Cohomology of compact connected Lie groups with Rank $\leqslant 3$

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Abstract

In this paper we will use spectral sequences to deduce the de Rham cohomology of Lie groups with rank $\leqslant 3$.

Contents

1 Introduction

A theorem due to Hopf states that for suitable[1] H-spaces X, its cohomology ring over fields F of characteristic 0 is the tensor product of exterior algebras and polynomial rings. In particular, for a compact connected Lie group G, this implies that $H^*(G;\mathbb{R})$ is an exterior algebra with odd-degree generators. It is also known that for a compact connected Lie group G with maximal torus T, Weyl group W, $H^*(G) \cong [H^*(T) \otimes H^*(G/T)]^W$, the W-invariants of $H^*(T) \otimes H^*(G/T)$ under an action of W. In this paper, we will use the Serre spectral sequence to show this result for compact connected Lie groups of rank $\leqslant 3$. We make no claim of originality in this paper.

Notation and conventions G is a compact and connected Lie group, T is its maximal torus, and W is the Weyl group of G. $l = \dim(T)$ and $v = \frac{1}{2}\dim(G/T)$. All cohomology below will be taken with real coefficients unless otherwise stated. e denotes the identity element of a group.

2 Preliminaries

We will give a brief overview of the preliminaries. For more details see [1].

Proposition 2.1. *W is finite.*

[1] Suitable here means that X is path-connected and the cohomology spaces $H^n(X;F)$ are finite-dimensional

Proof. Let $N(T)$ be the normaliser of T in G. We have a homomorphism $N(T) \to Aut(T)$ given by conjugation. Let C be the connected component of $N(T)$ containing T. Since $Aut(T) \cong GL(l, \mathbb{Z})$ which is discrete, the image of C must be the identity of $GL(l, \mathbb{Z})$. Hence any element of C commutes with T. If $C \neq T$, then there is an element $n \in C$ such that the one-parameter subgroup generated by n is not contained in T. As n commutes with T, the closure of the subgroup generated by T and the one-parameter subgroup is Abelian, hence a torus. However, it strictly contains T, which contradicts maximality. It follows that $C = T$, so $W = N(T)/T = N(T)/C$ is compact and discrete, hence finite. $\square$

Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{t}$ the Lie algebra of T. T acts on $\mathfrak{g}$ via the adjoint action AdT.

Proposition 2.2. *We have a decomposition* $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$*, where* $\mathfrak{m}$ *is invariant under* $Ad(T)$*. In addition,* $\mathfrak{m}$ *splits into the direct sum of* 2*-dimensional irreducible* $Ad(T)$*-representations* $\mathfrak{m} = \bigoplus_{i=1}^{v} m_i$.

Proof. Since G is compact, there exists a positive-definite bilinear form that is invariant under $Ad(G)$, and hence invariant under $Ad(T)$. $\mathfrak{t}$ is clearly $Ad(T)$-invariant, hence the orthogonal complement $\mathfrak{m}$.
If $X \in \mathfrak{g} \setminus \mathfrak{t}$ is fixed by $Ad(T)$, then the one-parameter subgroup generated by $\exp(tX) \notin T$ commutes with T. This implies that T is strictly contained in the torus generated by T and $\exp(tX)$, which is a contradiction. Hence every X in $\mathfrak{m}$ is not fixed by $\mathfrak{m}$, so $\mathfrak{p}$ decomposes into irreducible non-trivial representations of T, which are all 2-dimensional. $\square$

So we obtain an $Ad(G)$-invariant decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$. $w \in W$ now acts on $\mathfrak{t}$ and $\mathfrak{m}$ by $Ad(n)$, where n is some representative of w. When context is clear, we will abuse notation and $Ad(w)$ instead. The action on $\mathfrak{t}$ induces the dual action on $\mathfrak{t}^*$, given by $(w \cdot f)(X) = f(Ad(w)^{-1}X)$ for all $X \in \mathfrak{t}$.

Proposition 2.3. G/T *is a manifold.*

Proof. See [1, Proposition 15.10]. $\square$

Now let Φ be the set of roots of G. Write Φ^+ for the positive roots, and Σ for the simple roots, which are roots that cannot be expressed as the sum of other roots. For a real vector space V, define the complexification $V_{\mathbb{C}} := V \otimes \mathbb{C}$ of V. We have

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{i=1}^{v} \mathfrak{m}_{i\mathbb{C}}$$

Each $\mathfrak{m}_{i\mathbb{C}}$ further decomposes into 1-dimensional subspaces $\mathfrak{X}_\alpha$ and $\mathfrak{X}_{-\alpha}$, where $\mathfrak{X}_\alpha$ is the eigenspace $V(\alpha) := \{X \in \mathfrak{g} \colon Ad(t)X = \alpha(t)X\}$.

Theorem 2.4. *Let* $\alpha \in \Phi$*. Then there exists* X_α *such that* $H_\alpha := [X_\alpha, X_{-\alpha}] \in \mathfrak{t}$ *and the* X_α *span* $\mathfrak{m}$*. Furthermore, for* $\alpha, \beta \in \Phi$*,* $\alpha \neq -\beta$ *we have* $[X_\alpha, X_\beta] \in V(\alpha + \beta)$

Proof. See [1, Chapter 18]. $\square$

W is generated by the reflections s_α across the hyperplane orthogonal to α for $\alpha \in \Sigma$. We can then define the length of $w \in W$ $l(w)$ to be the minimal n such that w can be written as a product of simple reflections[2].
Now consider the complexification $G_{\mathbb{C}}$ of G. This is the complex Lie group that satisfies the following universal property for any complex Lie group H:

$$\begin{tikzcd} G \arrow[r, "i"] \arrow[dr, "\forall"'] & G_{\mathbb{C}} \arrow[d, dashed, "\exists!"] \\ & H \end{tikzcd}$$

Define $\mathfrak{n} := \bigoplus_{\alpha \in \Phi^+} \mathfrak{X}_\alpha$, $\mathfrak{b} := \mathfrak{n} \oplus \mathfrak{t}_{\mathbb{C}}$.

[2]with $l(e) = 0$

Theorem 2.5. *Let $N := \exp(\mathfrak{n}), B = T_{\mathbb{C}}N$. Then N and B are closed Lie subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{n}$ and $\mathfrak{b}$ respectively. Furthermore, There is an embedding $G_{\mathbb{C}} \to GL(n, \mathbb{C})$ for some n such that $G, T_{\mathbb{C}}, B$ consist of unitary, diagonal and upper triangular matrices respectively.*

B is called the standard Borel subgroup of $G_{\mathbb{C}}$

Proof. See [1, Theorem 26.2]. □

Lemma 2.6. $G_{\mathbb{C}}/B \cong G/T$.

Proof. Iwasawa decomposition implies that $G_{\mathbb{C}} = BG$, and $B \cap G = T$ (which can be seen by using the above embedding). Hence we have $G_{\mathbb{C}}/B \cong G/T$ □

3 section 3

The p^{th} symmetric and exterior power of $\mathfrak{t}^*$ are

$$\mathscr{S}^p := \frac{(\mathfrak{t}^*)^{\otimes p}}{\langle v_1 \otimes \cdots \otimes v_p - v_{\sigma 1} \otimes \cdots \otimes v_{\sigma p} \colon \sigma \in Sym(\{1, \cdots, p\}), v_i \in \mathfrak{t}^* \rangle}$$

and

$$\Lambda^p := \frac{(\mathfrak{t}^*)^{\otimes p}}{\langle v_1 \otimes \cdots \otimes v_p - sgn(\sigma) v_{\sigma 1} \otimes \cdots \otimes v_{\sigma p} \colon \sigma \in Sym(\{1, \cdots, p\}), v_i \in \mathfrak{t}^* \rangle}$$

respectively. Note that $\dim(\Lambda^p) = 0$ for $p > l$.
Define

$$\mathscr{S} := \bigoplus_{p=0}^{\infty} \mathscr{S}^p, \Lambda := \bigoplus_{p=0}^{l} \Lambda^p$$

Since $\mathscr{S}^p$ can be thought of as the homogenous polynomials on t^* of degree p, $\mathscr{S}$ can be identified with the polynomials on t^*, i.e. $\mathbb{R}[x_1, \cdots, x_l]$.
The action of W on $\mathfrak{t}^*$ extends to an action on $(\mathfrak{t}^*)^{\otimes p}$, which induces a well-defined action on $\mathscr{S}$ and Λ that preserves degrees.
We have the following theorem due to Chevalley:

Theorem 3.1. *(Chevalley): Let $\mathscr{S}^W$ be the W-invariant polynomials. Then there are algebraically independent homogenous polynomials $J_1, \cdots, J_l$ such that $\mathscr{S}^W = \mathbb{R}[J_1, \cdots J_l]$*

Proof. See [5, page 356]. □

Proposition 3.2. *Let $m_i = \deg(J_i) - 1$. Then $\sum_{i=1}^{l} m_i = v$ and $\prod_{i=1}^{l}(1 + m_i) = |W|$*

Proof. See [5, page 359] □

Now consider the action of $\mathfrak{t}$ on $\mathfrak{t}^*$ by

$$(X \cdot f)(Y) = \frac{d}{dt} f(Y + tX)\bigg|_{t=0}$$

This extends to an action of $\mathscr{D} := \bigoplus_{p=0}^{\infty} S^p$ where S^p is the pth symmetric power of $\mathfrak{t}$.
Define the harmonic polynomials as

$$\mathscr{H} := \left\{ f \in \mathscr{S} \colon \mathscr{D}^W f = 0 \right\}$$

Consider $\mathscr{H}^p := \mathscr{H} \cap \mathscr{S}^p$.

Proposition 3.3. $\mathscr{H} = \bigoplus_{p=0}^{\infty} \mathscr{H}^p$

Proof. Note that an element of $\mathscr{D}$ is W-invariant iff all of its homogenous components are. Consider $f \in \mathscr{H}$ of degree n. Write f as the sum of its homogenous components, $f = \sum_{k=0}^{n} f_k$, where f_k has degree k. Consider a homogenous element $D \in \mathscr{D}^W$ of degree q. This sends a polynomial in $\mathscr{S}^p$ to a polynomial in $\mathscr{S}^{p-q}$. $D(f) = \sum_{k=0}^{n} D(f_k) = 0$ as f is harmonic, and since non-zero homogenous polynomials of different degrees are linearly independent, the $D(f_k)$ must all be zero. Hence the f_k are harmonic. $\square$

Now let I be the ideal generated by the positive degree elements of $\mathscr{S}^W$.

Proposition 3.4. *$\mathscr{S} = \mathscr{H} \oplus I$ and there is an isomorphism of vector spaces $\mathscr{H} \otimes \mathscr{S}^W \xrightarrow{\sim} \mathscr{S}$ induced by multiplication.*

Proof. See [5, page 381(or 360 i cant tell)] $\square$

Proposition 3.5.

$$\sum_{p=0}^{\infty} \dim \mathscr{H}^p t^p = \prod_{i=1}^{l} (1 + t + \cdots + t^{\deg F_i - 1})$$

Hence $\dim \mathscr{H}^v = 1$ *and* $\dim \mathscr{H}^p = 0$ *for* $p > v$.

Proof. Since we have an isomorphism $\mathscr{S} \cong \mathscr{H} \otimes \mathscr{S}^W \cong \mathscr{H} \otimes \bigotimes_{i=1}^{l} \mathbb{R}[F_i]$, we have

$$\sum_{p=0}^{\infty} \dim(\mathscr{S}^p) t^p = \left(\sum_{p=0}^{\infty} \dim(\mathscr{H}^p) t^p \right) \prod_{i=1}^{l} \left(\sum_{p=0}^{\infty} t^{p \deg(F_i)} \right) = \sum_{p=0}^{\infty} \dim(\mathscr{H}^p) t^p \prod_{i=1}^{l} \frac{1}{1 - t^{\deg(F_i)}}$$

$\mathscr{S}^p$ is the vector space of homogenous polynomials of degree p, so it has basis $\left\{ x_1^{k_1} \cdots x_l^{k_l} : k_1 + \cdots + k_l = p \right\}$ which implies $\dim(\mathscr{S}^p) = \binom{l+p-1}{p}$. Hence

$$\sum_{p=0}^{\infty} \dim(\mathscr{H}^p) t^p = \prod_{i=1}^{l} \left(1 - t^{\deg(F_i)} \right) \sum_{p=0}^{\infty} \binom{l+p-1}{p} t^p = \left(\sum_{p=0}^{\infty} \binom{l+p-1}{p} t^p (1-t)^l \right) \prod_{i=1}^{l} (1 + t + \cdots + t^{\deg F_i - 1})$$

We recognize $\binom{l+p-1}{p} t^p (1-t)^l$ as the probability mass function of a negative binomial variable with parameters l and $1-t$, and hence the sum is 1. The result follows. $\square$

In particular, this implies that $\dim(\mathscr{H}^v) = 1$ and $\dim(\mathscr{H}^p) = 0$ for $p > v$.

Proposition 3.6. *Let*

$$\Pi := \prod_{\alpha \in \Phi^+} \alpha$$

Then Π *is harmonic.*

Proof. See [5, page 361]. $\square$

Theorem 3.7. *(Solomon): $(\mathscr{S} \otimes \Lambda)^W$ is a free $\mathscr{S}^W$ module with basis*

$$\left\{ dF_{i_1} \wedge \cdots \wedge dF_{i_q} : 1 \leqslant i_1 \leqslant \cdots \leqslant i_q \leqslant l, 1 \leqslant q \leqslant l \right\}$$

Proof. See [5, page 363] $\square$

Lemma 3.8. *Let G be a finite group, V be a finite-dimensional representation of G over a subfield $\mathbb{K} \subseteq \mathbb{C}$. Then* $\dim(V) = \dim([\mathbb{K}[G] \otimes V]^G)$

Proof. Assume $\mathbb{K} = \mathbb{C}$. The map $\pi := \frac{1}{|G|}\sum_{g\in G} g$ is the projection of V onto the G-invariants V^G. Since the trace of the projection map is equal to the dimension of the image, $\dim(V^G) = \mathrm{Tr}(\pi) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$ where χ_V is the character of the representation. This implies that

$$\dim([\mathbb{C}[G]\otimes V]^G) = \frac{1}{|G|}\sum_{g\in G}\chi_{\mathbb{C}[G]\otimes V}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_{\mathbb{C}[G]}(g)\chi_V(g)$$

Since $\chi_{\mathbb{C}[G]}(g) = \begin{cases} |G| \text{ if } g = e \\ 0 \text{ otherwise} \end{cases}$, we have

$$\dim([\mathbb{C}[G]\otimes V]^G) = \frac{1}{|G|}\chi_{\mathbb{C}[G]}(e)\chi_V(e) = \frac{1}{|G|}|G|\dim(V) = \dim(V)$$

So the claim is true when $\mathbb{K} = \mathbb{C}$. For arbitrary $\mathbb{K} \subseteq \mathbb{C}$, note that we have $\dim_{\mathbb{K}}(V) = \dim_{\mathbb{C}}(V\otimes\mathbb{C}), \dim_{\mathbb{K}}(V^G) = \dim_{\mathbb{C}}([V\otimes\mathbb{C}]^G)$, so the statement is true over arbitrary subfields of $\mathbb{C}$ as well. □

In particular, it is true over $\mathbb{R}$

Corollary 3.9. *$\{dJ'_{i_1}\wedge\cdots\wedge dJ'_{i_k} : 1 \leqslant k \leqslant l\}$ is a basis for the W-invariants of $(\mathscr{S}/I\otimes\Lambda)$ (as an $\mathbb{R}$-vector space), where dJ'_i is dJ_i with its coefficients reduced modulo I.*

Proof. Theorem 3.6 implies that $\{dJ'_{i_1}\wedge\cdots\wedge dJ'_{i_k} : 1 \leqslant k \leqslant l\}$ spans the W-invariants of $(\mathscr{S}/I\otimes\Lambda)$, and since $\dim([\mathscr{S}/I\otimes\Lambda]^W) = \dim(\Lambda) = 2^l$ by Lemma 3.8, it is indeed a basis. □

4 Left-invariant differential forms

For more details, see [2].

Let M be a manifold with an action τ of G on it, with each τ_g being a diffeomorphism. The left-invariant forms on M are the forms such that $\tau_g^*\omega := \omega$, τ_g^* the pullback of τ_g.

We can naturally identify the left-invariant forms on M with the dual space of the tangent space at a point.

Given a form ω on M, we can define a left-invariant form

$$\tilde{\omega} := \int_G L_g^*\omega \,\mathrm{d}\mu$$

where μ is the unique Haar measure on G with $\mu(G) = 1$. The map $\omega \mapsto \tilde{\omega}$ commutes with the exterior derivative, and hence induces a chain map on the chain complexes of the left-invariant differential forms and the ordinary differential forms.

Theorem 4.1. *The cohomology ring of the left-invariant differential forms is the same as the cohomology ring of the ordinary differential forms. In addition, the above map induces a chain homotopy between the two chain complexes.*

Proof. See [2] □

Hence we can identify the differential forms on G, T and G/T with elements on $\mathfrak{g}^*, \mathfrak{t}^*$, and T_T^* respectively, where T_T is the tangent space of G/T at $T \in G/T$. By considering the differential of the projection $G \to G/T$, we can identify T_T with $\mathfrak{m}$.

For a manifold M, smooth vector fields X_i, and an n-form ω on M, we have

$$\begin{aligned} d\omega(X_1,\cdots,X_{n+1}) = &\sum_{1\leqslant i\leqslant n+1}(-1)^i X_i(\omega(X_1,\cdots,\hat{X}_i,\cdots,X_{n+1}) \\ &+ \sum_{1\leqslant i<j\leqslant n+1}(-1)^{i+j}\omega([X_i,X_j],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{n+1}) \end{aligned}$$

where the hat denotes removal. So for left-invariant forms on we just have

$$d\omega(X_1,\cdots,X_{n+1}) = \sum_{1\leqslant i<j\leqslant n+1}(-1)^{i+j}\omega([X_i,X_j],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{n+1}) \qquad (*)$$

5 Structure of $H^*(T)$ and $H^*(G/T)$

We start by determining the structure of $H^*(T)$.

Lemma 5.1. *Let $T = (S^1)^l$ be a torus of dimension l. Then $H^*(T) \cong \Lambda[dx_1, \cdots, dx_l]$ where $dx_1, \cdots, dx_l$ are a basis for $H^1(T) \cong \mathfrak{t}^*$.*

Proof. This is a standard application of the Künneth formula. □

Now consider the complexification $G_{\mathbb{C}}$ of G.

Theorem 5.2. *Let W be the Weyl group G, and B the standard Borel subgroup of $G_{\mathbb{C}}$. Then we have a decomposition $G_{\mathbb{C}} = \coprod_{w \in W} BwB$.*

Proof. See [1, Chapter 27] □

This gives a decomposition $G_{\mathbb{C}}/B = \coprod_{w \in W} BwB/B$.

Theorem 5.3. *The Bruhat decomposition gives a cellular structure on $G_{\mathbb{C}}/B \cong G/T$, with cells BwB/B. Furthermore, the interior of BwB/B is homeomorphic to $\mathbb{C}^{l(w)}$ where $l(w)$ is the length of w.*

Proof. See [7] □

Since each cell is even-dimensional, cellular cohomology implies that $H^*(G/T)$ only has even-degree elements, and as an $\mathbb{R}$-vector space it has dimension W.
Let $w \in W$ act on G/T by $w \cdot gT = gn^{-1}T$ where n is some representative of W in the normalizer of T in G. (This action is chosen because it induces the adjoint action on $\mathfrak{m}$.)

Proposition 5.4. *The W-module structure of $H^*(G/T)$ induced by the action of W on $H^*(G/T)$ is isomorphic to $\mathbb{R}[W]$, the regular representation of W.*

Proof. Consider the action of W on G/T. It suffices to show that for any $w \in W$ the trace of the maps $w \colon H^*(G/T) \to H^*(G/T)$ and $W \colon \mathbb{R}[W] \to \mathbb{R}[W]$, since $H^*(G/T)$ and $\mathbb{R}[W]$ have the same dimension as $\mathbb{R}$-vector spaces. For any non-identity $w \in W$, w clearly has no fixed points. Since G/T is compact, by the Lefschetz fixed-point theorem we have

$$\sum_{k \geqslant 0} (-1)^k \operatorname{Tr}\big(w^* \colon : H^k(G/T) \to H^k(G/T)\big) = 0$$

Since there is no cohomology in the odd degrees, this implies that the trace of the induced map on cohomology w^* is 0.
For the identity $e \in W$, the trace is clearly $|W|$. Clearly this is the same for $\mathbb{R}[W]$, so they are isomorphic as W-modules. □

Corollary 5.5. *$H^*(G/T)$ has no non-trivial W-invariants in non-zero degrees.*

Proof. $H^*(G/T) \cong \mathbb{R}[W]$ as W-modules, and

$$w \cdot \sum_{w \in W} \alpha_w w = \sum_{w \in W} \alpha_w w \; \forall w \in W \Rightarrow \alpha_e = \alpha_w \; \forall w \in W$$

as W is a basis for $\mathbb{R}[W]$. Clearly $\sum_{w \in W} w$ is W-invariant, so the space of W-invariants in $\mathbb{R}[W]$ is 1-dimensional. Representing each class in $H^0(G/T)$ as locally constant smooth functions from G/T to $\mathbb{R}$, we see that for such functions f, w acts by $(w \cdot f)(gT) = f(w \cdot gT) = f(gT)$ as G/T is connected. So $H^0(G/T)$ is a 1-dimensional subspace with each of its elements being W-invariant, and hence is precisely the space of W-invariants. This implies that $H^{2n}(G/T)$ does not have any non-zero W-invariants for any $n \geqslant 1$. □

Now we have the following theorem due to Borel which will allow us to determine the structure of $H^*(G/T)$.

Theorem 5.6. *(Borel): There is a degree-doubling W-equivariant isomorphism*

$$\bar{c}\colon \mathscr{S}/I \to H^*(G/T)$$

Proof. We can treat the classes in $H^*(G/T)$ as left-invariant forms by the arguments in the previous section. Let $\lambda \in \mathfrak{t}^*$. We can extend it to an element of $\mathfrak{g}^*$ by setting it to be 0 on $\mathfrak{m}$. Now define a 2-form ω_λ on $\mathfrak{g}$ by $\omega(X,Y) = \lambda([X,Y])$. Restricting to G/T, we get a form ω_λ corresponding to a left-invariant form on G/T. $w \in W$ acts on both $\mathfrak{t}$ and $\mathfrak{m}$ by $Ad(w)$[3], so $\omega_{w\lambda}(X,Y) = \lambda(Ad(w)[X,Y]) = \lambda([Ad(w)X, Ad(w)Y]) = (w \cdot \omega_\lambda)(X,Y)$. Hence the map preserves the W-action.

Now $\delta\omega(X,Y,Z) = -\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X) = \lambda([Z,[X,Y]] + [Y,[Z,X]] + [X,[Y,Z]]) = 0$. This implies that $\delta\omega_\lambda = 0$, and hence $[\tilde{\omega_\lambda}] \in H^2(G/T)$, and we then define $c(\lambda) = [\tilde{\omega_\lambda}]$. This then extends to a degree-doubling homomorphism $c\colon \mathscr{S} \to H^*(G/T)$ which preserves the W-action. Since positive-degree elements are mapped to positive-degree elements and $H^{2n}(G/T)$ has no non-zero W-invariants, this implies that $c(I) = 0$. So c induces a homomorphism $\bar{c}\colon \mathscr{S}/I \to H^*(G/T)$. As $\mathscr{S} \cong I \oplus \mathscr{H}$ and $\dim(\mathscr{H}) = \dim(H^*(G/T)) = |W|$, it suffices to show that the restriction of c to $\mathscr{H}$ is injective. We will show this by a (backwards) induction argument.

To start off, we prove that $c\colon \mathscr{H}^v \to H^{2v}(G/T)$ is injective. As $\{\Pi\}$ is a basis for $\mathscr{H}^v$, it suffices to show that $c(\Pi) \neq 0$. For a positive root α_i, Write $\omega_i = \omega_{\alpha_i}$, so $c(\Pi) = c(\prod_{i=1}^v \alpha_i) = [\tilde{\omega_1} \wedge \cdots \wedge \tilde{\omega_v}]$. As G/T is orientable, for any non-zero left-invariant[4] $2v$-form ω we have

$$\int_{G/T} \omega \neq 0$$

Hence it suffices to show that $\tilde{\omega_1} \wedge \cdots \wedge \tilde{\omega_v} \neq 0$. We have

$$\begin{aligned}
&\omega_1 \wedge \cdots \wedge \omega_v(X_{\alpha_1}, X_{-\alpha_1}, \cdots, X_{\alpha_v}, X_{-\alpha_v}) \\
=&\frac{1}{(2v)!} \sum_{\sigma \in Sym(X_{\alpha_1}, X_{-\alpha_1}, \cdots, X_{\alpha_v}, X_{-\alpha_v})} sgn(\sigma)\omega_1(\sigma X_{\alpha_1}, \sigma X_{-\alpha_1}) \cdots \omega_v(\sigma X_{\alpha_n}, \sigma X_{-\alpha_n}) \\
=&\frac{1}{(2v)!} \sum_{\sigma \in Sym(X_{\alpha_1}, X_{-\alpha_1}, \cdots, X_{\alpha_v}, X_{-\alpha_v})} sgn(\sigma)\alpha_1([\sigma X_{\alpha_1}, \sigma X_{-\alpha_1}]) \cdots \alpha_v([\sigma X_{\alpha_n}, \sigma X_{-\alpha_n}])
\end{aligned}$$

Now $\alpha_i([\sigma X_{\alpha_i}, \sigma X_{-\alpha_i}]) = 0$ unless $[\sigma X_{\alpha_i}, \sigma X_{-\alpha_i}] \in \mathfrak{t}$, so $sgn(\sigma)\alpha_1([\sigma X_{\alpha_1}, \sigma X_{-\alpha_1}]) \cdots \alpha_v([\sigma X_{\alpha_n}, \sigma X_{-\alpha_n}])$ is non-zero only if σ permutes the pairs $\{X_{\alpha_i}, X_{-\alpha_i}\}$, and possibly switches the members of each pair. For such σ, we clearly have $sgn(\sigma) = (-1)^{\text{number of pair switches}}$, so

$$\begin{aligned}
&\omega_1 \wedge \cdots \wedge \omega_v(X_{\alpha_1}, X_{-\alpha_1}, \cdots, X_{\alpha_v}, X_{-\alpha_v}) \\
=&\frac{1}{(2v)!} \sum_{\sigma \in Sym(H_{-\alpha_1}, \cdots, H_{\alpha_v})} 2^v \alpha_1(\sigma H_{\alpha_1}) \cdots \alpha_v(\sigma H_{-\alpha_v}) \\
=&\frac{2^v}{(2v)!} \partial_1 \cdots \partial_v \Pi
\end{aligned}$$

where $\partial_i \in \mathscr{D}$ is the derivation extending $\lambda \mapsto \lambda(H_{\alpha_i})$. We have a perfect pairing

$$\mathscr{D} \otimes \mathscr{S} \to \mathbb{R}$$

given by $(D,f) \mapsto (Df)(0)$. This implies that there is some $D \in \mathscr{D}^k$ such that $Df \neq 0$. An irreducible W-module can only pair non-trivially with its dual, so $\mathscr{H}^v$ must pair with $\mathscr{D}^v$. Since Π affords the self-dual sign character ϵ, which occurs with multiplicity one in $\mathscr{D}^v$ and is afforded by $\partial_1 \cdots \partial_v$, this implies that $\partial_1 \cdots \partial_v \Pi \neq 0$.

Hence $c(\Pi) \neq 0$, so $c\colon \mathscr{H}^v \to H^{2v}(G/T)$ is injective.

Now assume that $c\colon \mathscr{H}^j \to H^{2j}(G/T)$ is injective for all $v \geqslant j > k$, and consider $c\colon \mathscr{H}^k \to H^{2k}(G/T)$. Define $V = \mathscr{H}^k \cap \mathrm{Ker}(c)$. As c preserves the W-action, V is W-invariant. As $v > k$, the sign character does not occur

[3]We will abuse notation and refer to w as both a representative of w in G and just w in W proper when the context is clear

[4]Note that a non-zero form that is not left-invariant does not necessarily have non-zero integral

in $\mathscr{H}^k$, and hence not in V. So there exists α such that the reflection s_α does not act by $-I$ on V. Fix this α. Decompose $V = V_+ \oplus V_-$ according to the eigenspaces of s_α. Since V is W-invariant, it is invariant under the action of s_α. In particular, for $f \in V$ we have $f + s_\alpha f \in V$, which is s_α-invariant. By assumption on α, if $V \neq 0$, then $V_+ \neq 0$. So take $f \in V_+$ with f s_α-invariant. Now $c(\alpha f) = c(\alpha)c(f) = 0$, and $\deg(\alpha f) = k$ so by hypothesis $\alpha f \in I$. Choose a basis $\{h_1, \cdots h_{|W|}\}$ for $\mathscr{H}$, with $h_1, \cdots h_r$ s_α-skew and the rest s_α-invariant for some r. Since $\mathscr{S} \cong \mathscr{H} \otimes \mathscr{S}^W$, we can write $\alpha f = \sum_i h_i \sigma_i$ where $\sigma_i \in \mathscr{S}^W$. αf is s_α-skew, so this sum only goes up to r. Considering α as an element of $\mathfrak{t}^*$, we have that for $i \leqslant r$, h_i must vanish on $\mathrm{Ker}(\alpha)$ as they are s_α skew and the vectors in $\mathrm{Ker}(\alpha)$ are s_α invariant, so $h_i = \alpha h_i'$ for some $h_i' \in \mathscr{S}$. This implies that $f = \sum_i h_i' \sigma_i$, so $f \in I$. But $f \in \mathscr{H}$, so $f = 0$. Hence $c\colon \mathscr{H}^k \to H^{2k}(G/T)$ is injective. This closes the induction, and so we are done. □

6 Spectral sequences

Consider a fibration $F \to X \to B$ with B path-connected. Elements of $\pi_1(B)$ are homotopies of maps $* \to B$ mapping to a fixed point x_0, which then lift to homotopy equivalences of the fiber $F_{x_0} \to F_{x_0}$ by the homotopy lifting property of the fibration. This induces an action of $\pi_1(B)$ on $H^*(F)$. We are interested in the case where this action is trivial.[5]

Theorem 6.1. *(Serre spectral sequence for cohomology): For a fibration $F \to X \to B$ with B path-connected and $\pi_1(B)$ acting trivially on $H^*(F;G)$, there is a spectral sequence $\{E_r^{p,q}, d_r\}$ with*
a) $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ and $E_{r+1}^{p,q} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$ at $E_r^{p,q}$
b) Stable terms $E_\infty^{p,n-p}$ isomorphic to successive quotients F_p^n/F_{p+1}^n in a filtration $0 = F_{n+1}^n \subseteq F_n^n \subseteq \cdots \subseteq F_0^n = H^n(X;G)$ of $H^n(X;G)$
c) $E_2^{p,q} \cong H^p(B;H^q(F;G))$

Proof. See [4, page 26] □

Now for our fibration $T \to G \to G/T$, since we can give G/T a cell structure with only even-dimensional cells, $\pi_1(G/T)$ is trivial. Hence we indeed have the Serre cohomological spectral sequence as described above.
Lemma: We have $E_2^{p,q} \cong H^p(G/T) \otimes H^q(T)$.
Proof: There is a canonical isomorphism $V \otimes W \to \mathrm{Hom}(\mathrm{Hom}(V,\mathbb{R}), W)$ given by $v \otimes w \mapsto (f \mapsto f(v)w)$. Hence

$$\begin{aligned} E_2^{p,q} &\cong H^p(G/T; H^q(T)) \\ &\cong \mathrm{Hom}(H_p(G/T), H^q(T)) \text{ by the universal coefficient theorem} \\ &\cong H^p(G/T) \otimes H^q(T) \text{ as } H^p(G/T) \cong \mathrm{Hom}(H_p(G/T), \mathbb{R}) \text{ by the universal coefficient theorem} \end{aligned}$$

Now let R be a commutative ring.

Theorem 6.2. *Each page $E_r^{*,*}$ can be provided with a bilinear product $E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t}$ such that*
a) For $x \in E_r^{p,q}, y \in E_r^{s,t}$, d_r satisfies $d_r(xy) = d_r(x)y + (-1)^{p+q} x d_r(y)$. This implies that the project on the $E_r^{*,*}$ page induces a product on the $E_{r+1}^{*,*}$ page, which coincides with the product on the $E_{r+1}^{*,*}$ page. The products on the $E_r^{*,*}$ pages for r finite also induce the product on $E_\infty^{*,*}$
b) The product $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$ is $(-1)^{qs}$ times the cup product

$$H^p(B; H^q(F;R)) \times H^s(B; H^t(F;R)) \to H^{p+s}(B; H^{q+t}(F;R))$$

where coefficients are multiplied via the cup product as well.
c) The cup product on $H^*(X;R)$ restricts to products $F_p^m \times F_s^n \to F_{p+s}^{m+n}$, which induce products $F_p^m/F_{p+1}^m \times F_s^n/F_{s+1}^n \to F_{p+s}^{m+n}/F_{p+s+1}^{m+n}$. This is precisely the product on the $E_r^{*,*}$ page.

[5]One can use local coefficients to deal with the case when the action is non-trivial

Proof. For a), refer to [6, page 11], for b) and c), refer to [4, page 28]. □

The nth transgression map

$$\begin{tikzcd} & H^n(X,F) \arrow[dll, dashed] & H^n(F) \arrow[l, "\delta"'] \\ H^n(B) & H^n(B,b) \arrow[l, "j^*"] \arrow[u, "p^*"] & \end{tikzcd}$$

is the map which maps elements of $\delta^{-1}(\mathrm{Im}\, p^*) \subset H^n(F)$ to $H^{n+1}(B)/j^*(\mathrm{Ker}\, p^*)$. In general, it will not map to $H^{n+1}(B)$ proper since p^* need not be an isomorphism.

Proposition 6.3. *The transgression map above is the differential $d_n^{n,0} : E_n^{n,0} \to E_n^{0,n+1}$.*

Proof. See [4, page 25] for the proof in the homology spectral sequence. The proof for the cohomology spectral sequence is analogous. □

Proposition 6.4. *The differential $d_2^{0,1} : H^1(T) \to H^2(G/T)$ is the map $[\lambda] \mapsto [\omega_\lambda]$ where $\omega_\lambda(X,Y) := -\tilde{\lambda}([X,Y])$, where $\tilde{\lambda}$ is λ extended to all of $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ by setting it to 0 on $\mathfrak{m}$.*

Proof. We know that $d_2^{0,1} : H^1(T) \to H^2(G/T)$ is the 2nd transgression map. Since G/T is a manifold, each point on G/T has a contractible neighbourhood, so by the homotopy lifting property of the fibration $T \to G \to G/T$ we know that T is a (weak) deformation retract of one of its neighbourhoods. Excision then implies that the map p^* above is an isomorphism. Hence we can identify $d_2 : H^1(T) \to H^2(G/T)$ with the coboundary map $\delta : H^1(T) \to H^2(G/T)$ in the cohomology long exact sequence associated to $T \to G \to G/T$.

$$\begin{tikzcd} & C^2(G/T) & C^2(G/T) \arrow[l, "j^*"'] \\ C^1(T) & C^1(G) \arrow[l, "i^*"'] \arrow[u, "\delta"] & \end{tikzcd}$$

Identifying with left-invariant forms, the maps i^* and j^* are restricting to $\mathfrak{t}$ and extending to all of $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{t}$ by setting it to be 0 on $\mathfrak{t}$. We have $\delta\lambda(X,Y) = -\lambda([X,Y])$, so the map map sends $[\lambda]$ to $[\omega_\lambda]$ above. □

Write $x_i := \omega_{dx_i}$, and for $\phi \in H^{2n}(G/T), \psi \in H^q(T), X_i, Y_i \in \mathfrak{m}, H_i \in \mathfrak{t}$, define

$$(\phi \otimes \psi)(X_1, Y_1, \cdots X_n, Y_n, H_1, \cdots, H_q) = \phi(X_1, Y_1, \cdots X_n, Y_n)\psi(H_1, \cdots, H_q)$$

Corollary 6.5. *The differential $d_2 : H^{2n}(G/T) \otimes H^q(T) \to H^{2n+2}(G/T) \otimes H^{q-1}(T)$ is the map $[\lambda] \mapsto [\omega_\lambda]$ where*

$$\begin{aligned} &\omega_\lambda(X_1, Y_1, \cdots, X_n, Y_n, X_{n+1}, Y_{n+1}, H_1, \cdots, H_{q-1}) \\ &= \frac{1}{n!} \sum_{\sigma \in Sym(X_1, Y_1, \cdots X_{n+1}, Y_{n+1})} sgn(\sigma)\lambda(\sigma X_1, \sigma Y_1, \cdots, \sigma X_n, \sigma Y_n, [\sigma X_{n+1}, \sigma Y_{n+1}]_{\mathfrak{t}}, H_1, \cdots, H_{q-1}) \end{aligned}$$

Proof. Consider the case when $n = 0$. $H^q(T)$ has basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_q} : 1 \leqslant i_1 < \cdots < i_q \leqslant l\}$, and we have $\omega_{dx_{i_1} \wedge \cdots \wedge dx_{i_q}} = x_{i_1} dx_{i_2} \wedge \cdots \wedge dx_{i_q} - x_{i_2} dx_{i_1} \wedge dx_{i_3} \wedge \cdots \wedge dx_{i_q} + \cdots + (-1)^{q+1} x_{i_q} dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}}$.
Now let $X, Y \in \mathfrak{m}, H_1, \cdots, H_{q-1} \in \mathfrak{t}$. Then

$$\begin{aligned} &\omega_{dx_{i_1} \wedge \cdots \wedge dx_{i_q}}(X, Y, H_1, \cdots, H_{q-1}) \\ =& x_{i_1}(X,Y) dx_{i_2} \wedge \cdots \wedge dx_{i_q}(H_1, \cdots, H_{q-1}) + \cdots + (-1)^{q+1} x_{i_q}(X,Y) dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}}(H_1, \cdots, H_{q-1}) \\ =& dx_{i_1}([X,Y]) dx_{i_2} \wedge \cdots \wedge dx_{i_q}(H_1, \cdots, H_{q-1}) + \cdots + (-1)^{q+1} dx_{i_q}([X,Y]) dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}}(H_1, \cdots, H_{q-1}) \\ =& dx_{i_1} \wedge \cdots \wedge dx_{i_q}([X,Y], H_1, \cdots, H_{q-1}) \end{aligned}$$

So the claim is true when $n=0$. Now for $p \in H^{2n}(G/T)$, $\omega_{pdx_{i_1}\wedge\cdots\wedge dx_{i_q}} = p\omega_{dx_{i_1}\wedge\cdots\wedge dx_{i_q}} = px_{i_1}dx_{i_2}\wedge\cdots\wedge dx_{i_q} + \cdots + (-1)^{q+1}px_{i_q}dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}}$. Since for $p \in H^{2n}(G/T), q \in H^{2m}(G/T), X_1, Y_1, \cdots X_{n+m}, Y_{n+m} \in \mathfrak{m}$ we have

$$\begin{aligned}
&pq(X_1, Y_1, \cdots, X_{n+m}, Y_{n+m})\\
=&\frac{1}{n!m!}\sum_{\sigma\in Sym(X_1,Y_1,\cdots,X_{n+m},Y_{n+m})} sgn(\sigma)p(\sigma X_1, \sigma Y_1, \cdots, \sigma X_n, \sigma Y_n)q(\sigma X_{n+1}, \sigma Y_{n+1}, \cdots, \sigma X_{n+m}, \sigma Y_{n+m})
\end{aligned}$$

the claim is true for $pdx_{i_1}\wedge\cdots\wedge dx_{i_q}$. The result then follows by the fact that the $pdx_{i_1}\wedge\cdots\wedge dx_{i_q}$ generate $H^{2n}(G/T)\otimes H^q(T)$. □

Proposition 6.6. *The differential $d_2 : H^{2n}(G/T)\otimes H^1(T) \to H^{2n+2}(G/T)$ is surjective.*

Proof. Identify $H^*(G/T)$ with $\mathscr{S}/I$. Then $d_2 : H^{2n}(G/T)\otimes H^q(T) \to H^{2n+2}(G/T)\otimes H^{q-1}(T)$ is the map $p(t)\otimes\lambda \mapsto p(t)d_2(\lambda)$. It is clear that $d_2^{0,1} : H^1(T) \to H^2(G/T)$ is (up to sign) the map $c : \mathscr{S}^1 \to H^2(G/T)$, so $d_2^{0,1}$ is surjective. From the identification of $H^*(G/T)$ with $\mathscr{S}/I$ we also know that $H^2(G/T)$ generates $H^*(G/T)$ (as an $\mathbb{R}$-algebra). The result then follows by induction. □

Proposition 6.7. *The differential $d_2^{2n,q} \colon H^{2n}(G/T)\otimes H^q(T) \to H^{2n+2}(G/T)\otimes H^{q-1}(T)$ is W-equivariant*

Proof. From the proof of Theorem 6.1 and Proposition 7.3, we see that $d_2^{0,1} \colon H^1(T) \to H^2(G/T)$ is W-equivariant, so for all $p \in H^{2n}(G/T)$

$$\begin{aligned}
&d_2^{2n,q}(w\cdot(pdx_{i_1}\wedge\cdots\wedge dx_{i_q}))\\
=&d_2^{2n,q}((w\cdot p)(w\cdot dx_{i_1})\wedge\cdots\wedge(w\cdot dx_{i_q}))\\
=&(w\cdot p)d_2^{2n,q}((w\cdot dx_{i_1})\wedge\cdots\wedge(w\cdot dx_{i_q}))\\
=&(w\cdot p)[d_2^{0,1}((w\cdot dx_{i_1}))(w\cdot dx_{i_2})\wedge\cdots\wedge(w\cdot dx_{i_q})+\\
&\cdots+(-1)^{q+1}d_2^{0,1}((w\cdot dx_{i_q}))(w\cdot dx_{i_1})\wedge\cdots\wedge(w\cdot dx_{i_{q-1}})]\\
=&w\cdot[p(d_2^{0,1}(dx_{i_1})dx_{i_2}\wedge\cdots\wedge dx_{i_q}+\cdots+(-1)^{q+1}d_2^{0,1}(dx_{i_q})dx_{i_1}\wedge\cdots\wedge dx_{i_{q-1}})]\\
=&w\cdot d_2^{2n,q}(pdx_{i_1}\wedge\cdots\wedge dx_{i_q})
\end{aligned}$$

The $pdx_{i_1}\wedge\cdots\wedge dx_{i_q}$ generate $H^{2n}(G/T)\otimes H^q(T)$, so the result is proved. □

Write the W-invariants of $H^{2n}(G/T)\otimes H^q(T)$ as $[H^{2n}(G/T)\otimes H^q(T)]^W$.

Proposition 6.8. $[H^{2n}(G/T)\otimes H^q(T)]^W \subseteq \mathrm{Ker}(d_2^{2n,q})$

Proof. There are no no-zero W-invariants in $H^{2n}(G/T)$ for $n>0$. As $d_2^{2n,1}$ is W-equivariant, this implies that $[H^{2n}(G/T)\otimes H^1(T)]^W \in \mathrm{Ker}(d_2^{2n,1})$ for all $n \geqslant 0$. Now apply Corollary 1 and the fact that $d_2^{2n,q}$ is a derivation. □

Proposition 6.9. $[H^{2n}(G/T)\otimes H^q(T)]^W \cap \mathrm{Im}(d_2^{2n-2,q+1}) = \{0\}$

Proof: Assume that $\lambda \in \mathrm{Im}(d_2^{2n-2,q+1})$ is W-invariant. Then there exists $\mu \in H^{2n-2}(G/T)\otimes H^{q+1}(T)$ such that $d_2^{2n-2,q+1}(\mu) = \lambda$. Hence

$$d_2^{2n-2,q+1}(\sum_{w\in W} w\mu) = \sum_{w\in W} wd_2^{2n-2,q+1}(\mu) = \sum_{w\in W} w\lambda = |W|\lambda$$

as λ is W-invariant. However $\sum_{w\in W} w\mu$ is W-invariant by construction, so we must have $|W|\lambda = 0$ by Proposition 6.8. Hence $\lambda = 0$.

Since $[H^{2n}(G/T)\otimes H^q(T)]^W, Im(d_2^{2n-2,q+1}) \subseteq Ker(d_2^{2n,q})$, this shows us that

$$[H^{2n}(G/T)\otimes H^q(T)]^W \oplus \mathrm{Im}(d_2^{2n-2,q+1}) \subseteq \mathrm{Ker}(d_2^{2n,q})$$

Proposition 6.10. *Assume that* $[H^{2n}(G/T)\otimes H^q(T)]^W \oplus Im(d_2^{2n-2,q+1}) = Ker(d_2^{2n,q})$.
Then $E_3^{p,q} \cong [H^p(G/T)\otimes H^q(T)]^W$, *and the spectral sequence degenerates at* E_3. *Moreover,* $H^*(G) \cong [H^*(T)\otimes H^*(G/T)]^W$.

Proof. The first statement is obvious. For the second, note that for $r \geqslant 3$ $d_r^{2n,1} : E_r^{2n,1} \to E_r^{2n+r,2-r}$ is the zero map, as $E_3^{2n+r,2-r} \cong 0$. Since d_3 is a derivation, and the products of the elements in $E_3^{2n,1}$ generate the E_3 page, all the differentials on the E_3 page are zero. The product on E_{r+1} is induced by the product on E_r, so by induction this implies that all the differentials on E_r for $r \geqslant 3$ are zero. This proves the second claim.
For the last claim, Theorem 7.1 (b) tells us that there is a filtration $0 = F_{n+1}^n \subseteq F_n^n \subseteq \cdots \subseteq F_0^n = H^n(G)$ such that $E_\infty^{p,n-p} \cong F_p^n/F_{p+1}^n$. Vector spaces are nice, so

$$H^n(G) \cong \bigoplus_{p=0}^n F_p^n/F_{p+1}^n \cong \bigoplus_{i=0}^n [H^p(G/T)\otimes H^{n-p}(T)]^W$$

which are the degree n terms in $[H^*(T)\otimes H^*(G/T)]^W$. The cup product in $H^*(G)$ induces the product on the E_∞ page by Theorem 7.2c), so the above isomorphism implies that the product on $[H^*(G/T)\otimes H^*(T)]^W$ gives us the product on $H^*(G)$. ☐

Hence it suffices to show that

$$[H^{2n}(G/T)\otimes H^q(T)]^W \oplus \mathrm{Im}(d_2^{2n-2,q+1}) = \mathrm{Ker}(d_2^{2n,q}) \text{ for all } n,q \geqslant 0. \qquad (**)$$

Clearly this is true when $q = 0$.

Lemma 6.11. *Let* $\lambda \in H^{2n}(G/T)\otimes H^q(T)$ *be such that* $\omega_\lambda = 0$. *Then* $[\lambda] \in Im(d_2^{2n-2,q+1})$.

Here ω_λ is as in Proposition 6.4.

Proof. Consider the alternating form μ defined by

$$\begin{aligned}
&\mu(X_1, X_2, \cdots, X_{2n-2}, X_{2n-1}, H_{\beta_0}, H_{\beta_1}, H_2, \cdots, H_q)\\
=&\frac{1}{n+1}(\lambda(X_1, X_2 \cdots, X_{2n-2}, X_{2n-1}, X_{\beta_0}, X_{-\beta_0}, H_{\beta_1}, H_2, \cdots, H_q)\\
&-\lambda(X_1, X_2, \cdots, X_{2n-2}, X_{2n-1}, X_{\beta_1}, X_{-\beta_1}, H_{\beta_0}, H_2, \cdots, H_q))
\end{aligned}$$

Note that if each X_i is an X_{α_i} for some root α_i and μ is non-zero, $X_i = X_\alpha$ iff $X_j = X_{-\alpha}$ for some j (i.e. we can pair the X_i up). For $X_i \in \mathfrak{m}, H_i \in \mathfrak{t}, \beta_0, \beta_1$ roots, Write $\delta_{G/T}$ and δ_T for the differentials in the de Rham complex for G/T and T respectively. Consider $(\delta_{G/T}\otimes Id)(\mu)(X_{\alpha_1}, \cdots, X_{\alpha_{2n-1}}, H_{\beta_1}, \cdots, H_{\beta_{q-1}})$, and assume it is non-zero. Then, WLOG we can assume

$$\mu(X_{\alpha_1}, X_{\alpha_2}, \cdots, X_{\alpha_{2n-3}}, [X_{\alpha_{2n-2}}, X_{\alpha_{2n-1}}]_{\mathfrak{m}}, H_{\beta_0}, \cdots, H_{\beta_q}) \neq 0$$

So we can take $\alpha_1 = -\alpha_2, \cdots \alpha_{2n-5} = \alpha_{2n-4}, \alpha_{2n-3} = -\alpha_{2n-2} - \alpha_{2n-3}$, with all the α_i distinct. Since the α_i are distinct, this implies that the terms with $1 \leqslant i \leqslant 2n-4$ in $*$ are all zero, since the X_{α_i} cannot pair up. So

$$\begin{aligned}
&(\delta_{G/T}\otimes Id)(\mu)(X_{\alpha_1}, X_{\alpha_2}, \cdots, X_{\alpha_{2n-1}}, H_{\beta_0}, \cdots, H_{\beta_q})\\
=&-\mu(X_{\alpha_1}, X_{\alpha_2}, \cdots, [X_{\alpha_{2n-3}}, X_{\alpha_{2n-2}}], X_{\alpha_{2n-1}}, H_{\beta_0}, \cdots, H_{\beta_q})\\
&+\mu(X_{\alpha_1}, X_{\alpha_2}, \cdots, [X_{\alpha_{2n-3}}, X_{\alpha_{2n-1}}], X_{\alpha_{2n-2}}, H_{\beta_0}, \cdots, H_{\beta_q})\\
&-\mu(X_{\alpha_1}, X_{\alpha_2}, \cdots, [X_{\alpha_{2n-2}}, X_{\alpha_{2n-1}}], X_{\alpha_{2n-3}}, H_{\beta_0}, \cdots, H_{\beta_q})
\end{aligned}$$

Since $\delta_{G/T}\otimes Id(\lambda) = 0$, the same logic implies that

$$\begin{aligned}
&-\lambda(X_{\alpha_1}, X_{\alpha_2}, \cdots, [X_{\alpha_{2n-3}}, X_{\alpha_{2n-2}}], X_{\alpha_{2n-1}}, X_{\beta_0}, X_{-\beta_0}, \cdots, H_{\beta_q})\\
&+\lambda(X_{\alpha_1}, X_{\alpha_2}, \cdots, [X_{\alpha_{2n-3}}, X_{\alpha_{2n-1}}], X_{\alpha_{2n-2}}, X_{\beta_0}, X_{-\beta_0}, \cdots, H_{\beta_q})\\
&-\lambda(X_{\alpha_1}, X_{\alpha_2}, \cdots, [X_{\alpha_{2n-2}}, X_{\alpha_{2n-1}}], X_{\alpha_{2n-3}}, X_{\beta_0}, X_{-\beta_0}, \cdots, H_{\beta_q})\\
&=0
\end{aligned}$$

So in fact we have $(\delta_{G/T}\otimes Id)(\mu)(X_{\alpha_1},X_{\alpha_2},\cdots,X_{\alpha_{2n-1}},H_{\beta_0},\cdots,H_{\beta_q})=0$, a contradiction. Hence $\delta_{G/T}\otimes Id(\mu)=0$. Similar reasoning implies that $Id\otimes\delta_T(\mu)=0$, so $\mu\in H^*(G/T)\otimes H^*(T)$.
Now

$$
\begin{aligned}
&\omega_\mu(X_{\alpha_1},X_{-\alpha_1}\cdots,X_{\alpha_{n-1}},X_{-\alpha_{n-1}},X_{\alpha_n},X_{-\alpha_n},H_{\beta_1},\cdots,H_{\beta_q})\\
=&\frac{1}{(n-1)!}\sum_{\sigma\in Sym(X_{\alpha_1},X_{-\alpha_1},\cdots,X_{\alpha_n},X_{-\alpha_n})} sgn(\sigma)\mu(\sigma X_{\alpha_1},\sigma X_{-\alpha_1},\cdots,\sigma X_{\alpha_{n-1}},\sigma X_{-\alpha_{n-1}},[\sigma X_{\alpha_n},\sigma X_{-\alpha_n}],H_{\beta_1},\cdots,H_{\beta_{q-1}})\\
=&\frac{1}{n+1}\Big(\sum_{\sigma\in Sym(X_{\alpha_1},X_{-\alpha_1},\cdots,X_{\alpha_n},X_{-\alpha_n})}\lambda(\sigma X_{\alpha_1},\sigma X_{-\alpha_1},\cdots,\sigma X_{\alpha_{n-1}},\sigma X_{-\alpha_{n-1}},\sigma X_{\alpha_n},\sigma X_{-\alpha_n},H_{\beta_1},\cdots,H_{\beta_{q-1}})\\
&-\sum_{\sigma\in Sym(X_{\alpha_1},X_{-\alpha_1},\cdots,X_{\alpha_n},X_{-\alpha_n})} sgn(\sigma)\lambda(\sigma X_{\alpha_1},\sigma X_{-\alpha_1},\cdots,\sigma X_{\alpha_{n-1}},\sigma X_{-\alpha_{n-1}},X_{\beta_1},X_{-\beta_1},[\sigma X_{\alpha_n},\sigma X_{-\alpha_n}],\cdots,H_{\beta_{q-1}})\Big)\\
=&\frac{1}{n+1}\big(n\lambda(X_{\alpha_1},X_{-\alpha_1},\cdots,X_{\alpha_n},X_{-\alpha_n},H_{\beta_1},\cdots,H_{\beta_q})+\lambda(X_{\alpha_1},X_{-\alpha_1},\cdots,X_{\alpha_n},X_{-\alpha_n},[X_{\beta_1},X_{-\beta_1}],\cdots,H_{\beta_q})\big)\\
=&\lambda(X_{\alpha_1},X_{-\alpha_1},\cdots,X_{\alpha_n},X_{-\alpha_n},H_{\beta_1},\cdots,H_{\beta_q})
\end{aligned}
$$

Hence $\omega_\mu=\lambda$, so the result is proved. ☐

Assume that $[H^{2n}(G/T)\otimes H^q(T)]^W\oplus \mathrm{Im}(d_2^{2n-2,q+1})=\mathrm{Ker}(d_2^{2n,q})$. Then for any $\lambda\in\mathrm{Ker}(d_2^{2n,q})$, there exists $\mu\in[H^{2n}(G/T)\otimes H^q(T)]^W,\upsilon\in\mathrm{Im}(d_2^{2n-2,q+1})$ such that $\lambda=\mu+\upsilon$. So $\lambda-w\lambda=\upsilon-w\upsilon$, which is in $Im(d_2^{2n-2,q+1})$ by the W-equivariance of d_2. Conversely, if $\lambda-w\lambda\in\mathrm{Im}(d_2^{2n-2,q+1})$ for all $\lambda\in\mathrm{Ker}(d_2^{2n,q})$, then $\lambda-\frac{1}{|W|}\sum_{w\in W}w\lambda\in Im(d_2^{2n-2,q+1})$, so $\lambda=\left(\frac{1}{|W|}\sum_{w\in W}w\lambda\right)+\left(\lambda-\frac{1}{|W|}\sum_{w\in W}w\lambda\right)$, i.e. $[H^{2n}(G/T)\otimes H^q(T)]^W\oplus\mathrm{Im}(d_2^{2n-2,q+1})=\mathrm{Ker}(d_2^{2n,q})$. So it suffices to show that $\alpha\in[\lambda-w\lambda]$ such that $\omega_\alpha=0$. Note that for $\lambda\in\mathrm{Ker}(d_2^{2n,q})$, $[\omega_\lambda]=[0]$ so ω_λ is a form with coefficients in I.

Proposition 6.12. $[H^{2n}(G/T)\otimes H^1(T)]^W\oplus\mathrm{Im}(d_2^{2n-2,2})=\mathrm{Ker}(d_2^{2n,1})$

Proof. Identify the elements of $H^{2n}(G/T)\otimes H^1(T)$ with polynomial-coefficient 1-forms on T. Let $\lambda\in Ker(d_2^{2n,1})$. Then $d_2^{2n,1}(\lambda)=\sum_i f_iJ_i=J+\sum_i\tilde{f}_iJ_i$, where $J\in\mathscr{S}^W$ and $\tilde{f}_i$ is f_i without the constant term. Then there exists υ_i such that $d_2(\upsilon_i)=f_i$, as the maps $d_2^{2n,1}:H^{2n}(G/T)\otimes H^1(T)\to H^{2n+2}(G/T)$ are surjective. Hence $\alpha=\lambda-\sum_{i=1}^l J_i\upsilon_i$ maps to J. Since $\alpha\in[\lambda]$, $\alpha-w\alpha\in[\lambda-w\lambda]$, and clearly $\omega_{\alpha-w\alpha}=0$. Hence $\alpha-w\alpha\in Im(d_2^{2n-2,2})$, and so we're done by Lemma 1. ☐

This proves the result for compact connected Lie groups of rank 1.[6]
Now, we show that the result is true for compact connected Lie groups of rank 2 and 3. We will use the following lemma:

Proposition 6.13. *The differentials $H^{2n}(G/T)\otimes H^q(T)\to H^{2n+2}(G/T)\otimes H^{q-1}(T)$ and $H^{2v-2n-2}(G/T)\otimes H^{l-q+1}(T)\to H^{2v-2n}(G/T)\otimes H^{l-q}(T)$ are dual, in the sense that the diagram*

$$
\begin{tikzcd}
H^{2n}(G/T)\otimes H^q(T) \arrow[r, "\sim"] \arrow[d, "d_2^{2n,q}"] & \mathrm{Hom}(H^{2v-2n}(G/T)\otimes H^{l-q}(T),\mathbb{R}) \arrow[d, "(d_2^{2v-2n+2,l-q+1})^*"] \\
H^{2n+2}(G/T)\otimes H^{q+1}(T) \arrow[r, "\sim"] & \mathrm{Hom}(H^{2v-2n+2}(G/T)\otimes H^{l-q+1}(T),\mathbb{R})
\end{tikzcd}
$$

commutes up to sign, where the horizontal isomorphisms are the maps

$$\upsilon\otimes\lambda\mapsto(\phi\otimes\psi\mapsto(\upsilon\smile\phi)(\lambda\smile\psi))$$

Proof. The horizontal maps are indeed isomorphisms, since the pairings $H^{2n}(G/T)\otimes H^{2v-2n}(G/T)\to H^{2v}(G/T)\cong\mathbb{R}$ and $H^q(T)\otimes H^{l-q}(T)\to H^l(G/T)\cong\mathbb{R}$ are perfect. See [3, page 250] for more details.

[6]Note that every rank 1 compact connected Lie group is either a torus or 3 dimensional.

Now consider a $pdx_{i_1} \wedge \cdots \wedge dx_{i_q} \in H^{2n}(G/T) \otimes H^q(T)$. Going through the top horizontal map then the right vertical map gives us the functional
$(rdx_{i'_1} \wedge \cdots dx_{i'_{l-q+1}} \mapsto prx_{i'_1} dx_{i_1} \wedge \cdots dx_{i_q} \wedge dx_{i'_2} \wedge \cdots \wedge dx_{i'_{l-q+1}} - prx_{i'_2} dx_{i_1} \wedge \cdots dx_{i_q} \wedge dx_{i'_1} \wedge dx_{i'_3} \wedge \cdots \wedge dx_{i'_{l-q+1}} + \cdots + (-1)^{l-q+2} prx_{i'_{l-q+1}} dx_{i_1} \wedge \cdots dx_{i_q} \wedge dx_{i'_1} \wedge \cdots \wedge dx_{i'_{l-q}})$, while going through the left vertical map then the bottom horizontal map gives us the functional
$(rdx_{i'_1} \wedge \cdots \wedge dx_{i'_{l-q+1}} \mapsto prx_{i_1} dx_{i_2} \wedge \cdots \wedge dx_{i_q} \wedge \cdots \wedge dx_{i'_{l-q+1}} - prx_{i_2} dx_{i_1} \wedge dx_{i_3} \wedge \cdots \wedge dx_{i_q} \wedge \cdots \wedge dx_{i'_{l-q+1}} + \cdots + (-1)^{q+1} prx_{i_q} dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}} \wedge \cdots \wedge dx_{i'_{l-q+1}}$.
By the pigeonhole principle, $|\{i'_1, \cdots i'_{l-q+1}\} \cap \{i_1 \cdots i_q\}| \geqslant 1$. If the inequality is strict, then both functionals map $rdx_{i'_1} \wedge \cdots \wedge dx_{i'_{l-q+1}}$ to 0. If we have equality, then $\{i'_1, \cdots i'_{l-q+1}\} \cap \{i_1 \cdots i_q\} = \{j\}$ for some $1 \leqslant j \leqslant l$. Then up to sign, both functionals map $rdx_{i'_1} \wedge \cdots \wedge dx_{i'_{l-q+1}}$ to $prx_j dx_{i_1} \wedge \cdots \wedge dx_{i_q} \wedge dx_{i'_1} \wedge \cdots \wedge \hat{dx_j} \wedge \cdots dx_{i'_{l-q+1}}$, where hat denotes removal. Hence the diagram is commutative up to sign. ☐

Lemma 6.14. $\dim([H^{2n}(G/T) \otimes H^q(T)]^W) = \dim([H^{2v-2n}(G/T) \otimes H^{l-q}(T)]^W)$

Proof. For a sequence $I = (i_1, \cdots, i_q)$ with $1 \leqslant i_1 < \cdots < i_q \leqslant l$, write dJ'_I for $dJ'_{i_1} \wedge \cdots dJ'_{i_q}$. Also define I' as the increasing sequence with terms in $\{1, \cdots, n\} \setminus \{i_1, \cdots, i_q\}$. A basis for $[H^{2n}(G/T) \otimes H^q(T)]^W$ is $\mathscr{B}_{2n,q} := \left\{ dJ_I \colon |I| = q, \sum_{j=1}^{q} \deg(J_{i_j}) - 1 = n \right\}$ by Corollary 1. Now consider the map $dJ'_I \mapsto dJ'_{I'}$. This is a map from $\mathscr{B}_{2n,q}$ to $\mathscr{B}_{2v-2n,l-q}$. Since the map $dJ'_{I'} \mapsto dJ'_{I''} = dJ_I$ is an inverse, this implies both sets have the same cardinality, and hence $\dim([H^{2n}(G/T) \otimes H^q(T)]^W) = \dim([H^{2v-2n}(G/T) \otimes H^{l-q}(T)]^W)$. ☐

We can now prove the result for Lie groups of rank 2 and 3. Since $d_2^{2n,q} \colon H^{2n}(G/T) \otimes H^q(T) \to H^{2n+2}(G/T) \otimes H^{q-1}(T)$ and $d_2^{2v-2n-2,l-q+1} \colon H^{2v-2n-2}(G/T) \otimes H^{l-q+1}(T) \to H^{2v-2n}(G/T) \otimes H^{l-q}(T)$ are dual, this means that $\dim(\mathrm{Im}(d_2^{2n-2,q+1})) = \dim(H^{2v-2n}(G/T) \otimes H^{l-q}(T)) - \dim(\mathrm{Ker}(d_2^{2v-2n,l-q}))$, so

$$\begin{aligned} &\dim(\mathrm{Ker}(d_2^{2n,q})) = \dim([H^{2n}(G/T) \otimes H^q(T)]^W) + \dim(\mathrm{Im}(d_2^{2n-2,q+1})) \\ \Longrightarrow\ &\dim(H^{2n}(G/T) \otimes H^q(T)) - \dim(\mathrm{Im}(d_2^{2n-2,q+1})) \\ &= \dim([H^{2v-2n}(G/T) \otimes H^{l-q}(T)]^W) + \dim(H^{2n}(G/T) \otimes H^q(T)) - \dim(\mathrm{Ker}(d_2^{2n,q})) \\ \Longrightarrow\ &\dim(\mathrm{Ker}(d_2^{2v-2n,l-q})) = \dim([H^{2v-2n}(G/T) \otimes H^{l-q}(T)]^W) + \dim(\mathrm{Im}(d_2^{2v-2n-2,l-q+1})) \end{aligned}$$

Hence for a rank l group, $(**)$ for q implies $(**)$ for $l - q$. Hence when $l = 2, 3$, $(**)$ for $q = 0, 1$ implies $(**)$ for $q = 2$ and $q = 2, 3$ respectively. This proves the result for rank 2 and 3 groups.

7 Computations

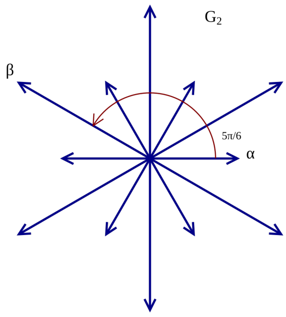
We will now digress and use spectral sequences to calculate $H^*(G_2)$. Let us first note down a useful result first

Theorem 7.1. *The partial derivatives of* Π *span* $\mathscr{H}$.

Proof. See [5, page 361] ☐

7.1 G_2

G_2 is a 14-dimensional compact connected Lie group of rank 2. It has Weyl group $W \cong D_{12}$, which is easily seen by noting that W is isomorphic to the group generated by the reflections of a hexagon.



We will refer to the short horizontal and long vertical root as x and $2y$ respectively. By Proposition 3.2 $m_1 + m_2 = 6, (m_1+1)(m_2+1) = |D_{12}| = 12$, so $m_1 = 1, m_2 = 5$. Hence the generators of the W-invariants as in Theorem 3.1 have degrees 2 and 6.

$$\begin{tikzcd}
 & \frac{1}{2}x - \frac{\sqrt{3}}{2}y & -\frac{1}{2}x - \frac{\sqrt{3}}{2}y \\
x & x \arrow[l, "s_y"'] \arrow[u, "s_{\frac{1}{2}x+\frac{\sqrt{3}}{2}y}"] \arrow[ur, "s_{\sqrt{3}x+y}"] \arrow[r, "s_x"] \arrow[d, "s_{-\frac{1}{2}x+\frac{\sqrt{3}}{2}y}"'] \arrow[dr, "s_{-\sqrt{3}x+y}"'] & -x \\
 & \frac{1}{2}x + \frac{\sqrt{3}}{2}y & -\frac{1}{2}x + \frac{\sqrt{3}}{2}y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -\frac{\sqrt{3}}{2}x - \frac{1}{2}y & -\frac{\sqrt{3}}{2}x + \frac{1}{2}y \\
-y & y \arrow[l, "s_y"'] \arrow[u, "s_{\frac{1}{2}x+\frac{\sqrt{3}}{2}y}"] \arrow[ur, "s_{\sqrt{3}x+y}"] \arrow[r, "s_x"] \arrow[d, "s_{-\frac{1}{2}x+\frac{\sqrt{3}}{2}y}"'] \arrow[dr, "s_{-\sqrt{3}x+y}"'] & y \\
 & \frac{\sqrt{3}}{2}x - \frac{1}{2}y & \frac{\sqrt{3}}{2}x + \frac{1}{2}y
\end{tikzcd}$$

If $ax^2 + bxy + cy^2$ is W-invariant, then we must have $b = 0$. We also must have

$$\begin{aligned}
ax^2 + cy^2 &= a\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)^2 + c\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)^2 \\
&= a\left(\frac{1}{4}x^2 + \frac{\sqrt{3}}{2}xy + \frac{3}{4}y^2\right) + c\left(\frac{3}{4}x^2 - \frac{\sqrt{3}}{2}xy + \frac{1}{4}y^2\right) \\
&\Rightarrow \frac{a}{4} + \frac{3c}{4} = a, \frac{3a}{4} + \frac{c}{4} = c \\
&\Rightarrow a = c
\end{aligned}$$

So J_1 must be a multiple of x^2+y^2. It is clear that x^2+y^2 is indeed invariant under the s_α, and hence W-invariant, so we can take $J_1 = x^2 + y^2$. Similarly, if $ax^6 + bx^5y + cx^4y^2 + dx^3y^3 + fx^2y^4 + exy^5 + gy^6$ is W-invariant, then we must have $b = d = e = 0$. We must also have

$$\begin{aligned}
&ax^6 + cx^4y^2 + fx^2y^4 + gy^6 \\
=&a\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)^6 + c\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)^4\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)^2 + e\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)^2\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)^4 + g\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)^6 \\
=&\frac{a+3c+9f+27g}{64}x^6 + \frac{3a+5c+3f-27g}{32}\sqrt{3}x^5y + \frac{45a+31c-27f+135g}{64}x^4y^2 \\
&+ \frac{15a+c-f-15g}{16}\sqrt{3}x^3y^3 + \frac{135a-27c+31f+45g}{64}x^2y^4 + \frac{27a-3c-5f-3g}{32}\sqrt{3}xy^5 + \frac{27a+9c+3f+g}{64}y^6
\end{aligned}$$

Which forces $c = 9g - 6a, f = 9a - 6g$, with no restriction on a or g. Setting $a = 0$ gives us the polynomial $gy^2(y^2 - 3x^2)^2$, which one can verify is W-invariant. So we can take $J_2 = y^2(y^2 - 3x^2)^2$.
The primordial harmonic Π is

$$\begin{aligned}
(x)(\sqrt{3}x + y)(\frac{1}{2}x + \frac{\sqrt{3}}{2}y)(2y)(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)(-\sqrt{3}x + y) &= \frac{1}{2}xy(y^2 - 3x^2)(3y^2 - x^2) \\
&= \frac{1}{2}(3x^5y - 10x^3y^3 + 3xy^5)
\end{aligned}$$

Theorem 8.1 then tells us that the generators for $\mathscr{H}$ are the below polynomials:

$$
\begin{tikzcd}
 & 3x^5y-10x^3y^3+3xy^5 \arrow[dl, "\frac{1}{3}\partial_x"'] \arrow[dr, "\frac{1}{3}\partial_y"] & \\
5x^4y-10x^2y^3+y^5 \arrow[d, "\frac{1}{20}\partial_x"'] \arrow[drr, "\frac{1}{5}\partial_y"] & & x^5-10x^3y^2+5xy^4 \arrow[d, "\frac{1}{5}\partial_x"] \arrow[dll, "-\frac{1}{20}\partial_y"'] \\
x^3y-xy^3 \arrow[d, "\partial_x"] \arrow[drr, "\partial_y"] & & x^4-6x^2y^2+y^4 \arrow[d, "\frac{1}{4}\partial_x"] \arrow[dll, "-\frac{1}{4}\partial_y"'] \\
3x^2y-y^3 \arrow[d, "\frac{1}{6}\partial_x"] \arrow[drr, "\frac{1}{3}\partial_y"] & & x^3-3xy^2 \arrow[d, "\frac{1}{3}\partial_x"] \arrow[dll, "-\frac{1}{6}\partial_y"'] \\
xy \arrow[d, "\partial_x"] \arrow[drr, "\partial_y"] & & x^2-y^2 \arrow[d, "\frac{1}{2}\partial_x"] \arrow[dll, "-\frac{1}{2}\partial_y"'] \\
y \arrow[dr, "\partial_y"] & & x \arrow[dl, "\partial_x"] \\
 & 1 &
\end{tikzcd}
$$

By Proposition 3.5, we have

$$
\begin{aligned}
\sum_{p=0}^{6} \dim(\mathscr{H}^p)t^p &= (1+t)(1+t+\cdots+t^5) \\
&= 1+2t+2t^2+2t^3+2t^4+2t^5+t^6
\end{aligned}
$$

Hence the E_2 page is

$$
\begin{tikzcd}
2 & \mathbb{R} \arrow[dr] & \mathbb{R}^2 \arrow[dr] & \mathbb{R}^2 \arrow[dr] & \mathbb{R}^2 \arrow[dr] & \mathbb{R}^2 \arrow[dr] & \mathbb{R}^2 \arrow[dr] & \mathbb{R} \\
1 & \mathbb{R}^2 \arrow[dr] & \mathbb{R}^4 \arrow[dr] & \mathbb{R}^4 \arrow[dr] & \mathbb{R}^4 \arrow[dr] & \mathbb{R}^4 \arrow[dr] & \mathbb{R}^4 \arrow[dr] & \mathbb{R}^2 \\
0 & \mathbb{R} & \mathbb{R}^2 & \mathbb{R}^2 & \mathbb{R}^2 & \mathbb{R}^2 & \mathbb{R}^2 & \mathbb{R} \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}
$$

The maps into the $q=0$ row are all surjective by Proposition 6.6, and hence by Proposition 6.13 the maps out of the $q=2$ row are all injective. This implies that the only non-zero entries on the E_3 page are $E_3^{0,0}, E_3^{5,1}, E_3^{1,1},$ and $E_3^{6,2}$. We have

$$
\begin{aligned}
\operatorname{Im}(d_2^{0,2}) &= \langle -ydx+xdy \rangle \\
\operatorname{Ker}(d_2^{1,1}) &= \langle -ydx+xdy, xdx+ydy \rangle \\
\operatorname{Im}(d_2^{4,2}) &= \big\langle (-x^3y^2+xy^4)dx+(x^4y-x^2y^3)dy, (-x^4y+6x^2y^3-y^5)dx+(x^5-6x^3y^2+xy^4)dy \big\rangle \\
&= \big\langle 2x^5dx+2y^5dy, -7y^5dx+7x^5dy \big\rangle = \big\langle x^5dx+y^5dy, -y^5dx+x^5dy \big\rangle \\
\operatorname{Ker}(d_2^{5,1}) &= \big\langle (x^5-10x^3y^2+5xy^4)dx, (5x^4y-10x^2y^3+y^5)dx-(x^5-10x^3y^2+5xy^4)dy, (5x^4y-10x^2y^3+y^5)dy \big\rangle \\
&= \big\langle 17x^5dx, 17y^5dx-17x^5dy, 17y^5dy \big\rangle \\
&= \big\langle -y^5dx+x^5dy, x^5dx+y^5dy, x^5dx-y^5dy \big\rangle
\end{aligned}
$$

noting that the W-invariant x^2+y^2 is zero in $H^*(G/T) \cong \mathscr{S}/I$. So the E_3 page is

2	0	0	0	0	0	0	$\mathbb{R}$
1	0	$\mathbb{R}(xdx + ydy)$	0	0	0	$\mathbb{R}(x^5dx - y^5dy)$	0
0	$\mathbb{R}$	0	0	0	0	0	0
	0	1	2	3	4	5	6

Clearly the spectral sequence degenerates on E_3, so $E_3 = E_\infty$. Hence $H^*(G_2)$ is isomorphic to the exterior algebra generated by $dJ_1 = 2(xdx + ydy)$ and $dJ_2 = (-2xy^4 + 6x^3y^2)dx + (y^5 - 4x^2y^3 + 3x^4y)dy = -8(x^5dx - y^5dy)$, which aligns with our expectations.

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