part the accompanying set of notes [Ser24]; here, we only highlight the construction of mirror manifolds via Fourier-Mukai transform.

2.1 The Strominger-Yau-Zaslow picture of mirror symmetry

Mirror symmetry describes a relation between Calabi-Yau manifolds that are regarded as physically equivalent as compactifications of string theory, see [Voi96; Hor+03; GJH03]. Though mirror symmetry is still poorly understood mathematically, it is expected that mirror symmetry should *exchange the symplectic and complex structures on a pair of mirror Calabi-Yau manifolds.* This vague statement should hopefully become clearer as we go through examples. In this section, we introduce a simple setting for mirror symmetry known as the semi-flat model. This allows us to motivate the deformed Hermitian Yang-Mills system from the equations defining special Lagrangian cycles, as was first explained in [LYZ00]. Another reference for this section is [Hor+03, Section 37.9], where this construction is referred to as *a geometric functor*.

2.1.1 The semi-flat model for mirror symmetry

The Strominger-Yau-Zaslow picture was first proposed in [SYZ96] as a geometric description of mirror symmetry. This eventually led to the formulation of the following conjecture:

Conjecture 2.1 (Strominger-Yau-Zaslow, or SYZ conjecture)**.** *Let M and $\check{M}$ be mirror Calabi-Yau manifolds, then there should exist a base space D and surjective maps p and $\check{p}$ such that the two fibrations in the diagram*

$$M \xrightarrow{p} D \xleftarrow{\check{p}} \check{M}$$

are nearly dual, in the sense that there exists a dense open subset $D' \subset D$ such that for all $d \in D'$, the fibres $p^{-1}(d)$ and $\check{p}^{-1}(d)$ are mutually dual tori.

This rough statement of the conjecture – too vague to be either true or false – is taken from [GJH03, Chapter 12], where a longer discussion of mirror symmetry in the spirit of Strominger, Yau and Zaslow may be found.

A successful approach to understand conjecture 2.1 is to work locally, in the so-called **semi-flat** model. Let D be an open domain in $\mathbb{R}^n$, so that D inherits global coordinates $(x^1, \cdots, x^n)$. The tangent and cotangent bundles of D are then canonically trivialized, as the (x^i) induce global coordinates (y^i) and $(\check{y}_i)$ on the tangent and cotangent fibres. With these coordinates, $\mathrm{T}D$ and $\mathrm{T}^\vee D$ are identified to $D \times \mathbb{R}^n$ and $D \times (\mathbb{R}^n)^\vee$. Since we want a torus fibration, it is natural to consider the quotient by a lattice $\mathbb{Z}^n \simeq \Lambda \subset \mathbb{R}^n$. We define:

$$M := \mathrm{T}D/\Lambda \qquad \text{and} \qquad \check{M} := \mathrm{T}^\vee D/\Lambda^\vee,$$

where $\Lambda^\vee = \mathrm{Hom}(\Lambda, 2\pi\,\mathbb{Z}) \subset (\mathbb{R}^n)^\vee$ is the dual lattice of Λ. This simple model reproduces – somewhat artificially – the mutually dual torus fibrations of conjecture 2.1. Of course, we still need to give M and $\check{M}$ more structure to turn them into Calabi-Yau manifolds. First of all, it is known [Can08, Chapter 2] that the cotangent bundle $\mathrm{T}^\vee D$ of any manifold inherits a natural symplectic form. In coordinates $(x^i, \check{y}_i)$, and using Einstein's summation convention, this is given by:

$$\check{\omega} := \mathrm{d}x^i \wedge \mathrm{d}\check{y}_i, \tag{2.1}$$

which is clearly $\Lambda^\vee$-invariant, and hence descends to a form on $\check{M}$, making it into a symplectic manifold.

On the other hand, the tangent bundle to any manifold admits a natural complex structure. In a chart (x^i, y^i), this is given by introducing the complex coordinates $z^i := x^i + \mathrm{i}y^i$. One may then check that the Cauchy-Riemann relations hold, making $\mathrm{T}D$ into a complex manifold. This structure is also invariant by Λ, so that we obtain a complex structure on M. Along with the complex structure on M, we also get a *global* holomorphic volume form $\Omega := \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$ as in equation (1.6).

We now have a canonical symplectic structure on $\check{M}$ and a canonical complex structure on M. In order to have Calabi-Yau manifolds, we will also need a complex structure on $\check{M}$ and a symplectic structure on M – and here we will have to make some choices. Let us proceed to construct a Kähler metric on M. The simplest way to construct a Kähler form is via a Kähler potential. Let ϕ be a real function on the base space D, then ϕ pulls back to a torus-invariant function on M, and we can define an associated $(1,1)$-form by:

$$\omega := 2\mathrm{i}\partial\overline{\partial}\phi = \frac{\mathrm{i}}{2}\phi_{ij}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j = \phi_{ij}\mathrm{d}x^i \wedge \mathrm{d}y^j, \tag{2.2}$$

where ϕ_{ij} are the components of the Hessian of ϕ as a function on D, which are given by:

$$\phi_{ij} = 4\frac{\partial^2\phi}{\partial z^i\,\partial\overline{z}^j} = \frac{\partial^2\phi}{\partial x^i\,\partial x^j},$$

and where the last equality comes from the fact that ϕ is invariant in the y^i directions. If ϕ is strictly convex, this defines a Kähler form on M.

For M to be a Calabi-Yau manifold, we now only need some compatibility between ω and Ω – namely the latter should be parallel for the Levi-Civita connection of the metric. As we know from the works of Calabi [Cal57], this is equivalent to $\det(\phi_{ij})$ being constant on D, which is a form of the **real Monge-Ampère equation**.

Now, since ϕ is strictly convex, we can identify $\mathrm{T}D$ to $\mathrm{T}^\vee D$ via a Legendre transformation – which is given in coordinates by the bijective map $\check{y}_i = \phi_{ij} y^j$. With this in mind, the cotangent bundle $\mathrm{T}^\vee D$ inherits a complex structure by setting $\mathrm{d}\check{z}_i := \mathrm{d}\check{x}_i + \mathrm{i}\mathrm{d}\check{y}_i$, where $\mathrm{d}\check{x}_i := \phi_{ij}\mathrm{d}x^j$ defines a Legendre-transformed coordinate on $\check{M}$. The corresponding holomorphic volume form is naturally defined by:

$$\check{\Omega} := \mathrm{d}\check{z}^1 \wedge \cdots \wedge \mathrm{d}\check{z}^n. \tag{2.3}$$

Let us finally observe that the symplectic form $\check{\omega}$ on $\check{M}$ given by equation (2.1) may be seen to be a Kähler form in these coordinates by writing:

$$\check{\omega} = \mathrm{d}x^i \wedge \mathrm{d}\check{y}_i = \phi^{ij}\mathrm{d}\check{x}_j \wedge \mathrm{d}\check{y}_i,$$

which makes manifest the duality with equation (2.2).

2.1.2 The Fourier-Mukai Transform

The main point of [LYZ00] is to relate data on $\check{M}$ to data on M via a duality known in algebraic geometry as the **Fourier-Mukai transform**. Consider dual lattices Λ and $\Lambda^\vee$ in $\mathbb{R}^n$ and $(\mathbb{R}^n)^\vee$, and denote the corresponding quotient tori by T and $\check{T}$. An element $\check{y} \in (\mathbb{R}^n)^\vee$ defines a morphism:

$$\begin{aligned} g_{\check{y}} \;:\; \Lambda &\longrightarrow \mathbb{R} \\ \lambda &\longmapsto \langle \check{y}, \lambda\rangle. \end{aligned}$$

By definition of the dual lattice, shifting $\check{y}$ by an element of $\Lambda^\vee$ amounts to shifting $g_{\check{y}}$ by an element of $2\pi\mathbb{Z}$, so that we obtain a map

$$\check{T} \longrightarrow \mathrm{Hom}(\Lambda, \mathrm{U}(1)) = \mathrm{Hom}(\pi_1(T), \mathrm{U}(1)),$$

where $\mathrm{U}(1)$ is identified to the unit circle $\mathbb{R}/2\pi\mathbb{Z}$. This is the **Fourier-Mukai transform**.

Now we know, see e.g. [DK90, Proposition 2.2.3], that there is a correspondence between flat connections on a manifold and representations of its fundamental group whereby the action of a loop is computed as the holonomy of the connection around this loop – the fact that this is homotopy invariant is a direct consequence of the flatness of the connection. Therefore, the Fourier-Mukai transform is a map from $\check{T}$ to the space of flat connections on a $\mathrm{U}(1)$-bundle on T. This is easily seen in coordinates: let a point $\check{y}$ in $\check{T}$ be parametrized locally by $\check{y}_1, \cdots, \check{y}_n$, then we should construct a form on T whose periods on closed loops yields the representation of $\pi_1(T)$ given by the Fourier-Mukai transform. This will be given in coordinates by the form $\check{y}_i\mathrm{d}y^i$. More precisely, identifying the Lie algebra $\mathfrak{u}(1)$ of $\mathrm{U}(1)$ to $\mathrm{i}\mathbb{R}$, the connection on T induced from $\check{y} \in \check{T}$ by Fourier-Mukai transform is given in coordinates by $\mathrm{D}_A := \mathrm{d} + A$ where

$$A = 2\pi\mathrm{i}\,\check{y}_i\mathrm{d}y^i \in \mathcal{A}^1(T, \mathfrak{u}(1)). \tag{2.4}$$

Here, the prefactor $2\pi\mathrm{i}$ ensures that the holonomy along a closed loop is trivial in $\mathrm{U}(1)$.

For now, all of this is happening at the level of a single torus fibre $\check{T}$. To get a global picture, we must consider the data of a point in each fibre of $\check{M}$, that is a section $s : D \to \check{M}$. Given such a section, we can perform a Fourier-Mukai transform pointwise, and define a connection form on M by the same formula (2.4). The only difference is that the $\check{y}_i$ are now seen as functions, more precisely as the components of the section s, which we denote by s_i. In particular, D_A is no longer a flat connection, and its curvature is given by:

$$\mathrm{F}_A = \mathrm{d}A = 2\pi\mathrm{i}\,\mathrm{d}s_i \wedge \mathrm{d}y^i = 2\pi\mathrm{i}\,\frac{\partial s_i}{\partial x^j}\mathrm{d}x^j \wedge \mathrm{d}y^i.$$

Introducing a normalized real curvature form α as in equation (1.9), we thus get

$$\alpha = -\frac{\partial s_i}{\partial x^j}\mathrm{d}x^j \wedge \mathrm{d}y^i. \tag{2.5}$$

2.1.3 Special Lagrangian sections and the deformed Hermitian Yang-Mills equations

Calibrated Geometry Let us first introduce some basic facts about calibrated geometry – see [GJH03] for more detail. Recall that if (M, g) is an oriented Riemannian manifold, then the metric g induces a volume n-form vol_M^g on M. If V is an oriented k-plane at some point $p \in M$, then g_p induces a scalar product on V, and thus a volume k-form vol_V^g on V. A *closed* differential form φ of degree k on M is said to be a **calibration** if for all $p \in M$, and every oriented k-plane at p, we have $\varphi|_V \leq \mathrm{vol}_V^g$. We also ask that equality is achieved for some oriented plane V at each $p \in M$, and we will say that such a k-plane is **calibrated**. This assumption implies in particular that the normalization of a calibration is uniquely determined.

Definition (Calibrated submanifold)**.** *Let (M, g, φ) be a Riemannian manifold equipped with a calibration φ. A submanifold $N \subset M$ is said to be* ***calibrated*** *by φ if we have the equality $\mathrm{vol}_N^g = \varphi|_N$, at every point of N.*

Examples. *Let us give some examples of calibrations that will be useful to our discussion.*

(i) If (M, ω) is a Kähler manifold of dimension n, then the form $\omega^k/k!$ is a calibration on M for all $1 \leq k \leq n$. The corresponding calibrated submanifolds are the complex submanifolds of M.

(ii) If (M, ω, Ω) is a Calabi-Yau manifold, then for any phase θ, the n-form $\mathrm{Re}(e^{-\mathrm{i}\theta}\Omega)$ is a calibration on M. The calibrated submanifolds for this form which are also Lagrangian for the Kähler form ω are the ***special Lagrangian submanifolds*** *of phase θ. See the excellent [Hit01] for a discussion of special Lagrangian submanifolds and SYZ mirror symmetry.*

Fourier-Mukai transform of calibrated sections Given a section s of $\check{M}$, we may see $s(D)$ as a submanifold of $\check{M}$. The Calabi-Yau structure on $\check{M}$ gives us a calibration $\mathrm{Re}(e^{-\mathrm{i}\theta}\check{\Omega})$ for every phase θ. We want to characterize those sections for which $s(D)$ is special Lagrangian – i.e. is calibrated by $\mathrm{Re}(e^{-\mathrm{i}\theta}\check{\Omega})$.

Let us first suppose that $s(D)$ is Lagrangian, i.e. that $\check{\omega}|_{s(D)} = 0$. Using equation (2.1), we have:

$$\check{\omega}|_{s(D)} = \mathrm{d}x^i \wedge \mathrm{d}s_i = \frac{\partial s_i}{\partial x^j}\mathrm{d}x^i \wedge \mathrm{d}x^j,$$

so that the Lagrangian condition becomes:

$$\frac{\partial s_i}{\partial x^j} - \frac{\partial s_j}{\partial x^i} = 0.$$

On the other hand, from equation (2.5), the above symmetry property yields:

$$\alpha = \frac{\mathrm{i}}{4}\frac{\partial s_i}{\partial x^j}\left(\mathrm{d}z^i + \mathrm{d}\bar{z}^i\right) \wedge \left(\mathrm{d}z^j - \mathrm{d}\bar{z}^j\right) = \frac{\mathrm{i}}{4}\frac{\partial s_i}{\partial x^j}\left(\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j - \mathrm{d}\bar{z}^i \wedge \mathrm{d}z^j\right),$$

so that form α is of degree $(1,1)$, and so is F_A. We have shown that $s(D)$ is Lagrangian if and only if the curvature of D_A is of type $(1,1)$, i.e. if and only if it induces a holomorphic structure.

Now, we look at what happens if $s(D)$ is special Lagrangian, i.e. if it is calibrated by $\mathrm{Re}(e^{-\mathrm{i}\theta}\check{\Omega})$. Recall that $\check{\Omega}$ is given by equation (2.3). We compute:

$$\mathrm{d}\check{z}_i|_{s(D)} = \mathrm{d}\check{x}_i|_{s(D)} + \mathrm{i}\,\mathrm{d}\check{y}_i|_{s(D)} = \left(\phi_{ij} + \mathrm{i}\frac{\partial s_i}{\partial x^j}\right)\mathrm{d}x^j,$$

so that we obtain:

$$e^{-\mathrm{i}\theta}\check{\Omega}\Big|_{s(D)} = \det\left(\phi_{ij} + \mathrm{i}\frac{\partial s_i}{\partial x^j}\right)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

We may already identify the coefficients of ω and F_A in the equation above. The condition that $s(D)$ be special Lagrangian turns out to be equivalent to the **deformed Hermitian Yang-Mills equations**:

$$\begin{gathered} \mathrm{F}_A^{0,2} = 0 \\ \left(\omega - \frac{1}{2\pi}\mathrm{F}_A\right)^n = (\omega + \mathrm{i}\alpha)^n = \tilde{r}e^{\mathrm{i}\theta}\,\omega^n, \end{gathered} \tag{2.6}$$

where $\tilde{r}$ is some positive function. The reason for this name is that the deformed Hermitian Yang-Mills equations reduce to the usual Hermitian Yang-Mills equations (1.10) on a line bundle in the so-called **large-volume limit**, whereby the Kähler form ω goes to infinity.

Regardless of its motivation, (2.6) can be seen as a system of equations for a connection on a line bundle L over M. Notice that, as for the Hermitian Yang-Mills equations, integration of the second line of (2.6) yields a *topological* obstruction in terms of the Kähler class:

$$(\kappa + \mathrm{i}\,c_1(L))^n \frown [M] = re^{\mathrm{i}\theta},$$

where r is a positive real number, so that θ is determined modulo 2π by the topology of the line bundle.

2.2 Stability conditions and Z-critical equations

The study of the deformed Hermitian Yang-Mills equations is driven by the hope to extend theorem 1.11 to this more difficult setting: there should exist a stability condition that ensures the existence of a solution to equation (2.6). This is the content of the **Collins-Jacob-Yau conjecture** [CJY20, Conjecture 1.5], which solves the problem under a technical hypothesis known as the **supercritical phase condition**. Our discussion of mirror symmetry relates this statement to the important **Thomas-Yau conjecture** in symplectic geometry [Tho01].

The **Z-critical equations** are a generalization of the deformed Hermitian Yang-Mills equations. They were introduced recently [DMS24] as part of a general approach to problems in complex geometry that relies on geometric invariant theory. The hope is that existence of solutions to these equations should depend on a class of categorical stability conditions introduced in [Bay09]. This approach also yields a natural generalization of the deformed Hermitian Yang-Mills equations to higher-rank vector bundles.

We will not say more here, and refer instead to [Ser24] for a review of stability conditions, an introduction to Z-critical connections, and some original results including a new interpretation of the supercritical phase condition, and a discussion of some links with mirror symmetry using generalized complex geometry.

3 Generalized geometry and the Hull-Strominger system

This section introduces the Hull-Strominger system, both as an equation for canonical metrics on non-Kähler Calabi-Yau manifolds, and as a pretext to talk about generalized geometry.

3.1 Introduction to generalized geometry

We first review the basics of the theory of generalized geometry, introduced by Hitchin in [Hit03]. We refer to [Hit11] for excellent lecture notes on the subject. In [Ser24], we also give a description of SYZ mirror symmetry using generalized geometry following [CG10], and go on to describe tentative mirrors to Z-critical connections.

3.1.1 Generalized tangent bundle

Let M be a manifold. Generalized geometry is the study of geometric structures on the **generalized tangent bundle** of M, defined simply as the sum of its tangent and cotangent bundles, $\mathbb{T}M := \mathrm{T}M \oplus \mathrm{T}^\vee M$. In particular, a section $\mathbb{X}$ of $\mathbb{T}M$ is given by a formal sum $\mathbb{X} = X + \xi$ of a vector and a one-form on M. The duality between the tangent and cotangent bundles leads to a natural non-degenerate metric of signature (n, n):

$$\langle \mathbb{X}, \mathbb{Y} \rangle = \langle X + \xi, Y + \eta \rangle := \frac{1}{2} \left(\xi(Y) + \eta(X) \right).$$

Another natural structure on the generalized tangent bundle is the **Courant bracket**, defined by:

$$[\![X + \xi, Y + \eta]\!] := [X, Y] + \mathcal{L}_X \eta - \iota_Y \mathrm{d}\xi, \tag{3.1}$$

where $[\cdot, \cdot]$ is the usual Lie bracket on $\mathfrak{X}(M)$. The Courant bracket should be thought of as the generalized geometry version of the classical Lie bracket. For instance, one can verify that it satisfies the Jacobi identity. However, it is *not* antisymmetric, since:

$$[\![\mathbb{X}, \mathbb{X}]\!] = \cancel{[X, X]} + \mathcal{L}_X \xi - \iota_X \mathrm{d}\xi = \mathrm{d}\iota_X \xi = \mathrm{d}\langle \mathbb{X}, \mathbb{X} \rangle, \tag{3.2}$$

from which we see that the failure to be antisymmetric is exact. These properties, along with some compatibility conditions between the Courant bracket and the pairing $\langle \cdot, \cdot \rangle$, will be the defining features of Courant algebroids in section 3.1.3.

The generalized tangent bundle $\mathbb{T}M$ has a natural action on the space of forms $\mathcal{A}^\bullet(M)$ given by:

$$\mathbb{X} \cdot \rho := \iota_X \rho + \xi \wedge \rho.$$

This extends to an action of the Clifford bundle $(\mathrm{Cl}(\mathbb{T}M), \langle \cdot, \cdot \rangle)$ associated to the canonical pairing on $\mathbb{T}M$, since:

$$\mathbb{X} \cdot (\mathbb{X} \cdot \rho) = \iota_X (\xi \wedge \rho) + \xi \wedge \iota_X \rho = (\iota_X \xi) \wedge \rho = \langle \mathbb{X}, \mathbb{X} \rangle \, \rho. \tag{3.3}$$

We refer to [LM89] for the relevant background on Clifford algebras. The $\cdot$ action is thus called the **Clifford action** of the generalized tangent bundle, for which $\bigwedge^\bullet \mathrm{T}^\vee M$ may thus be seen as a **spinor bundle**.

Given a spinor $\rho \in \mathcal{A}^\bullet(M)$, we define its **annihilating bundle** as:

$$\mathrm{Ann}(\rho) := \{ \mathbb{X} \in \mathbb{T}M \mid \mathbb{X} \cdot \rho = 0 \}. \tag{3.4}$$

Using equation (3.3), we may easily see that the canonical product $\langle \cdot, \cdot \rangle$ restricts to zero on the annihilating bundle, that is $\mathrm{Ann}(\rho)$ is an **isotropic** subbundle of $\mathbb{T}M$. Since the canonical product has signature (n, n), the annihilator bundle of a spinor has dimension at most n. This leads to the following definition:

Definition 3.1 (Pure spinor)**.** *A spinor $\rho \in \mathcal{A}^\bullet(M)$ is said to be a* ***pure spinor*** *if its annihilating bundle is maximally isotropic, that is if* $\mathrm{Ann}(\rho)$ *has rank n.*

3.1.2 Generalized geometric structures

Generalized metrics In classical Riemannian geometry, as we have mentioned before, a metric g on a manifold M gives rise to an isomorphism $g : \mathrm{T}M \xrightarrow{\sim} \mathrm{T}^\vee M$. The graph of this map

$$V_+ := \{ X + g(X), \, X \in \mathrm{T}M \} \subset \mathbb{T}M$$

may be seen as a subbundle of the generalized tangent bundle of M. On V_+, the bracket $\langle \cdot, \cdot \rangle$ restricts to:

$$\langle X + g(X), Y + g(Y) \rangle = \frac{1}{2} \left(g(X)(Y) + g(Y)(X) \right) = g(X, Y),$$

i.e. to the Riemannian metric we started with. The $\langle \cdot, \cdot \rangle$-orthogonal to V_+ is the bundle V_- of generalized vectors of the form $X - g(X)$.

Definition (Generalized metric)**.** *A* ***generalized metric*** *on M is the data of a $\langle \cdot, \cdot \rangle$-orthogonal splitting of the generalized tangent bundle $\mathbb{T}M = V_+ \oplus V_-$ for which the restriction of $\langle \cdot, \cdot \rangle$ to V_+ is positive definite, and such that V_+ projects isomorphically to $\mathrm{T}M$ via the canonical projection* $\mathrm{pr}_1 : \mathrm{T}M \oplus \mathrm{T}^\vee M \to \mathrm{T}M$.

A generalized metric can also be seen as a endomorphism $\mathbb{G}$ of $\mathbb{T}M$ such that $\mathbb{G}^2 = \mathrm{id}$, whose ± 1-eigenbundles correspond to $V_\pm$. This point of view is closer in spirit to the definition 3.2 of generalized complex structures.

Generalized complex structures We would now like to bring the complex structures of definition 1.7 to the context of generalized geometry.

Definition 3.2 (Generalized complex structure)**.** *A* ***generalized complex structure*** *on a manifold M is the data of an endomorphism $\mathbb{J}$ of $\mathbb{T}M$ such that $\mathbb{J}^2 = -\mathrm{id}_{\mathbb{T}M}$, and verifying:*

(i) ***Compatibility with*** $\langle\cdot,\cdot\rangle$ *– For all sections $\mathbb{X}$, $\mathbb{Y}$ of $\mathbb{T}M$, we have $\langle \mathbb{J}(\mathbb{X}), \mathbb{Y}\rangle + \langle \mathbb{X}, \mathbb{J}(\mathbb{Y})\rangle = 0$,*

(ii) ***Integrability*** *– The* i*-eigenbundle $\mathbb{T}^{1,0}M \subset \mathbb{T}M \otimes \mathbb{C}$ of $\mathbb{J}$ is closed under the Courant bracket.*

Given a generalized complex structure, let us define $\mathbb{T}^{0,1}M := \overline{\mathbb{T}^{1,0}M}$, i.e. the conjugate bundle in $\mathbb{T}M \otimes \mathbb{C}$. This corresponds to the $(-\mathrm{i})$-eigenbundle of $\mathbb{J}$. Item (i) in the definition above amounts to saying that $\mathbb{T}^{1,0}M$ and $\mathbb{T}^{0,1}M$ are isotropic subbundles of $\mathbb{T}M$. Since they are of dimension n, they are in fact maximally isotropic subbundles. In fact, it is easy to show that the data of a generalized complex structure is equivalent to the data of an integrable maximally isotropic subbundle $L \subset \mathbb{T}M \otimes \mathbb{C}$ such that $L \cap \overline{L} = \{0\}$. Such a bundle may always be written as the annihilating bundle of a *complex* pure spinor ρ, which is defined uniquely up to a multiplicative factor. This leads to the following definition:

Definition (Canonical bundle)**.** *Let $\mathbb{J}$ be a generalized complex structure on M, then the data at each point $p \in M$ of the complex line of pure spinors:*

$$\mathcal{K}_M^{\mathbb{J}} := \{\rho \in \mathcal{A}^\bullet(M,\mathbb{C}) \,|\, \forall \mathbb{X} \in \mathbb{T}^{1,0}M,\, \mathbb{X}\cdot\rho = 0\}$$

defines a complex line subbundle of $\mathcal{A}^\bullet(M,\mathbb{C})$. We call it the ***canonical bundle*** *associated to $\mathbb{J}$.*

Before giving a more precise description of pure spinors in the context of generalized complex structures in proposition 3.4, let us review a few examples.

Examples 3.3 (Generalized complex structures)**.**

(i) ***Classical complex structures*** *– Let J be a complex structure on M in the sense of definition 1.7. We define a generalized complex structure $\mathbb{J}_J$ on $\mathbb{T}M$ by:*

$$\mathbb{J}_J := \begin{pmatrix} -J & 0 \\ 0 & J^\vee \end{pmatrix},$$

where $J^\vee$ is the induced complex structure on forms. In this case, the holomorphic bundle $\mathbb{T}^{1,0}M$ is spanned by the basis $\left(\frac{\partial}{\partial \overline{z}^1}, \cdots, \frac{\partial}{\partial \overline{z}^d}, \mathrm{d}z^1, \cdots, \mathrm{d}z^d\right)$, where $n = 2d$ and (z^i) is a local holomorphic chart for (M,J). Integrability of $\mathbb{J}_J$ follows from that of J, and the canonical bundle is the usual $\mathcal{K}_M$ so that a local holomorphic volume form Ω is a pure spinor for $\mathbb{J}_J$.

(ii) ***Symplectic structures*** *– Let ω be a symplectic form on M, which we interpret as a bundle isomorphism $\omega : \mathrm{T}M \xrightarrow{\sim} \mathrm{T}^\vee M$. We define a generalized complex structure $\mathbb{J}_\omega$ on $\mathbb{T}M$ by:*

$$\mathbb{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Integrability amounts to ω being closed, and the canonical bundle is trivialized by the line directed by $e^{\mathrm{i}\omega}$.

The examples 3.3 are a rewriting of familiar structures in a new language, but generalized geometry has more to offer. In fact, one should think of these examples as extremal cases, whereas more general complex structures interpolate between symplectic and complex geometry. This is made precise by the following result of Gualtieri:

Proposition 3.4 (Classification of pure spinors)**.**

(i) A complex spinor $\rho \in \mathcal{A}^\bullet(M,\mathbb{C})$ is pure if and only if it is of the form $\rho = e^{B+\mathrm{i}\omega} \wedge \Omega$, where B and ω are real two-forms, and Ω is a decomposable complex k-form for some $1 \leq k \leq n$. The integer k is thus invariantly defined as the lowest degree appearing in the decomposition of ρ, and is called the ***type*** *of ρ.*

(ii) A complex pure spinor $\rho = e^{B+\mathrm{i}\omega} \wedge \Omega$ defines a generalized complex structure for which $\mathbb{T}^{1,0}M = \mathrm{Ann}(\rho)$ if and only if:

$$\mathrm{Ann}(\rho) \cap \overline{\mathrm{Ann}(\rho)} = 0 \quad \Leftrightarrow \quad \Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0,$$

and provided that there exists some section $\mathbb{X}$ of $\mathbb{T}M$ such that $\mathrm{d}\rho = \mathbb{X}\cdot\rho$ – this last requirement corresponds to the integrability condition.

See [Gua04, Proposition 2.25] for a proof. With this in mind, we see that a complex structure is a generalized complex structure of type n, while a symplectic structure is a generalized complex structure of type zero. The unification of complex and symplectic structures makes generalized complex geometry a nice framework to study mirror symmetry. The idea of [CG10] is that mirror symmetry should be expressed as a map between the spinor bundles of Calabi-Yau manifolds that exchanges the pure spinors associated to the complex and symplectic structures. This may be seen as a generalization of the picture presented in section 2, see also [Ser24].

3.1.3 Courant algebroids

A Courant algebroid is a vector bundle that carries structures similar to those of the generalized tangent bundle.

Definition 3.5 (Smooth Courant algebroid). *Let M be a smooth manifold, a* ***smooth Courant algebroid*** *is defined by a vector bundle E on M with bilinear maps:*

$$\langle \cdot, \cdot \rangle \,:\, E \otimes E \longrightarrow \mathbb{R} \qquad \text{and} \qquad [\![\cdot, \cdot]\!] \,:\, \mathcal{A}^0(E) \otimes \mathcal{A}^0(E) \longrightarrow \mathcal{A}^0(E)$$

such that $\langle \cdot, \cdot \rangle$ is non-degenerate and symmetric, together with a map $\pi : E \to \mathrm{T}M$ called the ***anchor map*** *verifying the following properties for all $u, v, w \in \mathcal{A}^0(E)$, $X \in \mathfrak{X}(M)$, and $f \in \mathcal{F}(M)$:*

(i) ***Jacobi identity*** – $[\![u, [\![v, w]\!]]\!] + [\![v, [\![w, u]\!]]\!] + [\![w, [\![u, v]\!]]\!] = 0$,

(ii) ***Compatibility with the anchor*** – $\pi\left([\![u, v]\!]\right) = [\pi(u), \pi(v)]$, *where $[\cdot, \cdot]$ is the natural bracket on $\mathfrak{X}(M)$,*

(iii) ***Leibniz rules*** – $\pi(u) \cdot \langle v, w \rangle = \langle [\![u, v]\!], w \rangle + \langle v, [\![u, w]\!] \rangle$ *and* $[\![u, fv]\!] = (\pi(u) \cdot f)\, v$,

(iv) ***Antisymmetry up to an exact term*** – $[\![u, u]\!] = 2\pi^\vee\left(\mathrm{d}\langle u, u \rangle\right)$, *where $\pi^\vee$ is the dual map $\pi^\vee : \mathrm{T}^\vee M \to E^\vee$, and where the dual $E^\vee$ is identified to E via $\langle \cdot, \cdot \rangle$.*

The generalized tangent bundle $\mathbb{T}M$ is of course an example of Courant algebroid where π is *half* the natural projection to $\mathrm{T}M$. The reason for this choice is largely a matter of convention, and matches with the normalization in equation (3.2). It is easy to show from the hypotheses of definition 3.5 that the sequence

$$0 \longrightarrow \mathrm{T}^\vee M \xrightarrow{\pi^\vee} E \xrightarrow{\pi} \mathrm{T}M \longrightarrow 0 \tag{3.5}$$

is in fact a complex, that is $\pi \circ \pi^\vee = 0$. If this complex is exact, then E is an **exact Courant algebroid**. Given an exact Courant algebroid E, the data of a section $s : \mathrm{T}M \to E$ of (3.5) yields an isomorphism $s \oplus \pi^\vee : \mathbb{T}M = \mathrm{T}M \oplus \mathrm{T}^\vee M \xrightarrow{\sim} E$ *as vector bundles.* Under this map, the Courant structure on E pulls back to a deformed version of the Courant bracket (3.1), the so-called **Dorfman bracket**:

$$[\![X + \xi, Y + \eta]\!]_H := [X, Y] + \mathcal{L}_X \eta - \iota_Y \mathrm{d}\xi + \iota_X \iota_Y H,$$

where $H \in \mathcal{A}^3(M)$ is a closed three-form given by $H(X, Y, Z) := \langle [s(X), s(Y)], s(Z) \rangle$ for $X, Y, Z \in \mathfrak{X}(M)$.

The above considerations allow us to classify smooth exact Courant algebroids on M. The upshot is that isomorphism classes of exact Courant algeboids correspond to the degree-three de Rham cohomology of M by considering the class of $H \in \mathcal{A}^3(M)$: this is known as the **Ševera classification**, see [Šev17].

The appearance of the closed form H allows for many exciting physical interpretations. In string theory, it is related to the so-called **Kalb-Ramond field**, i.e. the field strength of the locally defined B-field, that plays a rôle similar to that of the Faraday field strength F with respect to the local potential A – see [Koe11] for an account of generalized geometry with an eye towards physical applications. The form H also plays a major rôle in the interpretation of T-duality using generalized geometry in [CG10], as it allows for changes in topology between T-dual torus bundles. In the mathematical literature, H is seen as the curvature form for a connection on a **gerbe**, that is a higher generalization of line bundles. We highly recommend [Hit01] for an introduction to the differential geometry of gerbes with applications to the study of special Lagrangian submanifolds and mirror symmetry à la Strominger-Yau-Zaslow.

3.2 The Hull-Strominger system

The Hull-Strominger system first appeared in the physics literature as the equations of motions for the so-called heterotic string in the low-energy limit. Its solutions provide canonical metrics on non-Kähler manifolds – see [Yau09]. In this section, we follow the exposition of [Gar16; GM23].

3.2.1 Statement of the equations and immediate obstructions

Let (M, Ω) be a complex manifold, together with a non-vanishing holomorphic global section of the canonical bundle $\mathcal{K}_M$. This is essentially the definition of a Calabi-Yau manifold that we gave in section 1.3.1, except that we do not require M to be Kähler – in fact, we refer to M as a **non-Kähler Calabi-Yau manifold**. Let (E, h) be a Hermitian bundle on M, the **Hull-Strominger system** for a Hermitian metric ω on M, a unitary connection D_A on (E, h), and a unitary connection ∇ on $\mathcal{T}_M$ – whose curvature we denote by R_∇ – is:

$$\begin{aligned}
\Lambda_\omega \mathrm{F}_A = 0 \qquad \mathrm{F}_A^{0,2} &= 0 \\
\Lambda_\omega \mathrm{R}_\nabla = 0 \qquad \mathrm{R}_\nabla^{0,2} &= 0 && \text{(Hermitian Yang-Mills conditions)} \\
\mathrm{d}\left(\|\Omega\|_\omega\, \omega^{n-1}\right) &= 0 && \text{(Dilatino equation)} \\
\mathrm{i}\partial\overline{\partial}\omega - \alpha\left(\mathrm{tr}\, \mathrm{R}_\nabla \wedge \mathrm{R}_\nabla - \mathrm{tr}\, \mathrm{F}_A \wedge \mathrm{F}_A\right) &= 0 && \text{(Anomaly cancellation condition)}
\end{aligned} \tag{3.6}$$

for some real constant α. The names in parentheses are more or less standard in the literature – the anomaly cancellation condition is also referred to as the **heterotic Bianchi identity**.

We point out that multiple choices exist regarding the connection ∇ in the literature. Our conventions, whereby ∇ is a Hermitian Yang-Mills connection, are natural from a physical point of view, but some authors prefer to use the Chern connection of some Hermitian metric g on M. It is important to note that the holomorphic structure induced on $\mathrm{T}M$ by the connection ∇ – like in section 1.3.1 – does *not always* coincide with the natural holomorphic structure $\mathcal{T}_M$. We denote this non-canonical holomorphic structure by $\mathcal{T}_M^{\nabla}$.

As should by now be routine, we notice at once that the Hull-Stroming system (3.6) imposes *a priori* obstructions to the existence of solutions:

(i) The dilatino equation implies that the renormalized metric $\|\Omega\|_\omega^{\frac{1}{n-1}}\,\omega$ is balanced, and thus that (M,ω) is conformally balanced. This is a non-trivial requirement depending only the complex structure (M,J). We denote the balanced class of the renormalized metric by $\tau \in \mathrm{H}^{n-1,n-1}(M)$.

(ii) From theorem 1.11 and equation (1.11), the Hermitian Yang-Mills conditions imply that a solution to equations (3.6) can only exist if the holomorphic bundles $\mathcal{T}_M^{\nabla}$ and E are τ-polystable with vanishing degree. Notice by the way that the degree of $\mathcal{T}_M^{\nabla}$ is automatically zero as a consequence of $c_1(M) = 0$, since M has a trivial canonical bundle.

(iii) The anomaly cancellation condition imposes an equality of Chern characters $\mathrm{ch}_2(E) = \mathrm{ch}_2(X)$ in degree-four de Rham cohomology. In fact, we may say a bit more: since the two Chern-Weil representatives differ by a $\partial\overline{\partial}$-exact term, we actually have an equality the **Bott-Chern cohomology group** $\mathrm{H}^{2,2}_{\mathrm{BC}}(M)$, that is defined by replacing d-exact by $\partial\overline{\partial}$-exact forms – see for example [Bar23]. On non-Kähler manifolds where the $\partial\overline{\partial}$-lemma is not be verified [Voi96, Proposition 6.17], this is of course a stronger condition, since the map $\mathrm{H}^{2,2}_{\mathrm{BC}}(M) \to \mathrm{H}^4_{\mathrm{dR}}(M)$ has a kernel. Note by the way that, unlike de Rham cohomology, Bott-Chern cohomology is *not* a topological invariant, since it depends on the complex structure on M.

A conjecture of Yau [Yau10] assumed that the above necessary conditions were sufficient – in complex dimension three at least – for the existence of a solution to the Hull-Strominger system. It was recently shown to be false in general by expressing new obstructions in the form of Futaki-like invariants [GM23].

3.2.2 The generalized geometry perspective

We finish by giving a short explanation of the applications of generalized geometry to the study of the Hull-Strominger system.

The equations of motion for the heterotic string in the low energy limit – see e.g. [Gar16, equation (6.1)] – are written on a manifold M with a principal bundle P in terms of – among other fields – a connection A on P, a connection ∇ on $\mathrm{T}M$, and a Kalb-Ramond field strength H defined from a B-field $B \in \mathcal{A}^2(M)$ that is only locally defined. The origin of the anomaly cancellation condition in equation (3.6) comes from the physically motivated Green-Schwarz ansatz for H:

$$H = \mathrm{d}B - \alpha\left(\mathrm{CS}(\nabla) - \mathrm{CS}(A)\right), \tag{3.7}$$

where CS denotes the **Chern-Simons transgression form**, see [Nak03], defined so that:

$$\mathrm{dCS}(\nabla) = -\mathrm{tr}\,\mathrm{R}_\nabla \wedge \mathrm{R}_\nabla \qquad \text{and} \qquad \mathrm{dCS}(A) = -\mathrm{tr}\,\mathrm{F}_A \wedge \mathrm{F}_A.$$

These formulae can of course only hold locally, where cohomological restrictions vanish. Equation (3.7) implies

$$\mathrm{d}H = \alpha\left(\mathrm{tr}\,\mathrm{R}_\nabla \wedge \mathrm{R}_\nabla - \mathrm{tr}\,\mathrm{F}_A \wedge \mathrm{F}_A\right).$$

The upshot is that solutions to the equations of motion verifying the Green-Schwarz ansatz (3.7) end up being equivalent to solutions of the Hull-Strominger system for $H = \mathrm{d}^{\mathrm{c}}\omega$, where we recall that $\mathrm{d}^{\mathrm{c}} := \mathrm{i}\left(\overline{\partial} - \partial\right)$. We point out that the expression $\mathrm{d}^{\mathrm{c}}\omega$ is quite important in Hermitian geometry, as it is related to the torsion of the so-called **Bismut connection** – see [Bar23]. As is explained in [Gar16, Section 3], it turns out that the dilatino equation is in fact equivalent to a restriction of the holonomy of the Bismut connection, much like the Calabi-Yau condition corresponds to a restriction of the holonomy of the Levi-Civita connection.

Compared to our discussion in section 3.1.3, we notice that the form H is not closed. If we are to associate to H an algebroid structure, we thus have to go beyond exact Courant algebroids. The solution is to consider so-called **string algebroids**, that are modelled on a principal bundle with connection on M. Considering the principal bundle of split frames of the sum $E \oplus \mathrm{T}M$ yields the usual Hull-Strominger system. This is explained in [GM23], where the authors also give an interpretation in terms of an analogue of the Hermitian Yang-Mills equations (1.10) for generalized metrics.

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