Mémoire de Master 2

Special holonomy and construction of ALC G_2-manifolds

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Foreword

My first encounter with special holonomy took place about one year ago, when I had the chance to attend the conference given at the occasion of the 70th birthday of S.T. Yau, in Harvard. The first speaker was Sir Simon Donaldson, and this was the first time I heard of G_2-manifolds. Although I cannot pretend that I understood much of the talks, I knew this was the kind of mathematics I wanted to do.

A few months later, while studying for my masters, I wanted to learn more about these topics. This is what I explained to Pr. Olivier Biquard when I asked his advice about a potential subject for my master's thesis. On his recommendation, I contacted Pr. Jason Lotay in Oxford, who kindly accepted to meet me to discuss about my thesis, and to supervise me for this project. I therefore stayed in Oxford from March to July 2020.

I am very grateful to Pr. Jason Lotay, for his enthusiasm and the time he spent introducing me to this beautiful field of special holonomy and G_2-manifolds. I also want to thank Pr. Dominic Joyce, who accepted to be part of the committee for my thesis defense. Finally, I would like to thank Pr. Olivier Biquard for the many advices he gave me throughout the years I studied at ENS.

I also want to express my gratitude to Lea and Mathilde, who reviewed part of my thesis and provided helpful comments on english and grammar.

Due to the exceptional circumstances of Spring 2020, this project was conducted under difficult conditions, and I would like to thank all the staff of the North Oxford Overseas Center, where I lived during the five months I spent in Oxford. I am grateful for their providing such a comfortable and safe place to live. Finally, these acknowledgements would not be complete without mentioning the other residents of the NOOC, with whom I spent several months during the lockdown. I want to especially thank Davide, one of the very few who decided to remain against all odds, and with whom I had so many conversations (and ping pong games) that got us through these long weeks. Many thanks to Lea too, who despite her late arrival managed to join our traditional late night conversations as if she had always been part of it. I believe the three of us built this kind of friendship that only arises from hard times.

Introduction

The study of Ricci-flat and special holonomy metrics is of great importance in Riemannian geometry. To only mention some problems arising from physics, solutions of Einstein's equation for the evolution of space-time in the vacuum are Ricci-flat metrics, and the study of Calabi-Yau manifolds is central in String Theory. The study of special metrics gives rise to many interesting problems yet to be solved.

A central notion in the study of Riemannian manifolds and exceptional structures is holonomy, which is the group of all isometries of the tangent space that are realized by parallel transport along loops. On a Riemannian manifold, this group essentially determines the structures carried by the manifold that are compatible with the metric. Since the classification of the possible holonomy groups of irreducible, non-symmetric Riemannian manifolds by Berger in 1955 [3], it has been known that there are only seven classes of such metrics. Besides the generic case where the holonomy is the full orthogonal group, there are four infinite families of metrics, corresponding to Kähler, Calabi-Yau, hyperkähler and quaternionic-Kähler manifolds. The last two groups of the Berger's list are $G_2 \subset SO(7)$ and $\mathrm{Spin}(7) \subset SO(8)$, are called the exceptional holonomy groups, and only appear respectively in dimension 7 and dimension 8. At the time of Berger, no examples of complete metrics with exceptional holonomy were known and it took more than 30 years before Bryant gave the first explicit examples in 1989 [5]. The first compact examples of metrics with holonomy G_2 and $\mathrm{Spin}(7)$ were constructed by Joyce in 1996 [15, 16].

One reason that accounts for the difficulty of finding exceptional metrics is that it involves solving a PDE which is non-linear even in its highest-order terms. The four infinite families of special metrics are related to complex and quaternionic geometry, and thus one can use techniques from algebraic geometry to construct and study them - and yet this is not an easy problem. On the other hand, the exceptional holonomy groups are related to octonions, which are of much less help than complex numbers, and one can only use differential-geometric techniques. Many of the exceptional metrics constructed possess a large group of symmetry, in order to reduce the dimension of the problem from a PDE in dimension 7 or 8 to an ODE in dimension 1, which is much more tractable. Over the past decades, many new examples of manifolds with special holonomy have been found using various techniques but very little is known about the generic case of exceptional metrics.

Besides the problem of construction, there are many other unsolved questions about special holonomy manifolds. A first question is: given a particular manifold, does it admit a metric with exceptional holonomy? In other words, an important problem is to find obstructions to the existence of exceptional metrics. In dimension 7, G_2-metrics correspond to a particular type of 3-forms, called positive forms, and the positive 3-form associated to a G_2-metric must represent a non-trivial cohomology class. Hence, there is the obvious obstruction that the third cohomology group of a G_2-manifold cannot be trivial. Some other obstructions are known, but they are not strong enough to answer this question in all cases.

Another important problem in special geometry is: given two G_2-manifolds, can we decide whether they are isometric? Answering this question involves constructing invariants of G_2-manifolds. One possibility for constructing such invariants would be to study the moduli space of G_2-metrics on a given manifold. Since it is clear that two isometric G_2-manifolds should have the same moduli space, one can hope to extract invariants from the topology and the geometry of the moduli space. By a result of Joyce [17, §§10.3-10.4], the moduli space of G_2-metrics is smooth and the map that associates to a G_2-metric the cohomology class of the corresponding positive form is a local diffeomorphism. In particular, the dimension of the moduli space is equal to the third Betti number of the manifold, if it is non-empty. But this is a purely local result, which does not say anything about the global structure of the moduli space. In particular, not much is known about the different ways G_2-metrics can degenerate, which would be necessary to build a compacification of the moduli space. This is important, because in other geometric contexts such as Donaldson theory of 4-manifolds and the moduli space of solutions to the Yang-Mills equations, compacification is an important step towards extracting invariants from the moduli space.

In this thesis, we will look at a particular construction of complete G_2-metrics. The general aim of this master thesis was to work on one or several research papers, understanding them and writing a document explaining the purpose, context, background and, to some extent, the details of these articles. The thesis is mainly based on the article [11] by Lorenzo Foscolo, Mark Haskins and Johannes Nordström. They present a construction of G_2-metrics based on a single circle action, to reduce a 7-dimensional problem to a 6-dimensional one and build G_2-metrics on a S^1-bundle over a Calabi-Yau manifold. The analysis involved in this construction is quite sophisticated, in particular because the base manifold is not compact but has conical geometry at infinity. The G_2-metrics that are built in this way come in 1-parameter families of G_2-structure that collapse onto the base Calabi-Yau manifold with bounded curvature. The G_2-structures are constructed as power series expansion near the collapsed limit, using a technique that goes back to the Kodaira-Nirenberg-Spencer construction of deformations of complex structures [22]. Only here the analysis is much harder, because of the non-compactness.

Certainly, one of the main flaws of this thesis is that we chose an exposition that reflects more our own pathway towards understanding the results than the full context in which the article of Foscolo-Haskins-Nordström takes place. Hence, we spend quite some time on complex structures and the construction of Kodaira-Spencer, although this is not so relevant for special holonomy. However, this was an important step for understanding the idea behind the construction that is the aim of this thesis, so we decided to add it with the hope that it would fit well with the rest of our discussion. Another choice that we made was to give as many details as possible about the analysis on asymptotically conical manifolds and not to take it as a black box. Although we did admit few results from Lockhart-McOwen theory, we tried to give an account of the analytical results used in the construction as self-contained as possible in the time available. This is to a great extent justified by our own lack of knowledge in this kind of analysis, so we wanted take this opportunity to get more working familiarity with the theory of elliptic operators. The downside of this is that we did not dig into other aspects related to the Foscolo-Haskins-Nordström construction, and in particular we only briefly mention Sasaki-Einstein manifolds, K-stability and the resolution of Calabi-Yau cones.

This thesis is organized as follows. The first chapter contains background about principal bundles, holonomy and G-structures on manifolds. We wanted to underline in this chapter the relation between connections on principal bundles and its associated vector bundles, as well as the importance of the torsion-free condition for G-structures. We wrote the basic material in this chapter from memory, inspired from a course following the book of Kobayashi-Nomizu [20]. The material concerning G-structures and intrinsic torsion was written partly using Joyce's book [17], and partly by looking at the research literature.

The second chapter deals with special structures in geometry. In the first part, we introduce complex manifolds with the two aims of illustrating the meaning of the torsion-free condition for $GL(n, \mathbf{C})$-structures and preparing background for the fourth chapter, when we treat deformations of complex structures. This part was mainly written by memory from a course on complex geometry, with some help from Kodaira's book [21]. In the next two parts, we introduce $SU(3)$ and G_2-structures, which are the most important objects in this thesis. We mainly used the research literature to write these parts.

The third chapter treats analysis on Riemannian manifolds and is central in our thesis. We wanted to develop from scratch the theory of elliptic operators, with as few black boxes as possible. We start by local considerations, in order to state the central result of elliptic regularity and interior estimates, which is one of the few results that we take as a black box. Then we study compact manifolds, and prove the Fredholm property for elliptic operators, and apply it to the Laplacian operator. For this part, we used mainly Besse's book [4], as well as some material in the book of Joyce [17]. In a third part, we treat asymptotically conical manifolds, trying to give self-contained proofs of the results we will need in the fifth chapter, only admitting very few results from Lockhart-McOwen theory [25]. In the last part we focus on asymptotically conical manifolds. These last two parts were written mostly autonomously, looking at the research literature, for lack of other references on this subject.

In the fourth chapter we explain Kodaira-Spencer construction of analytic deformations of complex structures. We try to put the emphasis on the points of the construction that will be useful in the next chapter. This chapter was written using mainly Kodaira's book.

In the fifth chapter, we give an account of the construction of Foscolo-Haskins-Nordström of ALC G_2-metrics. It was difficult to make an exposition that was radically different from the one given in the paper, but we tried to emphasize some points that seemed important to us, and find proofs for some of the claims that were stated without details in the article.

In the last chapter, we give a few consequences and explicit examples deriving from the construction of Foscolo-Haskins-Nordström. We then say a few words about Sasaki-Einstein manifolds and the resolutions of Calabi-Yau cones, that provide a lot of examples in which the above construction can be applied and yields new G_2-manifolds.

Contents

Chapter 1

Holonomy groups

The aim of this chapter is to introduce holonomy groups, which are central in Riemannian geometry. On a vector bundle endowed with a connection, one can define parallel transport along a path on the base. The parallel transport along a loop defines an isomorphism of the fiber over the base point, and the group of all such isomorphisms is called the holonomy group of the connection. It is essentially independent of the choice of base point. On a Riemannian manifold with metric g, one is mainly concerned with the Levi-Civita connection, and its holonomy group is also called the holonomy group of g. To a great extent, the holonomy group of a Riemannian manifold determines its structure, and the existence of many interesting metrics (e.g., Kähler, Calabi-Yau, G_2) can be formulated in terms of holonomy.

The relation between holonomy and geometric structures on a manifold is best understood in terms of principal bundles, which we introduce in §1.1.1, as well as the corresponding notion of connection. In §1.1.2, we define parallel transport and holonomy on principal bundles, proving the main theorem of this chapter, which states that the holonomy group of a connection is the smallest structure group to which it can be reduced.

In §1.2.1, we explain the relation between principal and vector bundles, and between the corresponding notions of connection. Then we introduce the curvature in §1.2.2, and underline its relation with the holonomy algebra. The last important notions to be introduced in this chapter are G-structures and their intrinsic torsion, to which we give a general definition in §1.2.3.

Any Riemannian metric on an oriented manifold is associated to an $SO(n)$-structure, and therefore the interesting G-structures on a Riemannian manifolds are those with $G \subset SO(n)$. In §1.3.1, we give a more precise description of the intrinsic torsion for such a G-structure P, and in particular, we show that the torsion-free condition is equivalent to the fact that the Levi-Civita connection reduces to P. Lastly, in §1.3.2, we give a brief account of the classification of Riemannian holonomy group.

To finish with a word on references, we mostly used [20] for general background on principal and vector bundles, and [17] for the discussion on intrinsic torsion and the classification of Riemannian holonomy groups.

1.1 Principal bundles

The point of view of principal bundles is convenient when one wants to talk about connections, parallel transport and holonomy. It has the advantage of being in a way more general than the point of view of vector bundles, because as we shall see in the next part, a principal bundle may encode the structure of many vector bundles.

1.1.1 Connections on principal bundles

We begin with the definition and a few properties about principal bundles.

Definition 1.1.1. Let G be a Lie group and B be a manifold. A *principal G-bundle* over B is a manifold P, endowed with a right action of G on P and a map $\pi : P \to B$, such that:

(i) G acts freely on P and π identifies B with the quotient space P/G

(ii) P is locally trivial, that is, there exists an open cover $\{U_j\}$ of B, and for each j, a G-equivariant diffeomorphism $\varphi_j : \pi^{-1}(U_j) \longrightarrow U_j \times G$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_j) & \xrightarrow{\varphi_j} & U_j \times G \\ & {}_{\pi}\searrow \quad \swarrow_{\pi_1} & \\ & U_j & \end{array}$$

where π_1 is the projection on the first coordinate.

We will denote the action of g on $p \in P$ by $p \cdot g$. If U is an open set in B, we note $P_U = \pi^{-1}(U)$ and if $x \in B$, P_x will denote the fiber over x.

Since the action of G on P is free, a choice of $p \in P_x$ gives an identification of the fiber to G. Thus, the choice of a local trivialization is the same as a local section. Let U_j, U_j be two open sets in B over which P is trivial, and $U_{jk} = U_j \cap U_k$. Let $\varphi_{jk} = \varphi_j \circ \varphi_k^{-1} : U_{jk} \times G \longrightarrow U_{jk} \times G$ be the gluing function. This is a G-equivariant diffeomorphism and the following diagram commutes:

$$\begin{array}{ccc} U_{jk} \times G & \xrightarrow{\varphi_{jk}} & U_{jk} \times G \\ & {}_{\pi_1}\searrow \quad \swarrow_{\pi_1} & \\ & U_{jk} & \end{array}$$

Hence has the form $\varphi_{jk}(x,h) = (x, g_{jk}(x)h)$ with $g_{jk} : U_{jk} \to G$, called the transition function. The transition functions satisfy $g_{jj} = 1$, and the *cocycle relation*:

$$g_{jk}g_{kl} = g_{jl}$$

Conversely, an open covering $\{U_j\}$ of B, together with functions g_{jk} satisfying the above relations, defines a principal G-bundle over B.

Example 1.1.1. $B \times G$ is a principal G-bundle for the action $(x,h) \cdot g = (x, hg)$. It is called the *trivial bundle*. A principal bundle is isomorphic to the trivial bundle if and only if it admits a global section.

An important class of principal bundles is defined as follows. Let E be any vector bundle over B, of rank k. Let P be the set of pairs $(x, \mathcal{B})$, where $x \in B$ and $\mathcal{B}$ is a basis of the fiber E_x, with the obvious projection onto B. If $g \in GL(k, R)$, let $(x, \mathcal{B}) \cdot g = (x, \mathcal{B}')$, where $\mathcal{B}'$ is the basis that has matrix g in basis $\mathcal{B}$. This is a right action of $GL(k, \mathbf{R})$ on P, which endows P with a structure of $GL(k, \mathbf{R})$-bundle over B. It is called the *frame bundle* of E. An element $p \in P_x$ can be interpreted both as a choice of basis of E_x, or as a choice of identification of E_x with $\mathbf{R}^k$.

If, in addition, the vector bundle E has a metric, we can choose to only pick the orthonormal bases of each fiber, and and form a submanifold Q of P. By restriction of the action of $GL(k, \mathbf{R})$, the orthogonal group $O(k)$ acts freely on the right on Q, and Q is a principal $O(k)$-bundle over B. It is sometimes called the *orthogonal frame bundle* of E.

A general way to obtain principal bundles is the following:

Proposition 1.1.1. *If G is a Lie group acting freely, smoothly and properly on the right on a manifold P, then the quotient map $P \longrightarrow P/G$ is a principal G-bundle.*

If the action is not proper, P/G is generally not a manifold. In general the topology of the quotient need not even be Hausdorff.

Let $\pi : P \to B$ be a principal G-bundle, and $\mathfrak{g}$ be the Lie algebra of G. If $p \in P$, the action of G on p gives an injective map $\mathfrak{g} \to T_pP$. Let $\mathcal{V}_p$ be its image. $\mathcal{V}_p$ is called the *vertical space* at p. $\mathcal{V} = \{\mathcal{V}_p,\ p \in P\}$ is a smooth integrable distribution of P, and the leave through p is the fiber P_p. Then $\mathcal{V} = \ker d\pi$. Thus $d\pi$ induces canonical ismorphisms $T_pP/\mathcal{V}_p = T_{\pi(p)}B$ at each $p \in P$, and we have a splitting $T_pP = \mathcal{V}_p \oplus T_pB$. However, this splitting is not natural. Connections on a principal bundle correspond to splittings of TP that satisfy some conditions.

Definition 1.1.2. A *connection* D on P is defined by a smooth distribution $\mathcal{H}$ on P such that:

(i) $TP = \mathcal{V} \oplus \mathcal{H}$

(ii) $\mathcal{H}$ is invariant under the action of G

$\mathcal{H}_p \subset T_pP$ is called the *horizontal space* at p.

Remark 1.1.1. Condition (i) implies that for $p \in P_x$, $d\pi_p : \mathcal{H}_p \to T_xB$ is an isomorphism. If we denote R_g the action of $g \in G$ on P, condition (ii) means that $\mathcal{H}_{p\cdot g} = (R_g)_*\mathcal{H}_p$.

Given a connection with horizontal space $\mathcal{H}$, we can define its *connection form* $A \in \Omega^1(P, \mathfrak{g})$ on P by: (i') $A_p(\frac{d}{dt}p \cdot e^{t\xi}) = \xi$ on the vertical space, and $A_{\upharpoonright \mathcal{H}} = 0$ on the horizontal space. A is G-equivariant, that is, (ii') $R_g^* A = \mathrm{Ad}_{g^{-1}} A$. Note that we have the correspondence $\mathcal{H} = \ker A$.

Proposition 1.1.2. *This is a one-to-one correspondence between smooth distributions $\mathcal{H} \subset TP$ satisfying (i) and (ii) as in Definition 1.1.2, and connection forms $A \in \Omega^1(P, \mathfrak{g})$ satisfying (i') and (ii') as above.*

In particular, a connection can be defined by its connection form. If P is defined by an open covering $\{U_j\}$ and transition functions g_{jk}, then the connection form A restricts in local trivializations to $A_{\upharpoonright U_j} \in \Omega^1(U_j \times G, \mathfrak{g})$. Using G-equivariance, we only need to specify what it is on the section $U_j \times \{e\} \subset U_j \times G$, which gives a *local connection form* $A_j \in \Omega^1(U_j, \mathfrak{g})$. Local connection forms A_j, A_k satisfy the following transition relations on U_{jk}:

$$A_j = \mathrm{Ad}_{g_{jk}} A_k + g_{jk}^{-1} dg_{jk} \tag{1.1}$$

Here, $g^{-1}dg$ is a short hand for the Maurer-Cartan form of the Lie group G. In particular, for matrix groups we have:

$$A_j = g_{jk} A_k g_{jk}^{-1} + g_{jk}^{-1} dg_{jk} \tag{1.2}$$

Conversely, local forms $A_j \in \Omega^1(U_j, \mathfrak{g})$ that satisfy the above transition relations define a unique connection on P.

1.1.2 Parallel transport

Let $\pi : P \longrightarrow B$ be a principal G-bundle, with a connection ∇, of connection form A and horizontal space $\mathcal{H} = \ker A$. Let $\gamma : [0,1] \longrightarrow B$ be a piecewise C^1 curve. A *horizontal lift* of γ is a piecewise C^1 curve $\gamma' : [0,1] \longrightarrow P$ such that $\pi \circ \gamma' = \gamma$, and for all $t \in [0,1]$, $\dot{\gamma}'(t) \in \mathcal{H}_{\gamma'(t)}$. In a local trivialization $P_{U_j} \simeq U_j \times G$, any lift of γ has the form $\gamma' = (\gamma, \eta)$, with $\eta : [0,1] \to G$ the unknown. In this trivialization, we have:

$$A_{(\gamma,\eta)}(\dot{\gamma}, \dot{\eta}) = (L_\eta)_*^{-1} \dot{\eta} + \mathrm{Ad}_{\eta^{-1}} A_j(\dot{\gamma}) \tag{1.3}$$

so that the horizontal lift is locally defined by the following equation:

$$\dot{\eta}(t) + (R_{\eta(t)})_* A_j(\dot{\gamma}(t)) = 0 \tag{1.4}$$

This is a non-linear, first order ordinary differential equation, and thus given any lift $p \in P_{\gamma(0)}$ of $\gamma(0)$, there exists a unique horizontal lift γ' of γ with $\gamma'(0) = p$. In particular, if γ_1, γ_2 are two curves with $\gamma_1(1) = \gamma_2(0)$, and γ_1', γ_2' horizontal lifts with $\gamma_1'(1) = \gamma_2'(0)$, then $\gamma_1'\gamma_2'$ is a horizontal lift of $\gamma_1\gamma_2$.

Another important property is G-equivariance. Let γ be a curve in B, $p \in P_\gamma(0)$ and γ' the lift of γ with $\gamma'(0) = p$. Let $\gamma'' = \gamma' \cdot g$, with $g \in G$. In a local trivialization $U_j \times G$, we have $\gamma' = (\gamma, \eta')$ and $\gamma'' = (\gamma, \eta' g)$. Using equation (1.4), we get:

$$\dot{\eta}'' + (R_{\eta' g})_* A_j(\dot{\gamma}) = (R_g)_* \Big[\dot{\eta}' + (R_{\eta'})_* A_j(\dot{\gamma})\Big] = 0$$

Hence, γ'' is the horizontal lift of γ with $\gamma''(0) = p \cdot g$.

Let γ be a loop in B, based at x, and let $p \in P_x$. Let γ' be the horizontal lift of γ with $\gamma'(0) = p$. Then $\gamma'(1) \in P_x$, so that there exists a unique element, noted $Hol_p(\gamma)$, in G, such that $\gamma'(1) = p \cdot Hol_p(\gamma)$. Using G-equivariance, we get:

$$Hol_{p \cdot g}(\gamma) = g^{-1} Hol_p(\gamma) g, \quad \text{and} \quad Hol_p(\gamma_1\gamma_2) = Hol_p(\gamma_2) Hol_p(\gamma_1)$$

Thus, the set

$$\mathrm{Hol}_p(P, D) = \{Hol_p(\gamma), \gamma \text{ piecewise } C^1 \text{ loop based at } x\}$$

is a subgroup of G. Another choice of base point and of lift of this base point leads to a conjugated subgroup of G. This group, which is only defined up to conjugation, is called the *holonomy group* of P. It will be denoted $\mathrm{Hol}(P, D)$. The subgroup of $\mathrm{Hol}(P, D)$ generated by null homotopic loops will be denoted $\mathrm{Hol}^0(P, D)$.

Remark 1.1.2. Before stating the first important result about the holonomy group, we need to clarify some terminology. A *Lie subgroup* of G will mean an embedded Lie subgroup, not necessarily a closed one, that is, the embedding need not be proper.

Proposition 1.1.3. $\mathrm{Hol}(P, D)$ *is a Lie subgroup of* G*, and* $\mathrm{Hol}^0(P, D) \subset \mathrm{Hol}(P, D)$ *is the connected component of the identity. Its Lie algebra is noted* $\mathfrak{hol}(P, D)$ *and is called the* holonomy algebra *of* (P, D).

A proof is given in [17, Proposition 2.2.4, pp. 27-28]. If $p \in P$, we can also define $\mathfrak{hol}_p(P, D)$, the Lie algebra of $\mathrm{Hol}_p(P, D)$.

If P is the frame bundle $E \to B$ of a vector bundle with a connection, we will see below that the holonomy group is the minimal group to which we can reduce the structure of E.

Remark 1.1.3. In this chapter, submanifolds are not assumed to be closed in their ambient space. Submanifolds that are properly embedded in their ambient space will be called closed submanifolds.

Let $\pi : P \to B$ be a principal G-bundle and H a Lie subgroup of G. A *principal H-subbundle* of P is a submanifold $Q \subset P$, so that the right action of $H \subset G$ and the projection $\pi_{\restriction Q} : Q \to B$ gives Q the structure of a principal H-bundle. If P has a connection D with horizontal space $\mathcal{H}$, we say that D reduces to Q if $\mathcal{H}_q \subset T_q Q$ for all $q \in Q$. In that case, $\mathcal{H}_Q$ is the horizontal space of a connection D' on Q. If D reduces to Q, then the parallel transport in P preserves Q, since the lift of any vector $v \in T_x B$ lies in $\mathcal{H}_q \subset T_q Q$ for any $q \in Q$. In particular, if $q \in Q$, the holonomy group $\mathrm{Hol}_q(P, D)$ is a subgroup of H, and this is the holonomy group of Q for the connection D'. Thus, up to conjugation in G, $\mathrm{Hol}(P, D) \subset H$.

Conversely, if $\mathrm{Hol}(P, D)$ is conjugated to a subgroup of H, then there exists $q \in P$ such that $\mathrm{Hol}_q(P, D) \subset P$. Define:

$$Q = \{\gamma'(1) \cdot h, h \in H,\ \gamma' \text{ horizontal lift of } \gamma : [0, 1] \to P,\ \gamma'(0) = q\}$$

Then Q is a principal H-subbundle of P, and by construction it is preserved by parallel transport, so that the connection D reduces to Q. To summarize:

Theorem 1.1.4 (Reduction theorem) *Let P be a principal G-bundle with a connection D. Let H be a Lie subgroup of G. Then, there exists a principal H-bundle Q with a connection D' so that (P, D) reduces to (Q, D') if and only if* $\mathrm{Hol}(P, D)$ *is conjugated to a subgroup of H.*

Now, we will see that we can build new principal bundles out of old ones. Let P be a principal G-bundle, and $\Phi : G \to G'$ a Lie group morphism. Then $P \times G'$ is equipped with a right G-action by:

$$(p, h) \cdot g = (p \cdot g, \phi(g)^{-1} h)$$

This action commutes with the natural right G'-action $(p, h) \cdot g' = (p, hg')$. Then G' acts on the right on $P \times_\Phi G'$, the quotient of $P \times G'$ by the action of G. We have a commutative diagram:

$$\begin{array}{ccc} P \times G' & \longrightarrow & P \times_\Phi G' \\ \big\downarrow & & \big\downarrow \pi' \\ P & \xrightarrow{\ \pi\ } & B \end{array}$$

so that $\pi' : P \times_\Phi G' \to B$ is a principal G'-bundle over B. If P is defined by an open covering $\{U_j\}$ of B and transition functions $g_{jk} : U_{jk} \to G$, then P' is trivial over the U_j's, and has transition functions $g'_{jk} = \Phi(g_{jk}) : U_{jk} \to G'$. Moreover, suppose that P has a connection D with local connection forms A_j. Let $\phi : \mathfrak{g} \to \mathfrak{g}'$ be the morphism induced by Φ. Then the 1-forms

$$A'_j = \phi(A_j) \in \Omega^1(U_j, \mathfrak{g}')$$

also satisfy relation (1.1), so that they define a connection D' on $P \times_\Phi G'$. We say that D' is the connection induced by D on $P \times_\Phi G'$.

For instance, if $H \subset G$, we have a morphism $\iota : H \hookrightarrow G$. Then (P, D) reduces to (Q, D') if and only if we have an isomorphism $P \simeq Q \times_\iota G$, so that the connection induced by D' is D.

1.2 Vector bundles

In this part, we introduce holonomy for vector bundles, in the light of our preceding discussion of principal bundles. We show that the holonomy group of a connection is the smallest group to which the structure of the vector bundle, including its connection, can be reduced. Lastly we define the notion of curvature.

1.2.1 Connections on vector bundles

To fix notations, we begin by the definition of a connection on a vector bundle. If E is any vector bundle over a manifold B, we will denote $C^\infty(B, E)$, or simply $C^\infty(E)$, the vector space of smooth sections of E over B, and $\Omega^k(B, E) = C^\infty(B, \Lambda^k T^*B \otimes E)$, the vector space of E-valued k-forms on B.

Definition 1.2.1. Let $E \to B$ be a real vector bundle. A connection is a linear map $\nabla^E : C^\infty(E) \longrightarrow \Omega^1(B, E)$, that satisfies the following *Leibniz rule*:

$$\nabla^E(fS) = df \otimes S + f\nabla^E S$$

for all $f \in C^\infty(B, \mathbf{R})$ and $S \in C^\infty(E)$.

Suppose E is trivial over an open covering $\{U_j\}$ of B, with transition functions φ_{jk}, such that $E_{U_j} \simeq U_j \times \mathbf{R}^k$. A section S of E is locally given by $S_j : U_j \to \mathbf{R}^k$, with $S_j = \varphi_{jk} S_\beta$ on U_{jk}. By the Leibniz rule, we have:

$$(\nabla^E S)_j = dS_j + A_j^E S_\alpha \tag{1.5}$$

with $A_j^E \in \Omega^1(U_j, \mathrm{End}(\mathbf{R}^k))$, still called the local connection forms. They satisfy the same transition rule as in (1.2) for local connection forms on principal bundles:

$$A_j^E = \varphi_{jk} A_k^E \varphi_{jk}^{-1} + \varphi_{jk}^{-1} d\varphi_{jk} \tag{1.6}$$

the aim of this section will be to investigate the relation between connections on principal bundles and vector bundles.

Suppose we are given a vector bundle $\widetilde{E} \to P$ with a right G-action $v \mapsto v \cdot g$, such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\ \cdot g\ } & \widetilde{E} \\ \downarrow & & \downarrow \\ P & \xrightarrow{\ R_g\ } & P \end{array}$$

For instance, G acts on the right on TP by $v \cdot g = (R_g)_* v$. If P has a connection D, its horizontal space is invariant by this action, so that we can take $\widetilde{E} = \mathcal{H}$.

Another important example is the trivial bundle $P \times V$, where (ρ, V) is a (left) finite-dimensional representation of G, with the action:

$$(p, v) \cdot g = (p \cdot g, \rho(g)^{-1} v)$$

The quotient $E = \widetilde{E}/G$ has a natural projection $E \to B$ that give E the structure of a vector bundle over B. An element of E_x is an equivalence class $\{v \cdot g, g \in G\}$, where v is a vector in $\widetilde{E}_p$, for $p \in P_x$, so that the choice of an element $p \in P_x$ gives an identification of E_x with $\widetilde{E}_p$.

In particular, if $\widetilde{E} = P \times V$ as above, the choice of $p \in P_x$ gives an identification of E_x with V. If $v \in E_x$ is represented by $\widetilde{v}_p \in (P \times V)_p = V$, then, in $(P \times V)_{p \cdot g} = V$, it is represented by $\widetilde{v}_{p \cdot g} = \rho(g)\widetilde{v}_p$. If P is trivial over the U_j's and has transition functions g_{jk}, then E is trivial over the U_j's and has transition maps $\rho(g_{jk}) : U_{jk} \to GL(V)$. The vector bundle E will sometimes be noted $P \times_\rho V$.

Returning to the general case of a G-equivariant vector bundle $\widetilde{E} \to P$, let S be a smooth section of $E \to B$. Define a section $\widetilde{S}$ of $\widetilde{E}$, so that for any $p \in P$, $\widetilde{S}(p)$ is the representative of $S(\pi(p))$ in $\widetilde{E}_p$. It is a smooth section, and it is G-equivariant in the following sense:

$$\widetilde{S}(p \cdot g) = \widetilde{S}(p) \cdot g$$

Conversely, a G-equivariant section $\widetilde{S}$ of $\widetilde{E}$ determines a unique section of E. To summarize:

Proposition 1.2.1. *Let $\widetilde{E}$ be a G-equivariant vector bundle over P, and let $E = \widetilde{E}/G$. Then, sections of E are in one-to-one correspondence with G-equivariant sections of $\widetilde{E}$.*

The notion of G-equivariant bundles on P and the associated bundles on B are natural with respects to the algebraic operations on vector bundles, such as direct sum, dual, tensor product, exterior product, etc. For instance, if $\widetilde{E}$ and $\widetilde{E}'$ are G-equivariant vector bundles on P, with associated vector bundles E and E' on B, then $\widetilde{E} \oplus \widetilde{E}'$ is G-equivariant and the associated bundle on B is $E \oplus E'$. In particular, for representations (ρ, V) of G, the construction of $P \times_\rho V$ behaves naturally with respect to the operations on representations. Moreover, if $\sigma : V_1 \to V_2$ is a morphism of representations, the map

$$P \times V_1 \longrightarrow P \times V_2, \quad (p, v) \longmapsto (p, \sigma(v))$$

is G-equivariant, so that it induces a vector bundle morphism from $P \times_{\rho_1} V_1$ to $P \times_{\rho_2} V_2$ that has constant rank. Also, $\ker \sigma$ and $\operatorname{coker} \sigma$ are representations of G, noted respectively (ρ_1', V_1') and (ρ_2', V_2'). With these notations, σ induces an exact sequence of vector bundles:

$$0 \longrightarrow P \times_{\rho_1'} V_1' \longrightarrow P \times_{\rho_1} V_1 \longrightarrow P \times_{\rho_2} V_2 \longrightarrow P \times_{\rho_2'} V_2' \longrightarrow 0$$

Lastly, if a representation V has an additional structure that is preserved under the action of G, then the fibers of $P \times_\rho V$ are equipped with the same structure. For example, if V is an algebra, and the G-action preserves the product, then the associated vector bundle is a bundle of algebras. If V has an inner product preserved by G, then $P \times_\rho V$ is equipped with a natural metric.

In the remaining of this section, we will use the above construction of vector bundles from representations of G, in order to define connections on associated vector bundles from a connection on P. Let $\mathcal{V}^\perp \subset T^*P$ be the subbundle of 1-forms that vanish on $\mathcal{V}$. Since $\mathcal{V}$ is G-equivariant, it is a G-equivariant vector bundle, and so are the $\Lambda^k\mathcal{V}^\perp$'s. We have $\mathcal{V} = \ker(d\pi : TP \to TB)$, so that $\mathcal{V}^\perp/G \simeq T^*B$. Let V be a representation of G, and $E \to B$ the associated vector bundle. Then $\Lambda^k\mathcal{V}^\perp \otimes V$ is a G-equivariant vector bundle, and its associated vector bundle on B is $\Omega^k(B, E)$. Using Proposition 1.2.1, we get:

Proposition 1.2.2. *E-valued k-forms on B are in one-to-one correspondence with smooth sections of $\Lambda^k\mathcal{V}^\perp \otimes V$.*

Assume now that D is a connection on P and let $\mathrm{pr}_\mathcal{H}$ be the projection of TP onto $\mathcal{H}$, with respect to the decomposition $TP = \mathcal{V} \oplus \mathcal{H}$. Let S be a smooth section of $E = P \times_\rho V$, represented by $\widetilde{S} \in C^\infty(P, V)^G$. Since $P \times V$ is a trivial bundle, we have a well defined exterior derivative d defined on $\Omega(P, V)$. The 1-form $d\widetilde{S} \circ \mathrm{pr}_\mathcal{H}$ is a G-equivariant 1-form that vanishes on $\mathcal{V}$, so that it defines an element $\nabla^E S \in \Omega^1(B, E)$. The operator ∇^E is a connection, called the connection induced by D on E. Explicitly, if X is a vector field on B and $X^\mathcal{H}$ lift, we have:

$$\widetilde{\nabla^E_X S} = X^\mathcal{H} \cdot \widetilde{S}$$

Suppose the connection D is given by a local form A_j over U_j, and the section S is locally given by $S_j \in C^\infty(U_j, V)$. Then $\widetilde{S}_j(x, g) = \rho(g)S_j(x)$. The horizontal lift of X at (x, e) is $(X, A_j(X)) \in T_{(x,e)}(U_j \times G) \simeq T_xU_j \oplus \mathfrak{g}$.Thus we have:

$$(X^\mathcal{H} \cdot \widetilde{S}_j)_{(x,e)} = (X \cdot S_j)(x) + d\rho_e(A_j(X))S_j$$

so that the connection ∇^E has local connection form $A^E_j = d\rho_e(A_j)$.

We finish this section by a comparison of parallel transport on principal bundles and associated vector bundles. With the notations above, let γ be a smooth curve, and $(v_1, ..., v_k)$ a basis of V. A choice of $p \in P_{\gamma(0)}$ induces an identification $\varphi_p : E_{\gamma(0)} \to V$. Let $(e_1, ..., e_k)$ be the basis of $E_{\gamma(0)}$ image of $(v_1, ..., v_k)$ under this identification. By parallel transport along γ, it is extended in $(e_1(t), ..., e_k(t))$ basis of $E_{\gamma(t)}$ for all t. They satisfy $\frac{D}{dt}e_i = 0$. In a trivialization $U_j \times V$ near $\gamma(0)$, we have:

$$\frac{D}{dt}e_i = \dot{e}_i + A_j(\dot{\gamma})e_i$$

where we see the e_i's as elements in V by abuse of notation. Let $\zeta(t)$ be the matrix of $(e_1(t), ..., e_k(t))$ in the basis $(e_1(0), ..., e_k(0))$. It satisfies the equation:

$$\dot{\zeta} + (R_\zeta)_* A_j(\dot{\gamma}) = 0 \tag{1.7}$$

where R_ζ is just the multiplication by ζ on the right. Comparing with equation (1.4), we see that $\zeta = \rho(\eta)$, where $\eta(t) \in G$ is the local solution of the parallel transport along γ starting from p. In particular, if γ is a loop, then:

$$\varphi_{p \cdot Hol_\gamma(p)} = \rho(Hol_\gamma(p))\varphi_p$$

so that the holonomy group of E with respect to ∇^E, defined as the group of linear isomorphisms of a E_x obtained by parallel transport along loops, satisfies:

$$\mathrm{Hol}(E, \nabla^E) = \rho(\mathrm{Hol}(P, D))$$

In particular, if P is the frame bundle of E, the two notions of holonomy group coincide.

1.2.2 Curvature

Let P be a principal G-bundle, and D a connection on P, with connection form A. We can define an operator $\Omega^k(P) \to \Omega^{k+1}(P)$, that will be denoted d_A, by:

$$d_A\omega = d\omega \circ \Lambda^{k+1} \mathrm{pr}_{\mathcal{H}} \tag{1.8}$$

This operator does not necessarily satisfy $d_A^2 = 0$. We can define d_A in a similar way for differential forms valued in a trivial bundle. The curvature of the connection A, noted F_A, is defined by:

$$F_A = d_A A$$

By construction, $F_A \in C^\infty(P, \Lambda^2\mathcal{V}^\perp \otimes \mathfrak{g})$ is a G-equivariant 2-form, for the adjoint action of G on its Lie algebra. Let $\mathrm{Ad}\, P = P \times_{\mathrm{Ad}} \mathfrak{g}$. Then, by Proposition 1.2.2, we can see F_A as an element of $\Omega^2(B, \mathrm{Ad}\, P)$. In local coordinates, F_A can be locally written as:

$$F_j = dA_j + A_j \wedge A_j \tag{1.9}$$

where A_j is the local connection form. We say that the connection is flat if $F_A = 0$.

Proposition 1.2.3. *A connection on P is flat if and only if its horizontal space $\mathcal{H}$ is integrable.*

Proof. Let X, Y be vector fields on B. Then we have:

$$\begin{aligned} dA(X^{\mathcal{H}}, Y^{\mathcal{H}}) &= X^{\mathcal{H}} \cdot A(Y^{\mathcal{H}}) - Y^{\mathcal{H}} \cdot A(X^{\mathcal{H}}) - A([X^{\mathcal{H}}, Y^{\mathcal{H}}]) \\ &= -A([X^{\mathcal{H}}, Y^{\mathcal{H}}]) \end{aligned}$$

Then $F_A = 0$ if and only if $[X^{\mathcal{H}}, Y^{\mathcal{H}}]$ is a horizontal vector field, for all vector fields X, Y on B. By Frobenius theorem, this is the condition for $\mathcal{H}$ to be an integrable distribution. □

If D is a flat connection on P, then two loops based at x that are homotopic relative to x will be lifted with homotopic paths with the same end points, so that the holonomy group is a topological object. In particular, for a flat connection, the holonomy group is discrete, so that $\mathrm{Hol}^0(D) = \{1\}$. Conversely, if $\mathrm{Hol}^0(D)$ is trivial, we can make the reverse argument to show that the distribution $\mathcal{H}$ must be integrable. In particular, we have shown that:

Proposition 1.2.4. *Let D be a connection on a principal G-bundle P. Then D is flat if and only if $\mathrm{Hol}^0(D)$ is trivial.*

For vector bundles, there is a more convenient way to define the curvature of a connection.

Definition 1.2.2. Let $E \to B$ be a vector bundle, and ∇ a connection on E. The curvature tensor $R^\nabla \in \Omega^2(B, \mathrm{End}(E))$ is defined by:

$$R^\nabla(X, Y)S = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})S \tag{1.10}$$

In order to show that R^∇ is a well defined tensor, we need to show that it is $C^\infty(M)$-linear in X, Y, S. We will check this for X. Using Liebniz rule, we find that:

$$\nabla_Y \nabla_{fX} S = f \nabla_Y \nabla_X S + (Y \cdot f) \nabla_X S$$

Since $[fX, Y] = f[X, Y] - (Y \cdot f)X$, we have:

$$\nabla_{[fX,Y]} S = f \nabla_{[X,Y]} - (Y \cdot f) \nabla_X S$$

so that the terms in $(Y \cdot f)\nabla_X S$ cancel each others. Thus we have:

$$R^\nabla(fX, Y)S = fR^\nabla(X, Y)S$$

and similarly in Y and S. Thus R^∇ is a well defined tensor which lies pointwise in $\Lambda^2 T^*B \otimes \mathrm{End}(E)$. We will sometimes write $R^\nabla(X \wedge Y \otimes S)$ instead of $R^\nabla(X, Y)S$, and drop the exponent ∇ when it is clear which connection we are talking about.

If $E = P \times_\rho V$ and ∇ is induced by a connection D on P, the curvature on P and E are related in the following way. ρ induces a representation morphism $d\rho : \mathfrak{g} \to \mathrm{End}(V)$, which in turn induces a vector bundle morphism $c_\rho : \mathrm{Ad}(P) \to \mathrm{End}(E)$. Since F_A is in $\Omega^2(B, \mathrm{Ad}(P))$, we can apply c_ρ to the curvature of P. By an explicit computation, we can show that:

Proposition 1.2.5. *With the above notations, $R^\nabla = c_\rho(F_A)$.*

In the remaining of this section, we draw a more precise link between holonomy and curvature. Let E be a vector bundle on B with a connection ∇. For $x \in B$, $\mathfrak{hol}_x(\nabla)$ is a subspace of $\mathrm{End}(E_x)$. For a path γ, we will denote by P_γ the parallel transport of E along γ. The following theorem, proved by Ambrose and Singer in [1], shows that the curvature tensor determines the holonomy algebra:

Theorem 1.2.6 (Ambrose-Singer holonomy theorem) *With the above notations, $\mathfrak{hol}_x(\nabla)$ is the subspace of $\mathrm{End}(E_x)$ consisting of endomorphisms that can be written $P_\gamma^{-1} R^\nabla(X,Y) P\gamma$, where γ is a smooth path with $\gamma(0) = x$, and $X, Y \in T_{\gamma(1)}B$.*

In particular, Proposition 1.2.4 can be seen as a corollary of this theorem. If we know that a connection is induced by a G-bundle, the restriction that R^∇ lies pointwise in $\Lambda^2 T^*M \otimes \mathfrak{hol}(\nabla)$ gives additional constraints on the curvature tensor.

1.2.3 Intrinsic torsion

From now on, let M be a manifold of dimension n, and let F be its frame bundle. If G is a Lie subgroup of $GL(n, \mathbf{R})$, a G-structure on M is a principal G-subbundle of F. For instance, an orientation of M amounts to the choice of a $SL(n, \mathbf{R})$-structure on M. If G has a metric, the orthogonal frame bundle of M is an $O(n)$-structure on M, because we can use the Grahm-Schmidt orthonormalization process to define smooth local orthonormal frames. Conversely, suppose M has an $O(n)$-structure. This structure determines local trivializations $TU_j \simeq U_j \times \mathbf{R}^n$ with transition functions in $O(n)$. We can locally define a metric on $U_j \times \mathbf{R}^n$ via the standard metric of $\mathbf{R}^n$, and since it is preserved by the transition functions, it pulls back to a globally defined metric. To summarize, we proved that $O(n)$-structures on M are in one-to-one correspondence with metrics on M.

More generally, if S is any tensor on $\mathbf{R}^n$, and $G \subset GL(n, \mathbf{R})$ the subgroup that stabilizes this tensor, then G is a closed subgroup of $GL(n, \mathbf{R})$, and G-structures on M are in one-to-one correspondence with tensors on M that can be written as S in local trivializations. For instance, if $n = 2m$, almost complex structures on M correspond to $GL(m, \mathbf{C})$-structures on M.

Let ∇ be a connection on M, and let S be a tensor on S. If S is parallel, i.e. $\nabla S = 0$, then S is invariant by parallel transport. In particular, for $x \in M$, S_x in invariant under $\mathrm{Hol}_x(M, \nabla)$. Conversely, suppose S_x is a tensor on T_xM fixed by $\mathrm{Hol}_x(M, \nabla)$. Then, for all $y \in M$, we can choose a path γ form x to y, and define S_y as the parallel transport of S_x along γ. It does not depend on the choice of γ because of our assumptions. By construction, S is a smooth tensor that satisfies $\nabla S = 0$. Thus, we have proved the following theorem:

Theorem 1.2.7 *Let M be a smooth manifold, and ∇ a connection on M. Let S be a tensor on M. If $\nabla S = 0$, then for all $x \in M$, $\mathrm{Hol}_x(\nabla)$ fixes S_x.*

Conversely, let $x \in M$ and S_x be a tensor on T_xM, and suppose the latter is fixed by $\mathrm{Hol}_x(\nabla)$. Then, there exists a unique parallel tensor S on M taking the value S_x at x.

This theorem essentially says that the geometric structures on M that are compatible with a connection ∇ are determined by the holonomy group of the connection. Often, one also wants to satisfy analytical conditions on the structure of M. For instance, a complex manifold is a manifold endowed with an integrable almost complex structure, and a symplectic manifold is a manifold endowed with a non-degenerate 2-form ω that satisfies $d\omega = 0$. These conditions are related to the notion of torsion of a connection.

Definition 1.2.3. Let ∇ be a connection on M. If X, Y are vector fields on M, define:

$$T(\nabla)(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{1.11}$$

Then $T(\nabla)$ is a tensor in $\Lambda^2 T^*M \otimes TM$, called the *torsion* of ∇. If $T(\nabla)$ vanishes, ∇ is called *torsion-free*.

Since $T(\nabla)$ is skew-symmetric in X, Y, we only need to check that for all smooth functions f, we have $T(\nabla)(fX,Y) = fT(\nabla)(X,Y)$. This comes from the Liebniz rule for ∇, and the formula:

$$[fX, Y] = f[X,Y] - (Y \cdot f)X$$

Given a G-structure P on M, it is important to know whether it carries torsion-free connections. Let ∇, ∇' be two connections induced by P. Then $\alpha = \nabla' - \nabla$ is an element of $\Omega^1(M, \mathrm{Ad}(P))$, and by definition of the torsion, we have:

$$T(\nabla')(X,Y) = T(\nabla)(X,Y) + \alpha_X(Y) - \alpha_Y(X) \tag{1.12}$$

In particular, if ∇ is torsion free, then for all connections ∇' induced by P, there exists an $\mathrm{Ad}(P)$-valued 1-form α such that $T(\nabla')(X,Y) = \alpha_X(Y) - \alpha_Y(X)$. Conversely, if this equation holds for a connection ∇' on P, then the connection $\nabla = \nabla' - \alpha$ also reduces to P, and by equation (1.12), ∇ is torsion-free.

Using this, we define a tensor τ, which will be the obstruction for a G-structure to admit torsion-free connections. Let V denote the representation of G on $\mathbf{R}^n$ induced by the inclusion $G \subset GL(n, \mathbf{R})$. Let σ be the morphism of representations $V^* \otimes \mathfrak{g} \longrightarrow \Lambda^2 V^* \otimes V$ defined by:

$$\sigma(\alpha)(X,Y) = \alpha_X(Y) - \alpha_Y(X)$$

It induces a bundle morphism $c_\sigma : T^*M \otimes \mathrm{Ad}(P) \to \Lambda^2 T^*M \otimes TM$. Let $E_1 = \ker \sigma$ and $E_2 = \operatorname{coker} \sigma$. The torsion $T(\nabla)$ of a connection on P is in $\Lambda^2 T^*M \otimes TM$, and by equation (1.12), it depends on the choice of ∇ only up to a section of the image of c_σ. In particular, the image of $T(\nabla)$ in E_2, noted

$\tau(P)$, is independent of the choice of ∇. A G-structure P admits torsion-free connections if and only if $\tau(P) = 0$, and in that case, the set of torsion-free connections is an affine space modeled on $C^\infty(E_1)$. In particular, if σ is injective, then a torsion-free connection is necessarily unique. The tensor $\tau(P)$ is called the *intrinsic torsion* of P, and we say that P is torsion-free if $\tau(P) = 0$.

1.3 Riemannian holonomy groups

In this part, we say a few general words of the holonomy groups associated with the Levi-Civita connection of a metric, including the Berger's classification of complete irreducible Riemannian manifolds.

1.3.1 Intrinsic torsion for subgroups of $SO(n)$

We have seen at the beginning of §1.2.3 that an $O(n)$-structure on a manifolds M is equivalent to a Riemannian metric. If M is oriented, metric and a choice of orientation are equivalent to a $SO(n)$-structure. Any Riemannian manifold (M, g) is endowed with a unique torsion-free connection compatible with the metric g, called the *Levi-Civita connection*. In particular, any $SO(n)$-structure is torsion-free. The holonomy group of (M, g) is by definition the holonomy group of the Levi-Civita connection. The holonomy group of an oriented Riemannian manifold must be a subgroup of $SO(n)$.

On the other hand, let G be a subgroup of $SO(n)$, and P a G-structure on M. Here, implicitly, we choose a particular embedding $\iota : G \hookrightarrow SO(n)$. Then we define an $SO(n)$-structure $\tilde{P}$ by

$$\tilde{P} = \{u \cdot g,\ u \in P, g \in SO(n)\}$$

Then this $SO(n)$-structure $\tilde{P}$ is equivalent to a Riemannian metric g on M. Any connection on P is compatible with g. Hence, if P admits a torsion-free connection, this is necessarily the Levi-Civita connection of g. In particular, a G-structure admits at most one torsion-free connection. This is equivalent to the fact that the map σ_G defined in §1.2.3 is injective for $G \subset SO(n)$. Moreover, a G-structure P is torsion-free if and only if the Levi-Civita connection of G reduces to P. In this part, we discuss a more precise description of the intrinsic torsion of G-structures in the case where G is a subgroup of $SO(n)$.

We fix a subgroup G of $SO(n)$, a G-structure P, and a metric g induced by P as described above. Every connection ∇' induced by a G-structure P has torsion $T(\nabla')$ in $T^*M \otimes \mathfrak{so}(n)$ since ∇' preserves g, so that we actually have:

$$\tau(P) \in T^*M \otimes \frac{\mathfrak{so}(n)}{\mathfrak{g}}$$

More precisely, write $\nabla' = \nabla + \alpha$, where ∇' is any connection on M induced by P, and ∇ is the Levi-Civita connection of g. Then $T(\nabla') = c_\sigma(\alpha)$ since

∇ is torsion free, so that $T(\nabla')$ can be identified with the 1-form α since σ is injective. $\alpha \in \Omega^1(M, P \times_{\mathrm{Ad}_G} \mathfrak{so}(n))$, so that it can be identified with an 1-form $\widetilde{\alpha} \in \mathcal{V}^\perp \otimes \mathfrak{so}(n)$. The intrinsic torsion of P is then identified with $\widetilde{\alpha}^\perp$, the component of $\widetilde{\alpha}$ in $\mathcal{V}^\perp \otimes \mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ is the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{so}(n)$. Let $A' \in \Omega^1(P, \mathfrak{g})$ be the connection form of ∇' and $A \in \Omega^1(P, \mathfrak{so}(n))$ the restriction to P of the Levi-Civita connection form. We have $A' = A + \widetilde{\alpha}$, and since $A'^\perp = 0$, we get $A^\perp = -\widetilde{\alpha}^\perp$. Therefore, up to the sign, we can identify the intrinsic torsion of P with $A^\perp$, which can be seen downstairs as a section of $T^*M \otimes \mathfrak{g}^\perp$. This interpretation gives a more concrete construction of the intrinsic torsion of a G-structure when G is a subgroup of $SO(n)$. Moreover, it shows that in this case, P is torsion-free if and only if the Levi-Civita connection on M reduces to P. If the Levi-Civita connection has local forms A_j in local trivializations $P_{U_j} \simeq U_j \times G$ of P, then the local expression of $\tau(P)$ in the corresponding trivialization of $T^*M \otimes \mathfrak{g}^\perp$ is $A_j^\perp$, the orthogonal projection of A_j onto $\mathfrak{g}^\perp$.

Now, we would like to make the link between the intrinsic torsion of a G-structure P and the tensors invariants under G. Let (ρ, W) be a representation of $SO(n)$ and let $T_0 \in W$ invariant under G. The G-structure P then defines a tensor $T \in P \times_\rho W$. Let $\tau(P)$ be the intrinsic torsion of P. As explained above, $\tau(P)$ takes values in the bundle $T^*M \otimes \mathfrak{g}^\perp$, seen as a subbundle of $T^*M \otimes \mathfrak{so}(n)$. The representation $\rho : SO(n) \to GL(W)$ induces a representation morphism $d\rho_e : \mathfrak{so}(n) \to \mathrm{End}(W)$. Then, if X is a vector field on M, $\tau(P)_X$ naturally acts on $P \times_\rho W$ by $d\rho_e(\tau(P)_X)$. Thus, if S is a section of $P \times_\rho W$, $\tau(P)_* S$ defines an element of $T^*M \otimes W$. Note that ∇S is a section of the same bundle.

Proposition 1.3.1. *With the above notations, $\nabla T = \tau(P)_* T$, where ∇ is the Levi-Civita connection on M.*

Proof. This is a local statement, so that we can work in a local trivialization $P_{U_j} \simeq U_j \times G$, that induces local trivializations for the associated bundles. Let $A_j \in \Omega^1(U_j, \mathfrak{so}(n))$ be the local connection form of the Levi-Civita connection; it is well defined but this is not a connection form associated to the G-structure P, since it does not necessarily takes values in $\mathfrak{g}$. The local connection form of ∇ acting of $P \times_\rho W$ is given by $d\rho_e(A_j)$. Locally, $T = T_0$ in this trivialization, so that we have:

$$\nabla T = dT_0 + d\rho_e(A_j)T_0 = d\rho_e(A_j^\perp)T_0$$

where $A_j^\perp$ is the component of A_j in $\mathfrak{g}^\perp$. The second equality holds because T_0 is as constant, and since G stabilizes T_0, we have $d\rho_e(\mathfrak{g}) \cdot T_0 = 0$. But $A_j^\perp$ is the local expression of the intrinsic torsion, so that $d\rho_e(A_j^\perp)T_0 = \tau(P)_* T$. □

This proposition gives another way to see that if P is torsion-free, then the G-invariant tensors are parallel for the Levi-Civita connection. It turns out that in most cases of interest, the intrinsic torsion can be recovered from the covariant derivatives of G-invariant tensors. To see this, suppose (ρ^l, W^l) are irreducible representations of $SO(n)$, and choose $T^l \in W^l$, so that G is the subgroup of $SO(n)$ that stabilizes the T^l's. Then, the Lie algebra $\mathfrak{g}$ is the intersection of the kernels of the maps $\mathfrak{so}(n) \to \mathrm{End}(W^l), \xi \mapsto d\rho^l_e(\xi)T^l$. As a consequence, if $\mathfrak{m}$ is any complement of $\mathfrak{g}$ in $\mathfrak{so}(n)$, the linear map

$$\xi \in \mathfrak{m} \mapsto (d\rho^l_e(\xi)T^l)_l \in \bigoplus_l W^l$$

is injective. Thus, the following corollary is an immediate consequence of Proposition 1.3.1:

Corollary 1.3.2. *Let (M, g) be a Riemannian manifold, and P a G-structure on M, where $G \subset SO(n)$ is the stabilizer of tensors T^l in irreducible representations W^l of $SO(n)$. Then $\tau(P)$ is determined by the data of the ∇T^l's, where ∇ is the Levi-Civita connection of g.*

∇T^l is a section of $T^*M \otimes W^l$, but the representation $V^* \otimes W^l$ needs not be irreducible for G, so that the ∇T^l's can be decomposed into irreducible components. Often, some of these components are naturally isomorphic to each others, so that we do not need to specify all the components of all the ∇T^l's to recover the intrinsic torsion. For most groups of interest, the decomposition of the intrinsic torsion have been fully worked out, and we will give this decomposition without proof when needed, referring to the original articles.

1.3.2 The Berger's list

Let M be an oriented manifold with a metric g, and denote ∇ the Levi-Civita connection. The choice of metric g correspond to a $SO(n)$-structure P on M, and ∇ is the connection induced by the unique torsion-free connection on P. Let $R \in \Omega^2(M, \mathrm{End}(TM))$ the the curvature of ∇. Since g give a canonical isomorphism between TM and T^*M, we have $\mathrm{End}(TM) \simeq T^*M \otimes T^*M$. Moreover, we have seen in §1.2.2 that, since ∇ is induced by a connection on the $SO(n)$-structure P, R actually takes values in $\Lambda^2 T^*M \otimes \mathrm{Ad}(P)$, with fibers isomorphic to $\Lambda^2 V^*M \otimes \mathfrak{so}(n)$, where V is the natural representation of $SO(n)$ on $\mathbf{R}^n$. Through the isomorphism induced by the metric, $\mathfrak{so}(n) \simeq \Lambda^2 V^*$ as representations of $SO(n)$, so that we can see R as a section of the bundle $\Lambda^2 T^*M \otimes \Lambda^2 T^*M$. Let us write $R = R_{abcd} = g_{ae} R^e{}_{bcd}$ in index notations. Using the fact that ∇ is torsion free, we can prove the following result:

Theorem 1.3.3 *With the above notations, R and its covariant derivative ∇R satisfy the following equations:*

$$\begin{aligned} R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}, \\ R_{abcd} + R_{adbc} + R_{acdb} = 0, \\ \text{and} \qquad \nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0. \end{aligned} \tag{1.13}$$

In particular, the first equation implies that R is a section of $S^2\Lambda^2T^*M$. This is true for any metric, and by the Ambrose-Singer holonomy theorem, we obtain that the tensor R lives in $S^2\mathfrak{hol}(g)$, so that we can get additional constraints on the curvature tensor if we can further restrict the structure group of (M, g). Conversely, it also gives constraints on the Lie groups that can be the holonomy group of a Riemannian manifold. In the remaining of this part, we will describe without proof the classification of these groups.

If (M, g) is a Riemannian manifold of dimension n with Levi-Civita connection ∇, we call *holonomy representation* the representation of the holonomy group $\mathrm{Hol}_x(g)$ on T_xM. Up to isomorphism, it is independent of the choice of x, so that we can think of it as a representation of $\mathrm{Hol}(g) \subset SO(n)$ on $\mathbf{R}^n$. If M is isometric to a product of non-trivial Riemannian manifolds $(M_1, g_1) \times (M_2, g_2)$, then $\mathrm{Hol}(g) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2)$ and the representation of $\mathrm{Hol}(M)$ on $\mathbf{R}^n$ is reducible. Conversely, it turns out that if the representation of $\mathrm{Hol}(g)$ on $\mathbf{R}^n$ can be reduced as $\mathbf{R}^n = V_1 \oplus V_2$, then M is locally the product of two Riemannian manifolds M_1 and M_2, and $\mathrm{Hol}^0(g)$ is a product of groups H_1 and H_2, each of which acting trivially on one of the composants of $V_1 \oplus V_2$ [17, Propositions 3.2.2, 3.2.3 and 3.2.4, pp. 47-48]. If the holonomy representation is reducible, then (M, g) is *locally reducible*, that is, it is locally isometric to a Riemannian product. (M, g) is said to be irreducible if the holonomy representation is irreducible. In that case, both $\mathrm{Hol}(g)$ and $\mathrm{Hol}^0(g)$ act irreducibly on $\mathbf{R}^n$. It is then useful to assume that the holonomy representation is irreducible. In order to avoid the distinction between $\mathrm{Hol}(g)$ and $\mathrm{Hol}^0(g)$, we also assume that M is simply connected.

A particular class of Riemannian manifolds are called *symmetric spaces*. Locally symmetric spaces are defined by the property that, for all $x \in M$, the geodesic involution taking $v \in T_xM$ to $-v$ can be extended in a local isometry of M. Such spaces are also characterized by the property that the curvature tensor is parallel. If the geodesic involution at each point can be extended to a global isometry, the space is called symmetric. Such spaces have been introduced by Élie Cartan in 1925, who used his own classification of Lie algebras to classify symmetric spaces and their holonomy groups [7, 8].

For non-symmetric spaces, the classification of holonomy groups have been achieved by in 1955 by Berger [3], who proved the following theorem:

Theorem 1.3.4 (Berger) *Suppose M is a simply connected manifold of dimension n and g is a Riemannian metric on M, that is irreducible and non-symmetric. Then exactly one of the following cases holds.*

(i) $\mathrm{Hol}(g) = SO(n)$,

(ii) $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = U(m)$ *in* $SO(2m)$,

(iii) $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = SU(m)$ *in* $SO(2m)$,

(iv) $n = 4m$ *with* $m \geq 2$*, and* $\mathrm{Hol}(g) = Sp(m)$ *in* $SO(4m)$*,*

(v) $n = 4m$ *with* $m \geq 2$*, and* $\mathrm{Hol}(g) = Sp(m)Sp(1)$ *in* $SO(4m)$*,*

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$ *in* $SO(7)$*,*

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *in* $SO(8)$

Riemannian manifolds with holonomy $SU(m)$, $Sp(m)$, G_2 and $\mathrm{Spin}(7)$ have the special property that they are *Ricci-flat*. If R is the Riemann curvature tensor of a metric g, the Ricci curvature tensor can be defined as

$$\mathrm{Ric}(X,Y) = \sum_i R(X, e_i, e_i, Y)$$

where $\{e_i\}$ is a local orthonormal frame. The metric is called Ricci-flat if the Ricci curvature identically vanishes. More generally, a metric is called *Einstein* if there exists a scalar λ such that $\mathrm{Ric}_g = \lambda g$. Constructing manifolds with a complete Einstein or Ricci-flat metric is a very hard problem. An even harder problem contained in this one is the construction of complete manifolds with special holonomy. The purpose of the present thesis is to describe a particular construction.

Chapter 2

Special geometry

In this chapter, we introduce different types of geometric structures, that will be of interest to us throughout this thesis. In the light of the first chapter, we try to put the emphasis on the relations between linear representations, geometric structures, invariant tensors and the geometric meaning of the torsion-free condition.

In the first part of this chapter, we discuss complex manifolds from the point of view of $GL(n, \mathbf{C})$-structures. In §2.1.1, we introduce complex and almost-complex structures, and describe the decomposition of differential forms into different holomorphic and anti-holomorphic types. Throughout §2.2.1, we explain the Newlander-Nirenberg theorem, which implies that an almost complex structure is integrable if and only if the underlying $GL(n, \mathbf{C})$-structure is torsion-free. There are different ways of formulating this theorem, and we tried to choose the formulation that would be the more convenient in Chapter 4, when we talk about analytic deformations of complex structures.

In the second part we are concerned with $SU(3)$-structures. In §2.2.1, we give a brief account on Kähler and Calabi-Yau manifolds in all dimension. In §2.2.2, we give a more precise description in the case of (complex) dimension 3, and introduce stable 3-forms and the Hitchin's duality map, that will play an important role in Chapter 5. Lastly, in §2.2.3, we describe the relevant representations of $SU(3)$ and explain the structure of the intrinsic torsion of $SU(3)$-manifolds.

At last, we introduce G_2-structures, which are central in our thesis. In §2.3.1, we begin by some linear algebra and define positive 3-forms on an oriented 7-dimensional vector space. Then we give some details about the inclusion of holonomy groups $SU(3) \subset G_2$, since this is the viewpoint which will be the more useful to us. Then in §2.3.2, we describe some representations of G_2 and the intrinsic torsion of G_2-structures.

2.1 Complex manifolds

As smooth manifolds are defined as topological spaces locally homeomorphic to an open subset of $\mathbf{R}^n$, with smooth transition functions, a complex manifold is locally homeomorphic to $\mathbf{C}^n$ and has holomorphic transitions. A complex manifold is in particular a smooth one, and its complex structure is naturally associated to a $GL(n, \mathbf{C})$-structure. On the other hand, a $GL(n, \mathbf{C})$-structure on a smooth manifold defines what is called an almost complex structure. As we shall see, the condition for an almost complex structure to come from a complex structure is precisely that the associated $GL(n, \mathbf{C})$-structure be torsion-free.

2.1.1 Complex and almost complex structures

Definition 2.1.1. A complex manifold M of dimension n is a paracompact Hausdorff topological space equipped with a family of complex charts $\{(U_j, f_j)\}$, where the U_j's form an open covering of M, and $f_j : U_j \to \mathbf{C}^n$ are homeomorphisms onto an open subset of $\mathbf{C}^n$, that satisfy the following compatibility condition. Let $j \neq k$ so that $U_{jk} = U_j \cap U_k \neq \emptyset$, and consider the homeomorphism $f_{jk} : f_k(U_{jk}) \to f_j(U_{jk}) = f_j \circ f_k^{-1}$. The compatibility condition is that all f_{jk}'s must be holomorphic. The family $\{(U_j, f_j)\}$ is said called a *complex structure* on M.

A compact complex manifold of dimension n is then covered by a finite number of open subsets $U_1, ..., U_m$, where each U_j is homeomorphic to the polydisc $\{(z^1, ..., z^n) \in \mathbf{C}^n,\ |z_j| < 1, j = 1, ..., n\}$. Denoting by $z_j = (z_j^1, ..., z_j^n)$ the complex coordinates on U_j, the compatibility condition is that for all $j \neq k$ such that $U_{jk} \neq \emptyset$, we have $z_{jk}^\alpha = f_{jk}^\alpha(z_k)$, where the f_{jk}^α are holomorphic functions.

In this chapter, it will be convenient to adopt the following conventions for indices. Latin indices will refer to the coordinate chart, and in a given chart, greek indices refer the particular coordinate in $\mathbf{C}^n$. We also implicitely sum over all repeated up and down greek indices.

Obviously, a complex manifold of dimension n is in particular a smooth manifold of dimension $2n$, identifying $\mathbf{C}^n$ with $\mathbf{R}^{2n}$. Let J_0 the canonical complex structure of $\mathbf{R}^{2n} \simeq \mathbf{C}^n$, seen as an endomorphism of the tangent space of $\mathbf{C}^n$. If $f : U \to \mathbf{C}^n$ is any coordinate chart, we can define $J = f^* J_0$, which is an endomorphism of the tangent space to X over U, that satisfies $J^2 = -1$. Since the transitions between coordinate charts are holomorphic, J does not depend on any particular choice of complex coordinates. Therefore, J is a well defined section of the tangent space to the real manifold X.

Definition 2.1.2. Let M be a smooth manifold. An *almost complex structure* J on M is a smooth section of $\mathrm{End}(TM)$ that satisfies $J^2 = -\mathrm{Id}$. A manifold endowed with an almost complex structure is called an *almost complex manifold.*

By our discussion above, a complex manifold is also an almost complex manifold, with almost complex structure determined by the coordinates chart. Since any vector space equipped with an endomorphism that squares to $-\mathrm{Id}$

is even dimensional, it follows that any almost complex manifold has even dimension. An almost complex structure that comes from a complex structure on M is called *integrable*.

Lemma 2.1.1. *Let M be a smooth manifold of dimension $2n$. Then, an almost complex structure J on M is equivalent to a $GL(n,\mathbf{C})$-structure P on M.*

Proof. If P is a $GL(n,\mathbf{C})$-structure on M, then there exists trivializations $TU \simeq U\times\mathbf{R}^{2n}$ of the tangent spaces so that the transitions functions commute with J_0, the canonical complex structure of $\mathbf{R}^{2n}$. Hence, if we define J by $J_{|U} = J_0$ on $TU \simeq U \times \mathbf{R}^{2n}$, J is a globally well defined section of $\mathrm{End}(TM)$, and it is clear that $J^2 = -\,\mathrm{Id}$.

Conversely, let J be an almost complex structure on M. Let g be any Riemannian metric on M and h be defined by

$$h(u,v) = g(u,v) + g(Ju,Jv)$$

Then h is a Riemannian metric on M and J acts by isometries with respect to h. Let e_1 be any non-vanishing local section of TM and set $e_2 = Je_1$. Then the vector space spanned by (e_1,e_2) is locally a smooth subbundle of dimension 2, noted E, and its orthogonal with respect to h is a smooth subbundle of dimension $2n-2$, noted $E^\perp$. Since J preserves E and acts by isometries, J leaves $E^\perp$ invariant. We can then choose a locally non-vanishing section e_3 of the orthogonal, set $e_4 = Je_3$, and iterate this process to build a local frame $(e_1, e_2 = Je_1, ..., e_{2n-1}, e_{2n} = Je_{2n-1})$. In the local trivialization induced by this frame, J acts as the canonical complex structure J_0 on $\mathbf{R}^{2n}$. Hence, J induces a reduction of the frame bundle of TM to $GL(n,\mathbf{C})$. □

As a consequence of this lemma, the properties of most bundles over an almost complex manifold (M,J) depend on the representations of $GL(n,\mathbf{C})$. Let V be the canonical representation of $GL(n,\mathbf{C})$ onto $\mathbf{R}^{2n}$ induced by the identification of the complex structure J_0 with multiplication by i. Let $V_{\mathbf{C}} = V\otimes\mathbf{C}$ be the complexified representation, that is, this is the tensor product representation, where $GL(n,\mathbf{C})$ acts trivially on the factor $\mathbf{C}$. We consider $V_{\mathbf{C}}$ as a complex vector space where the multiplication by i is given by $i(X\otimes\lambda) = X\otimes(i\lambda)$, and drop the tensor product by writing $X\otimes\lambda = \lambda X$, for $\lambda\in\mathbf{C}$. The real endomorphism J of V extends to a complex endomorphism of $V_{\mathbf{C}}$, still denoted J, that still satisfies $J^2 = -\,\mathrm{Id}$. Hence we have a decomposition

$$V_{\mathbf{C}} = V^{1,0}\oplus V^{0,1}$$

where $V^{1,0} = \ker(J-i)$ and $V^{0,1} = \ker(J+i)$. Since the action of $GL(n,\mathbf{C})$ preserves J, $V^{1,0}$ and $V^{0,1}$ are representations of J. It is easy to check that have the explicit description:

$$V^{1,0} = \Big\{X - iJX,\ X\in V\Big\}$$
$$V^{0,1} = \Big\{X + iJX,\ X\in V\Big\}$$

The inclusion $V \hookrightarrow V_{\mathbf{C}} = V^{1,0} \oplus V^{0,1}$ can be decomposed as the sum $\pi^{1,0} \oplus \pi^{0,1}$, where we have:

$$\pi^{1,0} : X \in V \to \frac{1}{2}\Big(X - iJX\Big)$$

It is clear that $\pi^{1,0} : V \to V^{1,0}$ is an isomorphism of real representations. Since $\pi^{1,0}(JX) = J\pi^{1,0}(X) = i\pi^{1,0}(X)$, the complex vector space (V, J) is canonically isomorphic to $(V^{1,0}, i)$. Similarly, we have:

$$\pi^{0,1} : X \in V \to \frac{1}{2}\Big(X + iJX\Big) \in V^{0,1}$$

and $\pi^{0,1}$ is a complex antilinear isomorphism from (V, J) to $V^{0,1}$.

In the same way, we can consider, the complexification $V^*_{\mathbf{C}}$ of the dual representation, on which J acts as $J\alpha = \alpha \circ J^{-1} = -\alpha \circ J$. $V^*_{\mathbf{C}}$ splits as the direct sum $\Lambda^{1,0} \oplus \Lambda^{0,1}$, where

$$\Lambda^{1,0} = \Big\{\alpha - iJ\alpha,\ \alpha \in V^*\Big\}$$
$$\Lambda^{0,1} = \Big\{\alpha + iJ\alpha,\ \alpha \in V^*\Big\}$$

Moreover, one readily checks that $\Lambda^{1,0}$ and $\Lambda^{0,1}$ are the respective duals of $V^{1,0}$ and $V^{0,1}$, that is, any element $\eta \in \Lambda^{1,0}$ vanishes on $V^{0,1}$. Taking exterior products, we see that

$$(\Lambda^k V^*) \otimes \mathbf{C} = \Lambda^k_{\mathbf{C}} V^*_{\mathbf{C}} = \bigoplus_{p+q=k} \Lambda^{p,q}$$

where $\Lambda^{p,q} = (\Lambda^p_{\mathbf{C}} \Lambda^{1,0}) \wedge (\Lambda^q_{\mathbf{C}} \Lambda^{0,1})$. Elements of $\Lambda^{p,q}$ are called *(linear) forms of type* (p, q) or (p, q)*-forms*. Finally, note that the complex conjugation is well defined on $V_{\mathbf{C}}$ and $\Lambda^k V^*_{\mathbf{C}}$, since they are the complexification of real vector spaces, and we have $\overline{V^{1,0}} = V^{0,1}$ and $\overline{\Lambda^{p,q}} = \Lambda^{q,p}$.

All the above properties pass from linear representations of $GL(n, \mathbf{C})$ to almost complex manifolds. Hence, if (M, J) is an almost complex manifold of dimension $2n$, then $T^{\mathbf{C}}_M = TM \otimes \mathbf{C}$ splits as $T^{1,0}_M \oplus T^{0,1}_M$, and $\Omega^k(M, \mathbf{C}) = \bigoplus \Omega^{p,q}_M$. Sections of $\Omega^{p,q}_M$ are called *differential forms of type* (p, q), or simply (p, q)*-forms*.

For complex manifolds, one one can give a more concrete description of (p, q)-froms. Suppose M is a complex manifold, with associated almost complex structure J, and let $z = (z^1, ..., z^n)$ be a local complex chart. We write $z^\alpha = x^\alpha + iy^\alpha$. We know that $\Big(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\Big)$ is a local frame of the real tangent space $T^{\mathbf{R}}_M = TM$, and since $J\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial y^\alpha}$, $\Big(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\Big)$ is a complex basis frame of $(T^{\mathbf{R}}_M, J)$. Hence, if we define

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - i\frac{\partial}{\partial y^\alpha}\right), \qquad \frac{\partial}{\partial \overline{z}^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + i\frac{\partial}{\partial y^\alpha}\right)$$

then $\left(\frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^n}\right)$ is a local frame of $T_M^{1,0}$. Similarly, $T_M^{0,1}$ has local frame $\left(\frac{\partial}{\partial \overline{z}^1}, ..., \frac{\partial}{\partial \overline{z}^n}\right)$. We also define local 1-forms:

$$dz^\alpha = dx^\alpha + idy^\alpha, \qquad\qquad d\overline{z}^\alpha = dx^\alpha - idy^\alpha$$

Then elements of the form $dz^{\alpha_1} \wedge ... \wedge dz^{\alpha_p} \wedge d\overline{z}^{\beta_1} \wedge ... \wedge d\overline{z}^{\beta_q}$ form a local frame of $\Omega_M^{p,q}$. The operator d acts on complex-valued functions as

$$df = \frac{\partial f}{\partial z^\alpha} dz^\alpha + \frac{\partial f}{\partial \overline{z}^\beta} d\overline{z}^\beta$$

where we sum over repeated greek indices. We define the operator ∂ and $\overline{\partial}$ respectively as the composition of d with the projection of the space of 1-forms onto $\Omega_M^{1,0}$ and $\Omega_M^{0,1}$. In coordinates it is clear that

$$\partial f = \frac{\partial f}{\partial z^\alpha} dz^\alpha, \qquad\qquad \overline{\partial} f = \frac{\partial f}{\partial \overline{z}^\beta} d\overline{z}^\beta.$$

In general, we can easily see in coordinates that $d = \partial + \overline{\partial}$, where $\partial : \Omega_M^{p,q} \to \Omega_M^{p+1,q}$ and $\overline{\partial} : \Omega_M^{p,q} \to \Omega_M^{p,q+1}$ are defined in local coordinates by:

$$\partial(f_{A\overline{B}} dz^A \wedge d\overline{z}^B) = \frac{\partial f_{A\overline{B}}}{\partial z^\alpha} \, dz^\alpha \wedge dz^A \wedge d\overline{z}^B$$
$$\overline{\partial}(f_{A\overline{B}} dz^A \wedge d\overline{z}^B) = \frac{\partial f_{A\overline{B}}}{\partial \overline{z}^\beta} \, d\overline{z}^\beta \wedge dz^A \wedge d\overline{z}^B$$

where A, B are multi-indices, and if $A = (\alpha_1, ..., \alpha_p)$, we write dz^A as a shorthand for $dz^{\alpha_1} \wedge ... \wedge dz^{\alpha_p}$. Since $0 = d^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2$, we have the following relations:

$$\partial^2 = 0 = \overline{\partial}^2, \qquad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

For a general almost complex manifold (M, J) of dimension $2n$, if $\pi^{p,q}$ denotes the projection of $\Omega^{p+q}(M, \mathbf{C})$ onto $\Omega_M^{p,q}$, we can still define the operators $\partial_J = \pi^{p+1,q} \circ d : \Omega_M^{p,q} \to \Omega_M^{p+1,q}$ and $\overline{\partial}_J = \pi^{p,q+1} \circ d : \Omega_M^{p,q} \to \Omega_M^{p,q+1}$, but except in trivial cases where $p = q = 0$ or $p + q = 2n - 1$, $d \neq \partial_J + \overline{\partial}_J$. Moreover, ∂_J^2 and $\overline{\partial}_J^2$ need not to vanish.

2.1.2 The Newlander-Nirenberg theorem

Definition 2.1.3. Let (M, J) be an almost complex manifold. We define the *Nijenhuis tensor* $\mathcal{N} \in C^\infty(\Lambda^2 T^*M \otimes TM)$ by:

$$\mathcal{N}(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$$

It is straightforward to check that this expression defines a tensor. We can also see $\mathcal{N}$ as a tensor acting on complex-valued vector fields. As in the real case, a subbundle of $T_M^{\mathbf{C}}$ is called *integrable* if it is stable by Poisson bracket.

Lemma 2.1.2. *Let (M,J) be an almost complex manifold and $\mathcal{N}$ be the Nijenhuis tensor. Then $\mathcal{N} \equiv 0$ if and only if $T_M^{0,1}$ is integrable.*

Proof. Suppose the the Nijenhuis tensor vanishes identically, and let X, Y be local sections of $T_M^{0,1}$. Then we have:

$$0 = \mathcal{N}(X,Y) = -2[X,Y] + 2iJ[X,Y]$$

so that $J[X,Y] = -i[X;Y]$. Hence, $T_M^{0,1}$ is integrable.

Conversely, if $T_M^{0,1}$ is integrable, this is also true of $T_M^{1,0}$ by conjugation. It is then clear that $\mathcal{N}$ vanishes on $\Lambda^2 T_M^{1,0}$ and $\Lambda^2 T_M^{0,1}$. It remains to prove that $\mathcal{N}$ vanishes on cross term. Actually, this is always true: let X be a local section of $T_M^{1,0}$ and Y a local section of $T_M^{0,1}$. Then we have:

$$\mathcal{N}(X,Y) = [X,Y] + J[X,iY] - J[iX,Y] - [X,Y] = 0$$

Therefore $\mathcal{N}$ identically vanishes. □

On a complex manifold, it is straightforward to check in local coordinates that $T_M^{1,0}$ is integrable. Hence, $\mathcal{N} \equiv 0$ for an integrable complex structure. By a theorem of Newlander and Nirenberg, the converse is actually true. There are several ways to state this theorem, and we prefer use one way that we will be able to use again later on.

Let B be the open unit ball in $\mathbf{C}^n$, J_0 be the canonical complex structure of $\mathbf{C}^n$, and let $J = J(z)$ be an almost complex structure on B; we do not require J to be integrable. Remember that we can define vector fields in $T_B^{\mathbf{C}}$ by

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - i\frac{\partial}{\partial y^\alpha}\right), \qquad \frac{\partial}{\partial \overline{z}^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + i\frac{\partial}{\partial y^\alpha}\right)$$

that form a global frame of $T_B^{\mathbf{C}}$, with corresponding dual coframe given by:

$$dz^\alpha = dx^\alpha + idy^\alpha, \qquad d\overline{z}^\alpha = dx^\alpha - idy^\alpha$$

Even though J is not integrable, we can still define near 0 the complex vector fields

$$\frac{\partial}{\partial \zeta^\beta} = \frac{1}{2}\left(\frac{\partial}{\partial x^\beta} - iJ(z)\frac{\partial}{\partial x^\beta}\right), \qquad \frac{\partial}{\partial \overline{\zeta}^\beta} = \frac{1}{2}\left(\frac{\partial}{\partial x^\beta} + iJ(z)\frac{\partial}{\partial x^\beta}\right)$$

and the complex 1-forms

$$d\zeta^\beta = dx^\beta - iJ(z)dy^\beta, \qquad d\overline{\zeta}^\beta = dx^\beta + iJ(z)dy^\beta$$

even though there are a priori no complex functions $\zeta^1, ..., \zeta^n$ defined on B. Since $J(z)$ goes to J_0 when z goes to 0, these vector fields and 1-form form a frame and coframe of the complexified tangent space near 0. Moreover, if

f is a function defined on B, $\overline{\partial}_J f$ is the composition of d by the inclusion $T_J^{0,1} \hookrightarrow T_B^{\mathbf{C}}$, hence the formula

$$\overline{\partial}_J f = \frac{\partial f}{\partial \overline{\zeta}^\beta} \; d\overline{\zeta}^\beta$$

still holds near 0.Thus, a complex function defined near 0 is pseudo-holomorphic with respect to the almost-complex structure J if and only if $\frac{\partial f}{\partial \overline{\zeta}^\beta} = 0$ for all $\beta = 1, ..., n$. Near 0, we can also write

$$\begin{aligned}
\frac{\partial}{\partial z^\alpha} &= A^\beta{}_\alpha(z) \frac{\partial}{\partial \zeta^\beta} + A^{\overline{\beta}}{}_\alpha(z) \frac{\partial}{\partial \overline{\zeta}^\beta} \\
\frac{\partial}{\partial \overline{z}^\alpha} &= A^\beta{}_{\overline{\alpha}}(z) \frac{\partial}{\partial \zeta^\beta} + A^{\overline{\beta}}{}_{\overline{\alpha}}(z) \frac{\partial}{\partial \overline{\zeta}^\beta}
\end{aligned}$$

where the coefficients of the matrix $A(z)$ are smooth in the variable z, and $A^\beta{}_\alpha(0) = \delta^\beta{}_\alpha$. Hence, near 0, the matrix with coefficients $A^\beta{}_\alpha(z)$ are invertible, so that there exists coefficients $\psi^\lambda{}_{\overline{\alpha}}(z)$ satisfying

$$A^\beta{}_{\overline{\alpha}}(z) = \psi^\lambda{}_{\overline{\alpha}}(z) A^\beta{}_\lambda(z)$$

and since $A^\beta{}_{\overline{\alpha}}(0) = 0$, the coefficients $\psi^\lambda{}_{\overline{\alpha}}(z)$ go to 0 as z goes to zero. Define $\psi(z) \in T^{1,0} \otimes \Lambda^{0,1}$ by

$$\psi(z) = \psi^\lambda{}_{\overline{\alpha}}(z) \; \frac{\partial}{\partial z^\lambda} \otimes d\overline{z}^\alpha$$

Then we can compute:

$$\begin{aligned}
(\overline{\partial} - \psi) f &= \left(\frac{\partial f}{\partial \overline{z}^\alpha} - \psi^\lambda{}_{\overline{\alpha}} \frac{\partial f}{\partial z^\lambda} \right) d\overline{z}^\alpha \\
&= \left(A^\beta{}_{\overline{\alpha}} \frac{\partial f}{\partial \zeta^\beta} + A^{\overline{\beta}}{}_{\overline{\alpha}} \frac{\partial f}{\partial \overline{\zeta}^\beta} - \psi^\lambda{}_{\overline{\alpha}} A^{\overline{\beta}}{}_\lambda \frac{\partial f}{\partial \overline{\zeta}^\beta} - \psi^\lambda{}_{\overline{\alpha}} A^\beta{}_\lambda \frac{\partial f}{\partial \zeta^\beta} \right) d\overline{z}^\alpha \\
&= \left(A^{\overline{\beta}}{}_{\overline{\alpha}} - \psi^\lambda{}_{\overline{\alpha}} A^{\overline{\beta}}{}_\lambda \right) \frac{\partial f}{\partial \overline{\zeta}^\beta} \; d\overline{z}^\alpha
\end{aligned}$$

But the coefficients $A^{\overline{\beta}}{}_{\overline{\alpha}} - \psi^\lambda{}_{\overline{\alpha}} A^{\overline{\beta}}{}_\lambda$ go to $A^{\overline{\beta}}{}_{\overline{\alpha}}(0) = \delta^{\overline{\beta}}{}_{\overline{\alpha}}$ as z goes to 0. Hence, these coefficients form the entries of an invertible matrix when z is close enough to 0. Hence we have proven the following:

Proposition 2.1.3. *In a neighborhood of* 0*, a function f is J-holomorphic if and only if* $(\overline{\partial} - \psi) f = 0$.

Remark 2.1.1. In a sense, this propositions says that ψ measure how close the almost complex structure J is from J_0.

Now, we are ready to state the Newlander-Nirenberg theorem in the form we want:

Theorem 2.1.4 (Newlander-Nirenberg, [26]) *Let B be the unit ball in $\mathbf{C}^n$, equipped with the standard complex structure. Let $\psi = \psi^\lambda{}_{\overline{\alpha}} \frac{\partial}{\partial z^\lambda} \otimes d\overline{z}^\alpha$, and define the operators $L_1, ..., L_n$ by*

$$L_\nu = \frac{\partial}{\partial \overline{z}^\nu} - \psi^\lambda{}_{\overline{\nu}} \frac{\partial}{\partial z^\lambda}$$

Suppose that the operators $L_1, ..., L_n, \overline{L_1}, ..., \overline{L_n}$ are linearly independent and satisfy

$$L_\mu L_\nu - L_\nu L_\mu = 0, \qquad \mu, \nu = 1, ..., n$$

Then the system of equations

$$L_\nu f = 0, \qquad \nu = 1, ..., n$$

admits n independant solutions $\zeta^1, ..., \zeta^n : U \to \mathbf{C}$ defined on a neighborhood U of 0 in B.

Here, complex-valued functions $\zeta^1, ..., \zeta^m$ are said to be independent at a point if the differentials $d\zeta^1, ..., d\zeta^m, d\overline{\zeta}^1, ..., d\overline{\zeta}^m$ at this point span the cotangent space. It is clear that this is an open property. Since $\overline{\partial}_J f = \frac{1}{2}(df - iJdf)$, a function satisfies $\overline{\partial}_J f = 0$ if and only if

$$df = iJdf = -idf \circ J$$

that is, if and only if the 1-from df, seen as a map $(T_M^{\mathbf{R}}, J) \to \mathbf{C}$, is complex-linear. It is clear that there are at most n independent solutions of $\overline{\partial}_J f = 0$, and by the implicit mapping theorem, if there are n independent solutions $\zeta^1, ..., \zeta^n$, they form a local chart of M. Therefore, if (M, J) is an almost complex manifold of dimension $2n$ such that $T_M^{0,1}$ is integrable, we can cover M by open sets U_j with complex coordinates $z_j = (z_j^1, ..., z_j^n)$ such that the map $dz_j : (T_M^{\mathbf{R}}, J) \to \mathbf{C}^n$ is complex linear. Therefore, on $U_j \cap U_k$, the transition function f_{jk} defined by $f_{jk}(z_k) = z_j$ satisfies that $df_{jk} : \mathbf{C}^n \to \mathbf{C}^n$ is complex-linear. Hence, the transition functions are holomorphic, and the almost complex structure J is integrable.

Theorem 2.1.5 *Let (M, J) be an almost complex manifold of dimension $2n$. Then, we can identify the torsion of the underlying $GL(n, \mathbf{C})$-structure with the Nijenhuis tensor. In particular, an almost complex structure comes from a complex structures if and only if the corresponding $GL(n, \mathbf{C})$-structure is torsion-free.*

Proof. Let ∇ be any connection compatible with J. By definition of the torsion $T(\nabla)$, we have:

$$[X, Y] = \nabla_X Y - \nabla_Y X - T(\nabla)(X, Y)$$

Replacing the brackets in the expression defining $\mathcal{N}$, we obtain the following formula:

$$\mathcal{N}(X, Y) = T(\nabla)(X, Y) + JT(\nabla)(X, JY) + JT(\nabla)(JX, JY) - T(\nabla)(JX, JY)$$

Denotes by V the usual representation of $GL(2n, \mathbf{R})$ onto $\mathbf{R}^{2n}$. Recall that the canonical inclusion $GL(n, \mathbf{C}) \hookrightarrow GL(2n, \mathbf{R})$ is given by:

$$A + iB \in GL(n, \mathbf{C}) \to \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in GL(2n, \mathbf{R})$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}(2n, \mathbf{R})$. Then we have:

$$JAJ + A = \begin{pmatrix} a - d & b + c \\ b + c & d - a \end{pmatrix}$$

Hence, if we define $j(A) = JAJ + A$ for $A \in \mathfrak{gl}(2n, \mathbf{R})$, we have an exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{gl}(n, \mathbf{C}) \longrightarrow \mathfrak{gl}(2n, \mathbf{R}) \longrightarrow \mathfrak{m} \longrightarrow 0$$

where $\mathfrak{m} \subset \mathfrak{gl}(2n, \mathbf{R})$ is the vector spaces of matrices of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, for $a, b \in \mathfrak{gl}(n, \mathbf{R})$, and the arrow $\mathfrak{gl}(2n, \mathbf{R}) \to \mathfrak{m}$ is j. Taking tensor products, with V^*, we have:

$$0 \longrightarrow V^* \otimes \mathfrak{gl}(n, \mathbf{C}) \longrightarrow \otimes^2 V^* \otimes V = V^* \otimes \mathrm{End}(V) \longrightarrow V^* \otimes \mathfrak{m} \longrightarrow 0$$

Now we can define a map $s : V^* \otimes \mathrm{End}(V) \to \Lambda^2 V^* \otimes V$ by

$$s(\alpha)(X, Y) = \alpha_X(Y) - \alpha_Y(X)$$

for $\alpha \in V^* \otimes \mathrm{End}(V)$ and $X, Y \in V$. We can apply s to the above exact sequence. The resulting sequence is not necessarily exact, but it remains right-exact, so that get an exact sequence of vector spaces:

$$V^* \otimes \mathfrak{gl}(n, \mathbf{C}) \longrightarrow \Lambda^2 V^* \otimes V \longrightarrow s(V^* \otimes \mathfrak{m}) \longrightarrow 0$$

where the map $\sigma : V^* \otimes \mathfrak{gl}(n, \mathbf{C}) \to \Lambda^2 V^* \otimes V$ is given by $\sigma(\alpha)(X, Y) = \alpha_X(Y) - \alpha_Y(X)$ for all $\alpha \in V^* \otimes \mathfrak{gl}(n, C)$. Moreover, it is straightforward to see that the arrow $\Lambda^2 V^* \otimes V \longrightarrow s(V^* \otimes \mathfrak{m})$ is given by:

$$T \in \Lambda^2 V^* \otimes V \mapsto S(X, Y) = JT(X, JY) + T(X, Y) + JT(JX, Y) - T(JX, JY)$$

Everything we said passes at the level of vector bundles, and according to our discussion in §1.2.3, the intrinsic torsion of J is indeed identified with the Nijenhuis tensor. □

We finish the section by some terminology. If M, N are two complex manifolds, a smooth map $f : M \to N$ is said to be holomorphic if the differential df is complex linear with respect to the almost complex structures of M and N. It is equivalent to the fact that f is given by a holomorphic map in local complex coordinates. A biholomorphism is a diffeomorphism f such that f and f^{-1} are holomorphic. As for smooth manifolds, biholomorphic complex manifolds are identified.

2.2 $SU(3)$-structures

$SU(n)$-structures are, to the author's knowledge, the only type of structure in the Berger's list for which the question of knowing whether a given compact manifold admits a torsion-free structure is solved. A necessary condition for a manifold to admit a metric with holonomy $SU(n)$ is the the manifold must be Kähler, i.e. it must admit metrics with holonomy $U(n)$. Moreover, a Kähler manifold that admits a metric with holonomy $SU(n)$ must have trivial first Chern class. A conjecture of Calabi, proved later by Yau, states that conversely, any compact Kähler manifold with vanishing first Chern class admits metrics with restricted holonomy $SU(n)$. This is why such metrics are often called Calabi-Yau. In this part, we will be especially interested in the case $n = 3$ which will be useful to us.

2.2.1 Kähler and Calabi-Yau manifolds

Let (M, J) be an almost complex manifold. A metric g on M is said to be *hermitian* with repect to the almost complex structure J if J is an isometry with respect g. As we have seen in the proof of Lemma 2.1.1, near any point of M, there exists a smooth local frame $(e_1, ..., e_{2n})$ with dual coframe $(e^1, ..., e^{2n})$ in which (g, J) can be written:

$$J = \sum_{j=1}^{n} e_{2j} \otimes e^{2j-1} - e_{2j-1} \otimes e^{2i}, \qquad g = \sum_{j=1}^{2n} (e^j)^2 \tag{2.1}$$

Therefore, a couple (g, J) where J is an almost complex structure g is a hermitian metric on M is equivalent to a $U(n)$-structure, where we see $U(n)$ as the subgroup of $GL(n, \mathbf{C}) \hookrightarrow CL(2n, \mathbf{R})$, embedded in $GL(2n, \mathbf{R})$ by the usual identification $\mathbf{C}^n \simeq \mathbf{R}^{2n}$, that leaves invariant the canonical metric of $\mathbf{R}^{2n}$. We can also see $U(n)$ as the subgroup of $SO(2n)$ that leaves invariant the canonical almost complex structure J_0 or $\mathbf{R}^{2n}$. For this reason, it follows from Proposition 1.3.1 and Corollary 1.3.2 that a $U(n)$-structure (g, J) is torsion-free if and only if $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.

Definition 2.2.1. A manifold M endowed with a torsion-free $U(n)$-structure (g, J) is called a *Kähler manifold*. We define its associated *Kähler form* $\omega \in \Omega^2(M)$ by

$$\omega = g(J\cdot, \cdot)$$

A Käler manifold M is in particular a complex manifold. Indeed, if (g, J) is the couple defining the Kähler structure P, then the torsion-free condition is equivalent to the fact that the Levi-Civita connection of g reduces to P. But J defines a $GL(n, \mathbf{C})$-structure $\tilde{P}$, and P is a principal subbundle of P. In particular, the Levi-Civita connection also reduces to $\tilde{P}$, which is equivalent to the fact that J is integrable.

If (g, J) is a $U(n)$-structure, we can always define a 2-form $\omega \in \Omega^2(M)$ by $\omega = g(J\cdot, \cdot)$. In a local frame $(e_1, ..., e_{2n})$ where (g, J) take the form (2.1), ω can be written

$$\omega = \sum_{j=1}^{n} e^{2i-1} \wedge e^{2i}$$

This is a real 2-form, that extends to a complex 2-form, acting on the complexified tangent space $T_M^{\mathbf{C}}$. Thus, it must split into $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$ with respect to the decomposition $\Omega^2(M, \mathbf{C}) = \Omega_M^{2,0} \oplus \Omega_M^{1,1} \oplus \Omega_M^{0,2}$. Since ω is a real 2-form, it satisfies $\omega^{0,2} = \overline{\omega^{2,0}}$. Moreover if $X, Y \in T_M^{1,0}$, we have

$$\omega(X, Y) = g(JX, Y) = ig(X, Y) = g(X, JY) = \omega(Y, X) = -\omega(X, Y)$$

Therefore, $\omega(X, Y) = 0$. In particular, $\omega^{2,0} = 0 = \omega^{0,2}$. Thus, ω is a real form of type $(1, 1)$.

The intrinsic torsion of (g, J) can also be expressed in terms of the $(1, 1)$-form ω. Since $\nabla\omega = g(\nabla J\cdot, \cdot)$, it follows that the intrinsic torsion is determined by ∇J. In fact, we have the following proposition that describes the torsion of a $U(n)$-structure:

Proposition 2.2.1. *Let M be a manifold and (g, J) be a $U(n)$-structure on M, with associated $(1, 1)$-form ω. Then the intrinsic torsion of (g, J) is identified with the couple $(d\omega, \mathcal{N})$, where $\mathcal{N}$ is the Nijenhuis tensor of J. In particular, (g, J) is a Kähler structure on M if and only if J is integrable and $d\omega = 0$.*

Remark 2.2.1. A closed non-degenerate 2-form $\omega \in \Omega^2(M)$ is called a *symplectic form*. From this proposition, a complex manifold (M, J) is Kähler if and only if there exists a hermitian metric g on M such that the associated $(1, 1)$-form $\omega = g(J\cdot, \cdot)$ is symplectic.

Let V be the canonical representation of $U(n)$ on $\mathbf{R}^{2n} \simeq \mathbf{C}^n$, with basis $(e_1, ..., e_{2n})$, in which the canonical complex structure J_0 and hermitian metric g_0 are written

$$J_0 = \sum_{j=1}^{n} e_{2j} \otimes e^{2j-1} - e_{2j-1} \otimes e^{2i}, \qquad g_0 = \sum_{j=1}^{2n} (e^j)^2$$

The subgroup $SU(n) \hookrightarrow U(n)$ is defined as the subgroup of $U(n)$ that leaves invariant the following complex $(n, 0)$-form

$$\Omega_0 = (e^1 + ie^2) \wedge ... \wedge (e^{2n-1} + ie^{2n}) \tag{2.2}$$

which is a generator of $\Lambda^{n,0}V^*$.

Remark 2.2.2. On an almost-complex manifold (M, J), a nowhere vanishing section of $\Lambda_M^{n,0}$ is called a $(J$-$)$holomorphic volume form.

Then, an $SU(n)$-structure on a manifold M^{2n} is equivalent to a tuple (g, J, Ω), where g is an hermitian metric with respect to the almost complex structure J, and Ω is a J-holomorphic volume form, such that there exist smooth local frames $(e_1, ..., e_{2n})$ in which (g, J) have the form given in (2.1) and Ω can be written as in (2.2). If $\omega = g(J\cdot, \cdot)$ is the $(1,1)$-form associated with the couple (g, J), then the $SU(n)$-structure (g, J, Ω) is torsion-free if and only if $\nabla\omega = 0 = \nabla\Omega$. Another way to phrase it is to say that a metric g admits a complex structure J and a holomorphic volume form Ω such that (g, J, Ω) is a torsion-free $SU(n)$-structure if and only if the holonomy group of ∇ is (conjugated to) a subgroup of $SU(n)$.

A manifold M endowed with a torsion-free $SU(n)$-structure is in particular a Kähler manifold. On the other hand, given a Kähler manifold M, one can ask whether it admits torsion-free $SU(n)$-structure. A necessary condition is that the line bundle $\Lambda^{n,0}_M$, also called the *canonical bundle* of M, must be trivial, which imply that the first Chern class of M vanishes. It was proven by Yau that this condition is also sufficient, up to a universal cover. We briefly explains this result without proof.

Let M be a Kähler manifold with Kähler structure (g, J). As a first step, one can ask whether there exists a holomorphic volume form Ω such that (g, J, Ω) is a torsion-free $SU(n)$-structure. The answer is the following:

Lemma 2.2.2. *Let g be a Kähler metric on a complex manifold M. Then the restricted holonomy group $\mathrm{Hol}^0(g)$ is contained in $SU(n)$ if and only if g is Ricci-flat.*

Calabi made the following conjecture, that was proven later by Yau.

Theorem 2.2.3 (Calabi conjecture - Yau's Theorem) *Let (M, J) be a compact complex manifold, and g a Kähler metric on M, with Kähler form ω. Let ρ' be a real, closed $(1,1)$-form on M such that its cohomology class satisfies $[\rho'] = 2\pi c_1(M)$. Then there exists a unique Kähler metric g' on M with Kähler form ω', such that $[\omega'] = [\omega]$, and the Ricci form of g' is ρ'.*

For simply connected manifold with vanishing first Chern class, we can apply this theorem with $\rho' = 0$, and obtain that any Kähler class $[\omega]$, that is, any cohomology class in $H^2(M)$ which is the cohomology class of a Kähler form ω, there exists a unique Kähler metric g' with Kähler for ω', that has holonomy contained in $SU(n)$ and such that $[\omega'] = [\omega]$. Hence, Yau's theorem gives a construction of torsion-free $SU(n)$-structures. For this reason, manifolds that admit torsion-free $SU(n)$-structures are often called *Calabi-Yau manifolds*, although this terminology varies with the authors. The moduli space of Calabi-Yau structures over a Kähler manifold is fully understood: this is the space of Kähler classes in $H^2(M)$. However, deciding whether a given manifold admits Kähler structures is a very hard problem is general, as is deciding whether a given manifold admits complex structures.

2.2.2 Dimension 3 and Hitchin duality map

We turn to the particular case $n = 3$. Let V be a 6-dimensional vector space with complex structure J, hermitian metric g and associated real $(1,1)$-form

$\omega = g(J\cdot,\cdot)$. In a suitable basis $(e_1,...,e_6)$ of V with dual basis $(e^1,...,e^6)$, J, g and ω take the form:

$$J_0 = e^1 e_2 - e^2 e_1 + e^3 e_4 - e^4 e_3 + e^5 e_6 - e^6 e_5 \tag{2.3}$$

$$g_0 = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2 + (e^6)^2 \tag{2.4}$$

$$\omega_0 = e^{12} + e^{34} + e^{56} \tag{2.5}$$

As we have seen in §2.2.1, we can define a complex volume form

$$\Omega_0 = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) \tag{2.6}$$

Under the usual decomposition of $\Lambda^* V^* \otimes \mathbf{C}$ induced by J_0, Ω_0 is a generator of $\Lambda^{3,0}$. Since Ω_0 is a complex form, it can be written as $\Omega_0 = \operatorname{Re}\Omega_0 + i \operatorname{Im}\Omega_0$, where $\operatorname{Re}\Omega_0$ and $\operatorname{Im}\Omega_0$ are real 3-forms of type $(3,0)+(0,3)$. Explicitly, we have:

$$\begin{aligned} \operatorname{Re}\Omega_0 &= e^{135} - e^{146} - e^{245} - e^{236} \\ \operatorname{Im}\Omega_0 &= e^{136} + e^{145} + e^{235} - e^{246} \end{aligned} \tag{2.7}$$

Note that we have the relation $\operatorname{Im}\Omega_0 = -\operatorname{Re}\Omega_0(J_0\cdot,\cdot,\cdot)$. The closed subgroup of $GL(V)$ that fixes g_0, ω_0, J_0 and Ω_0 is isomorphic to $SU(3)$. Note that ω_0 and Ω_0 are subject to the following compatibility relations:

$$\begin{aligned} \omega_0 \wedge \Omega_0 &= 0, \\ \frac{1}{4}\operatorname{Re}\Omega_0 \wedge \operatorname{Im}\Omega_0 &= \frac{1}{6}\omega_0^3 \end{aligned}$$

As was noted by Hitchin in [14], a special feature of dimension 3 is that the orbit of $\operatorname{Re}\Omega_0$ under $GL(V)$ is open in $\Lambda^3 V^*$. Moreover, any real 3-form Ω in the orbit of $\operatorname{Re}\Omega_0$ determines a unique complex structure J on V such that, if $\tilde{\Omega} = -\Omega(J\cdot,\cdot,\cdot)$, then $\Omega + i\tilde{\Omega}$ is a generator of $\Lambda^{3,0}_J$, where we put the subscript J to specify that we take the decomposition of $\Lambda^3 V^* \otimes \mathbf{C}$ with respect to the complex structure J, and not the structure J_0. The 3-forms in the orbit of $\operatorname{Re}\Omega_0$ under $GL(V)$ are called *stable 3-forms*, and we will refer to the map $\Omega \to \tilde{\Omega}$ as the *Hitchin's duality map*. If Ω is a stable 3-form with associated complex structure J, and ω a 2-form such that the relations

$$\begin{aligned} \omega \wedge \Omega = 0 &= \omega \wedge \tilde{\Omega}, \\ \frac{1}{4}\Omega \wedge \tilde{\Omega} &= \frac{1}{6}\omega^3 \end{aligned} \tag{2.8}$$

are satisfied, then ω is a non-degenerate form of type $(1,1)$ with respect to J, and $g = \omega(\cdot, J\cdot)$ is a hermitian metric; moreover the subgroup of $GL(V)$ that stabilizes (ω,Ω) is isomorphic to $SU(3)$. Denote by $\mathcal{O}$ the set of couples (ω,Ω) that satisfy the compatibility equations and the open conditions (i.e. $\operatorname{Re}\Omega$ is a stable 3-form), and denote by $\mathfrak{c}$ a generic element of $\mathcal{O}$. By our discussion, $\mathcal{O}$ is the orbit of $\mathfrak{c}$ under $GL(V)$, and by the Hitchin's duality map and the algebraic relations above, there exists a smooth map $\mathfrak{c} \mapsto g_{\mathfrak{c}}$ that maps

$\mathfrak{c} \in \mathcal{O}$ to its associated inner product. It is clear, by the algebraic relations above, that this map is equivariant for the natural action of $GL(V)$, that is, $g_{\phi^*\mathfrak{c}} = \phi^*\mathfrak{c}$. This is important, because it implies that the action of $GL(V)$ on $\mathcal{O}$ is proper. Hence $\mathcal{O}$ is a smooth embedded submanifold of $\Lambda^2 V^* \oplus \Lambda^3 V^*$. The following consequence is purely technical but we will need it later on:

Lemma 2.2.4. *There exists an $\epsilon_1 > 0$ such the the following holds. Let $\mathfrak{c}_0 \in \mathcal{C}$, and let $\mathcal{U}(\mathfrak{c}_0, \epsilon_1)$ denote the open subset of $\mathcal{C}$ such that $|\mathfrak{c} - \mathfrak{c}_0| < \epsilon_1$ for the norm induced by the inner product $g_{\mathfrak{c}_0}$. Then there exists a smooth map*

$$F : \mathcal{U}(\mathfrak{c}_0, \epsilon_1) \to V^6$$

such that for all $\mathfrak{c} \in \mathcal{U}(c_0, \epsilon_1)$, $F(\mathfrak{c})$ is a basis of V in which the structure $\mathfrak{c} = (\omega, \Omega)$ can be written in its standard form.

Proof. Pick any basis $\mathcal{B}_0$ of V in which $\mathfrak{c}_0$ is written in its standard form, and let $\mathfrak{m} \subset \mathfrak{gl}(V)$ be any complement of the Lie algebra of the stabilizer of $\mathfrak{c}_0$. Then we have a diffeomorphism between a neighborhood of 0 in $\mathfrak{m}$ and an neighborhood of $\mathfrak{c}_0$ in $\mathcal{C}$ given by $\xi \in \mathfrak{m} \mapsto \exp(\xi)_* \mathfrak{c}_0$. A suitable map F is therefore given by $F(\exp(\xi)_* \mathfrak{c}_0) = \exp(\xi)_* \mathcal{B}_0$. The fact that ϵ_1 can be chosen independent of $\mathfrak{c}_0$ is clear. □

For manifolds, Lemma 2.2.4 implies that an $SU(3)$-structure on a manifold M^6 is equivalent a couple (ω, Ω), where $\omega \in \Omega^2(M)$ and $\Omega \in \Omega^3(M)$ is a stable 3-form, subject to the compatibility relations (2.8). We needed this lemma to conclude that since, contrary to $O(n)$ or $GL(n, \mathbf{C})$, we cannot not adapt a Graham-Schmidt orthonormalization process to construct smooth standard frames.

2.2.3 Representations of $SU(3)$ and intrinsic torsion

In the following, if W is a complex vector space, we will denote $[[W]]$ to denote the underlying real vector space and $[W]$ to denote a real form of W, i.e., $[W] \otimes \mathbf{C} = W$. Let $n \geq 1$ and V be the canonical representation of $GL(2n, \mathbf{R})$ on $\mathbf{R}^{2n}$. Let J be a complex structure on V, that leads to an inclusion $GL(n, \mathbf{C}) \subset GL(2n, \mathbf{R})$. We have seen in §2.1.1 that we have a decomposition

$$\Lambda^k V^* \otimes \mathbf{C} = \bigoplus_{p+q=k} \Lambda^{p,q}$$

of the complexified exterior product representations of $GL(n, \mathbf{C})$. The representations $\Lambda^{p,q}$ are irreducible, and $\Lambda^{p,q} = \overline{\Lambda^{q,p}}$. We can also give a decomposition of ΛV^* into irreducible representation, using the decomposition of the complexified exterior algebra. The real part map $\mathrm{Re} : \Lambda^k V^* \otimes \mathbf{C} \to \Lambda^k V^*$ is a morphism of real representations. A k-form η such that $\mathrm{Re}\, \eta = 0$ satisfies $\overline{\eta} = -\eta$, so that η is of symmetric type. For $p \neq q$, the restriction of Re to $\Lambda^{p,q}$ is then injective. It follows that $\mathrm{Re}\, \Lambda^{p,q} = \mathrm{Re}\, \Lambda^{q,p} \simeq [[\Lambda^{p,q}]]$ as real representations of $GL(n, \mathbf{C})$. Moreover, $\mathrm{Re}\, \Lambda^{p,p}$ is a real form of $\Lambda^{p,p}$. Hence the representation $\Lambda^k V^*$ of $GL(n, \mathbf{C})$ admits the following decomposition into irreducible representations:

$$\Lambda^k V^* = \bigoplus_{p<k/2} [[\Lambda^{p,k-p}]] \oplus [\Lambda^{k/2,k/2}]$$

Suppose now $n = 3$, and let $\omega \in \Lambda^2 V^*$ be a non-degenerate 2-form, $\mathrm{Re}\,\Omega \in \Lambda^3 V^*$ a stable 3-form inducing the complex structure J on V, such that (ω, Ω) satisfy the relations defining a $SU(3)$-structure:

$$\omega \wedge \Omega = 0, \qquad \frac{1}{4} \mathrm{Re}\,\Omega \wedge \mathrm{Im}\,\Omega = \frac{1}{6}\omega^3$$

Let g be the associated metric and $*$ the corresponding Hodge operator. ω is a real $(1,1)$-form invariant under $SU(3)$, so that we have the orthogonal decomposition $[\Lambda^{1,1}] = \mathbf{R} \oplus [\Lambda_0^{1,1}]$. The representation $[\Lambda_0^{1,1}]$ of $SU(3)$ is irreducible, and its elements are called *primitive* $(1,1)$*-forms*. This representation is naturally isomorphic to the adjoint representation via the metric g. Moreover, $[[\Lambda^{2,0}]]$ is also irreducible: it has dimension 6 and is isomorphic to $V^* \simeq V$ via $X \in V \mapsto X \lrcorner \mathrm{Re}\,\Omega = \mathrm{Re}\,\Omega(X, \cdot, \cdot) \in \Lambda^2 V^*$. Hence we have the following decomposition of $\Lambda^2 V^*$ into irreducible representations of $SU(3)$:

$$\Lambda^2 V^* = [\Lambda_0^{1,1}] \oplus V^* \oplus \mathbf{R}$$

Any element $\sigma \in \Lambda^2 V^*$ can be written in a unique way

$$\sigma = \lambda\omega + X \lrcorner \mathrm{Re}\,\Omega + \kappa$$

where $\lambda \in \mathbf{R}$, $X \in V$ and κ is a primitive $(1,1)$-form.

To decompose $\Lambda^3 V^*$, we first remark that $[[\Lambda^{3,0}]]$ has dimension 2, with orthonormal basis $(\mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega)$, since it is an orthonormal family. Moreover, the map

$$V^* \to \Lambda^2 V^*, \ \eta \to \eta \wedge \omega$$

is an injective map, and is a morphism of representations of $SU(3)$. Moreover, since ω has type $(1,1)$, it takes values in $[[\Lambda^{2,1}]]$. Therefore, V^* is a subrepresentation of $[[\Lambda^{2,1}]]$. Let Λ^3_{12} be its orthogonal complement. It is irreducible, and the index stands for its dimension, which is 12. Hence we have the following decomposition of $\Lambda^3 V^*$:

$$\Lambda^3 V^* = \Lambda^3_{12} \oplus V^* \oplus \mathbf{R} \oplus \mathbf{R}$$

so that any element $\alpha \in \Lambda^3 V^*$ can be uniquely written

$$\alpha = \nu + \eta \wedge \omega + \lambda\,\mathrm{Re}\,\Omega + \mu\,\mathrm{Im}\,\Omega$$

with $\nu \in \Lambda^3_{12}$, $\eta \in V^*$ and $\lambda, \mu \in \mathbf{R}$.

Since $SU(3)$ leaves g invariant, the operator $*$ is an representation isomorphism, so that we also have:

$$\Lambda^4 V^* = [\Lambda_0^{1,1}] \oplus V^* \oplus \mathbf{R}$$

As for $\Lambda^3 V^*$, any element β of $\Lambda^4 V^*$ can be uniquely written:

$$\beta = \kappa \wedge \omega + \mathrm{Re}\,\Omega \wedge \eta + \lambda\omega^2$$

where κ is a primitive $(1,1)$-form, η a 1-form and λ a scalar.

Since ω, ω^2 and Ω are invariant under $SU(3)$, the linear endomorphisms of the exterior algebra of V^* defined by wedging by these forms are representation morphisms. It is useful to have a precise description of these.

Proposition 2.2.5. *Let $\eta = \lambda\omega + X \lrcorner \operatorname{Re}\Omega + \kappa \in \Lambda^2 V^*$. Then we have:*

$$\eta \wedge \omega^2 = \lambda\omega^3$$
$$\eta \wedge \operatorname{Im}\Omega = \frac{1}{3} X \lrcorner \omega^3$$

In particular, η is a primitive $(1,1)$-form if and only if

$$\eta \wedge \omega^2 = 0 = \eta \wedge \operatorname{Im}\Omega$$

Proof. The map obtained by wedging by ω^2 gives a morphism

$$\Lambda^2 V^* \simeq \mathbf{R} \oplus V \oplus [\Lambda_0^{1,1}] \longrightarrow \Lambda^6 V^* \simeq \mathbf{R}$$

In particular since these representations are irreducible, the map vanishes on the component $V \oplus [\Lambda_0^{1,1}]$, which gives the first equality.

Similarly, wedging by $\operatorname{Im}\Omega$ is a morphism

$$\Lambda^2 V^* \simeq \mathbf{R} \oplus V \oplus [\Lambda_0^{1,1}] \longrightarrow \Lambda^5 V^* \simeq V$$

Then, it vanishes on the components $\mathbf{R} \oplus [\Lambda_0^{1,1}]$. Since $X \in V \mapsto \lrcorner\omega^3 \in \Lambda^5 V^*$ is an isomorphism, we must have $\eta \wedge \operatorname{Im}\Omega = C X \lrcorner \omega^3$ for some constant C. In order to determine this constant, we make an explicit computation using the expressions (2.7), and choosing $X = e_1$:

$$(e_1 \lrcorner \operatorname{Re}\Omega) \wedge \operatorname{Im}\Omega = 2e^{23456} = \frac{1}{3}\omega^3$$

which gives $C = \frac{1}{3}$. $\square$

It is also useful to write the operator $*$ in terms of the above decomposition of ΛV^*.

Proposition 2.2.6. *The Hodge operator $*$ acts in the following way.*

If $\alpha \in V^*$, then

$$*\alpha = -\frac{1}{2}\alpha \wedge \omega^2$$

For $\eta = \lambda\omega + X \lrcorner \operatorname{Re}\Omega + \kappa$, with $\lambda \in \mathbf{R}$, $X \in V$ and $\kappa \in [\Lambda_0^{1,1}]$,

$$*(\lambda\omega + X \lrcorner \operatorname{Re}\Omega + \kappa) = \frac{\lambda}{6}\omega^2 - X \wedge \operatorname{Re}\Omega - \kappa \wedge \omega$$

For $\rho = \nu + \eta \wedge \omega + f \operatorname{Re}\Omega + g \operatorname{Im}\Omega$ with $\nu \in \Lambda^3_{12}$, $\eta \in V^*$ and $f, g \in \mathbf{R}$,

$$*(\nu + \eta \wedge \omega + f \operatorname{Re}\Omega + g \operatorname{Im}\Omega) = *\nu - (J\eta) \wedge \omega + \frac{1}{4}(f \operatorname{Im}\Omega - g \operatorname{Re}\Omega)$$

Lastly, it will be useful later on to describe the linearization of the Hitchin's duality map. Here is the statement, and a proof is given in [10, Proposition 2.12].

Proposition 2.2.7. *Let* $\operatorname{Re}\Omega$ *be a stable form,* ω *a* 2*-form such that* (ω, Ω) *are compatible, and for* $\rho \in \Lambda^3 V^*$*, write* $\rho = \rho_6 + \rho_{1\oplus 1} + \rho_{12}$ *in the decomposition* $\Lambda^3 V^* \simeq V \oplus \mathbf{R} \oplus \mathbf{R} \oplus \Lambda^3_{12}$ *into irreducible* $SU(3)$*-representations induced by* (ω, Ω)*. Then the linearization of the Hitchin's duality map at* $\operatorname{Re}\Omega$ *is given by*

$$\hat{\rho} = *(\rho_6 + \rho_{1\oplus 1}) - *\rho_{12}$$

If B is a 6-manifold equipped with a $SU(3)$-structure (ω, Ω), the decompositions above pass to differential forms. As explained in §1.2.3, the intrinsic torsion can be identified with components of $\nabla\omega$ and $\nabla\Omega$. It turns out that it can be recovered solely from the antisymmetric part of those tensors, that is, from $d\omega$ and $d\Omega$:

Proposition 2.2.8 (Chiossi-Salamon [9],§1)**.** *Let* (ω, Ω) *be a* $SU(3)$*-structure on* B^6*. Then there exist functions* $w_1, \hat{w}_1$*, primitive* $(1,1)$*-forms* $w_2, \hat{w}_2$*, a* 3*-form* $w_3 \in \Omega^3_{12}$*, and* 1*-forms* w_4, w_5 *on* B *such that:*

$$\begin{aligned} d\omega &= 3w_1 \operatorname{Re}\Omega + 3\hat{w}_1 \operatorname{Im}\Omega + w_3 + w_4 \wedge \omega, \\ d \operatorname{Re}\Omega &= 2\hat{w}_1 \omega^2 + w_5 \wedge \operatorname{Re}\Omega + w_2 \wedge \omega, \\ d \operatorname{Im}\Omega &= -2w_1 \omega^2 + w_5 \wedge \operatorname{Im}\Omega + \hat{w}_2 \wedge \omega. \end{aligned}$$

Moreover, $(w_1, \hat{w}_1, w_2, \hat{w}_2, w_3, w_4, w_5)$ *is identified with the intrinsic torsion of the* $SU(3)$*-structure.*

Remark 2.2.3. If (ω, Ω) is torsion-free, the associated complex structure J is integrable, and $d\omega = 0$, so that B is a Kähler manifold. Moreover, Ω trivializes the canonical bundle of B, which implies that the first Chern class of B vanishes.

As we have seen above, an $SU(3)$-structure (ω, Ω) on a manifold B is torsion-free if and only if ω and Ω are parallel. In this case, the restricted holonomy group of the associated metric g is contained in $SU(3)$. Hence, according to the Berger's list, the restricted holonomy group is either trivial is the metric is flat, $SU(2)$ or the full $SU(3)$. Since the action of $SU(2)$ on $\mathbf{R}^6$ leaves invariant a line (even a plane), if the restricted holonomy group is contained in $SU(2)$, the Riemannian manifold (B, g) carries a non-trivial parallel 1-form. Therefore, B has full restricted holonomy $SU(3)$ if and only it does not carry any non-trivial parallel 1-form.

2.3 G_2-manifolds

One of the exceptional holonomy group in the Berger's list is the Lie group $G_2 \subset SO(7)$, which we introduce here, with an emphasis on its relation with the group $SU(3)$.

2.3.1 The holonomy group $G_2 \subset SO(7)$

Let V be a 7-dimensional real vector space with basis $(e_1, ..., e_7)$ and dual basis $(e^1, ..., e^7)$. We define the 3-form:

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{245} - e^{236} \tag{2.9}$$

We define G_2 as the subgroup of $GL(V)$ that leaves φ invariant. G_2 is closed. By an explicit computation we have

$$(v \lrcorner \varphi)^2 \wedge \varphi = -6|x|^2 e^{1234567}, \quad v \in V$$

where $|v|$ is the norm of v with respect to the inner product $g = \sum (e^i)^2$. Therefore, G_2 is a subgroup of $SO(7)$, and in particular, it is compact. It turns out that G_2 is a compact, connected, simply-connected subgroup of $SO(7)$ of dimension 14 [27]. Let $*$ be the Hodge operator associated with g. The action of G_2 on $\Lambda^* V^*$ commutes with $*$, so that $* : \Lambda^k V^* \mapsto \Lambda^{7-k} V^*$ is a representation morphism. In particular, G_2 also leaves invariant $*\varphi$. In coordinates we have:

$$* \varphi = e^{1367} + e^{1457} + e^{2357} - e^{2467} + e^{1234} + e^{1256} + e^{3456} \tag{2.10}$$

If we make the identification $V \simeq \mathbf{C}^3 \oplus \mathbf{R}$, with real basis $(e_1, ..., e_6)$ on $\mathbf{C}^3$ and (e_7) on $\mathbf{R}$, then we can define ω and Ω by equations (2.5) and (2.6), so that we have:

$$\begin{aligned} \varphi &= \omega \wedge e^7 + \operatorname{Re} \Omega, \\ *\varphi &= \operatorname{Im} \Omega \wedge e^7 + \frac{1}{2}\omega^2 \end{aligned} \tag{2.11}$$

The goal of the next lemmas is to make sense of this decomposition.

Lemma 2.3.1. *With the above notations, the stabilizer of e_7 in G_2 is $SU(3)$ acting on $\mathbf{C}^3 = (\mathbf{R}e^7)^\perp \subset \mathbf{R}^7$.*

Proof. Since $SU(3)$ leaves invariant e_7, g, ω and Ω, it leaves invariant φ, so that with the identification $\mathbf{C}^3 = (\mathbf{R}e^7)^\perp$, we have $SU(3) \subset G_2$. In particular, it is contained in the stabilizer of e_7 in G_2. Conversely, let $\phi \in G_2$ that stabilizes e_7. Since $\phi^* \varphi = \varphi$, we obtain:

$$\omega \wedge e^7 + \operatorname{Re} \Omega = (\phi^* \omega) \wedge e^7 + \phi^* \operatorname{Re} \Omega$$

Moreover, ϕ stabilizes the orthogonal space to e_7, so that $\phi^* \omega$ and $\phi^* \operatorname{Re} \Omega$ can be decomposed in terms that are wedge products of 1-forms $e^1, ..., e^6$. That forces $\phi^* \omega = \omega$ and $\phi^* \operatorname{Re} \Omega = \operatorname{Re} \Omega$. Then ϕ leaves invariant ω and $\operatorname{Re} \Omega$, so that by our discussion in §2.2.2, ϕ is an element of $SU(3)$. □

Lemma 2.3.2. *The action of G_2 on $S^6 \subset \mathbf{R}^7$ is transitive.*

Proof. The orbit of e_7 under G_2 is a closed, connected submanifold of S^6, of dimension $\dim G_2 - \dim SU(3)$. But $\dim G_2 = 14$ and $\dim SU(3) = 8$, so that the orbit of e_7 has dimension 6. Hence $G_2 \cdot e_7 = S^6$. □

From the above lemmas it immediately follows:

Corollary 2.3.3. *Let v be a unit vector in V and $\alpha = g(v, \cdot)$ its dual form. Then we can write*

$$\varphi = \omega \wedge \alpha + \operatorname{Re}\Omega \quad \textit{and} \quad *\varphi = \operatorname{Im}\Omega \wedge \alpha + \frac{1}{2}\omega^2$$

where $\omega \in \Lambda^2 V^*$ and $\Omega \in \Lambda^3 V^* \otimes \mathbf{C}$ satisfy

$$v \lrcorner \omega = 0 = v \lrcorner \Omega$$

Moreover, if we regard ω, Ω are forms acting on $\Omega^6 = (\mathbf{R}v)^\perp$, then $\operatorname{Re}\Omega$ is a stable 3-form, and (ω, Ω) are subject to the compatibility relations (2.8).

Let V be an oriented 7-dimensional vector space. The 3-forms that can be written in the form given by equation (2.9) in a suitable direct basis of V are called *positive* 3-forms. The set of positive 3-forms on V will be denoted $\mathcal{P}^3 V$. This is an open subset of $\Lambda^3 V^*$. Indeed, $\mathcal{P}^3 V$ is the orbit of any positive form under $GL_+(7, \mathbf{R})$. But $GL(7, \mathbf{R})$ has dimension 49, and the stabilizer of any positive form is G_2 that has dimension 14, so that $\mathcal{P}^3 V$ is a submanifold of $\Lambda^3 V^*$ of dimension $49 - 14 = 35 = \dim \Lambda^3 V^*$. As for $SU(3)$-structures, the fact that a positive form determines an $SU(3)$-structure implies that the action of $GL(V)$ on $\mathcal{P}^3 V$ is proper, and we can locally find a smooth map that associate to any positive form a basis of V in which it takes a standard form. Hence on a manifold, G_2-structure are equivalent to positive forms.

Now let φ be a positive 3-form on V, g_φ the inner product determined by φ and v a non-zero vector. According to Corollary 2.3.3, φ and $*\varphi$ can be written

$$\varphi = \omega \wedge \alpha + \operatorname{Re}\Omega, \quad *\varphi = \operatorname{Im}\Omega \wedge \alpha + \frac{1}{2}\omega^2$$

where ω and Ω define have the form (2.5) and (2.6) in a basis $(v_1, ..., v_6)$ of the hyperplane $W = (\mathbf{R}v)^\perp$, and $\alpha = g_\varphi(\cdot, \frac{v}{|v|})$. Let $h = g(v, v)^{-1}$ and $\theta = h^{\frac{1}{2}}\alpha$, so that θ satisfies $\theta(v) = 1$. Also define $\omega_h = h^{-\frac{1}{4}}\omega$ and $\Omega_h = h^{-\frac{3}{4}}\Omega$. Then it is straightforward to check that:

$$\varphi = \omega_h \wedge \theta + h^{\frac{3}{4}} \operatorname{Re}\Omega_h, \quad *\varphi = h^{\frac{1}{4}} \operatorname{Im}\Omega_h \wedge \theta + \frac{1}{2} h \omega_h^2 \tag{2.12}$$

Moreover, if we rescale the basis of W by setting $e_i = h^{-\frac{1}{4}} v_i$, then ω and Ω respectively satisfy (2.5) and (2.6) in the basis $(e_1, ..., e_6)$, and in particular (ω_h, Ω_h) satisfies the compatibility relations (2.8). Rescaling the metric on W so that $(e_1, ..., e_6)$ is orthonormal, we obtain an inner product g_W, so that the inner product g_φ on V satisfies:

$$g_\varphi = h^{\frac{1}{2}} g_W + h^{-1}\theta^2 \tag{2.13}$$

These relations will be an essential part of §5.1.1. At this point, the choice of parameter h and rescaling of all the forms is arbitrary, but it will make more sense later on.

If M is an oriented seven-dimensional manifold, a G_2 structure on M is equivalent to the choice of a 3-form φ on M, such that pointwise φ_x is a positive form on T_xM. The set of positive 3-forms on M, denoted $\mathcal{P}^3(M)$, is an open subset of $\Omega^3(M)$. Any positive form φ determines a Riemannian metric g_φ on M. A G_2-structure φ is torsion free if and only if $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection determined by the metric g_φ. In the next section, we discuss the intrinsic torsion of G_2-structures.

2.3.2 Representations of G_2 and intrinsic torsion

In this part, V is an oriented 7-dimensional vector space equipped with a positive 3-form. As representation of G_2, V is irreducible, and so is V^*. There is an injective map $v \in V \mapsto v \lrcorner \varphi \in \Lambda^2 V^*$, whose image is denoted Λ^2_7, the index referring to the dimension of the representation, and the exponent to the total representation which it belongs to. Here $v \lrcorner \varphi = \varphi(v, \cdot, \cdot)$. The orthogonal complement of Λ^2_7 in $\Lambda^2 V^*$ is isomorphic to the adjoint representation $\mathfrak{g}_2$ and is denoted Λ^2_{14}. Hence we have:

$$\Lambda^2 V^* = \Lambda^2_7 \oplus \Lambda^2_{14}$$

The map $GL(V) \to \Lambda^3 V^*, g \mapsto g_*\varphi$ induces a linear map $\delta : \mathrm{End}(V) \mapsto \Lambda^3 V^*$, which has kernel $\mathfrak{g}_2$. The map δ is a representation morphism for the action of G_2. Moreover, we know that the orbit of φ under $GL(V)$ is open, so that δ is surjective. We have $\mathrm{End}(V) \simeq V^* \otimes V^* \simeq \Lambda^2 V^* \oplus S^2 V^*$, and under this identification, the kernel of δ is Λ^2_{14}. Thus $\Lambda^3 V^* = \Lambda^2_7 \oplus S^2 V^*$. But the action of G_2 on $S^2 V^*$ leaves a metric invariant, so that $S^2 V^* = \mathbf{R} \oplus S^2_0 V^*$, and $S^2_0 V^*$ is irreducible. We denote Λ^3_7 the image of Λ^2_7 under δ, Λ^3_{27} the image of $S^2_0 V^*$, and Λ^3_1 the line generated by the image of g. Then we have:

$$\Lambda^3 V^* = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$$

Since $*$ defines an isomorphism from $\Lambda^k V^*$ to $\Lambda^{7-k} V^*$, we also have:

$$\begin{aligned}
\Lambda^4 V^* &= \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \\
\Lambda^5 V^* &= \Lambda^5_7 \oplus \Lambda^5_{14} \\
\Lambda^6 V^* &= \Lambda^6_7
\end{aligned}$$

There are canonical isomorphisms $\Lambda^1_7 \to \Lambda^4_7$ and $\Lambda^1_7 \to \Lambda^5_7$ by wedging respectively with φ and $*\varphi$. In particular, any 4-form α can be written uniquely

$$\alpha = \lambda * \varphi + \eta \wedge \varphi + *\psi$$

where λ is a scalar, η a 1-form and $\chi \in \Lambda^3_{27}$. Similarly, we can decompose 5-forms β in the following way:

$$\beta = \theta \wedge *\varphi + *\chi$$

where θ is a 1-form and $\chi \in \Lambda^2_{14}$.

Another isomorphism $\Lambda^2 V^* \to \Lambda^5 V^*$ given by wedging by φ. If we denote by π_l the projection of Λ^k onto the component Λ^k_l, then we have:

$$\eta \wedge \varphi = 2 * \pi_7(\eta) - *\pi_{14}(\eta)$$

As for the case of $SU(3)$, the above linear decomposition in irreducible representations pass to 7-manifold M with a G_2-structure φ. The intrinsic torsion is identified with the covariant derivative $\nabla\varphi$, where ∇ is the Levi-Civita connection associated with the metric g_φ. However, it is more convenient to express the intrinsic torsion in terms of differentials. The key point is the following result, based on [9, §2] and [17, Lemma 10.3.1]:

Proposition 2.3.4. *Let M^7 be a manifold equipped with a G_2-structure φ. Then there exist a function χ_1, a 2-form $\chi_2 \in \Omega^2_{14}$, a 3-form $\chi_3 \in \Omega^3_{27}$ and a 1-form χ_4 on M such that:*

$$\begin{aligned} d\varphi &= \chi_1 * \varphi + 3\chi_4 \wedge \varphi + *\chi_3, \\ d * \varphi &= 4\chi_4 \wedge *\varphi + *\chi_2. \end{aligned}$$

Moreover, $(\chi_1, \chi_2, \chi_3, \chi_4)$ is identified with the intrinsic torsion of the G_2-structure φ. In particular, it is torsion-free if and only if $d\varphi = 0$ and $d*\varphi = 0$.

Remark 2.3.1. In particular, it follows from this proposition that $\pi_7(d\varphi) = 0$ if and only if $\pi_7(d * \varphi) = 0$.

As we discussed for $SU(3)$, the metric associated with a torsion-free G_2-structure has restricted holonomy equal to either 1, $SU(2)$, $SU(3)$ or G_2 itself. Since the first three groups leave invariant at least a line in $\mathbf{R}^7$, it follows that a G_2-metric has full restricted holonomy G_2 if and only if there are no non-trivial parallel 1-forms.

Chapter 3

Analysis on Riemannian manifolds

In this chapter, we explain the main analytic tools that will be used in the constructions of Chapter 4 and Chapter 5. In both cases, a geometric problem is reduced to the resolution of a set of PDEs. In general, several problems arise when one wants to solve a PDE. First, showing the existence of a solution can be very difficult, especially since the space of smooth functions is not complete for any reasonable norm. Therefore, one must extend differential operators to more general space of functions with less regularity, in order to have some hope of using the general theory of Banach spaces and fixed points theorems. Even when this is achieved, showing that a solution to a PDE is smooth is troublesome. Lastly, one may be concerned with the uniqueness of solutions. All of these problems are extremely hard to tackle for general PDEs, even for linear ones.

However, for a particular kind of PDEs, that are called *elliptic*, these problems can be overcome. The key feature of an elliptic operator P is the existence of *interior estimates*, which insure that locally the norm of a function u and its first derivatives can be controlled by the norm of Pu and the L^2-norm of u. In particular, any solution to an elliptic PDE is at least as smooth as the data, and the kernel of an elliptic operator is composed of smooth functions. This result is known as *elliptic regularity.* On a compact space, these estimates can be made global, and the general theory of Banach spaces is enough to conclude that an elliptic operator is Fredholm, that is, it is invertible up to finite-dimensional kernel and cokernel. Fortunately, many operators of geometric interest, such as the operators $d + d^*$, the Laplacian Δ or the Dirac operator on a spin manifold, are elliptic. As a consequence of elliptic regularity, the Hodge theorem which gives an isomorphism between the space of harmonic forms and the real cohomology on a compact manifold is of great importance in geometry.

Over a non-compact manifold however, the interior estimates are not enough to understand the properties of elliptic operators, and even in simple cases like harmonic forms on $\mathbf{R}^2$, we see that the kernel of an elliptic operator may very well be infinite-dimensional, and there is no obvious relation between cohomology and harmonic forms. In the case where the non-compact manifold have controlled geometry at infinity however, it is possible to recover good Fredholm properties for elliptic operators. We will be mainly concerned with *asymptotically conical* (AC) manifolds, that are manifolds which geometry approaches the geometry of a cone at infinity, up to a decaying error term. On such manifolds, the Fredholm properties of the operators $d + d^*$ and Δ can be deduced from the corresponding operators on the cone, for which we can explicitly derive the structure of the kernel. As an application that will be useful to our purpose, we will see that we can relate cohomology classes on an AC manifold to L^2 closed and co-closed forms.

This chapter is structured as follows. The first part is devoted to analysis on $\mathbf{R}^n$, in order to introduce basic concepts. In §3.1.1, we introduce the Sobolev and Hölder norms, which are the basic norms that we will use in all our analysis. In §3.1.2, we give define differential operators, and discuss some basic properties, as formal adjoints and the action of differential operators on Sobolev and Hölder spaces. Lastly, in §3.1.3, we give interior estimates for elliptic operators, which are crucial in geometric analysis.

The second part of this chapter concerns compact manifolds. Mostly, we show how to properly patch up the local description of the first part in order to study differential operators between vector bundles on compact manifolds. In §3.2.1, we define Sobolev and Hölder norms on the space of sections of a vector bundle, and state the Sobolev and Kondrakov embedding theorems, that play a crucial role in the study of differential operators. In §3.2.2, we use the results of the previous part to prove that elliptic operators on a compact manifold are Fredholm, and describe the mapping properties of such operators. Finally, in §3.2.3, we apply these results to the operators $d + d^*$ and $dd^* + d^*d$, that are omnipresent in geometric analysis. One of the most important result of this part is the Hodge theorem, that identifies cohomology classes with harmonic forms. We also describe how to diagonalize the Laplacian operator, which will be important in order to study harmonic forms on cones.

In the third part, which is the most important to the purpose of this thesis, we try to give an exposition as self-contained as possible of the analysis on non-compact manifolds that have conical geometry at infinity. In §3.3.1, we give explicit formulas for the operators $d + d^*$ and $dd^* + d^*d$ on a cone, and a precise description of the kernels of these operators, in terms of eigenfunctions of the Laplacian of the boundary manifold. Then, we prove the Fredholm property for the operators $d + d^*$ and $dd^* + d^*d$ on manifolds that have one end isometric to a cone. In §3.3.2, we explain how these results carry on to manifolds that have one end asymptotic to a cone. Lastly, in §3.3.3, we discuss L^2-cohomology, which gives a correspondence between cohomology classes and L^2 closed and co-closed forms on an AC manifold.

The last part of this chapter is devoted to AC Calabi-Yau manifolds. In §3.4.1, we very briefly discuss Sasaki-Einstein manifolds, which are the boundary of Calabi-Yau cones, and give some results on the indicial roots of $d + d^*$ and $dd^* + d^*d$ on Calabi-Yau cones. Then, in §3.4.2, we use them to describe the kernel of some operators that will be of interest to us in Chapter 5. We finish in §3.4.3 by some further properties concerning deformations of $SU(3)$-structure. Mainly, we want to give precise statements in order to prove that all the objects naturally attached to an $SU(3)$-structure vary with the same regularity and asymptotic behavior as the $SU(3)$-structure. These results will also be useful in Chapter 5.

3.1 Analysis on $U \subset \mathbf{R}^n$

In this part we start by some basic analysis on open subset of $\mathbf{R}^n$, with the purpose of introducing Sobolev and Hölder spaces, and the classical interior estimates for elliptic operators.

3.1.1 Sobolev and Hölder spaces

Let U be an open set in $\mathbf{R}^n$, and let $m \geq 1$. We equip $\mathbf{R}^m$ with its canonical inner product $\langle \cdot, \cdot \rangle$, and denote by $| \cdot |$ the associated euclidian norm on $\mathbf{R}^m$. We define the space $C^k(U, \mathbf{R}^m)$ as the space of functions $f : U \to \mathbf{R}^m$ of class C^k on U, such that all the derivatives of f up to order k are bounded, equipped with the norm:

$$\|f\|_{C^k} = \sum_{|A| \leq k} \sup_{x \in U} |\partial_A f(x)|$$

This norm is the norm of uniform convergence of f and all its derivatives up to order k. Therefore, $C^{k,\alpha}(U, \mathbf{R}^m)$ is a Banach space. It can be equivalently defined as the closure the space $C_c^\infty(U, \mathbf{R}^m)$ of compactly-supported functions on U with respect to the C^k-norm.

Another important class of functions we need are Hölder functions. Let $\alpha \in (0, 1)$. If $f : U \to \mathbf{R}^m$ is any function, we say that f is an α-Hölder function if:

$$[f]_\alpha = \inf_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$$

Any Hölder function is continuous. If $k \geq 0$ is an integer, we say that $f : U \to \mathbf{R}^m$ is of class $C^{k,\alpha}$ on U if f is C^k, the derivatives of f up to order k are bounded on U, and the derivatives of f of order k are α-Hölder on U. The space $C^{k,\alpha}(U, \mathbf{R}^m)$ of functions of class $C^{k,\alpha}$ on U, equipped with the norm:

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + \sum_{|A| = k} [\partial_A f]_\alpha$$

As for $C^k(U, \mathbf{R}^m)$, $C^{k,\alpha}(U, \mathbf{R}^m)$ is also a Banach space, which can be defined as the closure of $C_c^\infty(U, \mathbf{R}^m)$ for the $C^{k,\alpha}$-norm. It is clear that if $k + \alpha \geq l + \beta$, we have a continuous embedding $C^{k,\alpha}(U, \mathbf{R}^m) \subset C^{l,\beta}(U, \mathbf{R}^m)$.

It is often useful to consider functions that are not as regular as $C^{k,\alpha}$ functions. Any locally integrable function $f : U \to \mathbf{R}^m$ is identified with the distribution on U defined by

$$(f,u) = \langle f,u\rangle_{L^2} = \int_U \langle f(x),u(x)\rangle d^n x$$

for any test function $u \in C^\infty_c(U,\mathbf{R}^m)$. If $A = (a_1,...,a_n)$ is a multi-index, we can define the *weak derivative* $\partial_A f$ as the distribution acting on $C^\infty_c(U,\mathbf{R}^m)$ by:

$$(\partial_A f,u) = (-1)^{|A|}\langle f,\partial_A u\rangle_{L^2}$$

By integration by parts, if f is a C^k function on U for some $k \geq 1$, then the strong and weak derivatives of f agree up to order k. Therefore, the weak derivative generalizes the usual notion of derivative to functions that are not necessarily smooth or even differentiable. In particular, we can define the weak derivative of an L^p-function for any $p \in [1,\infty]$.

Let $p \in [1,\infty)$ and $k \geq 0$ be an integer. The Sobolev space $L^p_k(U,\mathbf{R}^m)$ is the space of functions $f : U \to \mathbf{R}^m$ of class L^p on U such that all the weak derivatives of f up to order k are of class L^p. It is a Banach space equipped with the norm:

$$\|f\|_{L^p_k} = \left(\sum_{|A|\leq k} \|\partial_A f\|^p_{L^p}\right)^{\frac{1}{p}}$$

It can equivalently be defined as the closure of $C^\infty_c(U,\mathbf{R}^m)$ for the above norm.

3.1.2 Differential operators

Let $m,m' \geq 1$ be integers. A (smooth) differential operator of order k, taking functions with value in $\mathbf{R}^m$ to functions with values in $\mathbf{R}^{m'}$ is a map $P : C^\infty(U,\mathbf{R}^m) \to C^\infty(U,\mathbf{R}^{m'})$ of he form

$$Pf(x) = F(f(x),\partial_a f(x),...,\partial_{a_1...a_k} f(x))$$

where F is a smooth function of f and its derivatives up to order k. It is said to be linear if $f \mapsto Pf$ is a linear map. A linear differential operator of order k can be written

$$Pf(x) = A^{a_1...a_k}(x)\partial_{a_1...a_k} f(x) + ... + A^a(x)\partial_a f(x) + A(x)f(x)$$

where the $A^{a_1...a_j}(x)$'s are $m' \times m$ matrices depending smoothly on x. For any $x \in U$ and 1-form $\xi = \xi_a dx^a$, we can define a linear map

$$\sigma_\xi(P,x) = A^{a_1...a_k}\xi_{a_1}...\xi_{a_k}(x) : \mathbf{R}^m \to \mathbf{R}^{m'}$$

called the *principal symbol* of P.

Lemma 3.1.1. *Let $P : C^\infty(U, \mathbf{R}^m) \to C^\infty(U, \mathbf{R}^{m'})$ be a linear differential operator of order k. Then there exists a unique linear differential operator P^* : $C^\infty(U, \mathbf{R}^{m'}) \to C^\infty(U, \mathbf{R}^m)$ such that, for all compactly supported functions $u \in C^\infty(U, \mathbf{R}^m)$ and $v \in C^\infty(U, \mathbf{R}^{m'})$, we have*

$$\langle Pu, v\rangle_{L^2} = \langle u, P^* v\rangle_{L^2}$$

The operator P^* is called the formal adjoint *of P.*

Proof. If Q, Q' are two linear operators that satisfy this property, then for all $u \in C^\infty(U, \mathbf{R}^m)$ and $v \in C^\infty(U, \mathbf{R}^{m'})$, we have

$$\langle u, (Q - Q')v\rangle_{L^2} = 0$$

and thus $Q - Q' = 0$.

For the existence, we write:

$$Pu(x) = A^{a_1 \dots a_k}(x)\partial_{a_1 \dots a_k} u(x) + \dots + A^a(x)\partial_a u(x) + A(x)u(x)$$

By integration by parts,

$$\begin{aligned}\langle A^{a_1 \dots a_j}(x)\partial_{a_1 \dots a_j} u(x), v(x)\rangle &= \langle \partial_{a_1 \dots a_j} u(x), (A^{a_1, \dots, a_j}(x))^T v(x)\rangle_{L^2} \\ &= \langle u(x), (-1)^j \partial_{a_1 \dots a_j} \Big((A^{a_1, \dots, a_j}(x))^T v(x)\Big)\rangle\end{aligned}$$

We can then use the Leibniz rule and sum all of these equalities to define the operator P^* □

Remark 3.1.1. It follow from the proof that the principal symbols of P and P^* satisfy

$$\sigma_\xi(P^*, x) = (-1)^k \sigma_\xi(P, x)^T$$

Let $P : C^\infty(U, \mathbf{R}^m) \to C^\infty(U, \mathbf{R}^{m'})$ be a linear differential operator of order k. Using the operator P^*, we can define Pu when u is not a smooth function. As for weak derivatives, if u is a locally integrable function, we may define the distribution Pu by

$$(Pu, v) = \langle u, P^* v\rangle_{L^2}$$

for all compactly supported functions $v : U \to \mathbf{R}^{m'}$. If $u : U \to \mathbf{R}^m$ and $f : U \to \mathbf{R}^{m'}$ are locally integrable functions, we say that $Pu = f$ holds in the weak sense if

$$\langle u, P^* v\rangle_{L^2} = \langle f, v\rangle_{L^2}$$

for all $v \in C_c^\infty(U, \mathbf{R}^{m'})$. If u is a C^k function, the weak definition of Pu agrees with the strong definition, and $Pu = f$ holds weakly if and only if it holds in the strong sense.

In particular, the operator P acts on the spaces $L_l^p(U, \mathbf{R}^m)$, $C^l(U, \mathbf{R}^m)$ and $C^{l,\alpha}(U, \mathbf{R}^m)$. The following lemma shows that in the interior of U, P extends to a continuous operator between Sobolev and Hölder spaces.

Lemma 3.1.2. *Let $P : C^\infty(U, \mathbf{R}^m) \to C^\infty(U, \mathbf{R}^{m'})$ be a smooth linear differential operator of order k. Let $l \geq 0$, $\alpha \in (0,1)$ and V be an open subset of U with compact closure $\overline{V} \subset U$. Then there exists a constant C, such that for all $u \in C_c^\infty(V, \mathbf{R}^m)$:*

$$\|Pu\|_{C^{l,\alpha}} \leq C\|u\|_{C^{k+l,\alpha}}$$

In particular, P extends as a continuous operator $P : C^{k+l,\alpha}(V, \mathbf{R}^m) \to C^{l,\alpha}(V, \mathbf{R}^{m'})$. Similar estimates hold in C^l and L^p_l-norm.

Proof. We write explicitly the operator P as

$$Pu(x) = A^{a_1 \dots a_k}(x)\partial_{a_1 \dots a_k} u(x) + \dots + A^a(x)\partial_a u(x) + A(x)u(x)$$

Using the Liebniz rule, the derivatives of $Pu(x)$ up to order l must be a sum of terms of the form $\partial_{b_1 \dots b_p} A_{a_1 \dots a_j}(x)\partial_{b_{p+1} \dots b_{p+q} a_1 \dots a_p} u(x)$ where $p + q \leq l$. Since the functions $A^{a_1 \dots a_j}$ a smooth on U, all of their derivatives are bounded when restricted to any compact subset of U. A similar arguments works for Hölder α-norms. Hence the claimed inequalities. □

Remark 3.1.2. The reason why the estimates of Lemma 3.1.2 do not hold globally on U as a whole is that the coefficients of P might explode at infinity.

Remark 3.1.3. Since a linear differential operator can be considered as an operator acting on different spaces of functions $f : U \to \mathbf{R}^m$, we will sometimes write $P : \Gamma(U, \mathbf{R}^m) \to \Gamma(U, \mathbf{R}^{m'})$ when we do not want to specify the class of function on which P acts.

3.1.3 Elliptic regularity

In general, if P is a differential operator, solving the equation $Pu = f$ is a very difficult problem. We usually aim at finding smooth solutions, but since the space of smooth functions is not complete, it is more convenient to work with Sobolev or Hölder spaces. Solving a PDE is still very difficult in this setting, and when we can prove that weak solutions exist, those are not necessarily smooth or even differentiable. However, for a particular kind of PDEs, that are called *elliptic*, such problems do not happen.

Let $P : \Gamma(U, \mathbf{R}^m) \to \Gamma(U, \mathbf{R}^m)$ be a linear differential operator of order k. We say that P is elliptic at $x \in U$ if for all non-zero 1-forms $\xi \in (\mathbf{R}^n)^*$, the linear map $\sigma_\xi(P, x)$ is invertible. We say that P is an *elliptic operator* if it is elliptic at all $x \in U$. By Remark 3.1.1, an operator P is elliptic if and only if its formal adjoint P^* is elliptic.

Example 3.1.1. The Laplacian $\Delta = -\sum_a \frac{\partial^2}{\partial x^a}$ acting on functions of $\mathbf{R}^n$ is elliptic, since $\sigma_\xi(\Delta, x) = |\xi|^2 \operatorname{Id}$ for all $x \in \mathbf{R}^n$ and $\xi \in (\mathbf{R}^n)^*$. An example of operator that is not elliptic is the heat operator $P = \frac{\partial}{\partial t} + \Delta$ acting on functions on $(0, \infty) \times \mathbf{R}^n$, because $\sigma_{dt}(P, t, x) = 0$ for all $(t, x) \in (0, \infty) \times \mathbf{R}^n$.

The most important result of this part is the following, called *elliptic regularity.*

Theorem 3.1.3 (Elliptic regularity) *Let B_1 and B_2 be the balls of radius 1 and 2 respectively. Let $P : \Gamma(B_2, \mathbf{R}^m) \to \Gamma(B_2, \mathbf{R}^m)$ be a smooth elliptic operator of order k. Let $p \geq 1$, $l \geq 0$ and $\alpha \in (0,1)$. Suppose that $Pu = f$ holds in the weak sense on B_2, for $u, f \in L^1(B_2)$.*

If $f \in C^{l,\alpha}(B_2)$, then $u_{|B_1} \in C^{k+l,\alpha}(B_1)$ and there exists a constant $C > 0$, independent of u and f, such that

$$\|u_{|B_1}\|_{C^{k+l,\alpha}} \leq C(\|f\|_{C^{l,\alpha}} + \|u\|_{C^0}),$$
$$\|u_{|B_1}\|_{C^{k+l,\alpha}} \leq C(\|f\|_{C^{l,\alpha}} + \|u\|_{L^2})$$

If $f \in L^p_l(B_2)$, then $u_{|B_1} \in L^p_{k+l}(B_1)$ and there exists a constant $C' > 0$, independent of u and f, such that

$$\|u_{|B_1}\|_{L^p_{k+l}} \leq C'(\|f\|_{L^p_l} + \|u\|_{L^1})$$

In particular, if $f \in C^\infty(B_2)$ and $Pu = f$ holds on B_2, possibly in a weak sense, then $u_{|B_1} \in C^\infty(B_1)$.

This theorem is fundamental, and we will admit it. The above estimates are called *interior estimates*, and it is impossible to have similar estimates globally on U, because the behavior at the boundary of U cannot be controlled.

3.2 Analysis on compact manifolds

On compact manifolds, most of the local results of the last part can be made global by taking a finite open cover. As a result, elliptic operators on a compact manifold are Fredholm, and their mapping properties are fully understood. We shall also derive some more precise result about $d + d^*$ and the eigenvalues of the Laplacian $dd^* + d^*d$.

3.2.1 Sobolev and Hölder spaces: the return

Let (M^n, g) be a compact Riemannian manifold, and (E, h, ∇) a vector bundle of rank m, with a metric h and compatible connection ∇. Then we can cover M with a finite number of charts U_j, such that we have trivializations $E_{|U_j} \simeq U_j \times \mathbf{R}^m$. For instance, it will be sometimes convenient to choose a particular system of trivializations constructed in the following way. Let $x \in M$, and let $B(x, \varepsilon)$ be a geodesic ball of radius ε centered at x, for some small $\varepsilon > 0$. Choose a basis $e_1, ..., e_m$ of E_x. For all $y \in B(x, \varepsilon)$ and $i = 1, ..., m$, define $e_i(y)$ as the parallel transport of e_i along the only minimizing geodesic $(x, y) \subset B(x, \varepsilon)$ for the connection ∇. Then $(e_1, ..., e_n)$ is a local frame of E on $B(x, \varepsilon)$, that we can restrict to $B(x, \varepsilon/2)$. From this infinite system of local trivializations we can extract a finite system $\{U_j\}$. Let $g_{ij} : U_{ij} \to GL(m, \mathbf{R})$ be the transition functions of E.

Let $k \geq 0$ be an integer. A section $u : M \to E$ is of class C^k as a map $M \to E$, if and only if in local trivializations, the functions $u_j = u_{|U_j} : U_j \to$

$\mathbf{R}^m$ are of class C^k. Let $C^k(E)$ be the vector space of C^k sections of E. We equip $C^k(E)$ with the norm

$$\|u\|_{C^k} = \sum_{j=0}^{k} \sup_{x\in M} |\nabla^j u(x)|_{h\otimes g}$$

Here, the connection ∇ acting on $T^*M^{\otimes r} \otimes E$ is the tensor product of the Levi-Civita connection on T^*M and ∇ on E, and the norm on $T^*M^{\otimes r} \otimes E$ is the norm with respect to the metric $h \otimes g$. In local trivializations, this norm corresponds to what we expect:

Lemma 3.2.1. *Let $U \subset M$ be a coordinate chart on M, with coordinates $(x^1, ..., x^n)$, such that $E_{|U} \simeq U \times \mathbf{R}^m$. Using this trivialization, we identify sections of E over U with functions $U \to \mathbf{R}^m$. Thus, we can define a C^k norm as in §3.1.1, denoted $|\cdot|_{C^k}$ in this lemma only. Then, if V is an open subset of U with compact closure $\overline{V} \subset U$, the norm $\|\cdot\|_{C^k}$ and $|\cdot|_{C^k}$ are equivalent on V.*

Proof. We make the proof for $k = 1$, the proof is similar for $k \geq 2$. In trivializations, the expression of ∇ is

$$\nabla u = (\partial_a u(x) + A_a(x)u(x)) \otimes dx^a$$

Since $A_a(x)$ is bounded on any compact subset $\overline{V} \subset U$, it follows that there exists a constant C such that $\|u\|_{C^k} \leq C|u|_{C^k}$ on $\overline{V}$. Covering M by a finite number of such V, we have the inequality $\|\cdot\|_{C^k} \leq C|\cdot|_{C^k}$. The other inequality follow by the same argument, writing $\partial_a u(x) = \nabla_a u(x) - A_a(x)u(x)$. □

Any compact manifold can be covered by a finite number of open sets V with compact closure contained in a chart U. Hence, convergence in C^k-norm imply uniform convergence of a section and all of its derivatives up to order k in local trivializations over V. Therefore, $C^k(E)$ is complete. The advantage of defining the C^k-norm with covariant derivatives and not using local trivialization is that it is more intrinsic, and once a metric and compatible connection (h, ∇) are fixed on E, the C^k-norm does not depend on any choice. But whenever useful, we can also use the local definition of C^k-norm, which is equivalent by Lemma 3.2.1.

We now want to extend the notion of α-Hölder functions to sections of a vector bundle. As above, let (E, h) be a vector bundle over a compact Riemannian manifold (M, g), and suppose ∇ is a connection on E compatible with h. In most examples of interest, E will be a subbundle $TM^{\otimes s} \otimes T^*M^{\otimes r}$, and the connection will be the one induced by the Levi-Civita connection on TM. Let $x \in M$ and U be a geodesic chart around x. Then for any $y \in U$, there exists a unique geodesic $(x, y) \subset U$ joining x to y. If u is a section of E, we can define $|u(x) - u(y)|$ using parallel transport along this minimal geodesic. Thus we can define the α-Hölder norm of u by

$$[u]_\alpha = \min_{x \neq y} \frac{|u(x) - u(y)|}{d(x,y)^\alpha}$$

where x, y are taken close enough to both belong to the same geodesic chart. We say that a section is α-Hölder if $[u]_\alpha < \infty$. For any integer $k \geq 0$, we define the $C^{k,\alpha}$-norm on $CC^\infty(E)$ by:

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + [\nabla^k u]_\alpha$$

The Hölder space $C^{k,\alpha}(E)$ is defined as the closure of $C^\infty(E)$ for this norm. It consists of the C^k-sections u such that $\nabla^k u$ is α-Hölder. As for the C^k-norm, we can prove that locally, this definition of the $C^{k,\alpha}$-norm and the definition of §3.1.1 are equivalent.

Finally, we generalize the notion of Sobolev spaces to sections. A section $u : M \to E$ is said to be locally integrable if its local expressions in trivializations are locally integrable. This notions is independent of any choice. For a locally integrable section u, we define its L^p-norm, for $p \in [1, \infty)$, by

$$\|u\|_{L^p} = \left(\int_M |u|^p \operatorname{Vol}_g \right)^{\frac{1}{p}}$$

Here, Vol_g is the volume form of M associated with g, and $|\cdot|$ refers to the norm associated with the metric h on E. We say that u is of class L^p if $\|u\|_{L^p} < \infty$. We can define a notion of weak covariant derivative for L^p sections of E. If u is a L^p section of E, identified with locally integrable functions $u_j : U_j \to \mathbf{R}^m$ in local trivializations of E, where the U_j's are coordinate charts. Since we can locally express the Levi-Civita connection ∇ of g as $d + A_j$, we can define the weak covariant derivative of S by $(\nabla u)_j = du_j + A_j u_j$, where $d = \partial_i dx^i$ in the weak sense. Whenever these expressions define locally integrable functions, they all patch together to a well-defined locally integrable section of $T^*M \otimes E$. We define the Sobolev space $L^p_k(E)$ as the space of L^p sections of E such that all the covariant derivatives $\nabla^j u$ of u of order $j \leq k$ are L^p sections of $T^*M^{\otimes j} \otimes E$. It is again the closure of $C^\infty(E)$ for the norm:

$$\|u\|_{L^p_k} = \left(\sum_{j=0}^{k} \|\nabla^j u\|^p_{L^p} \right)^{\frac{1}{p}}$$

where the norm of the $\nabla^j u$'s is induced by the metric g on T^*M and h on E. Once again, the L^p_k-norm is locally equivalent to what we expect.

We finish by two important embedding results for Sobolev and Hölder spaces:

Theorem 3.2.2 (Sobolev Embedding Theorem) *Let M be a compact Riemannian manifold, (E, h, ∇) a vector bundle with metric and compatible connection, $k \geq l \geq 0$ integers, $q, r \geq 1$ real numbers, and $\alpha \in (0, 1)$.*

If $\frac{1}{q} \leq \frac{1}{r} + \frac{k-l}{n}$, then there is a continuous embedding $L^q_k(E) \hookrightarrow L^r_l(E)$ by inclusion.

If $\frac{1}{q} \leq \frac{k-l-\alpha}{n}$, then there is a continuous embedding $L^q_k(E) \hookrightarrow C^{l,\alpha}(E)$ by inclusion.

Theorem 3.2.3 (Kondrakov) *With the same notations as Theorem 3.2.2, if the inequalities are strict, then the above embeddings are compact.*

3.2.2 Elliptic operators

Let E, F be vector bundles over a possibly non-compact Riemannian manifold (M, g). A smooth differential operator P of order is a map from $C^\infty(E)$ to $C^\infty(F)$, such that $Pu(x) = f(x, u(x), \nabla u(x), ..., \nabla^k u(x))$ for a function f continuous on its arguments. We say that P is a linear differential operator if it is linear as a map from $C^\infty(E)$ to $C^\infty(F)$. In a local chart with coordinates $(x^1, .., x^n)$ over which E, F are trivial, a linear differential operator can be written:

$$Pu(x) = A^{a_1 \dots a_k}(x)\partial_{a_1 \dots a_k} u(x) + ... + A^a(x)\partial_a u(x) + A(x)u(x)$$

where the $A^{a_1 \dots a_j}$'s are matrices depending smoothly on x and we use Einstein's summation conventions. For a 1-form ξ written locally as $\xi = \xi_a dx^a$, the local expressions $A^{a_1 \dots a_k}\xi_{a_1}...\xi_{a_k}$ patch up together to define a bundle morphism $\sigma_\xi(P)$ from E to F, still called the *principal symbol* of P. A linear differential operator is said to be *elliptic* if for every non-zero 1-form ξ, the principal symbol $\sigma_\xi(P)$ is invertible.

Example 3.2.1. Consider the operator $d : \Omega(M) \to \Omega(M)$ acting on differential forms. In local coordinates we have

$$d(f dx^{a_1} \wedge ... \wedge dx^{a_k}) = dx^a \wedge (\partial_a f dx^{a_1} \wedge ... \wedge dx^{a_k})$$

so that for $\xi \in \Omega^1(M)$, the local expressions $\xi_a dx^a \wedge \cdot$ patch up together to the globally defined linear map $\xi \wedge \cdot = \sigma_\xi(d) : \Omega(M) \to \Omega(M)$. Since it vanishes for example on top-dimensional forms, d is not elliptic.

Example 3.2.2. Let (M, g) be a Riemannian manifold and consider the operator $d^* = (-1)^{kn+n+1} * d* : \Omega^k(M) \mapsto \Omega^{k-1}(M)$. Since d^* is a derivation of $\Omega(M)$ of degree -1, this is also true for its principal symbol, so that we only need to determine it on 1-forms. In local coordinates $(x^1, ..., x^n)$ we have:

$$\begin{aligned}
d^*(f dx^a) &= - * d * (f dx^a) \\
&= - * d((-1)^{b-1} g^{ab} f \sqrt{\det g}\ dx^1...\hat{dx^b}...dx^n) \\
&= - * (g^{ab}\partial_a f \sqrt{\det g}\ dx^1...dx^n + f\frac{\partial_b(g^{ab}\sqrt{\det g})}{\sqrt{\det g}}\sqrt{\det g}\ dx^1...dx^n) \\
&= -g^{ab}\partial_b f - \frac{\partial_b(g^{ab}\sqrt{\det g})}{\sqrt{\det g}} f
\end{aligned}$$

Hence the principal symbol of d^* has local expressions $\eta_b dx^b \mapsto -g^{ab}\xi_a\eta_b$ on 1-forms, which patch up globally to the derivation $-\xi\lrcorner\cdot = \sigma_\xi(d^*) : \Omega(M) \to \Omega(M)$.

It is clear that the principal symbol $\sigma(P)$ is linear in P, so that the principal symbol of the operator $d+d^*$ acing on $\Omega(M)$ is $\sigma_\xi(d+d^*) = \xi\wedge\cdot - \xi\lrcorner\cdot$. This is an invertible map on $\Omega(M)$ for any $\xi \neq 0$, so that $d+d^*$ is elliptic. Moreover, the principal symbol also preserves composition, and then the principal symbol $\sigma_\xi(\Delta)$ of the Laplacian $\Delta = dd^* + d^*d$ is the scalar multiplication by $|\xi|^2$. Hence the Laplacian is also elliptic. In the remaining §3.2.2, we state general results about elliptic operators, but we will mostly be interested in applying these results to $d + d^*$ and Δ.

Let $P : C^\infty(E) \to C^\infty(F)$ be a smooth linear differential operator of order k, where E, F are vector bundles over M equipped with metrics. Using Lemma 3.1.1 in local trivializations and checking that the expressions obtained patch up together to a globally well-defined operator, there exists a unique smooth linear differential operator P^* of order k, so that for all $u \in C^\infty_c(E)$ and $v \in C^\infty_c(F)$

$$\langle Pu, v\rangle_{L^2} = \langle u, P^*v\rangle_{L^2}$$

where the L^2 inner products are defined with respect to the metrics on the fibers. Once again, we call the operator P^* the *formal adjoint* of P. If $E = F$ and $P^* = P$, we say that P is *formally self-adjoint*. An operator is elliptic if and only if its formal adjoint is.

Example 3.2.3. The operator d^* is the formal adjoint of d. To see this, let $\alpha \in \Omega^{k-1}_c(M)$ and $\beta \in \Omega^k_c(M)$. Then

$$\langle d\alpha, \beta\rangle_{L^2} = \int_M d\alpha \wedge *\beta = (-1)^k \int_M \alpha \wedge d * \beta$$

by Stockes' theorem. But since $*^2 = (-1)^{(k-1)(n-k+1)}$ on $n-k+1$-forms, we find:

$$\langle d\alpha, \beta\rangle_{L^2} = (-1)^{kn+n+1}\langle \alpha, *d*\beta\rangle_{L^2}$$

Example 3.2.4. It is is straightforward that the operators $d + d^*$ and $\Delta = dd^* + d^*d$ are formally self-adjoint operators.

As for differential operators defined on an open subset of $\mathbf{R}^n$, we can use P^* to define the action of P on sections that are not smooth. If u is a locally integrable section of E, Pu is defined as the distribution acting on $C^\infty_c(F)$ by

$$(Pu, v) = \langle u, P^*v\rangle_{L^2}$$

so that, if f is a locally integrable section of F, we say that $Pu = f$ holds in the weak sense if

$$\langle u, P^*v\rangle_{L^2} = \langle f, v\rangle_{L^2}$$

for all $v \in C^\infty_c(F)$. We will often write $P : \Gamma(E) \to \Gamma(F)$ when we de not want to specify the regularity of sections on which P acts.

From now on, suppose that (M, g) is a compact Riemannian manifold, and let $P : \Gamma(E) \to \Gamma(F)$ be a smooth linear differential operator. By Lemma 3.2.1, we know that locally, the $C^{l,\alpha}$ and L^p_l-norms are equivalent to the usual norms for functions. Since we can cover M by a finite number of coordinates charts that trivialize E and F, the estimates of Lemma 3.1.2 imply the following result:

Lemma 3.2.4. *Let M be a compact manifold, E, F vector bundles over M. Let $P : \Gamma(E) \to \Gamma(F)$ be a smooth differential operator of order k. Let $l \geq 0$ be an integer and $\alpha \in (0, 1)$. Then P extends to a continuous linear operator $P : C^{k+l,\alpha}(E) \to C^{l,\alpha}(F)$ on Hölder spaces, and $P : L^p_{k+l}(E) \to L^p_l(F)$ on Sobolev spaces.*

In the same way, for elliptic operators, we can make the estimates of Theorem 3.1.3 global. This leads to the following result:

Theorem 3.2.5 (Elliptic regularity on compact manifolds) *Let (M, g) be a smooth, compact Riemannian manifold, E, F vector bundles of the same dimension, and let $P : C^\infty(E) \to C^\infty(F)$ be an elliptic operator of order k. Let $p \geq 1$, $l \geq 0$ and $\alpha \in (0, 1)$. Suppose $Pu = f$ holds weakly for $u \in L^1(E)$ and $f \in L^1(F)$.*

If $f \in L^p_l(F)$, then $u \in L^p_{k+l}(E)$ and there exists an constant $C > 0$, independent of u, f, such that

$$\|u\|_{L^p_{k+l}} \leq C(\|f\|_{L^p_l} + \|u\|_{L^1})$$

Suppose now that $f \in C^{l,\alpha}(F)$. Then $u \in C^{k+l,\alpha}$ and there exists a constant $C > 0$, independent of u, f, such that

$$\|u\|_{C^{k+l,\alpha}} \leq C(\|f\|_{C^{l,\alpha}} + \|u\|_{C^0})$$

In particular, if $f \in C^\infty(F)$ and $Pu = f$ holds on M in a weak sense, then $u \in C^\infty(E)$.

Remark 3.2.1. The estimates of the above theorem are often called *a priori estimates*.

Theorem 3.2.5 implies that the kernel of an elliptic operator P on a compact manifold is the same for the action of P on $C^\infty(E)$, $L^p_k(E)$, $C^k(E)$ or $C^{k,\alpha}(E)$, and is composed of smooth sections. It turns out that, as a consequence of the Kondrakov theorem, it is finite dimensional. Indeed, let $B = \{v \in C^{k,\alpha}(E), \|v\| \leq 1 \text{ and } Pv = 0\}$. Since the embedding $C^{k,\alpha}(E) \hookrightarrow C^k(E)$ is compact, the closure $\overline{B}$ of the image of B in $C^k(E)$ is compact. But $Pv = 0$ holds for $v \in B$, and since P is continuous in the C^k-norm, P vanishes identically on $\overline{B}$. By elliptic regularity, $\overline{B} \subset C^\infty(E)$, so that $\overline{B} = B \subset C^{k,\alpha}(E)$. Hence, B is compact, and $\ker P$ must be finite dimensional.

The following lemma is not hard, but important:

Lemma 3.2.6. *Let $P : \Gamma(E) \to \Gamma(F)$ be a linear differential operator of order k. Then, for all $u \in L^2_k(E)$ and $v \in L^2_k(F)$, we have*

$$\langle Pu, v\rangle_{L^2} = \langle u, P^* v\rangle_{L^2}$$

Proof. The equality is true by definition for $u \in C^\infty(E)$ and $v \in C^\infty(F)$. Now by P and P^* define continuous operators $L^2_k \to L^2$, and the L^2 inner product defines a continuous bilinear map $L^2_k \times L^2 \to \mathbf{R}$. Thus, by density of smooth sections in L^2_k-sections, the equality holds for u, v of class L^2_k. □

If $v \in C^\infty(E)$ is a smooth section, we can make the L^2-inner product of v with any section u of class L^p_k, C^k or $C^{k,\alpha}$, and it defines a continuous linear form on the spaces $L^p_k(E)$, $C^k(E)$ and $C^{k,\alpha}(E)$. This is clear for sections of class C^k and $C^{k,\alpha}$ since we can make the inner product of two continuous functions defined on a compact space, and for L^p_k, it comes from the Sobolev continuous embedding $L^p_k \hookrightarrow L^1$, which is trivial in that case. As a consequence, the L^2-orthogonal of $\ker P$ is a well defined closed subspace in $L^p_k(E)$, $C^k(E)$ or $C^{k,\alpha}(E)$.

Proposition 3.2.7. *Let $P : \Gamma(E) \to \Gamma(F)$ be a linear elliptic operator of order $k \geq 1$. Let $l \geq 0$ and $p \geq 1$. Then there exists a constant $C > 0$ such that for every $u \in L^p_{k+l}(E)$ L^2-orthogonal to $\ker P$, we have*

$$\|u\|_{L^p_{k+l}} \leq C\|Pu\|_{L^p_l}$$

If $\alpha \in (0,1)$, we have similar estimates in $C^{k+l,\alpha}$-norm.

Proof. By contradiction, assume there exists a sequence $\{u_n\}$ of unit vectors in $L^p_{k+l}(E)$ so that Pu_n goes to 0 in $L^p_l(E)$. By the Kondrakov theorem, the embedding $L^p_{k+l}(E) \hookrightarrow L^1(E)$ is compact, and thus there exists a subsequence $\{u_{n_j}\}$ that converges in L^1-norm. Hence u_{n_j} is a Cauchy sequence in $L^1(E)$. Moreover, Pu_{n_j} is Cauchy in $L^p_l(E)$. Using the a priori estimates of Theorem 3.2.5, the sequence u_{n_j} is Cauchy in $L^p_{k+l}(E)$, so that it converges to an element u, which must be non-zero by continuity of the norm. Since P is continuous, we have $Pu = 0$. But since $(\ker P)^\perp$ is closed, u is an element of $(\ker P)^\perp$, and u must be zero, which gives a contradiction.

The proof for $u \in C^{k+l,\alpha}(E)$ is similar and can be found in [17, Proposition 1.5.2, p. 17-18]. □

As a corollary of the estimates of the above proposition, we obtain the following result:

Corollary 3.2.8 (Fredholm alternative)**.** *Let $P : \Gamma(E) \to \Gamma(F)$ be a linear elliptic operator of order $k \geq 1$.*

For $p \geq 1$ and $l \geq 0$, the image of $P : L^p_{k+l}(E) \to L^p_l(E)$ is the L^2-orthogonal of $\ker P^*$ in $L^p_l(F)$. Moreover, P has a continuous inverse from $(\ker P^*)^\perp$ to $(\ker P)^\perp$.

For $\alpha \in (0,1)$ and $l \geq 0$, the image of $P : C^{k+l,\alpha}(E) \to C^{l,\alpha}$ is the L^2-orthogonal of $\ker P^*$ in $C^{l,\alpha}$. Moreover, P has a continuous inverse from $(\ker P^*)^\perp$ to $(\ker P)^\perp$.

Proof. Let $l \geq 0$ be an integer, $p \geq 1$ a real number and $\alpha \in (0,1)$. Let v_n be a sequence of $\operatorname{im} P$ that converges to an element $v \in C^{l,\alpha}(F)$. Then we can write uniquely $v_n = Pu_n$ with $u_n \in (\ker P)^\perp \subset C^{k+l,\alpha}(E)$. Since v_n converges, it is a Cauchy sequence, and by the estimates in Proposition 3.2.7, u_n is also a Cauchy sequence. Therefore, there exists a subsequence $\{u_{n_j}\}$ that converges to an element $u \in C^{k+l,\alpha}(E)$. Since $P : C^{k+l,\alpha}(E) \to C^{l,\alpha}(E)$ is continuous, we have $Pu = v$. Hence, the image of P is closed in $L^p_l(F)$, and the estimates of Proposition 3.2.7 imply that the linear map $P^{-1} : \operatorname{im} P \to (\ker P)^\perp$ is continuous. It remains to identify the image of P.

By the same argument, the image of $P : L^p_{k+l}(E) \to L^p_l(F)$ is a closed subspace of $L^p_l(F)$, and we have a continuous inverse $P^{-1} : \operatorname{im} P \subset L^p_l(F) \to (\ker P)^\perp \subset L^p_{k+l}(F)$.

Let $f \in C^{k,\alpha}(F)$ such that $\langle f, v\rangle_{L^2}$ for all $v \in \ker P^*$. We want to find $u \in C^{k+l,\alpha}(E)$ such that $Pu = f$, which is equivalent to

$$\langle u, P^* v\rangle_{L^2} = \langle f, v\rangle_{L^2}$$

for all $v \in C^\infty(F)$. By elliptic regularity, it suffices to find $u \in L^2(E)$ such that the above equality holds for all smooth sections v. Consider the continuous linear form $L : L^2_k(F) \to \mathbf{R}$ defined by

$$L(v) = \langle f, v\rangle_{L^2}, \qquad v \in L^2_k(F)$$

Let H be the L^2-orthogonal space of $\ker P^*$ in $L^2_k(F)$, and let H' be the image of $P^* : L^2_k(F) \to L^2(E)$. Then $P : H \to H'$ is a continuous isomorphism. L defines a continuous linear form on H', and by the Hahn-Banach theorem, we can extend it to a continuous linear form on $L^2(E)$. Hence, there exists $u \in L^2(E)$ such that

$$\langle u, P^* v\rangle_{L^2} = L(v) = \langle f, v\rangle_{L^2}$$

for all $v \in H$. This is also trivially true for $v \in \ker P^*$, and thus the equality holds for all $v \in L^2_k(F)$. Thus $Pu = f$, and by elliptic regularity $u \in C^{k+l,\alpha}(E)$.

In particular, if f is a smooth section of F, then there exists a unique smooth section u of E such that $Pu = f$ and $u \perp \ker P$. Moreover, by Proposition 3.2.7, $\|u\|_{L^p_{k+l}} \leq C\|f\|_{L^p_l}$ for some constant C independent of f. If $f \in L^p_l(F)$ is L^2-orthogonal to $\ker P^*$, it is limit of a sequence f_n of smooth functions orthogonal to $\ker P^*$. Let u_n be the solution of $Pu = f$ in the orthogonal of $\ker P$. Then $\{u_n\}$ is a Cauchy sequence in $L^p_{k+l}(E)$ by the above estimates, and therefore it admits a limit $u \in L^p_{k+l}(E)$. By continuity of $P : L^p_{k+l}(E) \to L^p_l(F)$, we have $Pu = f$, which completes the proof. □

As a consequence of Corollary 3.2.8, the mapping properties of an elliptic operator $P : \Gamma(E) \to \Gamma(F)$ are fully understood. Since P^* is also an elliptic operator, $\ker P^*$ is a finite dimensional subspace of $C^\infty(F)$. Therefore, the operator P acting on suitable Sobolev or Hölder spaces has finite dimensional kernel and cokernel: we say that P is a *Fredholm operator*. The index of P is by definition $\dim\ker P - \dim\operatorname{coker} P$.

Remark 3.2.2. As we have seen, the equation $Pu = f$ admits a solution if and only if f is orthogonal to the kernel of P^*. Moreover, this solution is unique if we require u to be orthogonal to the kernel of P. We can think of $\ker P^*$ as a space of obstructions to solving the equation $Pu = f$. Similarly, $\ker P$ is a source of non-uniqueness of solutions to the equation $Pu = f$, that makes it impossible to define an inverse to P on its image. Working on the orthogonal to $\ker P$ allows us to define a continuous inverse, because we have the estimates of Proposition 3.2.7. The fact that we need to work transversally to the kernel of P in order to have such control on the norm of solutions to $Pu = f$ will play an important role in all our constructions.

3.2.3 Hodge theory and diagonalization of the Laplacian

Let (M, g) be a compact Riemann manifold. The results of §3.2.2 apply in particular to the formally self-adjoint operators $d + d^*$ and the Laplacian $\Delta = dd^* + d^*d$. For any k-form η, we have

$$\langle dd^*\eta + d^*d\eta, \eta\rangle_{L^2} = \|d\eta\|_{L^2}^2 + \|d^*\eta\|_{L^2}^2$$

Therefore, $\Delta\eta = 0$ if and only if η is d- and d^*-closed.Such differential forms are called *harmonic*. We denote by $\mathcal{H}^k$ the space of harmonic k-forms. Applying the results above leads to the decomposition $\Omega^k(M) = \mathcal{H}^k \oplus \operatorname{im}(d + d^*)$. Moreover, since $d^2 = 0$, the images of d and d^* are orthogonal. This result is known as the Hodge decomposition:

Proposition 3.2.9 (Hodge decomposition)**.** *Let (M, g) be a compact Riemannian manifold of dimension n, and let $0 \leq k \leq n$. Then we have the following orthogonal decomposition of the space of k-forms:*

$$\Omega^k(M) = \mathcal{H}^k \oplus \operatorname{im} d \oplus \operatorname{im} d^*$$

The Hodge decomposition also imply that $\ker d = \mathcal{H}^k \oplus \operatorname{im} d$, so that $H^k(M) \simeq \mathcal{H}^k$.

Theorem 3.2.10 (Hodge Theorem) *Let (M, g) be a compact Riemann manifold of dimension n and $0 \leq k \leq n$. Then we have a natural isomorphism $H^k(M) \simeq \mathcal{H}^k$, so that any cohomology class admits a unique harmonic representative.*

It will also be useful to have results on the eigenfunctions of Δ. Since $\langle dd^*\eta + d^*d\eta, \eta\rangle_{L^2} = \|d\eta\|_{L^2}^2 + \|d^*\eta\|_{L^2}^2$, the Laplacian is a positive operator, so that it has non-negative eigenvalues. For $\mu \geq 0$, the operator $\Delta - \mu\,\mathrm{Id}$ is also elliptic, so that the eigenspace associated with the eigenvalue μ is finite dimensional and consists of smooth differential forms. We will need the following result about the spectral properties of the Laplacian:

Proposition 3.2.11. *Let (M, g) be a compact Riemannian manifold and $k \geq 0$. Then there exists a non-decreasing sequence $\{\mu_n\}$ of non-negative real numbers, and an Hilbert basis $\{u_n\}$ of $L^2\Omega^k(M)$, such that $\Delta u_n = \mu_n u_n$ for all n. Moreover, $\mu_n \to \infty$ as $n \to \infty$.*

Proof. Consider Δ as an operator $L^2_2\Omega^k(M) \to L^2\Omega^k(M)$. Let H be the orthogonal complement of $\mathcal{H}^k$ in $L^2\Omega^k(M)$. Since H is closed, this is a Hilbert space equipped with the L^2inner product. If $v \in H$, let Gv be the unique solution $u \in L^2_2\Omega^k(M)$ of $\Delta u = v$ which is orthogonal to $\mathcal{H}^k$. Since we have a compact embedding $L^2_2\Omega^k(M) \hookrightarrow L^2\Omega^k(M)$ and the image of G is contained in H, we can see G as a compact operator $H \to H$. Moreover, G is an injective map. We claim that G is a positive and self-adjoint operator on H.

Indeed, let $u, v \in H$. If v is smooth, and $w = Gv$, we have $\delta w = v$ and $G\delta w = w$. Using this, we compute:

$$\begin{aligned}
\langle Gu, v\rangle_{L^2} &= \langle Gu, \Delta w\rangle_{L^2} \\
&= \langle u, w\rangle_{L^2} \\
&= \langle u, G\Delta w\rangle_{L^2} \\
&= \langle u, Gv\rangle_{L^2}
\end{aligned}$$

Therefore, $\langle Gu, v\rangle_{L^2} = \langle u, Gv\rangle_{L^2}$ holds for all $u \in H$ and $v \in \Omega^k(M) \cap H$. By density of $\Omega^k(M)$ in H, we conclude that G is self-adjoint.

To see that G is positive, let $u \in \Omega^k(M)$ orthogonal to $\mathcal{H}^k(M)$, and write $u = \Delta v$ for a smooth function v orthogonal to $\mathcal{H}^k$. Then

$$\langle Gu, u\rangle_{L^2} = \langle G\Delta v, \Delta v\rangle_{L^2} = \langle v, \Delta v\rangle_{L^2} > 0$$

and again we conclude by density.

By basic functional analysis, there exists a non-increasing sequence $\{l_n\}_{n\geq 1}$ of positive numbers and a Hilbert basis $\{u_n\}$ of H such that $Gu_n = l_n u_n$ for all $n \geq 1$. If we set $\mu_n = \frac{1}{l_n}$, the we have $\Delta u_n = \mu_n u_n$. We can also set $\mu_0 = 0$ and choose an orthonormal basis of $\mathcal{H}^k$ if the space of harmonic k-forms is non-trivial, which completes the proof. □

Since the Laplacian commutes with the operators d and d^*, these operators leave invariant the eigenspaces of Δ. Combining Proposition 3.2.11 with the Hodge decomposition, we find that each eigenspace $\ker(\Delta - \mu) \cap \Omega^k(M)$ has an orthogonal decomposition $E^k_\mu \oplus F^k_\mu$ where $E^k_\mu \subset \operatorname{im} d$ and $F^k_\mu \subset \operatorname{im} d^*$.

Lemma 3.2.12. *Let $\mu > 0$ be an eigen value of the Laplacian, and $\lambda = \mu^{\frac{1}{2}}$. The operators $\lambda^{-1}d$ and $\lambda^{-1}d^*$ define isometries*

$$\lambda^{-1}d : F^{k-1}_\mu \to E^k_\mu, \qquad \lambda^{-1}d^* : E^k_\mu \to F^{k-1}_\mu$$

that are inverse of each other.

Proof. Let $u = d^*\eta \in F_\mu^{k-1}$. Then $\Delta u = \mu u$ gives $d^*dd^*\eta = \mu d^*\eta$, that is $d^*du = \mu u$. In the same way, if $v \in E_\mu^k$, we have $dd^*v = \mu v$. It remains to show that $\lambda^{-1}d$ and $\lambda^{-1}d^*$ preserve the L^2 inner product. If $u \in F_\mu^{k-1}$, then u is d^*-exact and we have

$$\|du\|_{L^2} = \langle du, du\rangle_{L^2} = \langle d^*du, u\rangle_{L^2} = \langle \Delta u, u\rangle_{L^2} = \lambda^2\|u\|_{L^2}^2$$

and similarly for d^* acting on E_μ^k, which completes the proof. □

Remark 3.2.3. An application of this lemma, for example, is that the eigenvalues of the Laplacian acting on 1-forms are determined by the eigenvalues of the scalar laplacian and the eigenvalues of the Laplacian acting on d^*-exact 1-forms.

The last important feature of the Laplacian is the existence of a *Weitzenböck formula.* Following the notations in [4, §1.I], define an operator Γ on k-forms by:

$$\Gamma\xi = \sum_p g^{ab} r_{i_p a}\xi_{i_1\ldots b\ldots i_k} - \sum_{p\neq q} g^{ac}g^{bd}R_{i_p a i_q b}\xi_{i_1..c..d..i_k}$$

Proposition 3.2.13 (Weitzenböck-Lichnerowicz formula)**.** *Let (M, g) be a Riemannian manifold, ∇ the Levi-Civita connection and ∇^* its formal adjoint. Then the Laplacian operator acting on k-forms satisfy:*

$$(dd^* + d^*d)\xi = \nabla^*\nabla\xi + \Gamma\xi$$

Suppose that M admits a torsion-free G-structure for a subgroup G of $SO(n)$, compatible with the metric g. Then we can split the space of differential forms into vector sub-bundles associated with irreducible representations of G. Since the operators ∇, ∇^* and Γ, which is build up from the curvature of ∇, only depend on the associated representation of G, the Laplacian $dd^* + d^*d$ preserves the type of form. Moreover, its action on the space of differential forms of a certain type only depends on the associated representation of G, and not on the degree of the forms for example. In particular, the kernel of the Laplacian splits into the direct sum of the kernels of Δ, acting of differential forms of each type. Combined with the Hodge theorem, this gives a splitting of the cohomology of M on a compact manifold.

3.3 Analysis on asymptotically conical manifolds

For non-compact manifolds, the all of the results above generally fail. To take the simplest case possible, consider the non-compact manifold $\mathbf{C}$ identified with $\mathbf{R}^2$ with standard metric. It is well known that the harmonic functions on $\mathbf{C}$ are the real part of holomorphic functions. In particular, the kernel of Δ is infinite-dimensional. On the other hand, the closed and co-closed functions are exactly the constants. Therefore, we see that the kernel of $d + d^*$

on function is strictly contained in the kernel of Δ, and that non-constant harmonic functions do not represent any cohomology class.

For general non-compact manifolds, not much can be said about the mapping properties of differential operators, even elliptic ones. However, we will see that for non-compact manifolds that have one end asymptotic to a Riemannian cone, we can construct spaces of sections on which the important operators, like $d+d^*$ and dd^*+d^*d, have good Fredholm properties. These spaces are called *weighted Sobolev and Hölder spaces*; they generalize the Sobolev and Hölder spaces of compact manifolds by adding a so-called weight to control the asymptotic decay rate of sections.

3.3.1 Riemannian cones

Definition 3.3.1. Let (Σ, g_Σ) be a compact connected Riemannian manifold. A *Riemannian cone* over Σ is a Riemannian manifold $C(\Sigma) = (R, \infty) \times \Sigma$ for some $R > 0$, equipped with a metric of the form

$$g = dr^2 + r^2 g_\Sigma$$

where r is the coordinate in (R, ∞).

Let (Σ, g_Σ) be a compact connected Riemannian manifold, let $C = (1, \infty) \times \Sigma$ be the Riemannian cone endowed with the metric $g = dr^2 + r^2 g_\Sigma$. Let ϕ_t be the flow of the vector field $r\frac{\partial}{\partial r}$. Explicitly, $\phi_t(r, x) = (e^t r, x)$. Let $p : E \to C$ be any vector bundle with bundle metric h and compatible connection ∇. For any $\rho > 1$, let $E^\rho \to \Sigma$ be the vector bundle defined by restriction of p over $\{\rho\} \times \Sigma$, and let h^ρ, ∇^ρ be the induced metric and connection. Then there is an isomorphism $(E, h, \nabla) \simeq p^*(E^\rho, h^\rho, \nabla^\rho)$, given by $\Phi(\rho e^t, x)u = (\phi_t)_* u$, where $u \in E^\rho_x$ and $(\phi_t)_*$ is the parallel transport of u along the integral curves of ϕ_t. Thus, any vector bundle with metric bundle and compatible connection on a cone is isomorphic to the pull back of a vector bundle over Σ. This allows us to define a translation operator $\mathfrak{T}(a) : E \to E = (\phi_a)_*$, which is a bundle maps that covers ϕ_a. The translation operator acts on sections as $\mathfrak{T}(a)^* S(p) = \mathfrak{T}(a)^{-1} S(\phi_a(p))$. Hence, it makes sense to talk about translation-invariant sections of E. We can also talk about homogeneous sections of E: they are sections that satisfy $\frac{d}{dt}\mathfrak{T}(t)^* S = \lambda S$ for some $\lambda \in \mathbf{R}$, called the order of S.

Example 3.3.1. Consider the vector bundle $E = TC$ over the cone $C = (1, \infty) \times \Sigma$. We have a natural identification $TC \simeq \mathbf{R} \oplus T\Sigma$, under which the conical metric has the expression $g = dr^2 + r^2 g_\Sigma$. We endow TC with the metric g and Levi-Civita connection ∇. It is straightforward to check that $\nabla_{\frac{\partial}{\partial r}}$ acts on a vector field $Y = f(r)\frac{\partial}{\partial r} + X(r)$ as

$$\nabla_{\frac{\partial}{\partial r}} Y = \partial_r f(r) \frac{\partial}{\partial r} + \Big(\partial_r + \frac{1}{r}\Big) X(r)$$

so that translation-invariant vector fields have the form $a\frac{\partial}{\partial r} + r^{-1} X$, for $a \in \mathbf{R}$ and $X \in T\Sigma$ independent of r.

In the same way, for any integer $k \leq n$, we have a natural identification $\Lambda^k T^*C \simeq \Lambda^{k-1}T^*\Sigma \oplus \Lambda^k T^*\Sigma$, since any $\eta \in \Lambda^k T^*C$ is of the form $\eta = r^k \left(\frac{dr}{r} \wedge \alpha + \beta\right)$, and the metric is given by $|\eta|^2 = |\alpha|^2 + |\beta|^2$. It is easy to check that the translation-invariants k-forms are of the form $\eta = r^k \left(\frac{dr}{r} \wedge \alpha + \beta\right)$ for α, β differential forms independent of r, and more generally the homogeneous forms of order λ are the forms that can be written

$$\eta = r^{k+\lambda} \left(\frac{dr}{r} \wedge \alpha + \beta\right)$$

In the remaining of this part, we will investigate the properties of the operators $d + d^*$ and $dd^* + d^*d$ on a Riemannian cone $C = (1, \infty) \times \Sigma$, in the notations of Example 3.3.1.

Let us derive explicit formulas for the operators d_C, d_C^* and Δ_C on the cone C. We will denote $*_C$ and $|\cdot|_C$ for the Hodge operator and the norm relative to the metric g on C. The notations without subscript will refer to objects on the manifold Σ. It is straightforward to check that

$$d_C \left[r^k \left(\frac{dr}{r} \wedge \alpha + \beta\right)\right] = r^{k+1} \left(\frac{dr}{r} \wedge A + B\right)$$

where we have explicit formulas for the r-dependent forms A, B:

$$\begin{cases} A = \dfrac{1}{r}\Big((r\partial_r + k)\beta - d\alpha\Big) \\ B = \dfrac{1}{r} d\beta \end{cases} \tag{3.1}$$

For the operator d_C^*, we first need to express $*_C$ in terms of the Hodge star on Σ. We find that:

$$*_C \left[r^k \left(\frac{dr}{r} \wedge \alpha + \beta\right)\right] = r^{n-k} \left((-1)^k \frac{dr}{r} \wedge *\beta + *\alpha\right) \tag{3.2}$$

Using this, a straightforward computation gives

$$d_C^* \left[r^k \left(\frac{dr}{r} \wedge \alpha + \beta\right)\right] = r^{k-1} \left(\frac{dr}{r} \wedge A + B\right)$$

with:

$$\begin{cases} A = -\dfrac{1}{r} d^*\alpha \\ B = \dfrac{1}{r}\Big(d^*\beta - (r\partial_r + (n-k))\alpha\Big) \end{cases} \tag{3.3}$$

Finally, we can use equations (3.1) and (3.3) to find the following expression for the laplacian:

$$\Delta_C \left[r^k \left(\frac{dr}{r} \wedge \alpha + \beta\right)\right] = r^k \left(\frac{dr}{r} \wedge A + B\right)$$

where A, B are given by:

$$\begin{cases} A = \frac{1}{r^2}\Big(\Delta\alpha - (r\partial_r + k - 2)(r\partial_r + n - k)\alpha - 2d^*\beta\Big) \\ B = \frac{1}{r^2}\Big(\Delta\beta - (r\partial_r + n - k - 2)(r\partial_r + k)\beta - 2d\alpha\Big) \end{cases} \tag{3.4}$$

It is convenient to make the change of variables $r = e^t$, which gives $\partial_t = r\partial_r$. We see that the rescaled operators $r(d_C + d_C^*)$ and $r^2\Delta_C$ are only expressed in terms of operators acting on Σ and polynomials in ∂_t. Hence, these operators are translation-invariant, that is, we have $r(d_C + d_C^*)(\mathfrak{T}(a)^*\eta) = \mathfrak{T}(a)^*(r(d_C + d_C^*)\eta)$ and $r^2\Delta_C(\mathfrak{T}(a)^*\eta) = \mathfrak{T}(a)^*(r^2\Delta_C\eta)$ for all differential form η on C.

Using equations (3.1) and (3.3) and the variable t instead of r, closed and co-closed k-forms are given by the equations:

$$\begin{cases} d^*\alpha = 0 = d\beta \\ d^*\beta = \Big(\partial_t + (n-k)\Big)\alpha \\ d\alpha = \Big(\partial_t + k\Big)\beta \end{cases} \tag{3.5}$$

In the same way, harmonic k-forms are defined by the equations

$$\begin{cases} \Delta\alpha - \Big(\partial_t + k - 2\Big)\Big(\partial_t + n - k\Big)\alpha - 2d^*\beta = 0 \\ \Delta\beta - \Big(\partial_t + n - k - 2\Big)\Big(\partial_t + k\Big)\beta - 2d\alpha = 0 \end{cases} \tag{3.6}$$

Note that these equations are translation invariant. We now use Lemma 3.2.12 to give explicitly the structures of closed and co-closed forms on the cone C. The first row in (3.5) implies that closed and co-closed k-forms can written

$$\alpha(t) = \sum_j c_j(t)\gamma_j + \sum_{n\geq 1} a_n(t)\alpha_n, \quad \beta = \sum_j c_j'(t)\gamma_j' + \sum_{n\geq 1} b_n(t)\beta_n$$

where γ_j, γ_j' are harmonic forms, and for $n \geq 1$, α_n, β_n are eigenvalues of the Laplacian, satisfying $d\alpha = \lambda_n\beta_n$ and $d^*\beta^n = \lambda_n\alpha_n$, for the eigenvalue $\lambda^2 > 0$ of the Laplacian. Separating variables in (3.5) yields a system of coupled ODEs for the coefficients of α and β.

For the coefficients c_j, c_j', we obtain:

$$(\partial_t + n - k)c_j(t) = 0 = (\partial_t + k)c_j'(t)$$

Therefore, $c_j(t) = e^{-(n-k)t}C_j$, $c_j'(t) = e^{-kt}C_j'$ for some constants $C_j, C_j' \in \mathbf{R}$.

For $n \geq 1$, (3.5) is equivalent to the fact that a_n and b_n satisfy the equation:

$$(\partial_t + k)(\partial_t + n - k)f = \lambda_n^2 f$$

The discriminant of this equation is $n^2 - 4k(n-k) + 4\lambda_n^2 > 0$ since $\lambda_n > 0$. Hence, solutions to this equations are of the form $Ae^{\lambda_n^- t} + Be^{\lambda_n^+ t}$, where

$$\lambda_n^\pm = -\frac{n}{2} \pm \sqrt{\frac{n^2}{4} - k(n-k) + \lambda_n^2}$$

Substituting again the variable $r = e^t$, we see that any closed and co-closed form on C can be written as an L^2-sum of terms of the form

$$\eta = Ar^{\lambda}\left(\frac{dr}{r} \wedge \alpha + \beta\right)$$

where $\alpha \in \Omega^{k-1}(\Sigma)$, $\beta \in \Omega^k(\Sigma)$. Such form η is called a homogeneous k-form of order λ, and if there exists a homogeneous k-form η of order λ such that $d_C\eta = 0 = d_C^*\eta$, we say that λ is an *indicial root* of the operator $d_C + d_C^*$ acting of k-forms. We denote by $\mathcal{D}(d_C + d_C^*)$ the set on indicial roots of $d + d^*$. The set of indicial roots of $d_C + d_C^*$ is well understood in terms of the eigenvalues of the Laplacian Δ on Σ.

Proposition 3.3.1. *Let C be the cone $(R, \infty) \times \Sigma$, where (Σ, g_Σ) is a compact connected Riemannian manifold, and $g = dr^2 + r^2 g_\Sigma$ be the conical metric on C. Let $0 < \lambda_1^2 \leq ... \leq \lambda_n^2 \leq ...$ be the non-decreasing sequence of positive eigenvalues of Δ_Σ acting on closed k-forms. Then any closed and co-closed k-form on C can be written as an L^2-sum of homogeneous k-forms. If $H^{k-1}(\Sigma) \neq 0$, then $-(n-k)$ is an indicial root of $d_C + d_C^*$, and if $H^k(\Sigma) \neq 0$, then $-k$ is an indicial root. Moreover, we have*

$$\mathcal{D}(d_C + d_C^*)\backslash\{-k, -(n-k)\} = \{\lambda_n^+, \lambda_n^-,\ n \geq 1\}\backslash\{-k, -(n-k)\}$$

where

$$\lambda_n^{\pm} = -\frac{n}{2} \pm \sqrt{\frac{n^2}{4} - k(n-k) + \lambda_n^2}$$

Remark 3.3.1. The set of indicial roots of $d_C + d_C^*$ is in particular discrete without accumulation points in $\mathbf{R}$. Moreover, it is symmetric with respect to the value $-\frac{n}{2}$. From our discussion above, $-\frac{n}{2}$ is an indicial root if and only if $k = \frac{n}{2}$ and either $H^{k-1}(\Sigma) \neq 0$ or $H^k(\Sigma) \neq 0$.

For harmonic k-forms, we can also solve the equation $\Delta_C\eta = 0$ in the same way, by separating variables. Using Lemma 3.2.12, can decompose α, β as

$$\alpha(t) = \sum_j c_j(t)\gamma_j + \sum_{n\geq 1} a_n(t)\alpha_n + \sum_{m\geq 1} \tilde{a}_m(t)\tilde{\alpha}_m$$
$$\beta(t) = \sum_j c_j'(t)\gamma_j' + \sum_{n\geq 1} b_n(t)\beta_n + \sum_{m\geq 1} \tilde{b}_m(t)\tilde{\beta}_m$$

where γ_j, γ_j' are harmonic, and

$$d\alpha_n = \lambda_n\beta_n, \quad d^*\beta_n = \lambda_n\alpha_n, \quad d\tilde{\alpha}_m = 0 = d^*\tilde{\beta}_m$$
$$(\Delta - \lambda_n^2)\alpha_n = 0 = (\Delta - \lambda_n^2)\beta_n, \quad \Delta\tilde{\alpha}_m = \mu_m^2\tilde{\alpha}_m, \quad \Delta\tilde{\beta}_m = \nu_m^2\tilde{\beta}_m$$

Using equations (3.6), we have

$$(\partial_t + k - 2)(\partial_t + n - k)c_j(t) = 0$$

and therefore, if $k \neq \frac{n}{2}+1$, then $c_j(t) = Ae^{-(k-2)t} + Be^{-(n-k)t}$ for some constants A, B. If $k = \frac{n}{2}+1$, then $c_j(t) = (At+b)e^{-(\frac{n}{2}-1)t}$ for some constant A, B. In the same way, if $k \neq \frac{n}{2}-1$, then $c_j'(t) = Ae^{-kt} + Be^{-(n-k-2)t}$, and if $k = \frac{n}{2}-1$, then $c_j(t) = (At+b)e^{-(\frac{n}{2}-1)t}$.

The coefficients $\tilde{a}_n$ satisfy

$$\mu_n^2 a - \Big(\partial_t + k - 2\Big)\Big(\partial_t + n - k\Big)a = 0$$

which has solutions of the form $a(t) = Ae^{\mu_m^+ t} + Be^{\mu_m^- t}$, where

$$\mu_m^{\pm} = -\frac{n}{2} + 1 \pm \sqrt{\left(\frac{n}{2}-1\right)^2 - (k-2)(n-k) + \mu_m^2}$$

In the same way, the coefficients $\tilde{b}_m$ satisfy

$$\nu_m^2 b(t) - (\partial_t + n - k - 2)(\partial_t + k)b(t) = 0$$

which have solutions of the form $b(t) = Ae^{\nu_m^+ t} + Be^{\nu_m^- t}$ with

$$\nu_m^{\pm} = -\frac{n}{2} + 1 \pm \sqrt{\left(\frac{n}{2}-1\right)^2 - k(n-k-2) + \nu_m^2}$$

Lastly, the coefficients (a_n, b_n) satisfy the following system:

$$\lambda_n^2 a(t) - (\partial_t + k - 2)(\partial_t + n - k) - 2\lambda_n b = 0$$
$$\lambda_n^2 b(t) - (\partial_t + n - k - 2)(\partial_r + k) - 2\lambda_n a = 0$$

Since solving this system requires in principle to find the roots of a degree 4 polynomial, it is not straightforward to compute explicit solutions. However, since closed and co-closed 1-form are in particular harmonic, we already know two of the roots, λ_n^+ and λ_n^-. Some thought regarding the symmetries of the equations show that the other two roots (with multiplicity) must be $-(n-2) - \lambda_n^{\pm} = \lambda_n^{\pm} + 2$. These roots are not necessarily simple, but we can only have a double root when $\lambda_n^+ = \lambda_n^- + 2$, which regarding the expression gives $\lambda_n^+ = -\frac{n-2}{2}$. Moreover, this case is excluded in particular when we know that the first eigenvalue of the Laplacian acting on closed k-forms (or co-closed $k-1$ forms) is strictly greater than 4, since it forces $\lambda_n^+ - \lambda_n^- \geq \lambda_n > 2$. It will be true in cases of interest to us.

We summarize these results in the following lemma, keeping in mind the more precise description above of the indicial roots of the Laplacian, which will be useful later on.

Lemma 3.3.2. *Let C be the cone $(R, \infty) \times \Sigma$, where (Σ, g_Σ) is a compact connected Riemannian manifold, and $g = dr^2 + r^2 g_\Sigma$ be the conical metric on C. Then any solution of the equation $\Delta_C \eta = 0$ can be written as an L^2-sum of terms of the form*

$$\gamma = \gamma_0 + \log r \ \gamma_1$$

where γ_0, γ_1 are homogeneous forms of order λ, and $\Delta_C \gamma_0 = 0 = \Delta_C \gamma_1$. Moreover, $\gamma_1 = 0$ unless $\lambda = -\frac{n-2}{2}$.

This result is very general. If E is a vector bundle with metric and compatible connection over a cone C and P is an elliptic operator of order k such that $r^k P$ is translation invariant, then any section u of E satisfying $Pu = 0$ can be written as a possibly infinite sum of terms of the form

$$\gamma = \sum_{j=0}^{m} (\log r)^j \gamma_j$$

where the γ_j's are homogeneous sections of the same order λ, where λ is an indicial root of P. Moreover, the set $\mathcal{D}(P)$ of indicial roots of P is discrete without accumulation points in $\mathbf{R}$, and for any indicial root $\lambda \in \mathcal{D}(P)$, the vector space of sections $\gamma = \sum (\log r)^j \gamma_j$ that satisfy $P\gamma = 0$, where the γ_j's are homogeneous sections of of order λ, is finite dimensional.

We now introduce some terminology, which is not standard, but will be useful.

Definition 3.3.2. Let (B, g) be a non-compact Riemannian manifold. We say that (B, g) is an exact asymptotically conical manifold if there exists a compact set $K \subset B$, $R > 0$ and a compact Riemannian manifold Σ and a diffeomorphism $f : (R, \infty) \times \Sigma \to B \backslash K$ such that $f^* g = g_C$, where g_C is the conical metric on $C = (R, \infty) \times \Sigma$.

Let B be an exact AC manifold, and $f : (R, \infty) \times \Sigma \to B \backslash K$ the identification of the previous definition. If $E \to B$ is a vector bundle with metric and compatible connection, then as we have seen, we have seen that we can define canonically a translation operator on $f^* E \to C$, where C is the cone $C = (R, \infty) \times \Sigma$. Therefore, given a differential operator $P : \Gamma(E) \to \Gamma(F)$ between two vector bundles E, F, implicitly equipped with metrics and compatible connections, we will say that P is translation invariant if, for any section u of E over $B \backslash K \simeq C$, and for all $c \geq 0$, we have $P(\mathfrak{T}(c)^* u) = \mathfrak{T}(c)^* P u$, where $\mathfrak{T}(c)$ denotes the canonical translation operators on $f^* E \to C$ and $f^* F \to C$.

Example 3.3.2. The operators $r(d + d^*)$ and $r^2 \Delta$ defined on an exact AC manifold B are translation-invariant.

To study the mapping properties of the operators $d + d^*$ and $dd^* + d^* d$ on an exact AC manifold B, we need to define good norms. Let $f : (R, \infty) \times \Sigma \to B \backslash K$ be the identification of Definition 3.3.2, C the Riemannian cone $(R, \infty) \times \Sigma$, and (E, h, ∇) be a vector bundle with metric and compatible connection over B. We extend the coordinate function r defined on C to a function on B by setting $r \equiv R$ on K. Any choice of continuous positive function on K would do as well. For $p \geq 1$, $k \geq 0$, $\alpha \in (0, 1)$ and $\nu \in \mathbf{R}$, we define $L^p_{k,\nu}$ and $C^{k,\alpha}_\nu$ norms on $C^\infty_c(E)$ by:

$$\|u\|_{L^p_{k,\nu}} = \left(\sum_{j=0}^{k} \| r^{-\frac{n}{p} - \nu + j} \nabla^j u \|^p_{L^p} \right)^{\frac{1}{p}}$$

and

$$\|u\|_{C^{k,\alpha}_\nu} = \sum_{j=0}^{k} \|r^{-\nu+j}\nabla^j u\|_{C^0} + [r^{-\nu+k}\nabla^k S]_\alpha$$

Let us briefly discuss the powers of r appearing in these norms. Here, ν is to be thought of as a weight, that squeezes sections at infinity, so that the bigger ν is, the smaller the $L^p_{k,\nu}$ and $C^{k,\alpha}_\nu$ norms are. The other exponents are chosen so that the $L^p_{k,0}$ and $C^{k,\alpha}_0$ norms are translation-invariant in the following sense. Let $A_1 = \{a \le r \le c\}$ be an annulus and for $c > 0$, $A_c = \{ca \le r \le cb\}$. Then the translation operator takes any section u of E over A_c to the section $\mathfrak{T}(c)^*u$ of E over A_1. Since $\mathfrak{T}(c)$ is an isometry fiberwise and $\mathfrak{T}(c)^* \circ (r^l\nabla^l) = (r^j\nabla^j) \circ \mathfrak{T}(c)^*$, it is clear that the $C^{k,\alpha}_0$-norm satisfies

$$\|\mathfrak{T}(c)^*u\|_{C^{k,\alpha}_0} = \|u\|_{C^{k,\alpha}_0} \tag{3.7}$$

where the left hand side is the norm of a section over the annulus A_1 and the right hand side over A_c. This equality is also true if $L^p_{k,0}$ norm, and the factor $r^{-\frac{n}{p}}$ is chosen to cancel out with the factor $|\det\phi_c|$ in the integration. For $\nu \ne 0$, the factor r^ν adds an extra factor c^ν, so that we have the equalities:

$$\|\mathfrak{T}(c)^*u\|_{C^{k,\alpha}_\nu} = c^\nu\|u\|_{C^{k,\alpha}_\nu}, \quad \|\mathfrak{T}(c)^*u\|_{L^p_{k,\nu}} = c^\nu\|u\|_{L^p_{k,\nu}} \tag{3.8}$$

We define the weighted Sobolev space $L^p_{k,\nu}(E)$ as the closure of $C^\infty_c(E)$ for the $L^p_{k,\nu}$-norm, and the weighted Hölder space $C^{k,\alpha}_\nu(E)$ as the closure of $C^\infty_c(E)$ for the $C^{k,\alpha}_\nu$-norm.

As for compact manifolds, we have the following important embedding theorem:

Theorem 3.3.3 *Let B be an exact AC manifold of dimension n and (E, h, ∇) a vector bundle over B, with metric and compatible connection. Then the weighted Sobolev and Hölder spaces satisfy the following embedding properties:*

(i) If $k \ge l \ge 0$, $k - \frac{n}{p} \ge l - \frac{n}{q}$, $\nu \le \nu'$ and $p \le q$, then there is a continuous embedding $L^p_{k,\nu} \subset L^q_{l,\nu'}$. If moreover the first three inequalities are all strict, then this embedding is compact.

(ii) If $k \ge l \ge 0$, $k - \frac{n}{p} \ge l - \frac{n}{q}$, $\nu < \nu'$ and $p > q$, then there is a continuous embedding $L^p_{k,\nu} \subset L^q_{l,\nu'}$. If moreover the first two inequalities are strict, then this embedding is compact.

(iii) If $\nu < \nu'$ and $k - \frac{n}{p} \ge l + \alpha$, there are have continuous embeddings $L^p_{k,\nu} \subset C^{k,\alpha}_\nu \subset L^q_{l,\nu'}$ for any q.

(iv) If $\nu \le \nu'$ and $k + \alpha \ge l + \beta$, then there are continuous embeddings $C^{k+1}_\nu \subset C^{k,\alpha}_\nu \subset C^{l,\beta}_{\nu'} \subset C^l_{\nu'}$. If moreover $\nu < \nu'$, then the embedding $C^{k,\alpha}_\nu \subset C^l_{\nu'}$ is compact.

(v) If $\nu_1 + \nu_2 \leq \nu$, then the product $C^{k,\alpha}_{\nu_1} \times C^{k,\alpha}_{\nu_2} \to C^{k,\alpha}_{\nu}$ is continuous.

Remark 3.3.2. We we mostly use this theorem for the compact embedding $C^{k,\alpha}_{\nu} \hookrightarrow L^2_{\nu'}$ which follows from (iii) and (iv), where $k \geq 0$ is an integer, $\alpha \in (0,1)$ and $\nu < \nu'$ are real weights.

As we will see in the next two lemmas, we have estimates in $L^p_{k,\nu}$ and $C^{k,\alpha}_{\nu}$ norms over an exact AC manifold similar to those given in Theorem 3.2.5 for compact manifolds.

Lemma 3.3.4. *Let (B,g) be an exact AC manifold, with an end $B \backslash K \simeq C = (R,\infty) \times \Sigma$, and let E, F vector bundles over B with metrics and compatible connections. Let $P : C^{\infty}(E) \to C^{\infty}(F)$ be a differential operator of order k, such that the operator $r^k P$ is translation invariant. Let $p \geq 1$, $k \geq 0$, $\alpha \in (0,1)$ and $\nu \in \mathbf{R}$. Then there exists a constant $C > 0$ such that*

$$\|Pu\|_{C^{k,\alpha}_{\nu}} \leq C\|u\|_{C^{k+l,\alpha}_{\nu+k}}, \quad \|Pu\|_{L^p_{l,\nu}} \leq C\|u\|_{L^p_{k+l,\nu+k}}$$

for all $u \in C^{\infty}_c(E)$. In particular, P extends as a continuous operator $P : C^{k+l,\alpha}_{\nu+k}(E) \to C^{l,\alpha}_{\nu}(F)$ and $P : L^p_{k+l,\nu+k}(E) \to L^p_{l,\nu}(F)$.

Proof. Let u be a compactly supported section of E. We will make the proof for the $C^{k,\alpha}_{\nu}$-norm, the proof for $L^p_{k,\nu}$ is similar.

We can cover the compact set K by a finite number of open charts that trivialize E and F, and then using the estimates of Lemma 3.1.2, the inequality

$$\|Pu_{|U}\|_{C^{k,\alpha}_{\nu}} \leq C\|u_{|U}\|_{C^{k+l,\alpha}_{\nu+k}}$$

holds on some neighborhood U of K in B, for a constant C independent of u.

To have estimates on the conical end, we use the following scaling argument. By compactness, we can cover the annulus $A_0 = \{R \leq r \leq 2R\}$ by a finite number of open charts $U_1, ..., U_m$, and as above obtain the inequality

$$\|Pu\|_{C^{l,\alpha}} \leq C\|u\|_{C^{k+l,\alpha}}$$

on A_0.

Cover the cone C by the annuli $A_m = \{2^m R \leq r \leq 2^{m+1}R\}$ for all $m \geq 0$. We can use the translation operators to compare the $C^{l,\alpha}_{\nu}$-norms defined on the annuli A_m. Using (3.8) we have the equality:

$$\|\mathfrak{T}(2^m)^* Pu\|_{C^{l,\alpha}_{\nu}} = 2^{\nu m}\|Pu\|_{C^{l,\alpha}_{\nu}}$$

But since $r^k P$ is translation-invariant, we have $\mathfrak{T}(c)^* \circ P = c^{-k} P \circ \mathfrak{T}(c)^*$, so that we obtain

$$\|P(\mathfrak{T}(2^m)^* u)\|_{C^{l,\alpha}_{\nu}} = 2^{(k+\nu)m}\|Pu\|_{C^{l,\alpha}_{\nu}}$$

Over the annulus A_0, the $C^{k,\alpha}$, $C^{l,\alpha}_{\nu}$ and $C^{k,\alpha}_{\nu+k}$ norms are equivalent, so that an equality of the form

$$\|P(\mathfrak{T}(2^m)^* u)\|_{C^{l,\alpha}_{\nu}} \leq C\|\mathfrak{T}(2^m)^* u\|_{C^{k+l,\alpha}_{\nu+k}}$$

holds for some constant C independent of u and m. Combining all these inequalities and using again (3.8), we obtain

$$\|Pu\|_{C^{l,\alpha}_{\nu}} \leq 2^{-(k+\nu)m} C \|\mathfrak{T}(2^m)^* u\|_{C^{k+l,\alpha}_{\nu+k}} = C\|u\|_{C^{k+l,\alpha}_{\nu+k}}$$

for all $m \geq 0$, for a constant C independent of u and m. □

In a similar way, we have the following a priori estimates:

Lemma 3.3.5. *Let (B, g) be an exact AC manifold, with an end $B \backslash K \simeq C = (R, \infty) \times \Sigma$, and let E, F vector bundles over B with metrics and compatible connections. Let Let $P : C^\infty(E) \to C^\infty(F)$ be a an elliptic operator of order k, such that the operator $r^k P$ is translation-invariant. Let $p \geq 1$, $k \geq 0$, $\alpha \in (0, 1)$ and $\nu \in \mathbf{R}$. Then there exists a constant $C > 0$ such that the following estimates hold:*

$$\|u\|_{L^p_{k+l,\nu+k}} \leq C\Big(\|Pu\|_{L^p_{l,\nu}} + \|u\|_{L^p_{0,\nu+k}}\Big), \quad \|u\|_{C^{k+l,\alpha}_{\nu+k}} \leq C\Big(\|Pu\|_{C^{l,\alpha}_{\nu}} + \|u\|_{C^{0,\alpha}_{\nu+k}}\Big),$$

$$\|u\|_{C^{k+l,\alpha}_{\nu+k}} \leq C\Big(\|Pu\|_{C^{l,\alpha}_{\nu}} + \|u\|_{L^2_{\nu+k}}\Big)$$

The proof of these estimate is similar to the proof of the previous lemma, using a scaling argument on the annuli $\{2^m R \leq r \leq 2^{m+1} R\}$ to make the local estimates of Proposition 3.1.3 global.

Finally, we have a result analogous to Lemma 3.2.6 for exact AC manifolds:

Lemma 3.3.6. *Let B be an exact AC manifold and let $P : \Gamma(E) \to \Gamma(F)$ be a linear differentail operator of order k, such that $r^k P$ is translation-invariant at infinity. Let $\nu, \nu' \in \mathbf{R}$ such that $\nu + \nu' \leq -n + k$. Then, for all $u \in L^2_{k,\nu}(E)$ and $v \in L^2_{k,\nu'}(F)$, we have*

$$\langle Pu, v\rangle_{L^2} = \langle u, P^* v\rangle_{L^2}$$

Proof. The equality $\langle Pu, v\rangle_{L^2} = \langle u, P^* v\rangle_{L^2}$ holds y definition of P^* if u, v are compactly supported. Moreover, it is clear that by definition of the weighted Sobolev spaces, the bilinear form

$$L^2_\mu \times L^2_{\mu'} \to \mathbf{R}, \ (u, v) \mapsto \langle u, v\rangle_{L^2}$$

is well defined and continuous provided $\mu + \mu' \leq -n$. Using a density argument, and the fact that $P : L^2_{k,\nu}(E) \to L^2_{\nu-k}(F)$, $P^* : L^2_{\nu'}(F) \to L^2_{\nu'-k}(E)$ are continuous, the equality holds for $u \in L^2_{k,\nu}(E)$ and $v \in L^2_{k,\nu'}(F)$. □

Remark 3.3.3. By analogy with the compact case, we may expect, as a result of the above lemma, that the obstructions to solving $Pu = f$, where f is a section of rate $\nu - k$, and we look for a solution u with decay rate ν, lie in $\ker P^* \cap C^\infty_{-n-\nu+k}$. This is indeed what happens, at least when ν is not an indicial root of P. The rest of this part is dedicated to proving this result.

Let B be an exact AC manifold, the end of which is identified with the cone $(R,\infty)\times\Sigma$, and $P:\Gamma(E)\to\Gamma(F)$ be an elliptic operator of order k such that r^kP is translation-invariant at infinity. Then, if $u\in C^\infty(E)$ satisfies $Pu=0$, it can be written outside of a compact set as an L^2-sum of terms of the form $\sum_j(\log r)^j u_j$ where the u_j are homogeneous sections of E of rate $\lambda_j\in\mathcal{D}(P)$. If ν is not an indicial root of P, then u is of class $C^{l,\alpha}_\nu$ or $L^p_{l,\nu}$ if and only if $\lambda_j<\nu$ for all j. Moreover, if $\lambda<\nu<\nu'$ and $u=r^\lambda u_\lambda$ is a homogeneous section of E of rate λ over $(R,\infty)\times\Sigma$, then we can explicitly compute

$$\|u\|_{C^{l,\alpha}_\nu}=R^{\nu'-\nu}\|u\|_{C^{l,\alpha}_{\nu'}}$$

Therefore, if there are no indicial roots of P in $[\nu,\nu']$, then the $C^{k,\alpha}_\nu$ and $C^{k,\alpha}_{\nu'}$ norms are equivalent on $\ker P\cap C^{k,\alpha}_\nu(E)=\ker P\cap C^{k,\alpha}_{\nu'}(E)$. Since for operators on compact manifold, the norm of a section u is controlled by the norm of Pu and the norm of the projection of u onto the kernel, we would expect the same thing to be true for AC manifolds, with the norm of the kernel controlled with rate ν'. This is stated in the following lemma, which follows from Lockhart-McOwen theory:

Lemma 3.3.7. *Let B be an exact AC manifold, the end of which is identified with the cone $C=(R,\infty)\times\Sigma$, E,F vector bundles over B, and $P:\Gamma(E)\to\Gamma(F)$ be an elliptic operator of order k such that r^kP is translation-invariant on C. Let $\alpha\in(0,1)$ and $\nu<\nu'\in\mathbf{R}$ such that $[\nu,\nu']$ does not contain any indicial root of P. Then, for all $u\in C^\infty_c(E)$, we have the following estimate:*

$$\|u\|_{C^{k,\alpha}_\nu}\le C\left(\|Pu\|_{C^{0,\alpha}_{\nu-k}}+\|u\|_{C^{k,\alpha}_{\nu'}}\right)$$

Now we can state the result that will be of use to us, which is an immediate consequence of Lemmas 3.3.5 and 3.3.7:

Lemma 3.3.8. *Let B be an exact AC manifold, the end of which is identified with the cone $C=(R,\infty)\times\Sigma$, E,F vector bundles over B, and $P:\Gamma(E)\to\Gamma(F)$ be an elliptic operator of order k such that r^kP is translation-invariant on C. Let $\alpha\in(0,1)$ and fix $\nu<\nu'$ such that $[\nu,\nu']$ does not contain any indicial root of P. Then, there exists a constant $C>0$ such that, for all $u\in C^\infty_c(E)$, we have:*

$$\|u\|_{C^{k,\alpha}_\nu}\le C\left(\|Pu\|_{C^{0,\alpha}_{\nu-k}}+\|u\|_{L^2_{\nu'}}\right)$$

That lemma in particularly imply that if ν is not an indicial root of P, then $\ker P\cap C^{k,\alpha}_\nu(E)$ is finite dimensional (so that it is finite dimensional for all $\nu\in\mathbf{R}$. Indeed, let $\nu'>\nu$ such that $[\nu,\nu']$ does not contain any indicial root of P, and B be the unit ball of $\ker P$ in $C^{k,\alpha}_\nu(E)$. Then, using the compact embedding $C^{k,\alpha}_\nu\hookrightarrow L^2_{\nu'}$, the closure $\overline{B}$ of B in $L^2_{\nu'}$ is compact. Therefore, if $\{u_n\}$ is a sequence in B, it admits a subsequence that is Cauchy in $L^2_{\nu'}$-norm. By the estimates of the above lemma, this subsequence is Cauchy in $C^{k,\alpha}_\nu$-norm, and therefore converges in $C^{k,\alpha}_\nu(E)$. Thus B is compact.

In order to prove Fredholm alternative for elliptic operators on compact manifolds, an important fact was Proposition 3.2.7, that gave estimates on the L^2-orthogonal complement of the kernel of the operators. For AC manifolds, we need analogous estimates. We will mostly use $C^{k,\alpha}_\nu$-norms. Recall that for $\nu' > \nu$, we have a continuous (in fact, compact) embedding $C^{k,\alpha}_\nu(E) \hookrightarrow L^2_{\nu'}(E)$, and therefore we can equip $C^{k,\alpha}_\nu$ with a continuous (and non-canonical) inner product. The precise inner product we use does not matter, but most importantly, any finite dimensional subspace of $C^{k,\alpha}_\nu(E)$ has a closed complement. In particular, we can choose a closed complement of $\ker P$ in $C^{k,\alpha}_\nu(E)$.

Proposition 3.3.9. *Let B be an exact AC manifold and $P : \Gamma(E) \to \Gamma(F)$ be an elliptic operator of order k such that $r^k P$ is translation-invariant at infinity. Let $l \geq 0$ be an integer, $\alpha \in (0,1)$, and $\nu \in \mathbf{R} \backslash \mathcal{D}(P)$. Let H be any closed subspace of $C^{k+l,\alpha}_\nu(E)$, complement of $\ker P \cap C^{k+l,\alpha}_\nu(E)$. Then there exists a constant C such that, for all $u \in H$,*

$$\|u\|_{C^{k+l,\alpha}_\nu} \leq C \|Pu\|_{C^{l,\alpha}_{\nu-k}}$$

Proof. The proof goes by contradiction. Assume that there exists a sequence $\{u_n\}$ of unit vectors in H such that Pu goes to 0 in $C^{l,\alpha}_{\nu-k}$-norm. Let $\nu' > \nu$ such that P has no indicial root in $[\nu, \nu']$. We have a compact embedding $C^{k+l,\alpha}_\nu(E) \hookrightarrow L^2_{\nu'}(E)$, and therefore there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ that converges in $L^2_{\nu'}$-norm. Hence $\{u_{n_j}\}$ is Cauchy in $L^2_{\nu'}(E)$. Therefore, using the estimate of Lemma 3.3.8, $\{u_{n_j}\}$ is a Cauchy sequence in $C^{k+l,\alpha}_\nu(E)$, and therefore it converges to an element $u \in C^{k+l,\alpha}_\nu$, and by continuity of P, we have $Pu = 0$. But $\{u_{n_j}\}$ is a sequence in H which is closed, such that u is a unit vector in H, which gives a contradiction. □

In the next proposition, we identify the image of P acting on weighted Hölder spaces:

Proposition 3.3.10. *Let $P : C^\infty(E) \to C^\infty(F)$ be an elliptic operator of order k over an exact AC manifold B, such that $r^k P$ is translation invariant at infinity. Let $l \geq 0$ be an integer, $\alpha \in (0,1)$ and $\nu \in \mathbf{R}$ such that ν is not an indicial root of P. We fix a closed complement H of $\ker P$ in $C^{k+l,\alpha}_\nu$. Then for $f \in C^{l,\alpha}_{\nu-k}$, the equation $Pu = f$ has a solution $u \in C^{k+l,\alpha}_\nu(E)$ if and only if*

$$\langle f, v \rangle_{L^2} = 0$$

for every $v \in \ker P^* \cap C^\infty_{-n-\nu+k}(F)$. Moreover, if this condition is satisfied, the solution is unique if we require $u \in H$, and we have the estimate:

$$\|u\|_{C^{k+l,\alpha}_\nu} \leq C \|f\|_{C^{l,\alpha}_{\nu-k}}.$$

Proof. Since ν is not an indicial root of P, then we can choose $\nu' > \nu$ such that P has no root in $[\nu, \nu']$. Then P^* has no root in $[-n-\nu'+k, -n-\nu+k]$, so that $\ker P^* \cap C^\infty_{-n-\nu+k}(F) = \ker P^* \cap C^\infty_{-n-\nu'+k}(F) = \ker P^* \cap L^2_{-n-\nu'-k}(F)$. If $u \in C^{k+l,\alpha}_\nu$ and $v \in \ker P^* \cap C^\infty_{-n-\nu+k}(F)$, we can therefore integrate by parts:

$$\langle Pu, v\rangle_{L^2} = \langle u, P^*v\rangle_{L^2} = 0$$

On the other hand, if $f \in C^{l,\alpha}_{\nu-k}(F)$ is L^2-orthogonal to $\ker P^* \cap C^\infty_{-n-\nu+k}(F)$, then using the same argument as in the proof of Fredholm alternative for compact manifolds (Corollary 3.2.8), we can write the continuous linear form

$$v \in L^2_{k,-n-\nu'+k} \mapsto \langle f, v\rangle_{L^2} \in \mathbf{R}$$

in the form $\langle f, v\rangle_{L^2} = \langle u, P^*v\rangle_{L^2}$ for some $u \in L^2_{\nu'}$. Hence by the estimates of Lemma 3.3.8, $u \in C^{k+l,\alpha}_\nu$ is a solution of $Pu = f$. The rest of the proposition immediately follows from Proposition 3.3.9. □

As a consequence of all the above results, if $P : \Gamma(E) \to \Gamma(F)$ is an elliptic operator of order k over an exact AC manifold, such that $r^k P$ is translation-invariant at infinity, then $P : C^{k+l,\alpha}_\nu(E) \to C^{l,\alpha}_\nu(F)$ and $P : L^p_{k+l,\nu}(E) \to L^p_{l,\nu}(F)$ are Fredholm for all $\nu \in \mathcal{D}(P)$, and the image of P is given by the sections orthogonal to $\ker P^* \cap C^\infty_{-n-\nu+k}(F)$. The crucial point to obtain the Fredholm property in the AC setting was Lemma 3.3.7, and the rest followed by abstract considerations of Banach space theory and compact embeddings.

In the next part, we will consider the case of an asymptotically conical but not *exact* manifold B, that is, the geometry of B approaches the geometry of a cone at infinity, up to a decaying error term. We want to adapt the Fredholm theory for elliptic operators, such as $d + d^*$ and the Laplacian $dd^* + d^*d$, in this setting. Obtaining a priori estimates is a straightforward adaptation of the estimates obtained in this part, that follow from scaling arguments on the end of B. The only difficulty will be to control the asymptotic behavior of sections in the kernel of the operators we consider, in order to prove that Lemma 3.3.7 still holds in this setting. When this is proven, then we can conclude that the operators on an AC manifold have the same Fredholm properties as the operators on exact AC manifolds.

Before turning to (non-exact) AC manifold, we want to note that by Lokhart-McOwen theory, the operators $P : C^{k+l,\alpha}_\nu(E) \to C^{l,\alpha}_\nu(F)$ and $P : L^p_{k+l,\nu}(E) \to L^p_{l,\nu}(F)$ are never Fredholm when $\nu \in \mathcal{D}(P)$, so that it is hopeless to study the mapping properties of P at an indicial root. But in some situations, we can say something about the jump of the kernel of the operator P when we cross an indicial root, which is often sufficient.

3.3.2 Asymptotically conical manifolds

We will now define *asymptotically conical* (AC) manifolds. These are manifolds which geometry approaches the geometry of a cone at infinity, up to correction terms that have a prescribed decay at infinity. In order to understand the Fredholm properties of the operators $d + d^*$ and the Laplacian

$dd^* + d^*d$ on AC manifolds, we will adapt the results obtained for exact AC manifolds.

Definition 3.3.3. Let (B^n, g) be a connected, non-compact complete Riemannian manifold. We say that B is asymptotically conical of rate $\mu < 0$ if there exists a compact set $K \subset B$, $R > 0$, a Riemannian cone $(C(\Sigma), g_C = dr^2 + r^2 g_\Sigma)$ over a compact connected Riemannian manifold (Σ^{n-1}, g_Σ), and a diffeomorphism $f : (R, \infty) \times \Sigma \to B\backslash K$, such that:

$$|\nabla^j_{g_C}(f^*g - g_C)|_{g_C} = O(r^{\mu - j})$$

for all $j \leq 0$.

We will be interested in vector bundles and operators which behavior is asymptotically the same as translation-invariant objects over a cone.

Definition 3.3.4. Let B be an AC manifold of rate $\mu < 0$ and $f : (R, \infty) \times \Sigma \to M\backslash K$ be an identification of the end of B as in Definition 3.3.3. Let (E, h, ∇) be a vector bundle over B with metric and compatible connection. We say that E is *admissible* if $f^*h = h_\infty + h'$ and $f^*\nabla = \nabla_\infty + a$, where $(h_\infty, \nabla_\infty)$ are a metric and compatible connection on the bundle $f^*E \to (R, \infty) \times \Sigma$ that are pulled back from Σ and h', a satisfy the following decay conditions:

$$|\nabla^j_\infty h'|_{g_C \otimes h_\infty} = O(r^\mu - j), \quad |\nabla^j_\infty a|_{g_C \otimes h_\infty} = O(r^{\mu - 1 - j})$$

for all $j \geq 0$.

Remark 3.3.4. Strictly speaking, we saw at the beginning of §3.3.1 that any vector bundle (E, h, ∇) with metric h and compatible connection ∇ over a cone $C(\Sigma)$ is pulled back from Σ. What is implicit in Definition 3.3.4 is that we choose a particular metric h_∞ and compatible connection ∇_∞ over the bundle $f^*E \to (R, \infty) \times \Sigma$ that we want to think of as the asymptotic limit of (h, ∇), so that the differences $h - h_\infty$ and $\nabla - \nabla_\infty$ decay fast enough. Hence, when we say that E is an admissible vector bundle, we make a choice of $(h_\infty, \nabla_\infty)$, and the notion of translation on $B\backslash K$ refers to translations by parallel transport for our particular choice of connection ∇_∞. In the same way, when we say that B is an AC Riemannian manifold, we make an implicit choice of diffeomorphism $f : (R, \infty) \times \Sigma \to B\backslash K$.

Example 3.3.3. Let (B, g) an AC manifold of rate $\mu < 0$ and $f : (R, \infty) \times \Sigma \to B\backslash K$ be the identification of Definition 3.3.3. Consider the vector bundle TB, endowed with the metric g and the Levi-Civita connection ∇. We choose the asymptotic metric to be g_C over $f^*TB \simeq TC$ and $\nabla_\infty = \nabla_{g_C}$ to be the Levi-Civita connection of g_C. By definition of an AC manifold, the condition $|\nabla^j_{g_C}(f^*g - g_C)|_{g_C} = O(r^{\mu - j})$ is satisfied. The 1-form $a = \nabla - \nabla_{g_C}$ depends in coordinates on the first derivatives of the difference $g - g_C$, so that the condition $|\nabla^j_{g_C} a|_{g_C} = O(r^{\mu - 1 - j})$ is also satisfied. Hence, TB is an admissible vector bundle. The notion of admissibility for vector bundles is compatible with algebraic operations, like direct sum or tensor product, so that more generally any subbundle of $TB^{\otimes s} \otimes T^*B^{\otimes r}$ is admissible.

Let B be an AC manifold of rate $\mu < 0$, and E be an admissible vector bundle over B. As in §3.3.1, we can define the weighted Sobolev and Hölder spaces of sections of E in the following way. Let $p \geq 1$, $k \geq 0$, $\alpha \in (0,1)$ and $\nu \in \mathbf{R}$. The $L^p_{k,\nu}$ norm on $C^\infty_c(E)$ is defined by:

$$\|u\|_{L^p_{k,\nu}} = \left(\sum_{j=0}^{k} \|r^{-\frac{n}{p}-\nu+j} \nabla^j u\|^p_{L^p} \right)^{\frac{1}{p}}$$

The weighted Sobolev space $L^p_{k,\nu}(E)$ is the closure of $C^\infty_c(E)$ for the $L^p_{k,\nu}$-norm. We also define $C^k_\nu(E)$ and the weighted Hölder space $C^{k,\alpha}_\nu(E)$ as the closure of $C^\infty_c(E)$ for the following norms:

$$\|u\|_{C^k_\nu} = \sum_{j=0}^{k} \|r^{-\nu+j}\nabla^j u\|_{C^0}, \qquad \|u\|_{C^{k,\alpha}_\nu} = \|u\|_{C^k_\nu} + [r^{-\nu+k}\nabla^k S]_\alpha$$

Finally, define $C^\infty_\nu(E)$ as the intersection of the $C^{k,\alpha}_\nu(E)$'s for all $k \geq 0$.

The embedding properties for weighted Sobolev and Hölder spaces over an AC manifolds are the same as the ones stated in Theorem 3.3.3 for exact AC manifolds. Recall that in particular we have a compact embedding $C^{k,\alpha}_\nu \subset L^2_{\nu'}$ whenever $\nu < \nu'$.

Lastly, we define admissible operators, which are elliptic operators that are almost translation invariant at infinity, up to a decaying error term.

Definition 3.3.5. Let B be an AC manifold, and E, F admissible vector bundles over B. Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic operator of order k. Let $f : (R,\infty) \times \Sigma \to B \backslash K$ be the identification of Definition 3.3.3. We say that P is an *admissible operator* if there exists and elliptic operator $P_\infty : C^\infty(f^*E) \to C^\infty(f^*F)$ of order k, such that $r^k P_\infty$ is translation invariant, and such that the following condition holds:

$$|\nabla^l_\infty (f^*(Pu) - P_\infty(f^*u))|_{g_C \otimes h_\infty} = O(r^{-k+\mu-l})$$

for all smooth sections u of E over $B\backslash K$.

Remark 3.3.5. From the local expression of the adjoint, it is clear that the adjoint P^* of an admissible operator P is admissible.

Example 3.3.4. The operators $d + d^*$ and $dd^* + d^*d$ on an AC manifold are admissible.

The a priori estimates for admissible operators over AC manifolds are the same as in §3.3.1, since we control the decay of $(P - P_\infty)u$ at infinity. Here is the statement:

Proposition 3.3.11. *Let B be an AC manifold, ad E, F admissible vector bundles over B. Let $P : C^\infty(E) \to C^\infty(F)$ be an admissible operator of order k. Then for every $l \geq 0$, $p \geq 1$, $\alpha \in (0,1)$ and $\nu \in \mathbf{R}$, P extends to a* *continuous operator* $P : L^p_{k+l,\nu+k}(E) \to L^p_{l,\nu}(F)$ *and* $P : C^{k+l,\alpha}_{\nu+k}(E) \to C^{l,\alpha}_{\nu}(F)$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|u\|_{L^p_{k+l,\nu+k}} \leq C\Big(\|Pu\|_{L^p_{l,\nu}} + \|u\|_{L^p_{0,\nu+k}}\Big), \quad \|u\|_{C^{k+l,\alpha}_{\nu+k}} \leq C\Big(\|Pu\|_{C^{l,\alpha}_{\nu}} + \|u\|_{C^{0,\alpha}_{\nu+k}}\Big),$$

$$\|u\|_{C^{k+l,\alpha}_{\nu+k}} \leq C\Big(\|Pu\|_{C^{l,\alpha}_{\nu}} + \|u\|_{L^2_{\nu+k}}\Big)$$

for all $u \in C^\infty_c(E)$.

Moreover, the fact that $\langle Pu, v\rangle_{L^2} = \langle u, P^* v\rangle_{L^2}$ if $u \in L^2_{k,\nu}$ and $v \in L^2_{k,\nu'}$ whenever $\nu + \nu' \leq -n + k$ remains true in the AC setting, the proof being essentially the same as for Lemma 3.3.6.

If ν is not an indicial root of P_∞, we will denote $K_\nu(P_\infty)$ the vector space of solutions to $P_\infty u = 0$ on the cone, with decay rate $< \nu$ at infinity. The following result, proven by Lockhart and Mc Owen in [25] and stated in the form of [11, Proposition B.12], is crucial to control the asymptotic decay of the solutions to the equation $Pu = f$. Similar statements can be found in [19, Proposition 4.21] and [18, Proposition 4.27].

Proposition 3.3.12. *Let* B *be an AC manifold of rate* $\mu < 0$*. Let* $P : C^\infty(E) \to C^\infty(F)$ *be an admissible operator of order* k *and* $p \geq 1$, $\alpha \in (0,1)$ *and* $\nu < \nu'$ *such that* $\nu, \nu' \notin \mathcal{D}(P_\infty)$ *and* $\nu' - \nu < |\mu|$*. Let* $u_1, ..., u_N$, $N = N(\nu, \nu')$*, be a basis of* $K_{\nu'}(P_\infty)/K_\nu(P_\infty)$.

Then there exists a compact set $K \subset B$ *such that for every* $f \in C^{0,\alpha}_{\nu-k}$ *with* $f = Du'$ *for some* $u' \in C^{k,\alpha}_{\nu'}$*, there exists* $(a_1, ..., a_N) \in \mathbf{R}^N$ *and* $u \in C^{k,\alpha}_\nu(K \backslash K)$ *such that* $u'_{|B\backslash K} = u + \sum a_j u_j$*. Moreover, there exists a constant* $C > 0$ *independent of* f, u, u', a *such that*

$$\|u\|_{C^{k,\alpha}_\nu(B\backslash K)} + \|a\| \leq C\Big(\|f\|_{C^{0,\alpha}_\nu} + \|u'\|_{C^{k,\alpha}_{\nu'}}\Big).$$

In particular, this propositions implies that the estimate of Lemma 3.3.7

$$\|u\|_{C^{k,\alpha}_\nu} \leq C\left(\|Pu\|_{C^{0,\alpha}_{\nu-k}} + \|u\|_{C^{k,\alpha}_{\nu'}}\right)$$

hold whenever $\nu < \nu'$ are close enough and there are no indicial roots in $[\nu, \nu']$. Hence the Fredholm theory of admissible operators is the same as the Fredholm theory of the asymptotic operator P_∞, as we derived in §3.3.1. In particular, if $P : \Gamma(E) \to \Gamma(F)$ is an admissible operator of order k, asymptotic to P_∞, and ν is not an indicial root of P_∞, then $P : C^{k+l,\alpha}_\nu(E) \to C^{l,\alpha}_{\nu-k}$ is Fredholm. Moreover, the image of P is the subspace of $C^{l,\alpha}_{\nu-k}$ which is L^2-orthogonal to $\ker P^* \cap C^\infty_{-n-\nu+k}(F)$. This space is therefore to be seen as a space of obstructions to solving the equation $Pu = f$, for $f \in C^{l,\alpha}_{\nu-k}$, with variable $u \in C^{k+l,\alpha}_\nu$. On the other hand, if f is in the image of P and we fix a closed complement H of $\ker P$ in $C^{k+l,\alpha}$, the equation $Pu = f$ has a unique solution in H, and we have the estimate

$$\|u\|_{C^{k+l,\alpha}_\nu} \leq C\|f\|_{C^{l,\alpha}_{\nu-k}}$$

for a constant C independent of f.

3.3.3 L^2-cohomology

Let B be an AC manifold. We can think of B as the interior of a compact manifold $\overline{B}$ with boundary Σ. The real cohomology of B can be computed via the de Rham complex $(\Omega^*(B), d)$:

$$0 \longrightarrow C^\infty(B) \xrightarrow{d} \Omega^1(B) \xrightarrow{d} \Omega^2(B) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(B) \longrightarrow 0$$

On the other hand, the (real) relative cohomology $H^*(\overline{B}, \Sigma)$ can be computed by the complex of compactly supported differential forms

$$0 \longrightarrow C^\infty_c(B) \xrightarrow{d} \Omega^1_c(B) \xrightarrow{d} \Omega^2_c(B) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n_c(B) \longrightarrow 0$$

and thus we will write $H^*_c(B) = H^*(\overline{B}, \Sigma)$. Therefore, the long exact sequence of the pair (B, Σ) can be written

$$\ldots \to H^{k-1}(\Sigma) \to H^k_c(B) \to H^k(B) \to H^k(\Sigma) \to \ldots$$

If $\nu \in \mathbf{R}$, we define $\Omega^k_\nu(B) = C^\infty_\nu \Omega^k(B)$. It is clear that for any $\nu \in \mathbf{R}$ we have a complex $(\Omega^*_{\nu-*}, d)$:

$$0 \longrightarrow C^\infty_\nu(B) \xrightarrow{d} \Omega^1_{\nu-1}(B) \xrightarrow{d} \Omega^2_{\nu-2}(B) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n_{\nu-n}(B) \longrightarrow 0$$

which gives cohomology groups $H^*_{\nu-*}(B)$, where we define

$$H^k_\nu(B) = \frac{\ker\left(d : \Omega^k_\nu(B) \to \Omega^{k+1}_{\nu-1}(B)\right)}{\operatorname{im}\left(d : \Omega^{k-1}_{\nu+1}(B) \to \Omega^k_\nu(B)\right)}$$

We would like to relate these cohomology groups to the more usual cohomology groups $H^*(B)$ and $H^*_c(B)$. One may expect that the complex $(\Omega^*_{\nu-*}(B), d)$ for $\nu < 0$ computes the compactly supported cohomology, and computes the total cohomology for $\nu > 0$. We will show that this is true at least for $\nu < 0$.

To see this, the short exact sequence of complexes

$$0 \to \Omega^*_c \to \Omega^*_{\nu-*} \to \Omega^*_{\nu-*}/\Omega^*_c \to 0$$

yields the usual long exact sequence in cohomology, and to show that the natural map $H^*_c(B) \to H^*_{\nu-*}(B)$ are isomorphisms is equivalent to show that the relative complex has no cohomology. More explicitly, we want to show that if $\sigma \in \Omega^k_{\nu-k}$ satisfies $d\sigma \in \Omega^k_c$, then there exists $\eta \in \Omega^{k-1}_{\nu-k+1}$ such that $\sigma - d\eta$ is compactly supported. This is the aim of the following lemma, of which a more detailed proof can be found in [18, Lemma 2.11].

Lemma 3.3.13. *Let B be an AC Riemannian manifold, asymptotic to the cone $C(\Sigma) = (R, \infty) \times \Sigma$. Let $\sigma \in \Omega^k(B)$ such that at infinity, we have*

$$|\sigma| = O(r^{\nu-k})$$

for some $\nu < 0$, and $d\sigma = 0$ on $C(\Sigma)$. Then there exists a $k-1$ form on $C(\Sigma)$ such that $\sigma = d\gamma$ and at infinity,

$$|\gamma| = O(r^{\nu-k+1})$$

Moreover, if $\sigma \in C^l_{\nu-k}$ for some $l \geq 1$ then $\gamma \in C^l_{\nu-k+1}$.

Proof. The k-form σ can be written on $C(\Sigma)$ as

$$\sigma = dr \wedge \alpha(r) + \beta(r)$$

where $\alpha(r) \in \Omega^{k-1}(\Sigma)$, $\beta(r) \in \Omega^k(\Sigma)$ satisfy

$$|\alpha| = O(r^{\nu-1}), \quad |\beta| = O(r^{r^\nu})$$

Moreover $d_B\eta = dr \wedge (\beta'(r) - d\alpha(r)) + d\beta(r) = 0$ gives $\beta'(r) = d\alpha(r)$ and $d\beta(r) = 0$. Since $\nu - 1 < -1$, $\alpha(r)$ is integrable, and thus we can define a $k-1$-form γ on $C(\Sigma)$ by radial integration:

$$\gamma(r) = \int_\infty^r \alpha(u)du, \quad r \geq R$$

Since $d\alpha$ decays even faster than α, $d\alpha(r)$ is also integrable, and thus we have

$$\begin{aligned} d\gamma(r) &= dr \wedge \alpha(r) + \int_\infty^r d\alpha(u)du \\ &= dr \wedge \alpha(r) + \int_\infty^r \beta'(u)du \\ &= dr \wedge \alpha(r) + \beta(r) = \sigma(r) \end{aligned}$$

The asymptotic behavior $|\gamma| = O(r^{\nu-k+1})$ is clear by the formula, as well as $\gamma \in C^l_{\nu-k+1}$ when $\sigma \in C^l_{\nu-k}$. □

As a consequence of the lemma, if $\sigma \in \Omega^k_{\nu-k}(B)$ and $d\sigma$ is compactly supported, then $\sigma = d\gamma$ at infinity, with $\gamma \in C^\infty_{\nu-k+1}$. If χ is a bump function equal to 1 at infinity, and we define $\eta = \chi\gamma$, then $\eta \in \Omega^k_{\nu-k+1}(B)$ and $\sigma - d\eta$ is compactly supported. Thus we obtain the following corollary:

Corollary 3.3.14. *Let B^n be an AC manifold. Then for all $0 \leq k \leq n$ and $\nu < -k$, we have a natural isomorphism*

$$H^k_c(B) \simeq \frac{\ker\left(d : \Omega^k_\nu(B) \to \Omega^{k+1}_{\nu-1}(B)\right)}{\operatorname{im}\left(d : \Omega^{k-1}_{\nu+1}(B) \to \Omega^k_\nu(B)\right)}$$

As for compact manifolds, we would like to identify the kernel of the operator $d + d^*$ with cohomology classes. To this end, let us define

$$\mathcal{H}^k_\nu = \ker(d + d^*) \cap \Omega^k_\nu(B), \quad L^2\mathcal{H}^k = \{\eta \in \Omega^k(B) \cap L^2,\ d\eta = 0 = d^*\eta\}$$

Since the set of indicial roots of $d+d^*$ is discrete, we have $L^2\mathcal{H}^k = \mathcal{H}^k_{-\frac{n}{2}-\delta}$ for all sufficiently small $\delta > 0$, and the $d + d^* : \Omega_{-\frac{n}{2}+1+\delta} \to \Omega_{-\frac{n}{2}+\delta}$ is Fredholm. This observation yields the following proposition:

Proposition 3.3.15. *Let B^n be an AC manifold and $k < \frac{n}{2}$. Then we have a natural isomorphism $L^2\mathcal{H}^k \simeq H^k_c(B)$.*

Proof. For sufficiently small $\delta > 0$, we have a natural isomorphism

$$H^k_c(B) \simeq \frac{\ker\left(d : \Omega^k_{-\frac{n}{2}+\delta}(B) \to \Omega^{k+1}_{-\frac{n}{2}-1+\delta}(B)\right)}{\operatorname{im}\left(d : \Omega^{k-1}_{-\frac{n}{2}+1+\delta}(B) \to \Omega^k_{\frac{n}{2}+\delta}(B)\right)} = H^k_{-\frac{n}{2}+\delta}(B)$$

Moreover, since $d + d^*$ is Fredholm, we have an L^2-orthogonal decomposition

$$\Omega^k_{-\frac{n}{2}+\delta}(B) = \left(d\Omega^{k-1}_{-\frac{n}{2}+1+\delta}(B) + d^*\Omega^{k+1}_{-\frac{n}{2}+1+\delta}(B)\right) \oplus L^2\mathcal{H}^k$$

Now consider the map that sends $\sigma \in L^2\mathcal{H}^k$ to its class in $H^k_{-\frac{n}{2}+\delta}(B)$. We want to show that this is an isomorphism. It is injective because the above decomposition is L^2-orthogonal. For surjectivity, let $\eta = d\alpha+d^*\beta+\gamma$ according to the above decomposition and assume $d\eta = 0$. Then in particular $dd^*\beta = 0$, and thus $d^*\beta$ is a closed and co-closed form in $C^\infty_{-\frac{n}{2}+\delta}$. Since $k < \frac{n}{2}$, $-\frac{n}{2}$ is not an indicial root of $d + d^*$, and therefore for δ sufficiently small, $d^*\beta$ is a L^2 closed and co-closed form, so that $d^*\beta = 0$ by orthogonality. Thus $\eta - d\alpha = \gamma \in L^2\mathcal{H}^k$, which completes the proof. □

By Poincaré duality, the Hodge star operator induces natural isomorphisms $H^k_c(B) \simeq H^{n-k}(B)$ and $L^2\mathcal{H}^k \simeq L^2\mathcal{H}^{n-k}$. Hence, as an immediate corollary of the previous proposition, we obtain:

Corollary 3.3.16. *Let B^n be an AC manifold and $k > \frac{n}{2}$. Then we have a natural isomorphism $L^2\mathcal{H}^k \simeq H^k(B)$.*

For the critical case $k = \frac{n}{2}$, we have $L^2\mathcal{H}^k \simeq \mathcal{H}^k_{-\frac{n}{2}-\delta}$ for some small $\delta > 0$. In the same way as above, we can then find a canonical primitive at infinity $\gamma(r)$ for any $\sigma \in L^2\mathcal{H}^k$. Therefore, the cohomology class $[\sigma] \in H^k(R)$ is in the cohomology class of the compactly supported k-form $\sigma - d(\chi(r)\gamma(r))$. From Lockhart-McOwen theory [24], this map $L^2\mathcal{H}^k \to \operatorname{im}(H^k_c(B) \to H^k(B))$ is an isomorphism, but we will not use this fact. Hence we have the following theorem:

Theorem 3.3.17 *We have natural isomorphisms:*

$$L^2\mathcal{H}^k \simeq \begin{cases} H^k_c(B) & \text{if } k < \dfrac{n}{2} \\ \operatorname{im}(H^k_c(B) \to H^k(B)) & \text{if } k = \dfrac{n}{2} \\ H^k(B) & \text{if } k > \dfrac{n}{2} \end{cases}$$

3.4 AC Calabi-Yau manifolds

In this part, we apply the analytical results on AC Riemannian manifolds that we derived above in the particular case where the AC metric is Calabi-Yau. An AC Calabi-Yau manifold is asymptotic to a Calabi-Yau cone $C(\Sigma)$, where Σ is a compact Riemannian manifold. The requirement that the cone carries a Calabi-Yau structure induces a special type of structure on Σ, called Sasaki-Einstein [28]. We will not say much about Sasaki-Einstein manifolds, but the important fact is that it is possible to give a lower bound for the first positive eigenvalue of the Laplacian on Σ, which in turn gives precious informations on the indicial roots of the operators $d + d^*$ and Δ, and other operators derived from those [11, §4]. At the end of this part, we also want to say a word about what happens when we vary the $SU(3)$-structure near a torsion-free one, and in particular focus on the regularity of all the objects naturally associated to an $SU(3)$-structure.

3.4.1 Calabi-Yau cones

Definition 3.4.1. Let B be a non-compact complete Calabi-Yau manifold, with Calabi-Yau structure (ω, Ω). We say that B is an AC Calabi-Yau manifold of rate $\mu < 0$ if there exists a compact set $K \subset B$ and a diffeomorphism $f : (R, \infty) \times \Sigma \to B \backslash K$, where (Σ, g) is a connected, compact Riemannian manifold, and the cone $C(\Sigma) = (R, \infty) \times \Sigma$ admits a Calabi-Yau structure (ω_C, Ω_C), such that

$$|\nabla_C^j (f^*\omega - \omega_C)|_{g_C} = O(r^{\mu - j}), \quad |\nabla_C (f^*\Omega - \Omega_C)|_{g_C} = O(r^{\mu - j})$$

for all $j \geq 0$. In particular, an AC Calabi-Yau manifold is an AC Riemannian manifold.

If (Σ, g_Σ) is a compact Riemannian manifold, it is called Sasakian if the conical metric $g_C = dr^2 + r^2 g_\Sigma$ on $C(\Sigma)$ is Kähler. It is easy to compute that the Ricci tensor of a conical metric is given by $\mathrm{Ric}_C = \mathrm{Ric}_\Sigma - 2(n-1)g_\Sigma$, where $2n$ is the dimension of the cone g_Σ. Since the restricted holonomy of a Kähler metric is contained in $SU(n)$ if and only if the metric is Ricci flat, it follows that the conical metric on $C(\Sigma)$ has restricted holonomy in $SU(n)$ if and only if $\mathrm{Ric}_\Sigma = 2(n-1)g_\Sigma$. Such a metric is called *Sasaki-Einstein*. Here we are interested in the case $n = 3$, that is, Σ has dimension 5.

In order to understand the kernel of the operators $d + d^*$ and $dd^* + d^*d$ on an AC Calabi-Yau manifold modeled on $C(\Sigma)$, we need some results about the eigenvalues of the Laplacian on Σ. We will describe the results, referring to [28] for results about Sasaki-Einstein manifolds, and to the article [11] for proofs of the result claimed.

In general, a Sasaki-Einstein manifold has a Killing field, called the *Reeb vector field*, that can be defined as $\xi = J(r\partial_r)$, where J is the complex structure on the cone $C(\Sigma)$. It reduces to a unit Killing field on Σ, which gives a locally free $\mathbf{R}$-action. If the Reeb vector field has a non-compact orbit, the Sasaki-Einstein manifold Σ is called *irregular*. If all the orbits of the Reeb flow are compact, then it defines a $U(1)$-action. Σ is called *regular* if this action is free, and *quasi-regular* otherwise. An example of regular Sasaki-Einstein manifold is the round 5-sphere S^5, for the euclidian metric on $C(S^5) = \mathbf{R}^6 \backslash \{0\}$. Actually, we will need to exclude this example, and assume that the universal cover of Σ is not isometric to S^5. This assumption is necessary to get the estimates of Proposition 3.4.1, for reasons explained in [11, §4.1.1].

An important thing to note is that every compact Sasaki-Einstein manifold has $H^1(\Sigma) = 0$. Indeed, it satisfies $\mathrm{Ric}_\Sigma = 2(n-1)g_\Sigma$, and so does its universal cover. In particular, in virtue of Myer's theorem, the universal cover of Σ is compact. Therefore, Σ has finite fundamental group, which implies $H^1(\Sigma) = 0$.

The following proposition gives the necessary results to control the eigenvalues of the Laplacian on a Sasaki-Einstein manifold [11, Proposition 4.9]:

Proposition 3.4.1. *Let Σ be a Sasaki-Einstein manifold and assume its universal cover is not isometric to the round* 5*-sphere.*

(i) The first eigenvalue of the scalar Laplacian is strictly greater than 5*.*

(ii) The first eigenvalue of the Laplacian acting on co-closed 1*-forms is greater or equal to* 8*, and the eigenspace with eigenvalue* 8 *consist in* 1*-forms dual to Killing vector fields.*

(iii) If Σ is regular then the first eigenvalue of the Laplacian acting on co-closed 2*-form is strictly greater than* 4*.*

By our discussion in §3.3.1, it has the following consequences. The kernel of $d+d^*$ acting on 1-forms on an AC manifold asymptotic to $C(\Sigma)$ is controlled by the spectrum of the scalar Laplacian on Σ. Thus -5 is an indicial root of $d+d^*$, corresponding to closed and co-closed 1-forms of the form αdr for some constant α. Since Σ satisfies $H^1(\Sigma) = 0$, then -1 is not an indicial root. Finally, since the first non-zero eigenvalue of the scalar Laplacian is strictly greater than 5, $d + d^*$ has no indicial root in $[-6, 0]\backslash\{-5\}$.

On 2-forms, the indicial roots are controlled by the spectrum of the Laplacian of Σ acting on co-closed 1-forms. Since $H^1(\Sigma) = 0$, then -4 is not an indicial root of $d+d^*$. If $H^2(\Sigma)$ is non-zero, then -2 is an indicial root, corresponding to differential forms $\tau \in \mathcal{H}^2(\Sigma)$ since the factor r^2 precisely cancels the scaling factor r^{-2} in the case. Moreover, since the first non-zero eigenvalue is at least 8, it follows that $d + d^*$ has no-indicial roots in $(-6, 0)\backslash\{-2\}$.

For 3-forms, it follows from the above proposition that, if we assume Σ to be regular, $d + d^*$ has no indicial roots in $[-5, -1]\backslash\{-3\}$, and we have a precise description of the closed and co-closed homogeneous forms of rate -3.

In the same way, we can use the description of harmonic forms on a cone in §3.3.1 and the lower bounds of Proposition 3.4.1 to obtain some information about the indicial roots of the Laplacian. We refer to the article [11, Propositions 4.12, 4.13] for a more precise description, and we just want to state the results that we will use later on. In particular, 0 is an indicial root of the Laplacian acting on the cone, corresponding to constant functions, and -4 is an indicial root corresponding to harmonic forms Kr^{-4}, and there are no indicial roots in $[-5,1]\backslash\{-4,0\}$. For 1-forms, there are in particular no harmonic 1-forms in $[-4,0]$.

3.4.2 Some analytical facts

In this part we collect some key analytical parts on AC Calabi-Yau manifolds. Throughout this part, we assume that B is an AC Calabi-Yau manifold asymptotic to a cone $C(\Sigma)$, where Σ is a Sasaki-Einstein manifold whose universal cover is not isometric to the round 5-sphere. In particular, it implies that B has finite fundamental group [11, Proposition 5.10] and is irreducible.

We begin by a refinement of Theorem 3.3.17 about the representability of cohomology classes.

Proposition 3.4.2. *Let B be an AC Calabi-Yau manifold asymptotic to $C(\Sigma)$ as above.*

(i) For all $\nu \in (-6,-2)$, we have natural isomorphisms $\mathcal{H}^2_\nu \simeq L^2\mathcal{H}^2 \simeq H^2_c(B)$.

(ii) For all $\nu \in (-2,0)$, the natural map $\mathcal{H}^2_\nu \to H^2(B)$ which sends any $\sigma \in \mathcal{H}^2_\nu$ to its cohomology class is an isomorphism. In particular, for every harmonic 2-form τ on Σ such that $[\tau] \in \mathrm{im}(H^2(B) \to H^2(\Sigma)) \subset H^2(\Sigma)$, there exists $\sigma \in \mathcal{H}^2_\nu(B)$ such that for some for some $\mu < 0$,

$$\sigma = \tau + O(r^{-2+\mu})$$

Proof. Part (i) follows from the isomorphisms $\mathcal{H}^2_{-3+\delta} \simeq L^2\mathcal{H}^2 \simeq H^2_c(B)$ that holds for $\delta > 0$ small enough and the fact that $d + d^*$ has no indicial roots in $(-6,-2)$.

For part (ii), we have $H^2_c(B) \simeq \mathcal{H}^2_{-2-\delta}$, and since $H^1(\Sigma) = 0$, the long exact sequence in cohomology gives

$$H^2(B) \simeq H^2_c(B) \oplus \mathrm{im}(H^2(B) \to H^2(\Sigma))$$

We need to understand what happens when we cross the indicial roots -2. In particular, if we have $\dim \mathcal{H}^2_{-2+\delta} - \dim \mathcal{H}^2_{-2-\delta} = \dim \mathrm{im}\, H^2(B) \to H^2(\Sigma)$ then the result follows. The equality on dimension is proven, in the different context of AC G_2-manifolds, in [19, Proposition 4.65]. □

Next, we will need to describe the kernel of a few operators on AC Calabi-Yau manifolds. First, as a consequence of the irreducibility of B, there are no decaying harmonic functions and 1-forms:

Proposition 3.4.3. *Let B be an AC Calabi-Yau manifold asymptotic to $C(\Sigma)$, where the universal cover of Σ is not isometric to the round 5-sphere. Then there are no harmonic functions and 1-forms in C^∞_ν for any $\nu < 0$.*

Proof. Let u be a harmonic function on B in C^∞_ν for some $\nu < -2$. Since $\Delta u \in C^\infty_{-2-\nu}$ we can integrate by parts to obtain $0 = \langle \Delta u, u\rangle_{L^2} = \|du\|^2_{L^2}$. Hence u is a constant function that decay, and thus $u = 0$. Moreover, we have seen above that the scalar Laplacian on B has no indicial root in $[-2, 0)$, so that there are no harmonic function in C^∞_ν for any $\nu < 0$.

For 1-form, we need to recall the Weitzenböck-Licnerowicz formula from Proposition 3.2.13: $\Delta\eta = \nabla^*\nabla\eta + \Gamma\eta$, where $\Gamma\eta_c = g^{ab}r_{ac}\eta_b$, where r_{ab} is the Ricci curvature tensor. Since B is Ricci-flat, $\Delta\eta = \nabla^*\nabla\eta$. If η is a harmonic 1-form in C^∞_ν for some $\nu < -2$, we may also integrate by parts and obtain $\nabla\eta = 0$. Since B is irreducible, it carries no non-trivial parallel 1-forms, and therefore $\eta = 0$. We conclude again by the fact that Δ acting on 1-forms has no indicial root in $[-2, 0)$. □

Lastly, we need some results on operators derived from the Laplacian. Recall from §2.2.3 that on a Calabi-Yau manifold (B, ω, Ω), the $SU(3)$-structure (ω, Ω) induces a decomposition of the bundle of differential forms on B. We would like to describe the operator $u \in C^\infty(B) \to \pi_1(d * d(u\omega)) \in \Omega^4(B)$, where π_1 is the projection on the component $\mathbf{R}\omega^2$ of $\Omega^4(B)$. Since we will need to work with varying $SU(3)$-structures, we would rather see it as an operator $C^\infty \to \Omega^4$ than an operator between functions.

It is easy to compute that:

$$d * d(u\omega) = d * (du \wedge \omega) = -d(Jdu \wedge \omega) = -(dJdu) \wedge \omega$$

so that $\pi_1(d * d(u\omega)) = -\pi_1(dJdu) \wedge \omega$. On the other hand, the Laplacian acts on functions as

$$d * du = \frac{1}{2} d(Jdu \wedge \omega^2) = \frac{1}{2}(dJdu) \wedge \omega^2$$

which gives $\pi_1 d * d(u\omega) = -\frac{1}{3}(\Delta u)\omega^2$. In particular, combined with Proposition 3.4.3, we obtain the following:

Proposition 3.4.4. *Let (B, ω, Ω) be an AC manifold. Then the operator $u \in C^\infty(B) \to \pi_1 d * d(u\omega) \in \Omega^4(B)$ has trivial kernel when restricted to decaying function. In particular if $k \geq 1$ is an integer $\alpha \in (0, 1)$ and $\nu < -1$, there exists a constant $C > 0$ such that, for all $u \in C^{k+1,\alpha}_{\nu+1}$, we have*

$$\|u\|_{C^{k+1,\alpha}_{\nu+1}} \leq C\|\pi_1 d * d(u\omega)\|_{C^{k-1,\alpha}_{\nu-1}}$$

More generally, we will need to consider the operator

$$u \in C^\infty(B),\ \gamma \in \Omega^1(B) \to \pi_{1\oplus 6} d * d(u\omega + \gamma \lrcorner \operatorname{Re}\Omega)$$

According to [11, Lemma 2.19], it is identified to

$$(u,\gamma) \mapsto \left(\frac{2}{3}\Delta f, dd^*\gamma + \frac{2}{3} d^* d\gamma\right)$$

This operator is elliptic and translation-invariant at infinity (up to scaling by r^2), and therefore it can be treated by the Lockhart-McOwen theory. It also has no non-trivial kernel elements $(u,\gamma) \in C^\infty_\nu$ for any $\nu < 0$. Hence we also have the following:

Proposition 3.4.5. *Let (B,ω,Ω) be an AC manifold. Then the operator*

$$(u,\gamma) \in C^\infty(B) \times \Omega^1(B) \longmapsto \pi_1 d * d(u\omega + \gamma \lrcorner \operatorname{Re}\Omega) \in \Omega^4(B)$$

has trivial kernel when restricted to decaying elements. In particular if $k \geq 1$ is an integer $\alpha \in (0,1)$ and $\nu < -1$, there exists a constant $C > 0$ such that, for all $u \in C^{k+1,\alpha}_{\nu+1}$, we have

$$\|(u,\gamma)\|_{C^{k+1,\alpha}_{\nu+1}} \leq C \|\pi_1 d * d(u\omega + \gamma \lrcorner \operatorname{Re}\Omega)\|_{C^{k-1,\alpha}_{\nu-1}}$$

3.4.3 Deformations of $SU(3)$-structures

If B is a Riemannian manifold and (E,h,∇) a vector bundle with metric and compatible connection, we define

$$|u|_{C^k}(x) = \sum_{j=0}^{k} |\nabla^j u(x)|$$

for all $x \in B$ and u local section of E defined in a neighborhood of x. Here, as usual, ∇ is the connection induced by the connection of E and the Levi-Civita connection of B, and the norms on the fibers are induced by h and the metric on B.

Proposition 3.4.6. *Let G be a subgroup of $SO(n)$, and let (V_1,ρ_1), (V_2,ρ_2) be two orthogonal representations of G. Let $F : V_1 \to V_2$ be a smooth map such that $F \circ \rho_1(h) = \rho_2(h) \circ F$ for all $h \in G$. Let $k \geq 0$ be an integer and $M > 0$ a real number. Then there exists a constant $C > 0$, that depends only on M and k, such that the following holds.*

Let P be a torsion-free G-structure on a manifold B^n, and let g be the induced metric and ∇ be the induced Levi-Civita connection. All the following considerations are local, so we do not assume that B is compact. Let $E_1 = P \times_{\rho_1} V_1$, $E_2 = P \times_{\rho_2} V_2$ and $F : E_1 \to E_2$ be the (non-linear) bundle map induced by $F : V_1 \to V_2$. Then, for all local sections u, u' of E_1 defined on an open subset $U \subset B$ that satisfy $\|u\|_{C^k}, \|u'\|_{C^k} \leq M$ on U, we have pointwise

$$|F(u') - F(u)|_{C^k}(x) \leq C |u' - u|_{C^k}(x)$$

for all $x \in U$.

Proof. We fix M and we will make an induction on k. Since F is smooth on V_1, then there exists a constant C such that for all u, u' in V_1 with $|u' - u| \leq M$, we have $|F(u') - F(u)| \leq C|u' - u|$. Working in a trivialization of P, that induces trivializations of V_1 and V_2, it is clear that this imply the proposition for $k = 0$, because in this trivialization the C^0 norm on B is the standard C^0-norm for functions $u : U \to V_1$, and the bundle map here is really just composition by $F : V_1 \to V_2$.

Suppose $k = 1$. By differentiating the equality $\rho_2(e^{tA})F(u) = F(\rho_1(e^{tA}u))$ at $t = 0$ for a fixed $u \in V_1$, we obtain

$$\rho_2(A)u = \mathrm{Jac}_F(u) \cdot \rho_1(A)u$$

where Jac_F is the Jacobian matrix of F. Suppose now u is a local section of a vector bundle E_1 as in the proposition. Then, locally the connection ∇ is written as $\nabla u = du + \rho_1(A)u$ where A is a local connection form. Here, implicitly we choose a trivialization of E_1 that comes from P. If v is a section of E_2, we have in the same trivialization $\nabla v = dv + \rho_2(A)v$. Substituting $v = F(u)$ and taking the equality above into account we obtain

$$\begin{aligned}\nabla F(u) &= dF(u) + \rho_2(A)F(u)\\ &= \mathrm{Jac}_F(u) \cdot (du + \rho_1(A)u)\\ &= \mathrm{Jac}_F(u) \cdot \nabla u\end{aligned}$$

Let V be the representation of G induced by the inclusion $G \subset SO(n)$. Define a map $F' : V_1 \oplus V^* \otimes V_1 \to V_2 \oplus V^* \otimes V_2$ by

$$F(u, \alpha \otimes v) = (F(u), \alpha \otimes \mathrm{Jac}_F(u) \cdot v)$$

Then, by differentiating the expression $F(\rho_1(h)(u + tv)) = \rho_1(h)F(u + tv)$, we see that F' is a smooth equivariant map for the action of G, and we may apply the case $k = 0$ to the sections $v = (u, \nabla u)$ and $v' = (u', \nabla u')$.

In general for $k \geq 2$, we may replace by induction F by F' and apply the result for rank $k - 1$ to the section $v = (u, \nabla u)$. □

Remark 3.4.1. At first sight, we did not use the torsion-free assumption in the proof. Actually, we did use it to run the induction, because then the Levi-Civita connection reduces to a G-connection induced by P.

These local estimates only depend of the fact that in good trivializations, the bundle map induced by F is essentially independent of the variable on the base manifold. In different settings, we can derive global estimates. For instance, on a compact manifold, the estimates would hold globally since they hold on any local trivialization. For AC manifolds, which is the case that we will use, note that the C^k_ν-norms that we defined are equivalent (with a constant that depends on the manifold) to the norm given by

$$\|u\| = \sum_{j=0}^{k} \sup_{x \in B} \left(r^{-\nu-j} |u|_{C^k}(x) \right)$$

Therefore, we can multiply by the weights in the pointwise estimates and take the sup to obtain estimates in weighted C^k-norms:

Corollary 3.4.7. *Let G be a subgroup of $SO(n)$, and G a torsion-free G-structure on a manifold B, such that the induced metric g is AC. Let $F : V_1 \to V_2$ be a smooth map such that $F \circ \rho_1(h) = \rho_2(h) \circ F$ for all $h \in G$. Let $k \geq 0$ be an integer, $M > 0$ and $\nu < 0$ a real weight. Then there exists a constant $C > 0$ such that for all sections $u, u' \in C^k_\nu$ such that $\|u\|, \|u'\|_{C^k_0} \leq M$, we have*

$$\|F(u') - F(u)\|_{C^k_\nu} \leq C\|u' - u\|_{C^k_\nu}$$

The result of Proposition 3.4.6 would still hold if F were defined not on the whole V_1, but on an orbit $\mathcal{O}$ of $GL(V)$. To adapt it we would need to fix u to a particular element $u_0 \in \mathcal{O}$ (which can always be achieved in a good choice of trivialization). Rather than taking any $M > 0$, we would need to take an $\epsilon > 0$ small enough to remain in a compact neighborhood of $u_0 \in \mathcal{O}$. This is essentially what we will use in the following Proposition.

Proposition 3.4.8. *Let (ρ, W) be a representation of $GL(6, \mathbf{R})$ and let $\Psi \in W$ be invariant under $SU(3)$. Let $k \geq 0$ be an integer. Then there exists $\epsilon_0 > 0$ and a constant $C > 0$ such that the following holds.*

Let B be a possibly non-compact manifold, with frame bundle F and vector bundle $E = F \times_\rho W$. Let $\mathfrak{c}_0 = (\omega_0, \Omega_0)$ be a torsion-free $SU(3)$-structure on B and (g_0, ∇_0) be the metric and Levi-Civita connection induced by $\mathfrak{c}_0$. Let ψ_0 be the section of E induced by $\mathfrak{c}_0$. Suppose $\mathfrak{c} = (\omega, \Omega)$ is another $SU(3)$-structure on B, not necessarily smooth but locally C^k, and assume $\|\mathfrak{c} - \mathfrak{c}_0\|_{C^k} \leq \epsilon$. Let ψ be the section of E induced by $\mathfrak{c}$. Then ψ is locally C^k and we have pointwise estimates

$$|\psi - \psi_0|_{C^k}(x) \leq C|\mathfrak{c} - \mathfrak{c}_0|_{C^k}(x)$$

Proof. We start by some linear algebra. Let (ω_0, Ω_0) be 2- and 3-forms on $V = \mathbf{R}^6$ satisfying the compatibility condition, and identify $SU(3)$ with the stabilizer of (ω_0, Ω_0). Let $\mathcal{O} = GL(6, \mathbf{R}) \cdot (\omega_0, \Omega_0)$ be its orbit under $GL(6, \mathbf{R})$. The map

$$GL(6, \mathbf{R}) \to W, \quad h \to \rho(h)\Psi$$

is invariant by left multiplication by an element of $SU(3)$, and therefore defines a smooth map $f : \mathcal{O} \to W$. If we denote by (ρ', W') the action of $GL(6, \mathbf{R})$ onto $W' = \Lambda^2 V^* \oplus \Lambda^3 V^*$, the map f satisfies

$$f(\rho'(h)\mathfrak{c}) = \rho(h) f(\mathfrak{c})$$

If ϵ_0 is chosen small enough so that the ball of radius ϵ_0 has compact closure in $\mathcal{O}$, the rest of the proof goes as for the proof of Proposition 3.4.6. □

For AC manifolds, we can also multiply these inequalities by the weights to obtain the following corollary.

Corollary 3.4.9. *Let (B, ω_0, Ω_0) be an AC Calabi-Yau manifold, with frame bundle F, and $\mathfrak{c}_0 = (\omega_0, \Omega_0)$. Let (W, ρ) be a representation of $GL(6, \mathbf{R})$ and $\Psi \in W$ invariant under $SU(3)$, and denote by ψ_0 the section of $E = F \times_\rho W$ induced by Ψ and $\mathfrak{c}_0$. Let $k \geq 0$ be an integer and $\nu \leq 0$ a real number. Then for $\epsilon_0 > 0$ small enough, there exist a constant C such that the following holds.*

Let $\mathfrak{c} = (\omega, \Omega)$ be another $SU(3)$-structure on B, such that $\|\mathfrak{c} - \mathfrak{c}_0\|_{C^k_0} \leq \epsilon_0$, and assume moreover that $\mathfrak{c} - \mathfrak{c}_0$ is in C^k_ν, for the AC metric g_0 induced by $\mathfrak{c}_0$. In particular, we do not require $\mathfrak{c}$ to be smooth. Denote by ψ the section of E induced by P and Ψ.

Then $\psi - \psi_0$ is a section of E of class C^k_ν, and moreover we have the following estimate:

$$\|\psi - \psi_0\|_{C^k_\nu} \leq C\|\mathfrak{c} - \mathfrak{c}_0\|_{C^k_\nu}$$

Remark 3.4.2. We first tried to prove a similar result for Hölder norms, since we usually work with these norms rather than the C^k-norms, but it did not quite work out. However, we will mostly use this result and the following corollaries in Proposition 5.2.4, where it is sufficient to have such a control on weighted C^k-norms, and the continuous injection $C^k_\nu \hookrightarrow C^{k-1,\alpha}_\nu$.

The particular case that we will use is the following. We have seen in §2.2.3 that an $SU(3)$-structure on a manifold induces a decomposition of the bundle of differential forms according to the irreducible representations of $SU(3)$. If V is the usual representation of $GL(6, \mathbf{R})$ on $\mathbf{R}^6$, and $W \subset \Lambda^k V^*$ is some irreducible subrepresentation, then the orthogonal projection $\pi : \Lambda^k V^* \to W$ correspond to an element of $\mathrm{End}(\Lambda^k V^*)$ fixed by the action of $SU(3)$. Hence if $\mathfrak{c}_0$ is an AC Calabi-Yau structure on a manifold B, and $\mathfrak{c}_0$ is another $SU(3)$-structure on B, close enough to $\mathfrak{c}_0$ in the relevant C^k_ν-norm, then the difference of projections $\pi - \pi_0$ with respect to $\mathfrak{c}$ and $\mathfrak{c}_0$ is also in C^k_ν.

One particular case related to Proposition 3.4.6 which will be of particular interest to us is the case where F is an analytic map. We state the result as claimed in the article [11, §8.2]. If F is analytic and $u \in V_1$ is written formally as an expansion $u = u_0 + \epsilon u_1 + \sum_{k \geq 2} \epsilon^k u_k$, then we can write

$$F(u) = F_0 + \epsilon\ L(u_1) + \sum_{k \geq 2} \epsilon^k (L(u_k) + F_k(u_1, ..., u_{k-1}))$$

where here we will assume for convenience $F_0 = F(u_0) = 0$, L is the linearization of F at u_0 and the F_k's are homogeneous polynomials of order k. We would like to make use of the analyticity of F, and thus $F - L$, to get good estimates on the F_k for $k \geq 2$.

Suppose the u_k are functions depending on some base variable x, and we have a connection written $\nabla u = du + \rho_1(A)u$ where A is a local connection form. We have seen that the equivariance of F implies that the usual identity $dF(u) = \mathrm{Jac}_F(U) \circ du$ becomes $\nabla F(u) = \mathrm{Jac}_F(u) \circ \nabla u$, and thus the identities involving the usual derivatives of functions will remain true for the covariant derivatives. Note that F_{k+1} depends on $u_1, ..., u_k$ since we substracted the linear part of F. Since $F-L$ is analytic in u, there exists a series $\sum C_m t^m$ with non-negative coefficients C_m such that we have inequalities

$$|F(u)-L(u)| \leq \sum_{m\geq 2} C_m |u|^m, \qquad |\nabla(F(u)-L(u))| \leq |\nabla u| \sum_{m\geq 2} m C_m |u|^{m-1}$$

and similarly for $\nabla^j(F(u)-L(u))$ for all $j \geq 2$, that hold whenever $|u|$, $|\nabla u|$, and the higher orders $\nabla^j u$ if necessary, are small enough. Thus if we expand the function u as $u(x) = u_0 + \epsilon u_1(x) + \sum_{m\geq 2} \epsilon^m u_m(x)$, we must have inequalities in weighted Hölder norms of the type:

$$\|F_{k+1}(u)\|_{C^{l,\alpha}_{\nu}} \leq Q \sum_{m\geq 2} C_m \left(\sum_{I\in\mathcal{I}_{m,k}} \|u_1\|_{C^{l,\alpha}_{\nu_1}} \ldots \|u_k\|_{C^{l,\alpha}_{\nu_k}} \right)$$

that holds for some constant Q (that does not depend on k), where the weights satisfy $\nu_1 i_1 + \ldots + \nu_k i_k \leq \nu$ for $(i_1, \ldots, i_k) \in \mathcal{I}_{m,k}$. Here we follow the conventions used in [11, Equation (8.6), p. 43] and denote by $\mathcal{I}_{m,k}$ the set of indices $I = (i_1, \ldots i_k)$ such that $i_1 + \ldots + i_k = m$ and $i_1 + 2i_2 + \ldots + k i_k = k+1$. The observation of Foscolo-Haskins-Nordström is that this holds when we choose $\nu_1 = -1$ and $\nu_2 = \ldots = \nu_k = \nu$ for $\nu \in [-2, 0]$ [11, §8.2, p. 44].

Chapter 4

Deformations of complex structures

Besides the intrinsic interest of the result described in this chapter (and our own curiosity about it), the reason for talking about deformations of complex structures is that the Kodaira-Spencer construction should serve as a model construction, in the compact setting, before turning to the more sophisticated construction of ALC G_2-metrics in the following chapter. The underlying problem to the construction described in this chapter is to understand the moduli space of complex structures on a given smooth (differentiable) manifold, that is, the space of integrable almost complex structures modulo the action of diffeomorphisms. Upon some topological assumptions that we will describe below, Kodaira-Spencer showed that this is a smooth complex manifold, and its dimension is the dimension of the first cohomology group of the sheaf of holomorphic vector fields. Many proofs of smoothness for general moduli spaces, defined as the space of solutions to a geometric PDE modulo the action of diffeomorphisms, rely on the fact that the linearization of this equation is Fredholm transversely to the infinitesimal action of vector fields. Unlike such arguments, Kodaira-Spencer gave an explicit construction of the deformations of a complex manifold, which gives a local description of the moduli space. This is the construction we want to explain here. We will not treat global aspects of the moduli space, which are not relevant to our purpose, but we refer to the beautiful book of Kodaira [23], which we mainly used to write this chapter.

In §4.1.1, we discuss holomorphic vector bundles and $\overline{\partial}$-operators. Then, in §4.1.2, we briefly talk about Dobeault cohomology, introducing the few notions that will be useful to us. In §4.2.1, we define families of complex structures as introduced by Kodaira. Then, in §4.2.2, we focus on the notion of derivative of a family of complex structures. Lastly, in §4.2.3, we describe in details the Kodaira-Spencer construction of analytic deformations of complex structures, since all the key points of the construction will be useful to understand the constructions of Chapter 5.

4.1 More complex manifolds

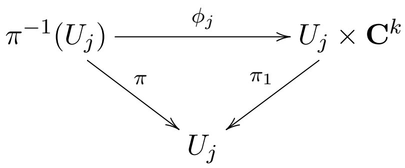
4.1.1 Holomorphic vector bundles

Definition 4.1.1. Let M be a complex manifold. A holomorphic vector bundle of rank k is a complex manifold E together with a holomorphic map $\pi : E \to M$ such that the following conditions hold:

(i) π is surjective, and for all $p \in E$ the differential $d\pi_p : T_pE \to T_{\pi(p)}M$ is onto,

(ii) for all $z \in M$, the fiber $E_z = \pi^{-1}(z)$ has the structure of a complex vector space of dimension k,

(iii) π is locally trivial, that is, there exists an open covering $\{U_j\}$ of M and biholomorphisms $\phi_j : U_j \times \mathbf{C}^k \to \phi^{-1}(U_j)$ such that the following diagram commutes :



and such that the ϕ_j's act linearly on the fibers.

As for smooth vector bundles, the gluing functions $\phi_{jk} = \phi_j \circ \phi_k^{-1}$, defined on $U_{jk} \times \mathbf{C}^k$, where $U_{jk} = U_j \cap U_k$, are of the form $\phi_{jk}(z, v) = (z, g_{jk}(z)v)$, where $g_{jk} : U_{jk} \to GL(n, \mathbf{C})$ are called the transition functions. The condition for the transition functions to define a holomorphic vector bundle is that they must be holomorphic. Moreover, a local (holomorphic) trivialization of a holomorphic vector bundle $E \to M$ is equivalent to a local frame that varies holomorphically.

Example 4.1.1. If M is a complex manifold of dimension n, $T_M^{1,0}$ naturally has the structure of a holomorphic vector bundle. M is covered by local coordinate charts (U_j, z_j), glued via holomorphic maps f_{jk} such that $z_j = f_{jk}(z_k)$. If we note $z_j = (z_j^1, ..., z_j^n)$, we have seen that $\left(\frac{\partial}{\partial z_j^1}, ..., \frac{\partial}{\partial z_j^n}\right)$ is a local frame of $T_M^{1,0}$. Differentiating the expression $z_j^\alpha = f_{jk}^\alpha(z_k)$, we obtain

$$\frac{\partial}{\partial z_j^\alpha} = \frac{\partial f_{jk}^\alpha}{\partial z_k^\beta} \frac{\partial}{\partial z_k^\beta}$$

Hence the transitions functions of $T_M^{1,0}$ are given by the matrix with entries $\frac{\partial f_{jk}^\alpha}{\partial z_k^\beta}(z)$, that are holomorphic functions of the variable z. Any local smooth section Z_j of $T_M^{1,0}$ on U_j can be written

$$Z_j = Z_j^\alpha(z) \frac{\partial}{\partial z_j^\alpha}$$

where the Z_j^α's are smooth complex-valued functions. Z_j is a holomorphic section if and only if the Z_j^α's are holomorphic.

Moreover, since we have a canonical complex-linear isomorphism $(T_M^{\mathbf{R}}, J) \simeq T_M^{1,0}$, so that we can consider $T_M^{\mathbf{R}}$ as a holomorphic vector bundle if it is useful. Explicitly, in local coordinates $z_j = x_j + iy_j$, this isomorphism sends $\frac{\partial}{\partial x_j^\alpha}$ to $\frac{\partial}{\partial z_j^\alpha}$ and $\frac{\partial}{\partial y_j^\alpha}$ to $i\frac{\partial}{\partial z_j^\alpha}$. Therefore, under this identification, a real vector field $Z_j = X_j^\alpha \frac{\partial}{\partial x_j^\alpha} + Y_j^\alpha \frac{\partial}{\partial y_j^\alpha}$ is a holomorphic section of $(T_M^{\mathbf{R}}, J)$ if and only if the functions $Z_j^\alpha = X_j^\alpha + iY_j^\alpha$ are holomorphic.

The notion of holomorphic vector bundle is compatible with the usual algebraic operations on vector bundles, like direct sum, dual, tensor product, etc. Therefore, if M is a complex manifold, the vector bundles $\Lambda_M^{p,0}$ have a natural structure of holomorphic vector bundles over M. However, for $q \neq 0$, there is no natural structure if holomorphic vector bundle on $\Lambda_M^{p,q}$.

Proposition 4.1.1. *Let $E \to M$ be a holomorphic vector bundle. Then there exists a unique differential operator $\overline{\partial}_E : C^\infty(E) \to C^\infty(\Lambda_M^{0,1} \otimes E)$ that satisfies the following properties:*

(i) for all functions f on M and smooth sections S of E, we have:

$$\overline{\partial}_E(fS) = \overline{\partial} f \otimes S + f\overline{\partial}_E S$$

(ii) if S is a local holomorphic section of E, then $\overline{\partial}_E S = 0$.

Proof. The uniqueness of the operator is clear. Indeed, in a local holomorphic trivialization $U_j \times \mathbf{C}^m$ of E, where the U_j's are coordinates charts on M, properties (i) and (ii) imply that a if S is a section of E, written locally as $S_j : U_j \to \mathbf{C}^m$ over U_j, we have must have

$$(\overline{\partial}_E S)_j = d\overline{z}_j^\alpha \otimes \frac{\partial S_j}{\partial \overline{z}_j^\alpha}$$

We need to check that these expressions patch up to a well defined section of E over M. The independance from the choice of holomorphic coordinates is clear, but we need to check the independance form the choice of trivialization. It comes from the fact that, since the g_{jk}'s are holomorphic, we have $\frac{\partial}{\partial \overline{z}^\alpha}(g_{jk}(z)S_k) = g_{jk}(z)\frac{\partial}{\partial \overline{z}^\alpha}S_k$. □

As for connections, the operator $\overline{\partial}_E : C^\infty(E) \to C^\infty(\Lambda^{0,1} \otimes E)$ has a unique extension to an operator $\overline{\partial}_E : C^\infty(\Lambda_M^{0,r} \otimes E) \to C^\infty(\Lambda_M^{0,r+1} \otimes E)$ that satisfies:

$$\overline{\partial}_E(\eta \wedge S) = \overline{\partial}\eta \wedge S + (-1)^p \eta \wedge \overline{\partial}_E S$$

for all $\eta \in C^\infty(\Lambda_M^{0,p})$ and $S \in C^\infty(\Lambda^{0,q} \otimes E)$. However, unlike connections, the identity $\overline{\partial}_E^2 = 0$ always holds. Indeed, this is a local statement, so that we can assume that the bundle E is trivial. On a trivial bundle, the operator $\overline{\partial}_E$ is just the operator $\overline{\partial} : \Omega_M^{0,r} \to \Omega_M^{0,r+1}$ acting on each components. Therefore, $\overline{\partial}^2 = 0$ imply $\overline{\partial}_E^2 = 0$.

4.1.2 Dolbeault cohomology

In this part, we want to say very few words about the cohomology of sheaves, and its relations to Cech and Dobeault cohomology in the case of holomorphic vector bundle. If $E \to M$ is any vector bundle, we have an associated sheaf $\mathcal{E}$ of holomorphic sections of E, i.e., for any $U \subset M$, $\mathcal{E}(U)$ is the complex vector space of local sections $s : U \to E$ that satisfy $\overline{\partial} s = 0$, and for any $V \subset U$ we have a natural map $\mathcal{E}(U) \to \mathcal{E}(V)$, $s \to s_{|V}$ of restriction. By the general theory of sheaves, we have cohomology groups $H^*(B, \mathcal{E})$ that we will not properly define here, but just explain how to compute. Moreover we will often denote $H^*(M, \mathcal{E}) = H^*(M, E)$ by abuse.

A first way to compute $H^*(M, \mathcal{E})$ is via the Dolbeault complex, which is the complex

$$0 \longrightarrow C^\infty(E) \xrightarrow{\overline{\partial}} C^\infty(T_M^{0,1} \otimes E) \xrightarrow{\overline{\partial}} ... \xrightarrow{\overline{\partial}} C^\infty(T_M^{0,n} \otimes E) \longrightarrow 0$$

where n is the complex dimension of M. The cohomology groups $H^*_{Db}(M, \mathcal{E})$ of this complex are isomorphic to the cohomology groups $H^*(M, \mathcal{E})$.

Another way to compute the cohomology of $\mathcal{E}$ is to use Cech cohomology. Let $\mathcal{U} = \{U_j\}_{j \in \mathcal{I}}$ be a finite open cover of M, and define a chain complex $C^*(\mathcal{U}, \mathcal{E})$ as follows. For all $p \geq 0$, let $C^p(\mathcal{U}, \mathcal{E})$ be the vector space of families $\{s_{j_1 \dots j_{p+1}}\}$, where $s_{j_1 \dots j_{p+1}} \in \mathcal{E}(\cap_{1 \leq i \leq p+1} U_{j_i})$, such that

$$s_{j_{\sigma(1)} \dots j_{\sigma(p+1)}} = (-1)^{|\sigma|} s_{j_1 \dots j_{p+1}}$$

for all permutations σ of $\{1, ..., p+1\}$, where $|\sigma|$ is the signature of σ. The differential of the chain complex $C^*(\mathcal{U}, \mathcal{E})$ are defined by $\delta\{s_{j_1 \dots j_p}\} = \{t_{j_1 \dots j_{p+1}}\}$ where

$$t_I = t_{j_1 \dots j_{p+1}} = \sum_{i=1}^{p+1} (-1)^i s_{j_1 \dots \hat{j}_i \dots j_{p+1} |U_I}$$

For instance, if $\{s_{ij}\}$ is a 1-cochain, then we have$\delta\{s_{ij}\} = \{t_{ijk}\}$ where

$$t_{ijk} = -s_{ij} + s_{ik} - s_{jk}$$

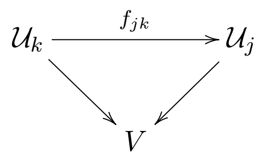
so that in particular, $\delta\{s_{ij}\} = 0$, i.e. $\{s_{ij}\}$ is a 1-cocycle, if and only if for all i, j, k we have $s_{ik} = s_{ij} + s_{jk}$. These differentials define a chain complex, with cohomology groups $H^*(\mathcal{U}, \mathcal{E})$. Moreover, there exists a natural map $H^*(\mathcal{U}, \mathcal{E}) \to H^*(M, \mathcal{E})$, which is not in general an isomorphism. Indeed, $H^*(M, \mathcal{E})$ is identified to the colimit of $H^*(M, \mathcal{U})$ for all coverings $\mathcal{U}$ (where the colimit is taken in a suitable sense).

4.2 Analytic deformations of complex structures

4.2.1 Families of complex structure

A natural question to ask is how to represent a family of complex structures. Since any compact complex manifold can be realized by gluing a finite number of polydiscs, the idea of Kodaira and Nirenberg was that deformations of the complex structure should be obtained by varying the transition functions. More precisely, let M be a compact manifold, and suppose M is covered by a finite number of open sets U_j, $j = 1, ..., n$, each of them identified to the coordinate polydisc $\{z_j \in \mathbf{C}^n, \ |z_j^1|, ..., |z_j^n| < 1\}$. The transition functions $f_{jk}(z_k) = z_j$, defined on some open subset of U_k, are holomorphic. Then, if t is a parameter taking values in an open subset of $V \subset \mathbf{R}^d$ for some $d > 0$, a smooth deformation of the complex structure of M is a family M_t of complex manifolds obtained by gluing the polydiscs U_j by transition functions $z_j = f_{jk}(z_k, t)$. Here, the functions $f_{jk}(z_k, t)$ defined on some open subset of $U_k \times V$ are smooth, and for fixed t, $f_{jk}(\cdot, t)$ is holomorphic. Moreover, we require that $f_{jk}(z_k, 0) = f_{jk}(z_k)$.

With the data above, if we let $\mathcal{U}_k = U_k \times V \subset \mathbf{C}^n \times \mathbf{R}^d$, the f_{jk}'s above are the gluing functions of a smooth manifold $\mathcal{M}$ of dimension $2n + d$. Moreover, every $\mathcal{U}_k$ has a natural projection $\mathcal{U}_k \to V$ such that the following diagram commutes:



Therefore, there is a natural map $\pi : \mathcal{M} \to V$. From the local expressions, we see that this map is smooth, surjective, and that this is a submersion. Moreover, this is a proper map. The fiber of $\mathcal{M}$ over t is identified with the complex manifold M_t obtained by gluing the polydiscs U_j with the functions $f_{jk}(\cdot, t)$.

More generally, we can define a smooth family $\{M_t\}$ of complex manifolds when t is a parameter taking value in any smooth manifold B in the following way:

Definition 4.2.1. Let B be a smooth, connected manifold and for every $t \in B$, let M_t be a compact, connected, complex manifold. We say that M_t is a smooth family of complex manifolds is there exists a smooth manifold $\mathcal{M}$ and a smooth proper submersion $\pi : \mathcal{M} \to B$, such that the following conditions hold:

(i) for all $t \in B$, $\pi^{-1}(t)$ is diffeomorphic to the smooth manifold underlying M_t,

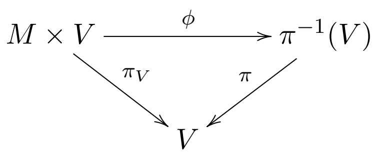
(ii) there are a locally finite open cover $\{\mathcal{U}_j\}$ of $\mathcal{M}$, and for all j, smooth functions $z_j^1, ..., z_j^n : \mathcal{U}_j \to \mathbf{C}$, such that for all $t \in B$, the functions

$$z_j^\alpha : \mathcal{U}_j \cap \pi^{-1}(t) \to \mathbf{C}$$

form a system of local coordinates for the complex manifold M_t, under the identification of (i).

An important fact is that if $\{M_t\}$ is a smooth family of complex manifolds, then all the M_t's are diffeomorphic to each other's. This is a consequence of the following lemma, of which the proof is almost as important as the result for our purpose.

Lemma 4.2.1 (Ehresmann's lemma)**.** *Let $\pi : \mathcal{M} \to B$ be a smooth, proper, surjective submersion. Then it is locally trivial: for every $t \in B$, there exists a smooth manifold M, a neighborhood V of t in B and a diffeomorphism $\phi : M \times V \to \pi^{-1}(V)$ such that the following diagram commutes:*



where π_V is the projection on the second coordinate.

Remark 4.2.1. A smooth surjective submersion if often called a (smooth) fibration.

Proof. This is a local statement, so hat we can assume that $B = I^d$, where $I = (-1, 1)$. The proof works by induction on d.

Suppose $d = 1$. By the submersion theorem, $\mathcal{M}$ is covered locally finite family of coordinate charts $(\mathcal{U}_j, x_j, t)$ in which $\pi(x_j, t) = t$. Therefore, $x_j : \mathcal{U}_j \cap \pi^{-1}(t) \to \mathbf{R}^n$ form a system coordinate of the fiber M_t for all $t \in I$. Without loss of generality, restricting ourselves to a smaller interval if needed, we can assume that the $\mathcal{U}_j$'s are in finite number, $j = 1, ..., m$. The manifold $\mathcal{M}$ is obtained by gluing the $\mathcal{U}_j \subset \mathbf{R}^n \times I$ along transition functions $(f_{jk}^\alpha(x_k, t), t) = (x_j, t)$. On each $\mathcal{U}_j$, we can define a vector field $\left(\frac{\partial}{\partial t}\right)_j$ as the pull-back of $\frac{\partial}{\partial t}$ under the map $(x_j, t) : \mathcal{U}_j \to U_j \times I$. Note that these vector fields flow transversally to the fibers, and satisfy $\pi_* \left(\frac{\partial}{\partial t}\right)_j = \frac{\partial}{\partial t}$. We have the relations:

$$\left(\frac{\partial}{\partial t}\right)_k = \frac{\partial f_{jk}^\alpha}{\partial t}(x_k, t)\frac{\partial}{\partial x_j^\alpha} + \left(\frac{\partial}{\partial t}\right)_j \tag{4.1}$$

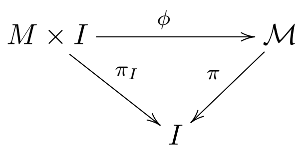
Let $\{\rho_j\}$ be partition of unity associated to the cover $\mathcal{U}_j$. Define the vector field:

$$X = \sum_{j=1}^m \rho_j \left(\frac{\partial}{\partial t}\right)_j$$

X is a smooth vector field on $\mathcal{M}$, and moreover using equation (4.1), we have $\pi_* X = \frac{\partial}{\partial t}$. Let ϕ_t be the flow of X, and let $M = \pi^{-1}(0)$. Then we can define:

$$\phi : M \times I \to \mathcal{M}, \ (x, t) \mapsto \phi_t(x)$$

By the general theory of ODEs, this is a diffeomorphism, and the condition $\pi_* X = \frac{\partial}{\partial t}$ ensures that the diagram



commutes. Therefore, the result holds for $d = 1$.

The generalization to all dimensions is not hard. Suppose we know the result is true up to dimension $d-1$ and let $\mathcal{M} \to I^d$ be a fibration satisfying the hypothesis of the lemma. We can consider the other fibration $\mathcal{M} \to I^d = I^{d-1} \times I \to I$. Using case $d = 1$, we know that this fibration is equivalent to the projection $\mathcal{N} \times I \to I$, where $\mathcal{N}$ is the fiber over 0. By restriction, the fibration $\mathcal{N} \to I^{d-1}$ satisfies the hypothesis of the lemma, and thus by induction, $\mathcal{N} \to I^{d-1}$ is equivalent to a projection $M \times I^{d-1} \to I^{d-1}$, where M is a smooth manifold. Then it is clear that $\mathcal{M} \to I^d$ is equivalent to the projection $M \times I^d \to I^d$, which completes the proof of Lemma 4.2.1. □

As stated above, a consequence of Ehresmann's lemma is that in a smooth family of compact complex manifolds, all manifolds are diffeomorphic to each others. Indeed, if $\{M_t\}$ are the fibers of a map $\mathcal{M} \to B$ as in Definition 4.2.1, then for any t_0 in B, the set of M_t diffeomorphic to M_{t_0} is open in B. That also implies that its complement is open, so that all fibers are diffeomorphic to M_{t_0} since B is connected.

However, if M is a complex manifold, and $\{M_t\}$ a smooth deformation of M, the complex manifolds M_t are not necessarily biholomorphic to M. It is interesting to try to run the same argument as for the smooth structure and see where it fails.

As explained above, we can take a fibration $\mathcal{M} \to I^d$ covered by a finite number of open sets $\mathcal{U}_j$ with functions $(z_j^1, ..., z_j^n, t)$ that form a system of local coordinates of the fibers. As in the proof of Ehresmann's lemma, we can restrict ourselves to the case $d = 1$, since this is the hard part of the proof, the generalization to all dimensions being straightforward. The transition functions are of the form $(z_j, t) = (f_{jk}(z_k, t), t)$, where the f_{jk} are smooth in both variables and holomorphic in the variable z_k. On each U_j, (z_j, t) are local coordinates of $\mathcal{M}$ where $t = t(p) = \pi(p)$ is the coordinate on I. The vector field $\left(\frac{\partial}{\partial t}\right)_j$ flows transversally to the fibers. We have the relations:

$$\left(\frac{\partial}{\partial t}\right)_k = \frac{\partial f_{jk}^\alpha}{\partial t}(z_k, t)\frac{\partial}{\partial z_j^\alpha} + \left(\frac{\partial}{\partial t}\right)_j \tag{4.2}$$

Hence, we can see the local vector fields

$$\theta_{jk} = \frac{\partial f_{jk}^\alpha}{\partial t}(z_k, t)\frac{\partial}{\partial z_j^\alpha}$$

as an obstruction for a global vector field $\frac{\partial}{\partial t}$ to be globally defined. Note that the vector fields θ_{jk}, seen for each $t \in I$ as a vector field locally defined on M_t, are holomorphic for the complex structure of M_t, since, they are manifestly holomorphic in the coordinates z_j. For smooth manifolds, we escaped the obstructions given by the θ_{jk} by using a partition on unity $\{\rho_j\}$ tu construct a smooth vector field $X = \sum \rho_j \left(\frac{\partial}{\partial t}\right)_j$ that flows transversally to the fibers. This flow generates a family of diffeomorphisms, which allowed us to conclude about local triviality in the smooth case. However, there is no reason that the flow of X would preserve the complex structure of the fibers. Indeed, in the chart $\mathcal{U}_j$, X has the expression

$$X_j = \left(\frac{\partial}{\partial t}\right)_j + \sum_{k \neq j} \rho_k \frac{\partial f_{jk}^\alpha}{\partial t} \frac{\partial}{\partial z_j^\alpha}$$

But the flow of $\sum \rho_k \frac{\partial f_{jk}^\alpha}{\partial t} \frac{\partial}{\partial z_j^\alpha}$ does not necessarily preserve the complex structure of $\mathbf{C}^n$. Indeed, we have the following result:

Lemma 4.2.2. *Let M be a complex manifold and let ϕ_t be a family of diffeomorphisms generated by a t-dependant real vector field X_t. Then ϕ_t is a family of biholomorhpisms if and only if, under the usual identification $T_M^{\mathbf{R}} \simeq T_M^{1,0}$, X_t is a holomorphic vector field for each t.*

Proof. Let J be the almost complex structure associated to the complex structure of M. Since we have $\frac{d}{dt}\phi_t^* J = \phi_t^* \mathcal{L}_{X_t} J$, where $\mathcal{L}$ denotes the Lie derivative, the condition for ϕ_t to act by biholomorphisms is $\mathcal{L}_{X_t} J = 0$ for all t. If X, Y are vector fields then

$$(\mathcal{L}_X J)Y = [X, JY] - J[X, Y]$$

If we choose local complex coordinates $z^1 = x^1 + iy^1, ..., z^n = x^n + iy^n$ and write $X = u^\alpha \frac{\partial}{\partial x^\alpha} + v^\alpha \frac{\partial}{\partial y^\alpha}$ and let $\frac{\partial}{\partial x^\beta}$ or $\frac{\partial}{\partial y^\beta}$, we have:

$$(\mathcal{L}_X J)\left(\frac{\partial}{\partial x^\beta}\right) = \left(-\frac{\partial u^\alpha}{\partial y^\beta} - \frac{\partial v^\alpha}{\partial y^\beta}\right)\frac{\partial}{\partial x^\alpha} + \left(\frac{\partial u^\alpha}{\partial x^\beta} - \frac{\partial v^\alpha}{\partial y^\beta}\right)\frac{\partial}{\partial y^\alpha}$$

Therefore, the condition $\mathcal{L}_X J = 0$ is equivalent to the Cauchy-Riemann equations for the functions $u^\alpha + iv^\alpha$. □

Therefore, the argument of the proof Ehresmann's lemma fails when we use the partition of unity, since we cannot insure that the resulting vector field will be holomorphic along the fibers. If we want to find a criterion for a deformation of a complex manifold M to be trivial, it must have to do with the possibility to extend the local vector field $\left(\frac{\partial}{\partial t}\right)_j$ to a globally defined vector field that is holomorphic along the fibers. That will be the goal of §4.2.2. In the remaining of this part, we define analytic deformations of complex manifolds, and make some further comments on the notion of family of complex manifolds.

Definition 4.2.2. Let B be a smooth, connected complex manifold and for every $t \in B$, let M_t be a compact, connected, complex manifold. We say that M_t is an analytic family of complex manifolds is there exists a complex manifold $\mathcal{M}$ and a holomorphic proper submersion $\pi : \mathcal{M} \to B$, such that for every $t \in B$, the fiber of $\mathcal{M}$ over t is biholomorphic to M_t.

Remark 4.2.2. An analytic family of complex manifolds is in particular a smooth family of complex manifolds. Indeed, if the submersion $\mathcal{M} \to B$ is holomorphic, condition (ii) automatically holds. Any holomorphic submersion can be locally written $(z, t) \mapsto t$ where $(z, t) \in \mathbf{C}^n \times \mathbf{C}^d$ are local complex coordinates on the total space and t is a local complex coordinate on the base.

Definition 4.2.3. Let $\mathcal{M}, \mathcal{N} \to B$ be smooth (respectively analytic) families of complex manifolds. We say that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if there exists a diffeomorphism (resp. biholomorphism) $\phi : \mathcal{M} \to \mathcal{N}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi} & \mathcal{N} \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

and such that for all $t \in B$ the restriction $\phi_t : M_t \to N_t$ of ϕ to the fiber over t is a biholomorphism.

A family of complex manifolds $\mathcal{M} \to B$ is called trivial if it is isomorphic to the trivial fibration $M \times B$, and locally trivial near a point $t_0 \in B$ if there exists a neighborhood V of t_0 in B such that the restriction of $\mathcal{M}$ over V is trivial.

4.2.2 The derivative of a family of complex structures

In this section we would like to find a good notion of derivative for a smooth or analytic family of complex structure. If $\mathcal{M} \to B$ is a smooth family of complex manifolds, and $\frac{\partial}{\partial t}$ is any tangent vector in $T_t B$, we would like to be able to define $\frac{\partial M_t}{\partial t}$, such that the map $\frac{\partial}{\partial t} \in T_t B \mapsto \frac{\partial M_t}{\partial t}$ is linear, and such that this map is zero on a neighborhood of t_0 if and only if the family M_t is trivial near t_0.

For now, assume that $B = I$ and that $\mathcal{M}$ is covered by a finite number of open sets $\mathcal{U}_j$, on, which we have complex valued functions z_j^α, $\alpha = 1, ..., n$ that form local coordinates for the fibers. We can further assume that (z_j, t), where $t = t(p) = \pi(p)$ is the coordinate on B, identifies $\mathcal{U}_j$ with $U_j \times I^d \subset \mathbf{C}^n \times \mathbf{R}^d$, where U_j is the polydisc $\{|z_j^1|, ..., |z_j^n| < 1\}$. We have transition functions f_{jk} defined on some open subset of $U_k \times I$, taking values in U_j, such that $(z_j, t) = (f_{jk}(z_k, t), t)$ when (z_j, t) and (z_k, t) represent the same point in $\mathcal{M}$.

We can define for all j the vector field $\left(\frac{\partial}{\partial t}\right)_j$ over $\mathcal{U}_j$ as the pull back of $\frac{\partial}{\partial t}$ under the map $(z_j, t) : \mathcal{U}_j \to U_j \times I$. The $\left(\frac{\partial}{\partial t}\right)_j$ are a priori not globally well defined, and satisfy equation (4.2)

$$\left(\frac{\partial}{\partial t}\right)_k = \theta_{jk} + \left(\frac{\partial}{\partial t}\right)_j \tag{4.3}$$

where the holomorphic vector fields

$$\theta_{jk}(t) = \frac{\partial f_{jk}^{\alpha}}{\partial t}(z_k, t)\frac{\partial}{\partial z_j^{\alpha}}$$

are defined on $M_t \cap \mathcal{U}_j \cap \mathcal{U}_k$, and represent an obstruction for a global vector field X acting by biholomorphisms on the fibers to exist. Let $\mathcal{U} = \{\mathcal{U}_j\}$. Since $\theta_{kj}(t) = -\theta_{jk}(t)$ and $\theta_{ik}(t) = \theta_{ij}(t) + \theta_{jk}(t)$ follow immediately from equation (4.3), $\{\theta_{jk}(t)\} \in Z^1(\mathcal{U}, T^{1,0}_{M_t})$ is a 1-cocycle. Hence, it defines a cohomology class $\theta(t)$ in $H^1(\mathcal{U}, T^{1,0}_{M_t}) \hookrightarrow H^1(M_t, T^{1,0}_{M_t})$. This cohomology class does not depend on the particular choice of coordinates we made [21, §4.2 (a)]. The invariance of the cohomology class under the choice of coordinates implies that two deformations of a complex manifold M that are locally isomorphic near M have the same derivative. Hence, if $\mathcal{M} \to B$ is trivial near t_0 in B, then $\theta(t) = 0$ near t_0. Actually, the converse turns out to be true, under a further condition.

To see which condition is needed, suppose that $\theta \equiv 0$. That means that for all $t \in I$, there exists a 0-cochain $\{\theta_j(t)\} \in C^0(\mathcal{U}, T^{1,0}_{M_t})$ such that $\theta_{jk}(t) = \theta_k(t) - \theta_j(t)$. Then, we can rewrite equation (4.3) as

$$\left(\frac{\partial}{\partial t}\right)_k - \theta_k(t) = \left(\frac{\partial}{\partial t}\right)_j - \theta_j(t)$$

so that the local expressions $\left(\frac{\partial}{\partial t}\right)_j - \theta_j(t)$ patch up to a globally well-defined vector field X, that flows transversally to the fibers, satisfy $\pi_* X = \frac{\partial}{\partial t}$, and is holomorphic along the fibers. However, without further assumptions, the local holomorphic vector fields $\theta_j(t)$ are not smooth in the variable t. In [23], Kodaira and Spencer proved that the assumption that the dimension of $H^1(M_t, T^{1,0}_{M_t})$ is constant is sufficient to insure that we can choose a smooth cocycle θ_j in the variable t. In that case, then, the family $\mathcal{M} \to I$ is trivial, with trivialization given by

$$\phi : M \times I \to \mathcal{M}, \ (z, t) \to \phi_t(z)$$

where ϕ is the flow of the smooth vector X.

As in the proof of Ehresmann's lemma, this generalizes without problem to all dimensions. If $\mathcal{M} \to B$ is a smooth of analytic family of complex manifolds, then M has a locally finite open cover $\mathcal{U}_j$ on which are defined coordinates $(z_j, t_1, ..., t_d)$, where the z_j's form a sytem of local coordinates on the fiber, and $t_1, ..., t_d$ are real coordinates on the base B for a smooth family, and complex coordinates for an analytic family. For any vector field $\frac{\partial}{\partial t} = u^{\nu}\frac{\partial}{\partial t^{\nu}}$ on the base, we can define the vectors $\left(\frac{\partial}{\partial t}\right)_j$ on $\mathcal{U}_j$ by pulling back $\frac{\partial}{\partial t}$ through the coordinate maps. These vector fields still satisfy the relations

$$\left(\frac{\partial}{\partial t}\right)_k = \theta_{jk}(t) + \left(\frac{\partial}{\partial t}\right)_j$$

where we write

$$\theta_{jk}(t) = \frac{\partial f^\alpha_{jk}}{\partial t}(z_k, t)\frac{\partial}{\partial z^\alpha_j}$$

The cocycle $\{\theta_{jk}(t)\}$ represents a cohomology class $\frac{\partial M_t}{\partial t} \in H^1(M_t, T^{1,0}_{M_t})$ that is independent of the choices made. This construction gives a well-defined linear map:

$$\rho_t : \frac{\partial}{\partial t} \in T_t B \mapsto \rho_t(\frac{\partial}{\partial t}) = \frac{\partial M_t}{\partial t}.$$

According to our discussion above, the following theorem holds:

Theorem 4.2.3 ([21], Theorem 4.3) *Let $\mathcal{M} \to B$ be a smooth or analytic family of complex manifolds. Suppose that* $\dim H^1(M_t, T^{1,0}_{M_t})$ *is constant. Then $\rho_t = 0$ identically if and only if $\mathcal{M} \to B$ is locally trivial.*

In this sense, ρ_t is a good notion of derivative for a family of complex structures ρ_t.

4.2.3 Existence theorem

We come to the problem that is of most interest to us, which is the following question: given a compact, connected, complex manifold M, can we find an analytic family $\mathcal{M} \to B$, where without restriction we assume that B is a polydisc in $\mathbf{C}^d$, such that $M_0 \simeq M$ and $\rho_0 : \mathbf{C}^d \to H^1(M, T^{1,0}_M)$ is an isomorphism? Kodaira and Spencer proved the following theorem:

Theorem 4.2.4 (Kodaira-Nirenber-Spencer, [22]) *Let M be a compact, connected, complex manifold and suppose that $H^2(M, T^{1,0}_M) = 0$. Then there exists an analytic family $\pi : \mathcal{M} \to B$ of complex manifolds, where $B \subset \mathbf{C}^d$ is a polydisc centered at 0, satisfying:*

1. *$\pi^{-1}(0) \simeq M$,*
2. *the map $\rho_0 : \mathbf{C}^d \to H^1(M, T^{1,0}_M)$ is an isomorphism.*

This section is devoted to the proof of this theorem, which is a prototype for the main construction that we study in this thesis.

As explained in [21, §5.3.(a)], for the proof of existence, it is not convenient to represent elements of $H^1(M, T^{1,0}_M)$ by cocycles $\{\theta_{jk}\}$ as we did before. Rather, it is more convenient to use Dobeault cohomology and see $H^1(M, T^{1,0}_M)$ as

$$H^1(M, T^{1,0}_M) = \frac{\ker\left(\overline{\partial} : C^\infty(\Lambda^{0,1}_M \otimes T^{1,0}_M) \to C^\infty(\Lambda^{0,2}_M \otimes T^{1,0}_M)\right)}{\operatorname{im}\left(\overline{\partial} : C^\infty(T^{1,0}_M) \to C^\infty(\Lambda^{0,1}_M \otimes T^{1,0}_M)\right)}$$

Hence, for $\frac{\partial}{\partial t} \in T_t B$, we want to identify $\frac{\partial M_t}{\partial t}$ as the cohomology class of a $\overline{\partial}$-closed $(0,1)$-form valued in $T^{1,0}_{M_t}$. In, we had identified a $T^{1,0}_M$-valued $(0,1)$-form representing how close two complex structures are from each others. This will be useful here.

We do this in the following way. According to Ehresmann's lemma, if $\{M_t\}$ is an analytic family of complex manifolds, parametrized by a parameter t taking value in a small polydisc in $\mathbf{C}^d$. According to Ehresmann's lemma, all the M_t's are diffeomorphic to $M = M_0$. Hence, rather than considering that the family M_t is obtained by patching polydiscs by t-dependent gluing functions, we consider the variation of the complex structure of M as a variation of the coordinates functions. Cover M with a finite number open subsets $U_j \subset M$, on which are defined coordinate functions $z_j : U_j \to \mathbf{C}^d$. For $t \in B$ small enough, we can consider the family $\mathcal{M} \to B$ as a complex manifold diffeomorphic to $M \times B$, where the complex structure is defined by local coordinates $(\zeta_j, t) : \mathcal{U}_j \to \mathbf{C}^n \times \mathbf{C}^d$, where $\zeta_j = \zeta_j(z,t)$ is a function of $(z,t) \in \mathcal{U}_j \subset U_j \times B$ and $\zeta_j(z,0) = z_j$.

If $z = (z^1, ..., z^n)$ are any complex coordinates on the complex manifold $M = M_0$, then $\left(\frac{\partial z_j}{\partial z^1}, ..., \frac{\partial z_j}{\partial z^n}\right)$ form a basis of $\mathbf{C}^n$, since z_j are complex coordinates in M. Hence for sufficiently small t, the family

$$\frac{\partial \zeta_j(z,t)}{\partial z^1}, ..., \frac{\partial \zeta_j(z,t)}{\partial z^n}$$

also form a basis of $\mathbf{C}^n$. In particular, there exists local functions $\psi^\lambda_{j,\overline{\nu}}(z,t)$ such that

$$\frac{\partial \zeta_j(z,t)}{\partial \overline{z}^\nu} = \psi^\lambda_{j,\overline{\nu}}(z,t) \frac{\partial \zeta_j(z,t)}{\partial z^\lambda} \tag{4.4}$$

Then we have a locally defined section $\psi_j(t)$ of $T^{1,0}_M \otimes \Lambda^{0,1}_M$ by

$$\psi_j(t) = \psi^\lambda_{j,\overline{\nu}}(z,t) \frac{\partial}{\partial z^\lambda} \otimes d\overline{z}^\nu$$

such that

$$\overline{\partial} \zeta_j(z,t) = \psi_j(t) \zeta_j(z,t)$$

where we see $\psi_j(t)$ as a local differential operator taking functions to $(0,1)$-forms. Actually, the ψ_j's are independent of the choice of coordinate $(z^1, ..., z^n)$ and $(\zeta^1_j, ..., \zeta^n_j)$ we have made [21, §5.3.(b), p. 262], and the ψ_j's patch up to a globally defined section ψ of $T^{1,0}_M \otimes \Lambda^{0,1}_M$. We can consider ψ as a differential operator $C^\infty(M) \to \Omega^{0,1}_M$. Recall from Proposition 2.1.3 that a function f satisfies $(\overline{\partial} - \psi(t))f = 0$ if and only if f is holomorphic for the complex structure of M_t.

We now show that if $\frac{\partial}{\partial t} \in T_0 B$, then $\frac{\partial \psi}{\partial t}$ is a $\overline{\partial}$-closed section of $\Lambda^{0,1}_M \otimes T^{1,0}_M$. Indeed, an explicit computation yields the formula:

$$\overline{\partial} \psi(t) = \frac{1}{2}[\psi(t), \psi(t)] \tag{4.5}$$

Here, $[\cdot,\cdot]$ is an extension of the Poisson bracket defined on $T^{1,0}_M$ to all $(0,p)$-forms valued in $T^{1,0}_M$. Explicitly, if $\psi = \psi^\lambda \frac{\partial}{\partial z^\lambda}$ and $\eta = \eta^\lambda \frac{\partial}{\partial z^\lambda}$, with $\psi^\lambda \in \Lambda^{0,p}_M$ and $\eta^\lambda \in \Lambda^{0,q}_M$ in coordinates, we have

$$[\psi,\eta] = \left(\psi^\mu \wedge \frac{\partial \eta^\lambda}{\partial z^\mu} - (-1)^{pq}\eta^\mu \wedge \frac{\partial \psi^\lambda}{\partial z^\mu}\right)\frac{\partial}{\partial z^\lambda} \tag{4.6}$$

Equation (4.5) immediately implies

$$\overline{\partial}\partial_t\psi(t) = \frac{1}{2}([\partial_t\psi(t),\psi(t)] + [\psi(t),\partial_t\psi(t)]) = 0$$

because of the antisymmetry properties of the bracket. Hence $\frac{\partial\psi}{\partial_t}$ is a good candidate to represent the derivative of the family of complex structures $\{M_t\}$ at $t=0$. We have the following result [21, Theorem 5.4]:

Theorem 4.2.5 *Let ρ_t be the derivative of the family $\{M_t\}$. Then, if $\frac{\partial}{\partial t} \in T_0B$, then the the cohomology class $\rho_0(\partial/\partial t) \in H^1(M, T^{0,1}_M)$ is represented by the $\overline{\partial}$-closed $(0,1)$-form $-\frac{\partial\psi}{\partial t}$.*

Therefore, in order to prove Theorem 4.2.4, we need to build a family $\psi(t) \in C^\infty(\Lambda^{0,1}_M \otimes T^{1,0}_M)$ that is smooth in the variable $t = (t_1, ..., t_d) \in B$, and such that the integrability condition

$$\overline{\partial}\psi(t) = \frac{1}{2}[\psi(t),\psi(t)]$$

and such that the cohomology classes of $\frac{\partial\psi}{\partial t_1}_{|t=0}, ..., \frac{\partial\psi}{\partial t_d}_{|t=0}$ form a basis of $H^1(M, T^{1,0}_M)$. Because of Newlander-Nirenberg theorem, this is also sufficient. Indeed, if we can construct such a family $\psi(t)$, we define operators $L_1, ..., Ln, \tilde{L}_1, ..., \tilde{L}_d$ by

$$L_\nu d\overline{z}^\nu + \tilde{L}_\mu d\overline{t}_\mu = \overline{\partial}_J - \psi + \overline{\partial}_{J_0}$$

where J denotes the complex structure of M and J_0 the canonical complex structure of $B \subset \mathbf{C}^n$. Then, these operators satisfy the hypothesis of Theorem 2.1.4, which shows that the almost complex structure $J(t) \oplus J_0$ on $M \times B$ is integrable, and thus defines an analytic family of complex structures.

For constructing such a family $\psi(t)$, it is important to be able to solve $\overline{\partial}\psi = \Psi$ with a good control on the Hölder norm of the solution. By Hodge theorem, since $H^2(M, T^{1,0}_M) = 0$, then the elliptic operator $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : C^{k+2,\alpha}(\Lambda^{0,1} \otimes T^{1,0}_M) \to C^{k,\alpha}(\Lambda^{0,1}_M \otimes T^{1,0}_M)$ is an isomorphism. Let G be its inverse, sometimes called the Green function of the operator $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. By elliptic regularity there exists a constant C such that

$$\|G\Psi\|_{C^{k+2,\alpha}} \le C\|\Psi\|_{C^{k,\alpha}}$$

for all $\Psi \in C^\infty(\Lambda_M^{0,1} \otimes T_M^{1,0})$. Suppose Ψ is $\overline{\partial}$-closed. We can use G to construct a canonical solution of the equation $\overline{\partial}\psi = \Psi$. Applying $\overline{\partial}$ to the equation

$$\Psi = (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})G\Psi$$

gives

$$\overline{\partial}\overline{\partial}^*\overline{\partial}G\Psi = 0$$

In particular,

$$0 = \langle \overline{\partial}G\Psi, \overline{\partial}\overline{\partial}^*\overline{\partial}G\Psi\rangle_{L^2} = \|\overline{\partial}^*\overline{\partial}G\Psi\|_{L^2}^2$$

so that in fact we have

$$\Psi = \overline{\partial}\overline{\partial}^*G\Psi$$

Hence $\psi = \overline{\partial}^*G\Psi$ is a solution of $\overline{\partial}\psi = \Psi$. Moreover, since $\overline{\partial}^* : C^{k+1,\alpha}(\Lambda_M^{0,1} \otimes T_M^{1,0}) \to C^{k,\alpha}(\Lambda_M^{0,1} \otimes T_M^{1,0})$ is continuous, there exists a constant C such that

$$\|\overline{\partial}^*G\Psi\|_{C^{k+1,\alpha}} \leq C\|\Psi\|_{C^{k,\alpha}}$$

Now suppose that there is another $\overline{\partial}^*$-exact form η that satisfies $\overline{\partial}\eta = \Psi$. Then the $(0,1)$-form $\psi - \eta$ is $\overline{\partial}$-closed. Since $H^2(M, T_M^{1,0}) = 0$, then $C^\infty(\Lambda_M^{0,2} \otimes T_M^{1,0}) = \operatorname{im}\overline{\partial} \oplus \operatorname{im}\overline{\partial}^*$, we can write in a unique way $\psi = \overline{\partial}^*\alpha$, $\eta = \overline{\partial}^*\beta$, where α, β are $\overline{\partial}$-exact. Then we have

$$0 = \overline{\partial}\overline{\partial}^*(\alpha - \beta) = (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})(\alpha - \beta)$$

so that $\alpha = \beta$ and $\psi = \eta$. We summarize our results in the following proposition:

Proposition 4.2.6. *Let M be a compact complex manifold such that*

$$H^2(M, T_M^{1,0}) = 0$$

Then, for all $\Psi \in C^\infty(\Lambda_M^{0,2} \otimes T_M^{1,0})$, there exists a unique $\overline{\partial}^*$-exact section ψ of $\Lambda_M^{0,1} \otimes T_M^{1,0}$ such that

$$\overline{\partial}\psi = \Psi$$

Moreover, for all $k \geq 0$ and $\alpha \in (0,1)$, there exists a constant C such that

$$\|\psi\|_{C^{k+1,\alpha}} \leq C\|\Psi\|_{C^{k,\alpha}}$$

Remark 4.2.3. This statement really contains two important facts, the uniqueness of solutions to $\overline{\partial}\psi = \Psi$ that satisfy the additional constraint of being $\overline{\partial}^*$-exact, and the control of the norm of such solutions. This two facts are both important and related. Indeed, adding any $\overline{\partial}$-closed 1-form to a solution of $\overline{\partial}\psi = \Psi$ leads to another solution, so that we do not have uniqueness of the solution. Moreover, we can add a $\overline{\partial}$-closed 1-form with arbitrary big $C^{k+1,\alpha}$-norm, so that for a general solution the estimate $\|\psi\|_{C^{k+1,\alpha}} \leq C\|\Psi\|_{C^{k,\alpha}}$ does not hold. Looking for a $\overline{\partial}^*$-exact solution gives us both uniqueness of the solution and a control on its norm.

The construction of $\psi(t)$ goes in two steps. We are looking for a $\psi(t)$ defined as a power series expansion near 0:

$$\psi(t) = \psi(0) + \sum_k \psi_k t_k + ... + \sum_{k_1+...+k_d=m} \psi_{k_1,...,k_d} t_1^{k_1} ... t_d^{k_d} + ...$$

such that $\psi(0) = 0$, $\psi_1, ..., \psi_d$ form a basis of $H^1(M, T_M^{1,0})$ and $\psi(t)$ is subject to the equation:

$$\overline{\partial}\psi(t) = \frac{1}{2}[\psi(t), \psi(t)] \tag{4.7}$$

We will show that we can construct a series that formally satisfies these constraints, and then show that it converges.

We construct $\psi(t)$ by solving (4.7) order by order. We will denote by ψ^m the expansion of ψ up to order m, and by $\psi^{[m]}$ the homogeneous part of order m. If P and Q are two power series in $t_1, ..., t_d$, we write $P \equiv_m Q$ when P and Q coincide at least up to order m. The important point is that, since the term of order 0 in the expansion of $\psi(t)$ vanishes, then $[\psi, \psi]^m = [\psi^{m-1}, \psi^{m-1}]$. Therefore, formally (4.7) reduces to:

$$\overline{\partial}\psi^m = \frac{1}{2}[\psi^{m-1}, \psi^{m-1}], \qquad m = 1, 2, ... \tag{4.8}$$

Let $\psi_1, ..., \psi_d$ be $\overline{\partial}$-closed sections of $\Lambda_M^{0,1} \otimes T_M^{1,0}$ which cohomology classes form a basis of $H^1(M, T_M^{1,0})$, and define:

$$\psi^1(t) = \psi^{[1]}(t) = \sum_k \psi_k t_k \tag{4.9}$$

To find the term $\psi^{[2]}$, we must solve (4.8) for $m = 2$, which is equivalent to

$$\overline{\partial}\psi_{kl} = [\psi_k, \psi_l], \qquad k, l = 1, ..., d \tag{4.10}$$

Since $\overline{\partial}[\psi_k, \psi_l] = [\overline{\partial}\psi_k, \psi_l] + [\psi_k, \overline{\partial}\psi_l] = 0$, the $(0, 2)$-form $[\psi_k, \psi_l]$ is $\overline{\partial}$-closed. By Proposition 4.2.6, the equation $\overline{\partial}\psi_{kl} = [\psi_k, \psi_l]$ admits a unique $\overline{\partial}^*$-exact solution ψ_{kl}.

For $m \geq 2$, suppose we have found a series ψ^m that satisfy (4.7) up to terms of order m. Then, we want to find a term $\psi^{[m+1]}$ homogeneous of order $m + 1$ that satisfies

$$\overline{\partial}\psi^{[m+1]} \equiv_{m+1} \frac{1}{2}[\psi^m, \psi^m] - \overline{\partial}\psi^m \tag{4.11}$$

Since ψ^m satisfies

$$\overline{\partial}\psi^m = \frac{1}{2}[\psi^{m-1}, \psi^{m-1}]$$

then the terms of order equal or less to m in the right hand side of (4.11) vanish. Denote by $\Psi^{[m+1]}$ the homogeneous part of degree $m + 1$ of $\frac{1}{2}[\psi^m, \psi^m]$. Then equation (4.11) reduces to

$$\overline{\partial}\psi^{[m+1]} = \Psi^{[m+1]}$$

Since
$$\overline{\partial}\left(\frac{1}{2}[\psi^m,\psi^m]-\overline{\partial}\psi^m\right)=\frac{1}{2}\left([\overline{\partial}\psi^m,\psi^m]+[\psi^m,\overline{\partial}\psi^m]\right)=0$$

then the coefficients of the $\Psi^{[m+1]}$ are also $\overline{\partial}$-closed. Using again Proposition 4.2.6, we find a unique solution of

$$\overline{\partial}\psi^{[m+1]}=\Psi^{[m+1]}$$

with $\overline{\partial}^*$-exact coefficients. Therefore, imposing the first order terms $\psi_1,...,\psi_d$, we find a unique power series expansion $\psi(t)$ that satisfies equation (4.8) for all $m\geq 1$, and such that the coefficients of the terms of order more or equal to 2 are $\overline{\partial}^*$-exact.

The second step of the construction is to prove that the series $\psi(t)$ converges. Now fix a integer $k\geq 2$. To this aim, we define by $|\psi|_{k,\alpha}(t)$ the power series

$$|\psi|_{k,\alpha}(t)=\sum_{k_1,...,k_d}\|\psi_{k_1...k_d}\|_{C^{k,\alpha}}t_1^{k_1}...t_d^{k_d}$$

and if $P=\sum P_{k_1...k_d}t_1^{k_1}...t_d^{k_d}$ and $Q=\sum Q_{k_1...k_d}t_1^{k_1}...t_d^{k_d}$ are two power series with non-negative scalar coefficients, we write $P(t)\ll Q(t)$ if $P_{k_1...k_d}\leq Q_{k_1...k_d}$ for all $k_1,...,k_d$.

We will first prove that the series

$$\psi(t)=\sum_{k_1,...,k_d}\psi_{k_1...k_d}t_1^{k_1}...t_d^{k_d}$$

converges in the $C^{k,\alpha}$-norm, in a sufficiently small neighborhood of 0. It is sufficient to show that there exists a series

$$A(t)=\sum_{k_1,...,k_d}a_{k_1,...,k_d}t_1^{k_1}...t_d^{k_d}$$

where the coefficients $a_{k_1,...,k_d}$ are non-negative scalars, that has positive radius of convergence, and such that

$$|\psi|_{k,\alpha}(t)\ll A(t)$$

Indeed, suppose that these inequalities hold, and let us prove that the series $\psi(t)$ converges to a section of $\Lambda_M^{0,1}\otimes T_M^{1,0}$ over $M\times B_\epsilon$, that is of class $C^{k,\alpha}$ in both variables (z,t). In a local trivialization over $U\subset M$, $\Lambda_U^{0,1}\otimes T_U^{1,0}\simeq U\times\mathbf{C}^{2n}$, and the $C^{k,\alpha}$-norm defined with respect to a hermitian metric h and compatible connection ∇ is equivalent to the usual $C^{k,\alpha}$-norm for functions $U\to\mathbf{C}^{2n}$. $\Psi(t)$ is identified with $\Psi_U(t):U\to\mathbf{C}^{2n}$ over U, and we can write

$$\Psi_U(t)=\sum_{k_1,...,k_d}(\psi_U)_{k_1...k_d}t_1^{k_1}...t_d^{k_d}$$

If the inequality $|\Psi_U|_{C^{k,\alpha}}(t) \ll A(t)$ hold and $A(t)$ has positive radius of convergence, then $|\Psi_U|_{C^{k,\alpha}}(t)$ and all its formal derivatives (in the variable t) up to order k have positive radius of convergence. In particular, all the derivatives of the series Ψ_U in both variables (z,t) uniformly converge on $U \times B_\epsilon$ up to order k. Since we can cover M with a finite number of such trivializations, we obtain that $\Psi(t)$ is a section of $\Lambda^{0,1}_M \otimes T^{1,0}_M$ which is of class $C^{k,\alpha}$ in both variable (z,t). Moreover, the derivatives of ψ up to order k coincide with the limit of the series of formal derivatives of ψ. In the same way, we can prove the convergence of the series of derivatives of any order of $\Psi(t)$ with respect to t, and then $\Psi(t)$ is analytic in the variable t.

We choose the series

$$A(t) = \frac{b}{16c} \sum_{k=1}^{\infty} \frac{c^k (t_1 + ... + t_d)^k}{k^2}$$

that has radius of convergence $1/c$. It has the following important property:

Lemma 4.2.7. *The series $A(t)$ satisfies the inequality*

$$A(t)^2 \ll \frac{b}{c} A(t).$$

Proof. We have the expression:

$$A(t)^2 = \frac{b^2}{16^2 c^2} \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m-1} \frac{m^2}{k^2 (m-k)^2} \right) \frac{c^m (t_1 + ... + t_d)^m}{m^2}$$

and since

$$\begin{aligned} \sum_{k=1}^{m-1} \frac{m^2}{k^2(m-k)^2} &= \frac{1}{m^2} \sum_{k=1}^{m-1} \left(\frac{m}{k} + \frac{m}{m-k} \right)^2 \\ &= \frac{1}{m^2} \sum_{k=1}^{m-1} 2\frac{m^2}{k^2} + 4\frac{m}{k} \\ &\leq \frac{\pi^2}{3} + 8\frac{\log m}{m} < 16 \end{aligned}$$

we obtain the claimed inequality. □

Lemma 4.2.8. *There exists $b, c > 0$ such that*

$$|\psi|_{k,\alpha}(t) \ll A(t)$$

Proof. We prove by induction that $|\psi|^m_{k,\alpha}(t) \ll A(t)$ for an appropriate choice of constants b and c. For $m = 1$, the linear term of ψ is $\psi_1 t_1 + ... + \psi_d t_d$, and the linear term of $A(t)$ is $\frac{b}{16}(t_1 + ... + t_d)$. We choose b so that $b > \|\psi_k\|_{C^{k,\alpha}}$ for all $k = 1, ..., d$. With this choice, $|\psi|^1_{k,\alpha}(t) \ll A(t)$ holds.

Suppose $|\psi|^m_{k,\alpha}(t) \ll A(t)$ holds for some $m \geq 1$. The term $\psi^{[m+1]}$ is the unique $\overline{\partial}^*$-exact solution of

$$\overline{\partial}\psi^{[m+1]} = \frac{1}{2}[\psi^m, \psi^m]^{[m+1]}$$

and by Proposition 4.2.6, we have

$$|\psi|^{[m+1]}_{k,\alpha} \leq C|[\psi^m, \psi^m]|^{[m+1]}_{k-1,\alpha}$$

for some constant C independent of m. But if η, ξ are any $T^{1,0}_M$-valued differential forms, it is clear from the expression of the bracket that

$$\|[\eta, \xi]\|_{C^{k-1,\alpha}} \leq C'\|\eta\|_{C^{k,\alpha}}\|\xi\|_{C^{k,\alpha}}$$

It follows that

$$|\psi|^{[m+1]}_{k,\alpha} \leq CC'(|\psi|^m_{k,\alpha}(t))^2$$

By induction, $|\psi|_{k,\alpha}(t) \ll A(t)$, and by Lemma 4.2.7, we conclude that

$$|\psi|^{[m+1]}_{k,\alpha} \ll \frac{bCC'}{c}A(t)$$

Where C, C' are independent of m. Hence, if we choose any $c > bCC'$, $|\psi|^{[m+1]}_{k,\alpha} \ll A(t)$. Since $|\psi|^{m+1}_{k,\alpha} = |\psi|^m_{k,\alpha} + |\psi|^{[m+1]}_{k,\alpha}$ and $|\psi|^m_{k,\alpha} \ll A(t)$ holds by induction, we conclude that

$$|\psi|^{m+1}_{k,\alpha}(t) \ll A(t)$$

and the proof is complete. □

So far, we have proven that for any $k \geq 2$, the series $\psi(t)$ converges to a section of $\Lambda^{0,1}_M \otimes T^{1,0}_M$ that is of class $C^{k,\alpha}$ in both variables $(z, t) \in M \times B_\epsilon$ for some small $\epsilon > 0$ that depends on k. Moreover, this series is analytic in the variable t. But since ϵ depends on k, that does not insures that $\psi(t)$ is smooth in both variables. The usual way to prove smoothness is to show that it is in the kernel of an elliptic operator.

Since $k \geq 2$, the derivatives of $\psi(t)$ up to order 2 coincide with the formal series of derivatives of $\psi(t)$. Hence $\psi(t)$ satisfies the equation

$$\overline{\partial}\psi(t) = \frac{1}{2}[\psi(t), \psi(t)]$$

since it only involves derivatives of order 1, and that this equation formally holds. Moreover, the coefficients of order not less than 2 are $\overline{\partial}^*$-exact, and the terms of order 1 are $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$-harmonic, and in particular $\overline{\partial}^*$-closed. Thus, formally, $\overline{\partial}^*\psi(t) = 0$, and since it only involves derivatives of order 1, the $(0, 1)$-form $\psi(t)$ satisfies

$$\overline{\partial}^*\psi(t) = 0$$

Therefore, we have

$$(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\psi(t) = \frac{1}{2}[\psi(t), \psi(t)]$$

Moreover, $\psi(t)$ is holomorphic in t, so that

$$\left(-\sum_{k=1}^{d} \frac{\partial^2}{\partial t_k \partial \bar{t}_k} + \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}\right)\psi(t) - \frac{1}{2}\overline{\partial}^*[\psi(t), \psi(t)] = 0$$

This is not a linear equation, but it is quasilinear, that is, linear in the highest order terms, and it is elliptic. Although we have not treated non-linear elliptic operators, a similar result of elliptic regularity applies to quasi-linear operators. Therefore, $\psi(t)$ is smooth in both variables. That concludes the proof of Theorem 4.2.4.

Chapter 5

Main construction

Constructing complete G_2-holonomy metrics, or more generally Ricci-flat metrics, is a very hard problem. The first examples of complete non-compact G_2-manifolds have been constructed by Bryant and Salamon in 1989 [5]. These examples are asymptotically conical, and admit a cohomogeneity one group action, that is, there is a Lie group action that preserves the G_2-structure such that generic orbits have codimension one. This high degree of symmetry allows to reduce the torsion-free condition, which is a non-linear PDE, to a mere ODE in one dimension, which is much more manageable.

By contrast with the cohomogeneity one case, where one can use the many symmetries of the problem to reduce the torsion-free condition to an ODE, the Foscolo-Haskins-Nordström construction that we study here uses only a circle action, to reduce the 7-dimensional problem to a 6-dimensional one. As a starting observation, we have seen in §2.3.1 that $SU(3)$ is the subgroup of G_2 that leaves invariant a vector. Therefore, S^1-invariant G_2-structures admit a reduction to $SU(3)$. When the action is free, and therefore determines a principal circle bundle $M^7 \to B^6$, the base manifold is thus endowed with an $SU(3)$-structure. In [2], Apostolov and Salamon showed that the torsion-free condition of an S^1-invariant G_2-structure on M was equivalent to a set of non-linear PDEs involving the corresponding $SU(3)$-structure on B. The Apostolov-Salamon equations are very difficult to study in general, but in the adiabatic limit where the size of the fibers of $M \to B$ go to zero, the Apostolov-Salamon equations reduce to a torsion-free condition for the $SU(3)$-structure of B. Hence, S^1-invariant G_2-structures should be able to collapse on Calabi-Yau manifolds.

The idea of Foscolo-Haskins-Nordström in [11] is to reverse this argument. Namely, starting from a Calabi-Yau 3-fold B^6 and a fixed circle bundle $M^7 \to B$, they build a 1-parameter family of S^1-invariant irreducible G_2-metrics g_ϵ on M, that collapse to the Calabi-Yau structure of B in the limit $\epsilon \to 0$. This is done by perturbing the Calabi-Yau structure on B, and looking for a solution to the Apostolov-Salamon equations as a power expansion series, as for the Kodaira-Nirenberg-Spencer construction of deformations of complex structure.

One complication with respect of the latter construction is that, if we want to obtain irreducible G_2-structures on M, we cannot choose a compact space as our base space, because of the Cheeger-Gromoll theorem which implies that a compact G_2-manifolds that admits a non-vanishing Killing field must be reducible.

The base space chosen in [11] is an AC Calabi-Yau manifold, and the metrics constructed on the total space M are ALC. Besides the above-mentioned interest of ALC metrics in the cohomogeneity one case, several other justifications for this choice can be given. The first, practical reason, is that on AC manifolds, the operators $d + d^*$ and $\Delta = dd^* + d^*d$ admit a good Fredholm theory, as we explained in Chapter 3. These properties are essential in order to be able to find a power series expansion that solves the Apostolov-Salamon equations order by order with estimates. Another, perhaps more important reason, is that there are a lot of tools available for constructing AC Calabi-Yau manifolds, so that the Foscolo-Haskins-Nordström construction gives rise to an important number of examples of G_2-holonomy metrics. In particular, they construct infinitely many diffeomorphism types of complete ALC G_2-metrics, whereas only a finite number was known before. They also use their construction to provide examples of families of complete ALC G_2-metrics of arbitrarily high dimension.

This chapter is organized as follows. In §5.1, we derive the Apostolov-Salamon equations for an S^1-invariant G_2-metric, in the particular case where the circle action is free, and give some properties of the associated $SU(3)$-structure on the base. Of particular importance is the fact that the intrinsic torsion of an $SU(3)$-structure coming from the reduction of a G_2-structure along a circle action is very constrained, and is almost torsion-free, although it cannot be Calabi-Yau if the G_2-structure is irreducible. In §5.1.2, we explain what is the adiabatic limit of the Apostolov-Salamon equations, and say a word on the expected form of solutions in this limit. Throughout §5.1.3, we discuss the strategy of the Foscolo-Haskins-Nordström construction, without entering into details, and try to give a more precise description of the expected difficulties and the way the authors propose to overcome them. We especially emphasize the troubles arising from the diffeomorphism and gauge invariance of the Apostolov-Salamon equations. As for the Kodaira-Spencer construction of analytic deformations of complex structure, the construction we study works by looking for solutions as a power series expansion. We finish the first part of this chapter by showing how, after the problems coming from gauge and diffeomorphism invariance are solved, the resolution of the Apostolov-Salamon equations is reduced to solving two systems of PDEs, one for the first order, and the second for higher order terms.

The second part of this chapter is devoted to explaining some (but not all) of the technical details of the construction. As it was difficult to take a step back from the paper of Foscolo-Haskins-Nordström and give a really personal account of their proof, we tried to explain in details some of the claims that were stated without detailed proof, and prefer to refer to the article rather than copying the arguments when we thought we could not do better. At some points however, our explanations follow very closely the treatment of the article, when we judged that it was interesting to write it down, in order to illustrate the use of the analytical tools of Chapter 3. In §5.2.1, we explain how to break the diffeomorphism and gauge invariance of the Apostolov-Salamon equations, in order to transform the equations we want to solve into an elliptic problem, for which we have tools to find solutions with estimates. In §5.2.2, we give the details of the resolution of the equations at first order. In §5.2.3, we give ideas on the resolution of the equations at higher order. Especially, we emphasize how to add variables to turn the system into an elliptic one, and use the relations existing between the different components of the intrinsic torsion of an $SU(3)$-structure to conclude that, whenever the additional parameters are small, they must actually vanish. Then we say a word about the convergence of the series and how to prove regularity of the solutions constructed.

5.1 The Apostolov-Salamon equations

In [2, §1], Apostolov and Salamon studied G_2-manifolds M that admit a non-trivial Killing vector field v of the G_2 structure, i.e., $\mathcal{L}_v\varphi = 0$. In the case where M is a circle bundle over a 6-manifold, the quotient $B = M/S^1$ is naturally equipped with an $SU(3)$-structure (ω, Ω). The G_2 structure on the total space M also determines a function h on B that measures the size of the fibers, and dual to the Killing field there is a connection form θ. Conversely, a tuple $(\omega, \Omega, h, \theta)$ where (ω, Ω) is a $SU(3)$-structure on B, h a smooth positive function and θ a connection form on $M \to B$ determines a G_2-structure on the total space M. Apostolov-Salamon showed that the condition that this structure is torsion-free is equivalent to a set of equations involving ω, Ω, h and θ, which we will call the *Apostolov-Salamon equations*.

These equations are very hard to study, because of their non-linearity. However, in the adiabatic limit where the size of the fibers shrink to zero, the linearized equations become more manageable, and makes it possible to use an argument à la Kodaira-Spencer to build torsion-free G_2-structures on M from its collapsed limit, which as we shall see is a Calabi-Yau manifold. One point that complicates the analysis is that, when one seeks metrics with full holonomy G_2, the Cheeger-Gromoll theorem rules out the existence of a non-trivial Killing field. Therefore, one has to work with non-compact manifolds.

5.1.1 $SU(3)$-reduction of G_2-holonomy metrics

In this part, we fix a a principal S^1-bundle $M^7 \to B^6$. Let v be the vector field that generates the S^1-action. We scale v so that the integral curves of v are 2π-periodic. We identify $\mathbf{R}$ to the Lie algebra of $S^1 = \mathbf{R}/2\pi\mathbf{Z}$.

Suppose first that g is a metric on the total space M that is preserved by the S^1-action. Define a smooth function on M by $h = g(v,v)^{-1}$. Since v is Killing, the function h is constant along the fibers, so that it descend to a function on the base space B, that we still denote h. Let θ be the 1-form defined by $\theta = hg(\cdot, v)$. Our choice of scaling gives $\theta(v) = 1$, and since g is invariant under the action of S^1, θ is also S^1-invariant. Since the Lie algebra of S^1 is trivially commutative, θ is thus an equivariant 1-form. Hence θ is a connection form on the circle bundle M. The associated horizontal space is the orthogonal hyperplane to the vector field v. If we let $\widetilde{g} = h^{-\frac{1}{2}} g_{|v^\perp}$, we can write:

$$g = h^{\frac{1}{2}}\widetilde{g} + h^{-1}\theta^2$$

Since g and h are S^1-invariant, $\widetilde{g}$ is also invariant. Moreover, $\widetilde{g}(v, \cdot) = 0$, so that $\widetilde{g}$ is a section of $S^2\mathcal{V}^\perp$, and hence it descends to a metric g_B on the base.

Assume now that φ is a G_2-structure on the total space M, invariant by S^1. Let g_φ be the associated metric on M, and $*_\varphi$ the associated Hodge operator. We define the function h and connection form θ as above. From §2.3.1, there exist a 2-form ω and a complex 3-form Ω, acting on the orthogonal space to v, so that we can write:

$$\varphi = \omega \wedge \theta + h^{\frac{3}{4}} \operatorname{Re}\Omega, \qquad *_\varphi\varphi = h^{\frac{1}{4}} \operatorname{Im}\Omega \wedge \theta + \frac{1}{2}h\omega^2$$

Moreover, on $(\mathbf{R}v)^\perp$, $\operatorname{Re}\Omega$ is a stable 3-form and (ω, Ω) satisfy the compatibility relations of an $SU(3)$-structure. Since it is clear by construction that $\omega \in \Lambda^2\mathcal{V}^\perp$ and $\operatorname{Re}\Omega, \operatorname{Im}\Omega \in \Lambda^3\mathcal{V}^\perp$, we only need to check that they are S^1-invariant to show that they descend to forms on B. Since $\mathcal{L}_v\varphi = 0 = \mathcal{L}_v\theta$, we have:

$$0 = (\mathcal{L}_v\omega) \wedge \theta + \mathcal{L}_v(h^{\frac{3}{4}} \operatorname{Re}\Omega)$$

Since the decomposition $TM = \mathbf{R}v \oplus (\mathbf{R}v)^\perp$ is trivially preserved by v, the forms $\mathcal{L}_v\omega$ and $\mathcal{L}_v\Omega$ still act on the orthogonal space to v. Wedging by θ in the equation above yields $\mathcal{L}_v(h^{\frac{3}{4}} \operatorname{Re}\Omega) \wedge \theta = 0$. But wedging by θ is an injective map on $\Lambda^*\mathcal{V}^\perp$, so that we get $\mathcal{L}_v(h^{\frac{3}{4}} \operatorname{Re}\Omega) = 0$ and $\mathcal{L}_v\omega \wedge \theta = 0$. Applying again the same argument and using that h is constant along the fibers, we obtain that $\mathcal{L}_v\omega = 0$ and $\mathcal{L}_v \operatorname{Re}\Omega = 0$. A similar argument shows that $\mathcal{L}_v \operatorname{Im}\Omega = 0$.

Hence, the $SU(3)$-reduction (ω, Ω) of a G_2-structure on M along a circle action descends to an $SU(3)$-structure on B, that comes with a positive function h and a connection form θ. It is clear that we can reverse the process. Starting with a tuple $(\omega, \Omega, h, \theta)$ on B, where (ω, Ω) is a $SU(3)$-structure on B, h a positive function and θ a connection form on a circle bundle $M \to B$, in order to construct an invariant G_2-structure φ on M.

In the above setup, the condition that the G_2-structure φ determined by $(\omega, \Omega, h, \theta)$ is torsion-free can be expressed as a set of non-linear PDEs:

Proposition 5.1.1 (Apostolov-Salamon [2]). *The S^1-invariant G_2-structure φ determined by the tuple $(\omega, \Omega, h, \theta)$ on B is torsion-free if and only if:*

$$\begin{gathered} d\omega = 0, \quad d(h^{\frac{3}{4}} \operatorname{Re}\Omega) = -d\theta \wedge \omega, \\ d(h^{\frac{1}{4}} \operatorname{Im}\Omega) = 0, \quad \frac{1}{2} dh \wedge \omega^2 = h^{\frac{1}{4}} d\theta \wedge \operatorname{Im}\Omega \end{gathered} \tag{5.1}$$

We will refer to these equations as the *Apostolov-Salamon equations*.

Proof. We know from §2.3.2 that the condition for φ to be torsion-free is $d\varphi = 0$ and $d * \varphi = 0$. We can explicitly compute:

$$d\varphi = d\omega \wedge \theta + \omega \wedge d\theta + d(h^{\frac{3}{4}} \operatorname{Re}\Omega)$$

But by construction, $v \lrcorner \theta = 1$, $v \lrcorner \omega = 0 = v \lrcorner \operatorname{Re}\Omega$, and by S^1 invariance, $v \lrcorner d\omega = -d(v \lrcorner \omega) = 0$ and $v \lrcorner d(h^{\frac{3}{4}} \operatorname{Re}\Omega) = -d(h^{\frac{3}{4}} v \lrcorner \operatorname{Re}\Omega) = 0$. Then, applying $v \lrcorner \cdot$ in the equation above yields $v \lrcorner d\varphi = d\omega$, so that the first two equations in (5.1) are equivalent to the condition $d\varphi = 0$.

For the second row of equations, we have:

$$d * \varphi = d(h^{\frac{1}{4}} \operatorname{Im}\Omega) \wedge \theta + h^{\frac{1}{4}} \operatorname{Im}\Omega \wedge d\theta + \frac{1}{2} dh \wedge \omega^2 + h\omega \wedge d\omega$$

The same argument as above gives $v \lrcorner d * \varphi = d(h^{\frac{1}{4}} \operatorname{Im}\Omega)$, and since we already derived the condition $d\omega = 0$, we get the last two equations. □

Remark 5.1.1. Since $d\theta$ is just the curvature of the connection form θ, this is a well defined element of $\Omega^2(B)$. It is a closed form, and the cohomology class $[d\theta]$ is the first Chern class of the circle bundle $M \to B$, noted $c_1(M)$. We derived equations (5.1) by working upstairs on the total space M, but since all the forms involved are S^1-invariant and descend downstairs to B, we can think about the Apostolov-Salamon equations as a set of PDEs on differential forms defined on B.

Remark 5.1.2. As we have seen in the proof, the first row of equations is equivalent to the condition $d\varphi = 0$. Hence, if φ is closed, ω is a symplectic form on B according to the first equation, and the second one gives the topological condition $c_1(M) \cup [\omega] = 0$ in $H^4(B)$.

An interesting question to ask about the $SU(3)$-structure (ω, Ω) determined by an S^1-invariant G_2-structure is how far it is from being torsion-free. From Proposition 2.2.8, the intrinsic torsion is identified with different components of $d\omega$, $d\operatorname{Re}\Omega$ and $d\operatorname{Im}\Omega$, that we denoted $(w_1, \hat{w}_1, w_2, \hat{w}_2, w_3, w_4, w_5)$. From the Apostolov-Salamon equations we have $d\omega = 0$, which implies that $w_1 = \hat{w}_1 = w_3 = w_4 = 0$, and

$$d\operatorname{Re}\Omega = -\frac{3}{4} h^{-1} dh \wedge \operatorname{Re}\Omega - h^{-\frac{3}{4}} d\theta \wedge \omega, \quad d\operatorname{Im}\Omega = -\frac{1}{4} h^{-1} dh \wedge \operatorname{Im}\Omega$$

which gives $\hat{w}_2 = 0$ and $w_5 = -\frac{1}{4}h^{-1}dh$. We know that $d\theta$ can be decomposed as

$$d\theta = f\omega + X\lrcorner \operatorname{Re}\Omega + \kappa_0$$

where f is a function on B, X a vector field and κ_0 is the projection of $d\theta$ onto the space of primitive $(1,1)$-forms. Therefore we have

$$d\theta \wedge \omega = f\omega^2 + (X\lrcorner \operatorname{Re}\Omega) \wedge \omega + \kappa_0 \wedge \omega$$

which imply that $0 = \hat{w}_1 = f$ and $\omega_2 = -h^{-\frac{3}{4}}\kappa_0$. Moreover, the relations between the different components of the torsion force

$$\frac{1}{2}h^{-\frac{1}{4}}dh \wedge \operatorname{Re}\Omega = -(X\lrcorner \operatorname{Re}\Omega) \wedge \omega = -\operatorname{Re}\Omega \wedge (X\lrcorner\omega)$$

where the equality on the right holds because $\operatorname{Re}\Omega \wedge \omega = 0$. Since wedging by $\operatorname{Re}\Omega$ is an injective map on 1-forms, X is the hamiltonian vector field associated with the function $\frac{2}{3}h^{\frac{3}{4}}$. More explicitly we have:

$$d\theta = -\frac{1}{2}h^{-\frac{1}{4}}(Jdh)\lrcorner \operatorname{Re}\Omega + \kappa_0 \tag{5.2}$$

In particular, if the $SU(3)$-structure on the base is torsion-free, then the associated G_2-structure on the total space is torsion-free if and only if h is a constant and θ is a flat connection. Therefore, the metric on the total space is locally just a product metric on $B \times S^1$, which has restricted holonomy group contained in $SU(3)$. Therefore, if we look for irreducible torsion-free G_2-metrics, one must allow a base which is not Calabi-Yau.

From the above computations, we can extract the useful necessary conditions

$$d\theta \wedge \omega^2 = 0, \quad d\left(\frac{4}{3}h^{\frac{3}{4}}\right) = *(d\theta \wedge \operatorname{Re}\Omega) \tag{5.3}$$

which, by Proposition 2.2.6 are equivalent to equation (5.2). On the other hand, suppose that we are given an $SU(3)$-structure on the base satisfying $w_1 = \hat{\omega}_1 = \hat{w}_2 = w_3 = w_4 = 0$, together with a function h and a primitive $(1,1)$-form κ_0 such that

$$w_5 = -\frac{1}{4}h^{-1}dh, \quad w_2 = -h^{-\frac{3}{4}}\kappa_0$$

Then, assuming the the expression $-\frac{1}{2}h^{-\frac{1}{4}}(Jdh)\lrcorner \operatorname{Re}\Omega + \kappa_0$ is the curvature of a connection form θ on a circle bundle $M \to B$, then the G_2-structure determined by $(\omega, \Omega, h, \theta)$ is torsion-free [11, Lemma 3.5].

5.1.2 Adiabatic limit

As in the previous part, let us fix a principal circle bundle $M^7 \to B^6$. We showed that an S^1-invariant G_2-structure φ on total space was equivalent to the data of $(\omega, \Omega, h, \theta)$, where (ω, Ω) is an $SU(3)$-structure, θ a connection form, and h a positive function. The role of the function h here is that it measures the (inverse of the) size of the fibers, that is, the norm of the Killing vector field v associated with the S^1-action. We might as well make a different choice of scaling, by setting $h_\epsilon = \epsilon^2 h$ for some constant ϵ. Rescaling the $SU(3)$-structure by h_ϵ rather than h to obtain a new $SU(3)$-structure $(\omega_\epsilon, \Omega_\epsilon)$, we get $\omega_\epsilon = \epsilon^{-1}\omega$ and $\Omega_\epsilon = \epsilon^{-\frac{3}{2}}\Omega$. The scaling of θ is fixed by the fact that we want it to be a connection form, and thus we set $\theta_\epsilon = \theta$. $(\omega_\epsilon, \Omega_\epsilon)$ is still an $SU(3)$-structure, and the G_2-structure φ is written:

$$\varphi = \omega_\epsilon \wedge (\epsilon\theta_\epsilon) + h_\epsilon^{\frac{3}{4}} \operatorname{Re}\Omega_\epsilon, \quad *_\varphi\varphi = h_\epsilon^{\frac{1}{4}} \operatorname{Im}\Omega_\epsilon \wedge (\epsilon\theta_\epsilon) + \frac{1}{2} h_\epsilon \omega_\epsilon^2$$

and the associated metric is

$$g_\varphi = h_\epsilon^{\frac{1}{2}} \tilde{g}_\epsilon + h_\epsilon^{-1}(\epsilon\theta_\epsilon)^2$$

where $\tilde{g}_\epsilon$ is the metric on the base determined by the $SU(3)$-structure $(\omega_\epsilon, \Omega_\epsilon)$. Here we want to think of ϵ as a parameter that represents what we could think of the average size of the fibers with respect to the metric g_φ, and h_ϵ is the variation with respect to this average.

The Apostolov-Salamon equations (5.1) can be re-written with this scaling as:

$$\begin{aligned} d\omega_\epsilon = 0, \quad d(h_\epsilon^{\frac{3}{4}} \operatorname{Re}\Omega_\epsilon) = -\epsilon d\theta_\epsilon \wedge \omega_\epsilon, \\ d(h_\epsilon^{\frac{1}{4}} \operatorname{Im}\Omega_\epsilon) = 0, \quad \frac{1}{2} dh_\epsilon \wedge \omega_\epsilon^2 = \epsilon h_\epsilon^{\frac{1}{4}} d\theta_\epsilon \wedge \operatorname{Im}\Omega_\epsilon \end{aligned} \tag{5.4}$$

The idea of Foscolo-Haskins-Nordström in [11] is to build a family of G_2-metrics on the total space with fibers that shrink to zero; hence we could hope to find a 1-parameter family of solutions to the system (5.4) where ϵ goes to zero, but $(\omega_\epsilon, \Omega_\epsilon, h_\epsilon, \theta_\epsilon)$ have a finite, non-singular limit $(\omega_0, \Omega_0, h_0, \theta_0)$. We can think of it as the limit of the G_2-structures φ_ϵ that collapses onto (ω_0, Ω_0) when the fibers shrink to zero length. Since the equations

$$\omega_\epsilon \wedge \Omega_\epsilon = 0, \qquad \frac{1}{4} \operatorname{Re}\Omega_\epsilon \wedge \operatorname{Im}\Omega_\epsilon = \frac{1}{6}\omega_\epsilon^3$$

that define an $SU(3)$-structure on the base B still hold in the limit $\epsilon \to 0$, we assume that the limit (ω_0, Ω_0) is a $SU(3)$-structure on B.

If we take the formal limit $\epsilon = 0$ in these equations, we obtain

$$\begin{aligned} d\omega_0 = 0, \quad d(h_0^{\frac{3}{4}} \operatorname{Re}\Omega_0) = 0, \\ d(h_0^{\frac{1}{4}} \operatorname{Im}\Omega_0) = 0, \quad dh_0 \wedge \omega_0^2 = 0 \end{aligned} \tag{5.5}$$

Since wedging by ω_0^2 is an injective map on 1-forms, the last equation implies that h_0 is constant. We will assume $h_0 \equiv 1$ from now on. The other equations then imply $d\omega_0 = 0 = d\Omega_0$, which is the condition for (ω_0, Ω_0) to be a torsion free $SU(3)$-structure on B. Thus, we look for a family of solutions $(\omega_\epsilon, \Omega_\epsilon, h_\epsilon, \theta_\epsilon)$ that collapses to a Calabi-Yau structure (ω_0, Ω_0) on the base B.

In order to build solutions to equations (5.4), the Foscolo-Haskins-Nordström construction proceeds in the same way as for the construction of deformations of a complex manifold in Chapter 4. We start by a fixed circle bundle $M \to B$ over an AC Calabi-Yau 3-fold (B, ω_0, Ω_0). We are looking for a solution $(\omega_\epsilon, \Omega_\epsilon, h_\epsilon, \theta_\epsilon)$ where the 2-form ω_ϵ, the 3-form Ω_ϵ, the function h_ϵ and the connection form θ_ϵ are written as power series expansions:

$$\omega_\epsilon = \omega_0 + \sum_{k=1}^{\infty} \epsilon^k \sigma_k, \qquad \operatorname{Re}\Omega_\epsilon = \operatorname{Re}\Omega_0 + \sum_{k=1}^{\infty} \epsilon^k \rho_k$$
$$h_\epsilon = 1 + \sum_{k=1}^{\infty} \epsilon^k h_k, \qquad \epsilon\theta_\epsilon = \epsilon\theta + \sum_{k=2}^{\infty} \epsilon^k \gamma_k$$

Let us briefly discuss the coefficients appearing in the expansion. Here, the coefficients σ_k are 2-forms and the coefficients ρ_k are 3-forms. They must satisfy some constraints so that $(\omega_\epsilon, \Omega_\epsilon)$ defines an $SU(3)$-structure on B. If ϵ is small enough, ω_ϵ will be non-degenerate and Ω_ϵ will be a stable 3-form, since these conditions are open. Hence, these conditions will be automatically satisfied if we can prove that the series converges at least in C^0-norm. We also want the algebraic relations

$$\omega_\epsilon \wedge \Omega_\epsilon = 0, \qquad \operatorname{Re}\frac{1}{4}\Omega_\epsilon \wedge \operatorname{Im}\Omega_\epsilon = \frac{1}{6}\omega_\epsilon^3$$

to hold at all orders. For the coefficients of h, h_k must be functions on B, and the only constraint is that h must be a positive function, which will be satisfied for ϵ small enough if the series converges in C^0-norm. Finally, θ is a connection form on M and the γ_k's are real 2-forms on B, identified to 2-forms on M that vanish along the fibers. Note that we want to think of θ as a term of order 1.

The aim is to build ALC G_2-structures on M, and hence we will look for a power series expansion with decaying coefficients. More precisely, we want the $\sigma_k, \rho_k, h_k, \gamma_k$ to be smooth with finite $C^{l,\alpha}_\nu$-norm, for some $l \geq 1$, $\alpha \in (0,1)$ and $\nu < 0$, to be specified later. We also want to prove the convergence of the series in $C^{l,\alpha}_\nu$ norm, as in Chapter 4. In particular, the series will converge in C^0-norm, and the open conditions on $(\omega_\epsilon, \Omega_\epsilon, h_\epsilon, \theta_\epsilon)$ will be automatically satisfied.

5.1.3 Overview of the construction

As for Kodaira-Nirenberg-Spencer construction of analytic deformations of complex structure, the Forscolo-Haskins-Nordström construction of solutions to the Apostolov-Salamon equations goes in two steps: solving equations (5.4) formally order by order, and then proving convergence. For deformations of complex structures, the key point was Proposition 4.2.6, that allowed us to find solutions order by order with a good control on Hölder norms. In Remark 4.2.3, we pointed out that solutions of $\overline{\partial}\psi = \Psi$ are not unique, so that without imposing further constraints, we cannot control the norm of solutions, which makes it impossible to prove the convergence of the series. We looked for $\overline{\partial}^*$-exact solutions to work transversally to the space of $\overline{\partial}$-closed 1-form. This allowed us to find solutions with good estimates in Hölder norm, that eventually makes it possible to prove that the series we built converges. For solving the Apostolov-Salamon equations, we also need to impose constraints on the solutions in order to avoid the different sources of non-uniqueness and obtain estimates on the Hölder norm of solutions.

In order to see which conditions to impose, let us write $\Psi = (\omega, \Omega, h, \theta)$ the tuple of variables of the Apostolov-Salamon equations, and A for the differential operator such that $A(\Psi) = 0$ are the Apostolov-Salamon equations. Formally we have an expansion

$$\Psi_\epsilon = \Psi_0 + \epsilon\psi_1 + \sum_{k\geq 2} \epsilon^k \psi_k$$

where $\Psi_0 = (\omega_0, \Omega_0)$ is our limit Calabi-Yau structure, and the components $\psi_k = (\sigma_k, \rho_k, h_k, \gamma_k)$ have been described above. The equation we want to solve will be written as:

$$A(\Psi_\epsilon) = A(\Psi_0) + \epsilon L(\psi_1) + \sum_{k\geq 2} \epsilon^k (L(\psi_k) - P_k(\psi_1, ..., \psi_{k-1})) = 0$$

Here, $A(\Psi_0) = 0$, L is the linearization of the operator A, and P_k, for $k \geq 2$, are differential operators polynomial in the variables $\psi_1, ..., \psi_{k-1}$ and their first order derivatives. As in Chapter 4, we want to solve these equations iteratively, by solving first $L(\psi_1) = 0$, and then, if $\psi_1, ..., \psi_{k-1}$ are constructed, solving

$$L(\psi_k) = P_k(\psi_1, ..., \psi_{k-1})$$

A first issue is the diffeomorphism invariance of the Apostolov-Salamon equations. It implies that the kernel of L is infinite-dimensional, and hence the equation $L(\psi_k) = P_k(\psi_1, ..., \psi_{k-1})$ has no chance of being elliptic. Hence, we need to impose conditions on the coefficients ψ_k in the expansion to break diffeomorphism invariance and recover an equation that can be seen as an elliptic problem, with an operator that has good Fredholm properties. Another source of infinite-dimensional kernel to L is the gauge invariance of the Apostolov-Salamon equations: they only involve the curvature of the connection form, that does not depend on the choice of gauge.

But even when we break diffeomorphism and gauge invariance, and transform our problem into an elliptic one, the equation $L(\psi_k) = P_k(\psi_1, ..., \psi_{k-1})$ does not necessarily admit any solution, and if it does, it may not be unique, because of a residual finite-dimensional kernel of L. As we said above, non-uniqueness of the solutions is problematic because it prevents a good control on the Hölder norm of solutions, which is crucial in order to prove convergence of the series. Moreover, in order to solve $L(\psi_k) = P_k(\psi_1, ..., \psi_{k-1})$, we must prove that $P_k(\psi_1, ..., \psi_{k-1})$ is in the image of L, which often amounts the same as proving that it is orthogonal to some finite-dimensional space of obstructions arising from the cokernel of L. In order to check that this condition is satisfied, it is easier to make the simplest choice possible for the ψ_k's, and assume from the beginning that they are orthogonal to the kernel of L. Thus, we need to impose further constraints to insure that we work transversely to the kernel of the linearization of the Apostolov-Salamon equations.

Let us explicitly write the linearization L of the Apostolov-Salamon equations, with variables $\sigma_1 \in \Omega^2(B)$, $\rho_1 \in \Omega^3(B)$, $h_1 \in C^\infty(B)$, and the connection form θ on B, and make an attempt to solve the Apostolov-Salamon equations at first order in ϵ. The aim is to find constraints on σ_1, ρ_1 and h_1 that insure that the linearized equation can be seen as an elliptic problem, so that we can solve it using the analytical tools of Chapter 3. The linearization of the Apostolov-Salamon equations near (ω_0, Ω_0) is given by

$$\begin{gathered} d\sigma_1 = 0, \qquad \frac{3}{4} dh_1 \wedge \operatorname{Re}\Omega_0 + d\rho_1 = -d\theta \wedge \omega_0, \\ \frac{1}{4} dh_1 \wedge \operatorname{Im}\Omega_0 + d\hat{\rho}_1 = 0, \qquad \frac{1}{2} dh_1 \wedge \omega_0^2 = d\theta \wedge \operatorname{Im}\Omega_0 \end{gathered} \tag{5.6}$$

We see that σ_1 only appears in the first equation $d\sigma_1 = 0$, which means that we could take any closed 1-form σ_1, and yields an infinite-dimensional kernel for L. Hence, in order to avoid terms that have a norm that we cannot control, we would like to make the following assumption:

Assumption 1 We look for a solution with $\sigma_1 \equiv 0$.

Assuming that $\sigma_1 = 0$, the algebraic constraints for $(0, \rho_1)$ to be an infinitesimal deformation of the $SU(3)$-structure (ω_0, Ω_0) are the following:

$$\begin{gathered} \omega_0 \wedge \rho_1 = 0 = \omega_0 \wedge \hat{\rho}_1 \\ \operatorname{Re}\Omega_0 \wedge \hat{\rho}_1 + \rho_1 \wedge \operatorname{Im}\Omega_0 = 0 \end{gathered} \tag{5.7}$$

From equation (5.3), the connection form θ must also satisfy:

$$d\theta \wedge \omega_0^2 = 0 \tag{5.8}$$

As we have seen in §2.2.3, the 3-form ρ_1 can be uniquely written

$$\rho_1 = \nu_1 + \omega_0 \wedge \eta_1 + f_1 \operatorname{Re}\Omega_0 + g_1 \operatorname{Im}\Omega_0$$

with $\nu_1 \in \Omega^3_{12}(B)$, $\eta_1 \in \Omega^1(B)$, $f_1, g_1 \in C^\infty(B)$, and the first line in (5.7) is equivalent to

$$\omega_0^2 \wedge \eta_1 = 0$$

which implies $\eta_1 = 0$. We also can compute explicitly:

$$\operatorname{Re}\Omega_0 \wedge \hat{\rho}_1 + \rho_1 \wedge \operatorname{Im}\Omega_0 = \frac{5}{4} f_1 \operatorname{Re}\Omega_0 \wedge \operatorname{Im}\Omega_0$$

Thus the second line in (5.7) gives $f_1 = 0$. Therefore, ρ_1 must be written in the form

$$\rho_1 = \nu_1 + g_1 \operatorname{Im}\Omega_0$$

for a unique function g_1 on B and $\nu_1 \in \Omega^3_{12}(B)$.

Now let us try to solve equations (5.6) with these constraints. From Proposition 2.2.6, the 1-form dh_1 satisfies

$$*(dh_1 \wedge \omega_0^2) = 2dh_1$$

and hence the last equation in (5.6) gives $dh_1 = *(d\theta \wedge \operatorname{Re}\Omega_0)$. In particular, $d^*dh_1 = 0$, so that h_1 is a harmonic function. Since we are looking for a decaying function, Proposition 3.4.3 implies that h_1 must vanish identically. Therefore, we make without restriction the following additional assumption:

Assumption 2 We look for a solution with $h_1 \equiv 0$.

With this assumption the system (5.6) simplifies as

$$\begin{aligned} d\rho_1 = -d\theta \wedge \omega_0, \quad d\hat{\rho}_1 = 0 \\ d\theta \wedge \operatorname{Im}\Omega_0 = 0 = d\theta \wedge \omega_0^2 \end{aligned} \tag{5.9}$$

where θ is a connection form on M and $\rho_1 = \nu_1 + g_1 \operatorname{Im}\Omega_0$ is a 3-form on B. By Proposition 2.2.5, the two bottom equations are equivalent to the fact that $d\theta$ is a primitive $(1,1)$-form. Therefore, using Proposition 2.2.7, we have $\hat{\rho}_1 = *\rho_1$, and we can rewrite the top equations

$$(d + d^*)\rho_1 = -d\theta \wedge \omega_0 = *d\theta \tag{5.10}$$

Hence, by making the choice that $\sigma_1 = 0 = h_1$, we have managed to reduce the resolution of the Apostolov-Salamon equations at order 1 to an elliptic problem, at least in the variable ρ_1.

Suppose we have been able to solve the Apostolov-Salamon equations up to terms of order $k \geq 2$. Then, in order to solve them at order k, we must solve the following system:

$$\begin{aligned} d\sigma_k = 0, \quad \frac{1}{2}dh_k \wedge \omega_0^2 - d\gamma_k \wedge \operatorname{Im}\Omega_0 = \alpha_{1,k}, \\ \frac{3}{4}dh_k \wedge \operatorname{Re}\Omega_0 + d\rho_k + d\gamma_k \wedge \omega_0 = \alpha_{2,k}, \quad \frac{1}{4}dh_k \wedge \operatorname{Im}\Omega_0 + d\hat{\rho}_k = \alpha_{3,k} \end{aligned} \tag{5.11}$$

where $\alpha_{1,k}, \alpha_{2,k}, \alpha_{3,k}$ are differential forms that depend on $\sigma_j, \rho_j, h_j, \gamma_j$ for $j \leq k-1$. As before, we can choose σ_k to be any closed 2-form, which leads to an infinite-dimensional space of solutions, with all the problems that it creates. Then, in order to simplify the equations, we would like to make the following assumption:

Assumption 3 We look for a solution with $\sigma_k \equiv 0$ for $k \geq 1$, that is, we fix the symplectic form in the $SU(3)$-structure to $\omega_\epsilon \equiv \omega_0$.

Another source problem to get uniqueness of solutions is that the system only depends on $d\gamma_k$, so that it remains unchanged if we add to γ_k a closed 1-form. If we were on a compact manifold, the Hodge theorem would imply that imposing the additional condition $d^*\gamma_k = 0$ breaks this invariance, up to a finite-dimensional space of harmonic 1-forms. By Proposition 3.4.3, there are no closed and co-closed decaying 1-forms, so that in our setting imposing $d^*\gamma_k = 0$ completely breaks the gauge invariance.

Assumption 4 We look for solutions with $d^*\gamma_k = 0$ for $k \geq 1$.

A natural question to ask is whether the assumption we have made are just convenient in order to solve the equations, and have no other justification. It turns out that we can give a geometric interpretation to these assumptions, which are essentially satisfied up to diffeomorphism and gauge invariance. This is important for the following reason. Fix a S^1-bundle $M^7 \to B^6$ over a 6-dimensional AC manifold B. We can ask whether there exists S^1-invariant torsion-free G_2-structures on M. The construction we study gives a 1-parameter family of such G_2-structures, that in addition are ALC. Hence, to answer the question of existence, we may impose as many constraints as we find convenient to build G_2-structures. But if we want to study the moduli-space of G_2-structures that are invariant under the circle action and ALC, then imposing too many constraints makes it impossible to conclude anything. Indeed, the moduli space is $\mathcal{M} = \mathcal{S}/\mathcal{G}$, where $\mathcal{S}$ is the set of solutions to the Apostolov-Salamon equations $A(\Psi) = 0$, and $\mathcal{G}$ is a infinite-dimensional Lie-group, acting by diffeomorphism and gauge transformations that preserve the ALC structure. Therefore, in order to say something about the local structure of the moduli space, we would like to impose constraints that are satisfied up to the action of $\mathcal{G}$.

At least to our knowledge, it is not known in general that this moduli space is smooth, but it is expected to be. This moduli space has a boundary, which represents the different ways by which G_2-structures on M can degenerate. As we discussed above, circle-invariant torsion-free G_2-structures should be able to collapse on AC Calabi-Yau structures on the base M, and therefore we expect part of the boundary of $\mathcal{M}$ to be composed of the moduli space of AC Calabi-Yau structures on the base B. The structure of the moduli space $\mathcal{M}$ near a boundary point represented by an AC Calabi-Yau structure (ω_0, Ω_0) on B is modeled on the kernel of the linearization L of the Apostolov-Salamon equations at (ω_0, Ω_0), modulo the tangent space to the action of diffeomorphisms and gauge transformations. Since we already know, or at least strongly suspect, that the boundary of $\mathcal{M}$ near (ω_0, Ω_0) contains the moduli space of AC Calabi-Yau structures on B, we may further restrict to the part of the kernel of L which is transversal to the space of infinitesimal Calabi-Yau deformations of (ω_0, Ω_0).

With the preceding discussion in mind, we will see that, working transversally to diffeomorphism and gauge invariance, and to a possibly non-empty moduli space of AC Calabi-Yau deformations of (ω_0, Ω_0) on the base, all of our assumptions are satisfied, and further we may take $\rho_k = \nu_k + \alpha_{0,k} \operatorname{Re}\Omega_0$, with $\nu_k \in \Omega^3_{12}(B)$ and $\alpha_{0,k} \in C^\infty(B)$. Therefore, the problem of finding a family $(\omega_\epsilon, \Omega_\epsilon, h_\epsilon, \theta_\epsilon)$ that solves the Apostolov-Salamon equations is essentially reduced to the resolution of the following two linear systems.

At order 1, we want to solve the system

$$d\rho = *d\theta, \quad d^*\rho = 0 \tag{5.12}$$

where ρ is a 3-form on B of the form $\rho = \pi_{12}(\rho) + g \operatorname{Im}\Omega_0$ and θ a connection form on B such that $d\theta$ is a primitive $(1,1)$-form.

At order $k \geq 2$, we want to solve the system

$$\begin{gathered} d^*\gamma = 0, \quad d\gamma \wedge \omega_0^2 = 0, \quad \frac{1}{2} dh \wedge \omega_0^2 - d\gamma \wedge \operatorname{Im}\Omega_0 = \alpha_1, \\ d\rho + \frac{3}{4} dh \wedge \operatorname{Re}\Omega_0 + d\gamma \wedge \omega_0 = \alpha_2, \quad d\rho + \frac{1}{4} dh \wedge \operatorname{Im}\Omega_0 = \alpha_3 \end{gathered} \tag{5.13}$$

where the unknowns are $\rho = \alpha_0 \operatorname{Re}\Omega_0 + \Omega^3_{12}$, $\gamma \in \Omega^2(B)$ and $h \in C^\infty(B)$, and $\alpha_1, \alpha_2, \alpha_3$ fixed differential forms.

In both cases we want to find solutions of class $C^{l,\alpha}_\nu$ norm for some $\nu < 0$, $l \geq 1$ and $\alpha \in (0,1)$. Moreover, for the second system, we would like to control the norm of the solution (ρ, h, γ) by the norm of $\alpha_1, \alpha_2, \alpha_3$, in order to be able to prove that the series $(\Omega_\epsilon, h_\epsilon, \theta_\epsilon)$ has positive radius of convergence.

5.2 Existence of solutions in the adiabatic limit

In this section, we describe in more details the construction of Foscolo-Haskins-Nordström, with a particular emphasis on the use of the analytical tools that we explained in Chapter 3. The setting is the following. We fix a principal circle bundle $M^7 \to B^6$ satisfying the following assumptions. B is an AC Calabi-Yau manifold with Calabi-Yau structure (ω_0, Ω_0), asymptotic to a cone $C(\Sigma)$, where Σ is a compact Sasaki-Einstein manifold. In order to be able to use the results of §3.4.2, we assume that the universal cover of Σ is not isometric to the round 5-sphere. As already noted, it implies in particular that B has finite fundamental group, and is irreducible. By [12, Lemma 2.18], the universal cover of B is still AC Calabi-Yau and has only one end. Therefore, since we care about the restricted holonomy group, we may as well assume from the beginning that B is simply connected. The simply-connectedness is useful because in that context, the circle bundles over B are classified by their first Chern class. In view of Remark 5.1.2, we assume that M is non-trivial, i.e., $c_1(M) \neq 0$, and satisfies the following necessary condition:

$$c_1(M) \cup [\omega_0] = 0 \in H^4(B) \tag{5.14}$$

Later on when we refer to the Calabi-Yau manifold B, we will always assume that these conditions are satisfied, unless otherwise noted. Although some of the results hold in a more general context, we prefer to assume all of this from the beginning for clarity.

5.2.1 Gauge fixing

The aim of this section is to give a geometric interpretation to the constraints imposed on the coefficients of the expansion of $(\omega_\epsilon, \Omega_\epsilon, h_\epsilon, \theta_\epsilon)$, before solving the Apostolov-Salamon equations at all orders in the next sections. In view of our discussion in §5.1.3, we want conditions to insure that $(\omega_\epsilon, \Omega_\epsilon)$ is transversal to Calabi-Yau deformations of (ω_0, Ω_0), which includes deformations by the action of diffeomorphisms and deformations along a possibly non-trivial moduli space of Calabi-Yau structures on B.

As we explained in §5.1.3, if we want to transform the linearization of the Apostolov-Salamon equations into an elliptic problem, we may assume that $\omega_\epsilon \equiv \omega_0$. That is, we look for deformations of the $SU(3)$-structure (ω_0, Ω_0) on the base B where the symplectic is fixed, and we only vary the stable 3-form Ω_ϵ. In order to solve the equations, we might as well just accept this assumption, but it is interesting to interpret this condition geometrically. As noted in [11, §7.1], that we essentially follow here, any cohomology class near $[\omega_0] \in H^2(B)$ is the cohomology class of a Ricci-flat Kähler form. Hence, fixing the cohomology class of the symplectic forms ω_ϵ to $[\omega_\epsilon] \equiv [\omega_0] \in H^2(B)$ means that we look for deformations of the $SU(3)$-structure on B that are transverse to Calabi-Yau deformations. Once the cohomology class of the symplectic forms ω_ϵ is fixed, the condition that $\omega_\epsilon \equiv \omega_0$ is satisfied up to the action of diffeomorphisms. Indeed, it is known that on compact manifolds, any 1-parameter family of symplectic forms that represent a fixed cohomology class are related by a path of diffeomorphisms. This comes from the so-called *Moser trick*, which can be adapted to the AC setting in the following way:

Lemma 5.2.1. *Let (B, ω_0, Ω_0) be an AC Calabi-Yau manifold, and suppose that $\{\omega_\epsilon\}_{0 \leq \epsilon < \epsilon_0}$ is a smooth family of symplectic forms on B. Suppose moreover that $[\omega_\epsilon] \equiv [\omega_0]$, and that for all ϵ, $\omega_\epsilon - \omega_0 \in C^{k,\alpha}_\nu$, for some $k \geq 1$, $\alpha \in (0,1)$ and $\nu \in (-5,-1)$ independent of ϵ.*

Then there exists a family ϕ_ϵ of diffeomorphisms of B, that preserve the asymptotic structure of B, and such that ${\phi_\epsilon}^* \omega_\epsilon = \omega_0$ for all $\epsilon \in [0, \epsilon_0)$.

Proof. If ϕ_ϵ is any family of diffeomorphisms on B and X_ϵ is the family of vector fields defined by

$$\frac{d}{d\epsilon}\phi_\epsilon = X_\epsilon \circ \phi_\epsilon$$

then we have the formula

$$\frac{d}{d\epsilon}{\phi_\epsilon}^*\omega_\epsilon = {\phi_\epsilon}^*\left(\mathcal{L}_{X_\epsilon}\omega_\epsilon + \frac{d\omega_\epsilon}{d\epsilon}\right) = {\phi_\epsilon}^*\left(d(X_\epsilon \lrcorner \omega_\epsilon) + \frac{d\omega_\epsilon}{d\epsilon}\right)$$

Therefore it is sufficient to find a family of vector fields X_ϵ such that $d(X_\epsilon \lrcorner \omega_\epsilon) + \frac{d\omega_\epsilon}{d\epsilon} = 0$. To insure that this family of vector fields can be integrated and defines a family of diffeomorphisms that leaves invariant the asymptotic structure of B, we want X_ϵ to decay at infnity.

By our assumptions, $\frac{d\omega_\epsilon}{d\epsilon}$ is a smooth family of exact 2-forms of class $C^{k,\alpha}_\nu$. Our aim is to find a smooth family of primitives. Let σ be an exact 2-form of class $C^{k,\alpha}_\nu$. Then $d^*\sigma \in C^{k-1,\alpha}_{\nu-1}$. By our choice of ν, the Laplacian $\Delta : C^{k+1,\alpha}_{\nu+1} \to C^{k-1,\alpha}_{\nu-1}$ acting on 1-forms in an isomorphism. Indeed, $\nu+1$ and $-5-\nu$ are negative weights, so that by Proposition 3.4.3, there are no non-trivial harmonic 1-forms in $C^\infty_{\nu+1}$ and $C^\infty_{-5-\nu}$. Let $G : C^{k-1,\alpha}_{\nu-1} \to C^{k+1,\alpha}_{\nu+1}$ be the inverse of Δ, and $\gamma = Gd^*\sigma$. We have

$$\Delta d^*\gamma = d^*dd^*\gamma = d^*\Delta\gamma = (d^*)^2\sigma = 0$$

and by, $d^*\gamma = 0$. We want to prove that $d\gamma = \sigma$. The 2-form $d\gamma - \sigma$ is exact, and in particular closed. Since γ is co-closed ,we have

$$d^*d\gamma - d^*\sigma = \Delta\gamma - d^*\sigma = 0$$

and therefore $d\gamma - \sigma \in \mathcal{H}^2_\nu$. Moreover, it represents the trivial cohomology class, thus we conclude by that $d\gamma = \sigma$.

Set $\gamma_\epsilon = Gd^*\left(\frac{d\omega_\epsilon}{d\epsilon}\right)$, so that $d\gamma_\epsilon = \frac{d\omega_\epsilon}{d\epsilon}$. Then γ_ϵ is of class $C^{k+1,\alpha}_{\nu+1}$, and moreover from its expression it is clear that γ_ϵ is smooth in the parameter ϵ. The equation

$$X_\epsilon \lrcorner \omega_\epsilon = \gamma_\epsilon$$

defines a unique smooth family of vector fields X_ϵ, and since $\gamma_\epsilon \in C^{k+1,\alpha}_{\nu+1}$, and that ω_ϵ is asymptotic to ω_0, then X_ϵ is of class $C^{k+1,\alpha}_{\nu+1}$. Since $\nu+1<0$, we can integrate X_ϵ to a smooth family of diffeomorphisms ϕ_ϵ that preserves the asymptotic structure of B. By construction, it is clear that we have ${\phi_\epsilon}^*\omega_\epsilon = \omega_0$. □

Remark 5.2.1. From this proof, we can extract the useful fact than any exact 2-form in $C^{k,\alpha}_\nu$ admits a unique co-closed primitive in $C^{k+1,\alpha}_{\nu+1}$.

Therefore, if we seek a solution $(\omega_\epsilon, \Omega_\epsilon, h_\epsilon, \theta_\epsilon)$ to the Apostolov-Salamon equations such that $[\omega_\epsilon] \equiv [\omega_0]$, then we may assume without loss of generality that $\omega_\epsilon \equiv \omega_0$, since it is satisfied up to the action of diffeomorphisms. This condition is not enough to completely break the diffeomorphism invariance of the Apostolov-Salamon equations, and we still need a condition to insure that we work transversally to the action of diffeomorphisms that preserve ω_0.

At the end of [11, §7.1], Foscolo-Haskins-Nordström argue that we can assume further

$$\operatorname{Re}\Omega_\epsilon \wedge \operatorname{Re}\Omega_0 = 0$$

This equation means that $\operatorname{Re}\Omega_\epsilon$ has no component of the form $f \operatorname{Im}\Omega_0$. However, this condition can only be imposed at each order of the expansion, and not globally. Indeed, in order to prove that this condition is satisfied, we would need to take at deformation $(\omega_0, \Omega_\epsilon)$ of the $SU(3)$-structure (ω_0, Ω_0), and find a path of diffeomorphisms such that ${\phi_\epsilon}^* \operatorname{Re}\Omega_\epsilon \wedge \operatorname{Re}\Omega_0 = 0$. By differentiating, this condition is equivalent to

$$\left(\mathcal{L}_{X_\epsilon} \operatorname{Re}\Omega_\epsilon + \frac{d \operatorname{Re}\Omega_\epsilon}{d\epsilon}\right) \wedge (\phi_\epsilon^{-1})^* \operatorname{Re}\Omega_0 = 0 \tag{5.15}$$

The remaining $(\phi_\epsilon^{-1})^*$ in the second factor makes it impossible to apply a similar argument to Moser trick, because we cannot reduces our problem to something linear in X_ϵ. What is proven in [11, §7.1] is that this condition can be satisfied at $\epsilon = 0$:

Lemma 5.2.2. *Let (B, ω_0, Ω_0) be an AC Calabi-Yau manifold, and ρ a smooth 3-form of class $C^{k,\alpha}_\nu$ for some $k \geq 1$, $\alpha \in (0,1)$ and $\nu \in (-1,-5)$. Suppose that $(0, \rho)$ is an infinitesimal deformation of the $SU(3)$ structure (ω_0, Ω_0), that is,*

$$\omega_0 \wedge \rho = 0 = \rho \wedge \operatorname{Im}\Omega_0 + \operatorname{Re}\Omega_0 \wedge \hat{\rho}$$

Then, there exists a unique smooth vector field X of class $C^{k+1,\alpha}_{\nu+1}$ on B, such that $\mathcal{L}_X \omega_0 = 0$ and $(\rho + \mathcal{L}_X \operatorname{Re}\Omega_0) \wedge \operatorname{Re}\Omega_0 = 0$.

Remark 5.2.2. We tried to prove that this lemma implied that equation (5.15) had a solution ϕ_ϵ, so that the condition $\operatorname{Re}\Omega_\epsilon \wedge \operatorname{Re}\omega_0$ was indeed satisfied up to the action of diffeomorphisms. However, we could not make it work, and we are not sure that it is actually possible. In any case, this assumption is sufficient for the construction. Lemma 5.2.2 at least gives a geometric interpretation to the condition $\operatorname{Re}\Omega_\epsilon \wedge \operatorname{Re}\omega_0 = 0$: it insures the transversality to the action of diffeomorphisms preserving the symplectic form ω_0.

Therefre, in order to work tranversely to the kernel of the linearization of the Apostolov-Salamon equation, we will impose that the coefficients of $\operatorname{Re}\Omega_\epsilon = \operatorname{Re}\Omega_0 + \sum_k \epsilon^k \rho_k$ satisfy

$$\rho_k \wedge \operatorname{Re}\Omega_0 = 0$$

in addition to the algebraic relations arising from the expansion of the compatibility equations equations on $(\omega_0, \Omega_\epsilon)$.

The last invariance that we want to break is the invariance by gauge transformations on the circle bundle $M \to B$. If $\epsilon\theta_\epsilon = \epsilon\theta + \sum_{k\geq 2} \epsilon^k \gamma_k$, this gauge invariance just means that only $d\gamma_k$ appears in the Apostolov-Salamon equations. In the proof of Lemma 5.2.1, we have seen that any exact 2-form of class $C^{k,\alpha}_\nu$ for $\nu \in (-5,-1)$ has a unique co-closed primitive of class $C^{k+1,\alpha}_{\nu+1}$. Therefore, we can impose the condition $d^*\gamma_k = 0$.

5.2.2 The linearized equation

Now that we have justified our assumptions on the expected form of the expansion of $\Psi_\epsilon = (\omega_\epsilon, \Omega_\epsilon, h_\epsilon, \theta_\epsilon)$, we are back to the problem of finding a solution to the Apostolov-Salamon equations in the adiabatic limit. Remember that we have:

$$\omega_\epsilon \equiv \omega_0, \quad \operatorname{Re}\Omega_\epsilon = \operatorname{Re}\Omega_0 + \epsilon\rho_1 + \sum_{k\geq 2} \epsilon^k \rho_k,$$
$$h_\epsilon = 1 + \sum_{k\geq 2} \epsilon^k h_k, \quad \epsilon\theta_\epsilon = \epsilon\theta + \sum_{k\geq 2} \epsilon^k \gamma_k$$

where $\rho_1 \in \Omega^3_{12}(B)$, $\rho_k \in \Omega^3_{12}(B) \oplus \mathbf{R}\operatorname{Re}\Omega_0$ for $k \geq 2$, h_k are functions on B for $k \geq 2$, θ is a connection form on $M \to B$, and for $k \geq 2$, γ_k is a co-closed 1-form on B. Moreover, we are seeking a solution where Ψ_ϵ is $C^{k,\alpha}_\nu$-close to its collapsed limit (ω_0, Ω_0), for some $k \geq 1$, $\alpha \in (0,1)$, and $\nu < 0$, to be determined later.

By the Hitchin's duality map, we can deduce the expression of $\operatorname{Im}\Omega_\epsilon$ from $\operatorname{Re}\Omega_\epsilon$. We have

$$\operatorname{Im}\Omega_\epsilon = \operatorname{Im}\Omega_0 + \epsilon\hat{\rho}_1 + \sum_{k\geq 2} \epsilon^k (\hat{\rho}_k + Q_k(\rho_1, ..., \rho_{k-1}))$$

for some Q_k polynomial in its arguments $\rho_1, ..., \rho_{k-1}$ for $k \geq 2$. We also set $Q_1 = 1$. The algebraic constraint $\omega_\epsilon \wedge \operatorname{Re}\Omega_\epsilon = 0$ is satisfied by our choice of ρ_k, and the constraint

$$\frac{1}{4}\operatorname{Re}\Omega_\epsilon \wedge \operatorname{Im}\Omega_\epsilon = \frac{1}{6}\omega_0^3$$

is equivalent to imposing at all orders

$$\alpha_{0,k}\operatorname{Re}\Omega_0 \wedge \operatorname{Im}\Omega_0 + \operatorname{Re}\Omega_0 \wedge Q_k + \sum_{m=1}^{k-1} \hat{\rho}_{k-m} \wedge (\rho_m + Q_m) = 0$$

which determines $\alpha_{0,k}$ as a polynomial function of $\rho_1, ..., \rho_{k-1}$ [11, Equation 8.2]. Hence $\alpha_{0,k}$ is not to be considered as a variable of our problem, the only variable being the Ω^3_{12} projection of ρ_k.

Remark 5.2.3. Suppose that ν is not an indicial root of $d+d^*$ acting on k forms. Then for sufficiently small $\delta < 0$, we have $\mathcal{H}^k_\nu = \mathcal{H}^k_{\nu-\delta}$. Therefore, considering the inclusion $C^{0,\alpha} \subset L^2_\nu$, the inner product with any element of $\mathcal{H}^k_\nu$ defines a continuous linear form on $C^{0,\alpha}$. Following the conventions of Foscolo-Haskins-Nordström, we will denote by $\mathcal{W}^k_\nu$ the closed subspace of $C^{0,\alpha}_\nu$ L^2_ν-orthogonal to $\mathcal{H}^k_\nu$. This is a closed complement of $\mathcal{H}^k_\nu$.

As we have seen in §5.1.3, the resolution of the Apostolov-Salamon equations at order 1 is equivalent to solving the system

$$d\rho = *d\theta, \quad d^*\rho = 0 \tag{5.16}$$

where θ is a connection form such that the curvature $d\theta$ is a primitive $(1,1)$-form and in view of §5.2.1, we impose $\rho \in \Omega^3_{12}(B)$. Here is the main existence theorem for solutions of this system [11, Theorem 6.1]:

Theorem 5.2.3 (Foscolo-Haskins-Nordström) *Let (B, ω_0, Ω_0) be a simply connected irreducible AC Calabi-Yau manifold and $M \to B$ a non-trivial principal circle bundle. Assume that*

$$c_1(M) \cup [\omega_0] = 0 \in H^4(B)$$

Fix $\nu = -1 + \delta$ for some small $\delta > 0$. Then there exists a unique solution (θ, ρ) of (5.16) such that $d\theta \in C^\infty_{-2}$ and $\rho \in \Omega^3_{12} \cap C^\infty_{-1} \cap \mathcal{W}^3_{-1+\delta}$.

The rest of this part is dedicated to the proof of this theorem.

Since M is a non-trivial circle bundle, and we assumed that B is simply connected, it follows that $c_1(M)$ is a non-trivial cohomology class in $H^2(B)$. We want to represent this cohomology class by a closed and co-closed 2-form in $\mathcal{H}^2_{-2+\delta}(B)$ for $\delta > 0$ arbitrarily small. We know by Proposition 3.4.2 (i) that $\mathcal{H}^2_{-2+\delta} \simeq L^2\mathcal{H}^k \simeq H^2_c(B)$ for every $\delta > 0$ small enough. Therefore, we cannot a priori represent $c_1(M)$ by a closed and co-closed form with decay rate better than -2.

As a consequence of Proposition 3.4.2 (ii), the cohomology class $c_1(M) \in H^2(B)$ can be represented by a unique closed and co-closed form $\kappa \in \mathcal{H}^2_{-2+\delta}$. Moreover, we can write at infinity

$$\kappa = \tau + O(r^{-2-\mu})$$

for some $\mu < 0$, where $\tau \in H^2(\Sigma)$ is the image of $c_1(M)$ by the map $H^2(B) \to H^2(\Sigma)$. Since κ is closed and co-closed, it is in particular harmonic. As we have seen in §2.2.3, we have a decomposition

$$\Omega^2(B) \simeq \mathbf{R} \oplus \Omega^1(B) \oplus \Omega^{1,1}_0(B)$$

with respect to the Calabi-Yau structure (ω_0, Ω_0). By the Weitzenböck formula, the Laplacian Δ preserves this decomposition. Since there are no non-trivial decaying harmonic functions and 1-forms, the $\mathbf{R} \oplus \Omega^1(B)$ part of κ must vanish, which insures that κ is a primitive $(1,1)$-form.

If θ' is any hermitian connection on the principal circle bundle $M \to B$, then $c_1(M) = [d\theta'] = [\kappa]$, which means that there exists $a \in \Omega^1(B)$ such that

$$\kappa = d\theta' + da = d\theta$$

if θ is the connection form $\theta = \theta' + a$. Moreover, if θ'' is another connection form on M such that $d\theta'' = \kappa = d\theta$, then $d(\theta'' - \theta) = 0$, and since $H^1(B) = 0$, there exists $f \in C^\infty(B)$ such that $\theta'' = \theta + df$. Hence θ is uniquely defined up to gauge transformations on $M \to B$.

Now we would like to solve the equation

$$(d + d^*)\rho = *\kappa$$

where $\rho \in C^\infty_{-1+\delta}$, for δ arbitrarily small. Any solution is a 3-form, because $*\kappa$ is closed and co-closed, so that any solution ρ must be harmonic, and there are no decaying harmonic 1- and 5-forms. The obstructions to solving this equation lie in $\mathcal{H}^4_{-4-\delta}$. From Proposition 2.2.6, we have

$$\kappa \wedge \omega_0 = - * \kappa$$

because κ is a primitive $(1,1)$-form. Therefore, taking the cohomology classes in this identity, the assumption $c_1(M) \cup [\omega_0] = 0$ implies that $*\kappa$ is exact. We want to prove that $*\kappa$ is orthogonal to $\mathcal{H}^4_{-4-\delta}$, which by duality is the same as proving that κ is orthogonal to $\mathcal{H}^2_{-4-\delta} \simeq L^2\mathcal{H}^2 \simeq \mathcal{H}^2_{-6+\delta}$. As we have seen at the end of §3.3.3, we have an isomorphism $L^2\mathcal{H}^2 \simeq H^2_c(B)$, that can be realized by writing any $\sigma \in L^2\mathcal{H}^k$ as

$$\sigma = \sigma_c + d\gamma$$

where σ_c is a compactly supported closed form and γ is a 1-form defined on the end of B, is defined by radial integration. Since $\sigma \in C^\infty_{-6+\delta}$, we have $\gamma \in C^\infty_{-5+\delta}$. Thus, if we write $*\kappa = d\eta$, we have

$$*\kappa \wedge \sigma = d(\eta \wedge \sigma + *\kappa \wedge \gamma)$$

and therefore

$$\begin{aligned} \langle \kappa, \sigma \rangle_{L^2} &= \int_B *\kappa \wedge \sigma \\ &= \int_B d(\eta \wedge \sigma + *\kappa \wedge \gamma) \\ &= \lim_{r \to \infty} \int_\Sigma *\kappa(r) \wedge \gamma(r) \\ &= 0 \end{aligned}$$

and thus κ is orthogonal to $L^2\mathcal{H}^2$. We could not have concluded this directly from the exactness of κ, because we need to control the decay rate of the primitive, whereas by radial integration we can control the decay of the primitive at infinity.

It follows that for any sufficiently small $\delta > 0$, we can solve the equation

$$(d + d^*)\rho = *d\theta$$

with $\rho \in \Omega^3(B) \cap C^\infty_{-1+\delta}$. Moreover, the solution is unique if we impose the condition. Moreover, since ρ is harmonic and decays, ρ must be in $\Omega^3_{12}(B)$, and smooth by elliptic regularity.

The precise asymptotic behavior $\rho \in C^\infty_{-1}$ comes from solving the equation $(d + d^*)\rho = *\kappa$ on the cone $C(\Sigma)$ itself rather than B [11, p. 36].

Let us make an informal comment on Theorem 5.2.3. As we have seen in §5.2.1, the assumption we made to reduce the linearized Apostolov-Salamon equations to the system

$$d\rho = *d\theta, \quad d^*\rho = 0$$

are satisfied up to the action of diffeomorphisms and gauge transformations, at least if we restrict ourselves to deformations $(\omega_\epsilon, \Omega_\epsilon)$ that are transverse to Calabi-Yau deformations of (ω_0, Ω_0). In terms of moduli space, it would mean that we can only collapse on the moduli space of AC Calabi-Yau structures on B from one direction, so we might expect the moduli space to be locally of dimension one more than the moduli space of Calabi-Yau structures on B.

5.2.3 Construction of solutions

Once the linearized equations is solved, we need to solve the Apostolov-Salamon equations iteratively at higher order. Supposed that they are solved up to order $m-1$ for some $m \geq 2$, and therefore we have a truncated expansion

$$\operatorname{Re}\Omega_\epsilon = \operatorname{Re}\Omega_0 + \epsilon\rho + \sum_{k=2}^{m-1} \epsilon^k \rho_k,$$
$$h = 1 + \sum_{k=2}^{m-1} \epsilon^k h_k, \quad \epsilon\theta_\epsilon = \epsilon\theta + \sum_{k=2}^{m-1} \epsilon^k \gamma_k$$

which solves the equations up to order $m-1$ in ϵ. Here, $(\rho_1, \gamma_1) = (\rho, \theta)$ is the solution to the linearized problem determined in Theorem 5.2.3 and

$$(\rho_k, h_k, \gamma_k)$$

are of class $C^{l,\alpha}_\nu$ for some fixed $l \geq 1$, $\alpha \in (0,1)$, and $\nu \in (-2,-1)$, away from some set of indicial roots. We will justify later the choice of weight made by Foscolo-Haskins-Nordström. Moreover, in view of §5.2.1, recall that we assume that

$$d^*\gamma_k = 0, \qquad \rho_k = \alpha_{0,k} + \pi_{12}\rho_k$$

We seek a solution (ρ_m, h_m, γ_m) of class $C^{l,\alpha}_\nu$ at order $m+1$ satisfying the constraints as above. According to the system (5.13), this is equivalent to solving

$$\begin{gathered} d^*\gamma_m = 0, \quad d\gamma_m \wedge \omega_0^2 = 0, \\ \frac{1}{2} dh_m \wedge \omega_0^2 - d\gamma_m \wedge \operatorname{Im}\Omega_0 = \alpha_{1,m}, \\ d\rho_m + \frac{3}{4} dh_m \wedge \operatorname{Re}\Omega_0 + d\gamma_m \wedge \omega_0 = \alpha_{2,m}, \\ d\rho_m + \frac{1}{4} dh_m \wedge \operatorname{Im}\Omega_0 = \alpha_{3,m} \end{gathered} \tag{5.17}$$

According to [11, Equation 8.4], the forms $\alpha_{i,m}$, for $i = 1, 2, 3$, are explicitly given as functions of (ρ_k, h_k, γ_k), $k = 1, ..., m-1$, by

$$\begin{aligned}
\alpha_{1,m} &= \sum_{k=1}^{m-1} (\epsilon d\theta_\epsilon)^{[k]} \wedge (h_\epsilon^{\frac{1}{4}} \operatorname{Im} \Omega_\epsilon)^{[m-1-k]} \\
\alpha_{2,m} &= -d \left(\sum_{k=1}^{m-1} (h_\epsilon^{\frac{3}{4}})^{[k]} (\operatorname{Re} \Omega_\epsilon)^{[m-1-k]} \right) \\
\alpha_{3,m} &= -d \left(Q_k + \sum_{k=1}^{m-1} (h_\epsilon^{\frac{1}{4}})^{[k]} (\operatorname{Im} \Omega_\epsilon)^{[m-1-k]} \right)
\end{aligned} \tag{5.18}$$

Here we stick to our conventions of §4.2.3, and for an expansion Ψ_ϵ we denote by $(\Psi_\epsilon)^{[k]}$ the homogeneous part of order k, which is slightly different from the conventions used by Foscolo-Haskins-Nordström. By induction, $\alpha_{1,m}$ is closed and $\alpha_{2,m}, \alpha_{3,m}$ are exact.

Unfortunately, the system (5.2.3) cannot be solved as such: one has to add variables in order to turn it into an elliptic problem. The situation is reminiscent of Joyce's proof of smoothness of the moduli space of torsion-free G_2-structures over a compact manifold [17, §10.3-10.4]. The rough idea is to take advantage of the relations existing between the different components of the intrinsic torsion. In particular, we noted in Remark 2.3.1 that for a G_2-structure φ, $\pi_7(d\varphi) = 0$ if and only if $\pi_7(d * \varphi) = 0$. In Joyce's argument, one shows that if the part $d\varphi = 0$ of the intrinsic torsion vanishes, then rather than solving $d * \varphi = 0$, we may add additional parameters, say χ, and solve $d\Theta(\varphi) = \chi$. If the parameters χ are small enough, they are forced to vanish and φ is torsion-free.

In our situation, we already know that some components of the intrinsic torsion automatically vanish. Thus it is possible add free parameters to the system (5.2.3), in order to find solutions with estimates. If the series converges, then these parameters have to vanish for ϵ small enough. Here is the precise statement:

Proposition 5.2.4. *Let $\mathfrak{c}_0 = (\omega_0, \Omega_0)$ be an AC Calabi-Yau structure on a 6-fold B and denote by g_0 and ∇_0 the induced metric and Levi-Civita connection. Fix $k \geq 1$, $\alpha \in (0, 1)$ and $\nu < -1$. Then there exists a constant ϵ_0 such that the following holds. Let $\mathfrak{c} = (\omega, \Omega)$ be a second $SU(3)$-structure on B, such that $\|\mathfrak{c} - \mathfrak{c}_0\|_{C_0^k} < \epsilon_0$. We do not require $\mathfrak{c}$ to be smooth. Suppose that there exists a function h and an integral closed 2-form $\kappa = d\theta$ on B such that*

$$\kappa \wedge \omega^2 = 0, \qquad \frac{1}{2} dh \wedge \omega^2 = h^{\frac{1}{4}} \wedge \operatorname{Im} \Omega$$

Moreover, assume the existence of functions u, v and a vector field X in $C_{\nu+1}^{k+1,\alpha}$ such that

$$d\omega = 0, \quad d(h^{\frac{3}{4}} \operatorname{Re} \Omega) + \kappa \wedge \omega = d * d(u\omega_0),$$
$$d(h^{\frac{1}{4}} \operatorname{Im} \Omega) = d * d(X \lrcorner \operatorname{Re} \Omega_0 + v\omega_0)$$

where $*$ is computed with respect to the metric g_0. Then $u = v = 0 = X$, i.e. $(\omega, \Omega, h, \theta)$ is a solution of the Apostolov-Salamon equations.

Remark 5.2.4. Our statement slightly differs from the one of the article. In the article, the right hand side of the last two equations is respectively $d{*}d(u\omega)$ and $d{*}d(X \lrcorner \operatorname{Re}\Omega + v\omega)$ instead of $d{*}d(u\omega_0)$ and $d{*}d(X \lrcorner \operatorname{Re}\Omega_0 + v\omega_0)$. The proof is exactly the same as the proof of Proposition 7.4 in [11], which only uses the fact that the projection of the bundle of differential forms onto forms of particular type with respect to $\mathfrak{c}$ and $\mathfrak{c}_0$ are close enough in the relevant weighted Hölder norm. However, we believe that the fact that the term added in the article is non-linear makes it impossible to control iteratively the Hölder norm of the terms of the expansion, especially because of the term $d * d(X \lrcorner \operatorname{Re}\Omega)$, which involves two derivatives (since ω is set to ω_0, the other two terms are actually linear). If $\operatorname{Re}\Omega - \operatorname{Re}\Omega_0$ is in $C^{k,\alpha}_{\nu}$, it gives terms in $C^{k-2,\alpha}_{\nu-2}$, which makes the assumptions of the following theorem fail, and causes a loss of regularity when we solve the equations iteratively. However, we believe that adding only a linear term on the right hand side is enough to conclude and overcome these difficulties, by the exact same argument as used in the article.

Proof. We first prove that $u = 0$. The equation involving $\operatorname{Re}\Omega$ can be written

$$d \operatorname{Re}\Omega = -\frac{3}{4} h^{-1} dh \wedge \operatorname{Re}\Omega - h^{-\frac{3}{4}} \kappa \wedge \omega + d * d(u\omega_0)$$

Since $d\omega = 0$, the algebraic relations between the components of the intrinsic torsion of (ω, Ω) imply that $d \operatorname{Re}\Omega$ can be written

$$d \operatorname{Re}\Omega = w_5 \wedge \operatorname{Re}\Omega + w_2 \wedge \omega$$

where w_5 is a 1-form and w_2 a primitive $(1,1)$-form for the decomposition induced by $\mathfrak{c}$. Since $\kappa \wedge \omega^2 = 0$ by assumption, it follows that $\kappa = \kappa_0 + Y \lrcorner \operatorname{Re}\Omega$ for some primitive $(1,1)$-form κ_0 (with respect to $\mathfrak{c}$) and Y a vector field. In particular

$$\kappa \wedge \omega = \kappa_0 \wedge \omega - (Y \lrcorner \omega) \wedge \operatorname{Re}\Omega$$

Therefore $\pi_1'(d \operatorname{Re}\Omega) = 0$ implies $\pi_1'(d * d(u\omega_0)) = 0$, where the projection π_1' is relative to the decomposition induced by the $SU(3)$-structure (ω, Ω). From Corollary 3.4.9 and the continuous injection $C^k_0 \hookrightarrow C^{k-1,\alpha}_0$, for ϵ_0 small enough, there exists a constant $C_1 > 0$ such that, if for any function $f \in C^{k+1,\alpha}_{\nu+1}$, we have the inequality

$$\|\pi_1'(d * d(f\omega_0)) - \pi_1(d * d(f\omega_0))\|_{C^{k-1,\alpha}_{\nu-1}} \leq C_1 \|f\|_{C^{k+1,\alpha}_{\nu+1}} \|\mathfrak{c} - \mathfrak{c}_0\|_{C^{k-1,\alpha}_0}$$

On the other hand, according to Proposition 3.4.4, since $\nu < -1$, then $\pi_1(d * d(f\omega_0)) = 0$ implies $f = 0$, and

$$\|f\|_{C^{k+1,\alpha}_{\nu+1}} \leq C_2 \|\pi_1(d * d(f\omega_0))\|_{C^{k-1,\alpha}_{\nu-1}}$$

Therefore, taking into account these two inequalities, it follows that for small enough $\|\mathfrak{c}-\mathfrak{c}_0\|_{C^k_0}$, $\pi'_1(d*d(u\omega_0))=0$ also implies $u=0$.

Now we prove $v=0=X$. By the algebraic relations between ω and Ω such as explained in §2.2.3, the equation $\frac{1}{2}dh\wedge\omega^2=h^{\frac{1}{4}}\kappa\wedge\operatorname{Im}\Omega$ is equivalent to:

$$\pi'_6(\kappa)=-\frac{1}{2}h^{-\frac{1}{4}}(Jdh)\lrcorner\operatorname{Re}\Omega$$

Inserting this into the equation $d(h^{\frac{3}{4}}\operatorname{Re}\Omega)+\kappa\wedge\omega=0$ yields

$$\pi'_6(d\operatorname{Re}\Omega)=-\frac{1}{4}h^{-1}dh\wedge\operatorname{Re}\Omega$$

Since $d\omega=0$ by assumption, it follows from the relations between the components of the intrinsic torsion of (ω,Ω) that

$$\pi'_{1\oplus 6}\left(h^{\frac{1}{4}}d\operatorname{Im}\Omega+\frac{1}{4}h^{-\frac{3}{4}}dh\wedge\operatorname{Im}\Omega\right)=0$$

which leads to $\pi'_{1\oplus 6}d*d(X\lrcorner\operatorname{Re}\Omega_0+v\omega_0)=0$. By a similar argument as above, Corollary 3.4.9 and Proposition 3.4.5 we can conclude $X=0=v$. □

The strategy of Foscolo-Haskins-Norström is then to solve with additional parameters, say u_m,v_m,X_m, to be carried at each step. Therefore, we end up with a tuple $(\omega_\epsilon,\Omega_\epsilon,h_\epsilon,\theta_\epsilon)$ and additional parameters $(u_\epsilon,v_\epsilon,X_\epsilon)$, that satisfy at all order the modified version of the Apostolov-Salamon equations as in the proposition above. Whenever the series converges in the appropriate norms, the parameters $(u_\epsilon,v_\epsilon,X_\epsilon)$ are forced to vanish for ϵ small enough, and therefore $(\omega_\epsilon,\Omega_\epsilon,h_\epsilon,\theta_\epsilon)$ is a genuine solution of the Apostolov-Salamon equations.

The main analytical theorem of the article is the following:

Theorem 5.2.5 *Let (B,ω_0,Ω_0) be an AC Calabi-Yau 3-fold. Fix $k\geq 1$, $\alpha\in(0,1)$, $\delta>0$ small enough and $\nu\in(-3-\delta,-1)$ away from a discrete set of indicial roots. Then there exists a constant $C>0$ such that the following holds.*

Let $\alpha_0\in C^{k,\alpha}_\nu$ be a function, $\alpha_1\in C^{k-1,\alpha}_{\nu-1}$ a closed 5-form, and $\alpha_2=d\beta_2$, $\alpha_3=d\beta_3$ exact 4-forms with $\beta_2,\beta_3\in C^{k,\alpha}_\nu$. Then there exists a unique function h, 1-form γ, 3-form ρ of the form $\frac{1}{2}\alpha_0\operatorname{Re}\Omega_0+\Omega^3_{12}$, functions f_1,f_2 and vector field X such that

$$d^*\gamma=0,\quad d\gamma\wedge\omega_0=0,\quad \frac{1}{2}dh\wedge\omega_0^2-d\gamma\wedge\operatorname{Im}\Omega_0=\alpha_1$$
$$d\rho+\frac{3}{4}dh\wedge\operatorname{Re}\Omega_0+d\gamma\wedge\Omega_0+d*d(f_1\omega_0)=\alpha_2$$
$$d\hat{\rho}+\frac{1}{4}dh\wedge\operatorname{Im}\Omega_0+d*d(X\lrcorner\operatorname{Re}\Omega_0+f_2\omega_0)=\alpha_3$$

Moreover we have the following estimates:

$$\|(h,\gamma,\rho)\|_{C^{k,\alpha}_\nu}+\|(f_1,f_2,X)\|_{C^{k+1,\alpha}_{\nu+1}}\leq C\left(\|(\alpha_0,\beta_2,\beta_3)\|_{C^{k,\alpha}_\nu}+\|\alpha_1\|_{C^{k-1,\alpha}_{\nu-1}}\right)$$

The proof of this theorem makes extensive use of the analysis we explained in Chapter 3 and used throughout this chapter. In particular, the Dirac operator which we did not introduce plays an important role in this analysis, and much of §5 in [11] is devoted to understanding its indicial roots and the asymptotic behavior of solutions. It also justify the choice of weight $\nu < -1$. The assumption $\nu > -2$ will be useful for the proof of convergence.

Once the power series expansion $(\omega_\epsilon, \Omega_\epsilon, h_\epsilon, \theta_\epsilon, u_\epsilon, v_\epsilon, X_\epsilon)$ is constructed using Theorem 5.2.5, the proof of convergence goes as the Kodaira-Spencer's argument that we have seen in Chapter 4. In particular we used the series

$$A(\epsilon) = \frac{b}{16c} \sum_{k=1}^{\infty} \frac{c^k \epsilon^k}{k^2}$$

that has radius of convergence $1/c$ and satisfies the important property

$$A(\epsilon)^m \ll \left(\frac{b}{c}\right)^{m-1} A(\epsilon) \tag{5.19}$$

Let $\psi = (\rho, \gamma, h)$ be the triple of variables of the Apostolov-Salamon equations and $Y = (u, v, X)$ be the triples of variables added by Proposition 5.2.4. Fix $l \geq 1$ and $\alpha \in (0, 1)$ and let $B(\epsilon)$ be the series

$$B(\epsilon) = \epsilon \|\psi_1\|_{C^{l,\alpha}_{-1}} + \sum_{m \geq 2} \epsilon^k \left(\|\psi_m\|_{C^{l,\alpha}_\nu} + \|Y_m\|_{C^{l+1,\alpha}_{\nu+1}} \right)$$

The aim is to show that there exists $b, c > 0$ such that $B(\epsilon) \ll A(\epsilon)$ holds, i.e., the coefficients of $B(\epsilon)$ are bounded by the coefficients of $A(\epsilon)$.

At order 1, we can just set $\frac{b}{16} = \|\psi_1\|_{C^{l,\alpha}_{-1}}$. Now assume that the inequality holds up to order m. As noted in [11, §8.2], when we solve iteratively $L(\psi_{m+1}) = P_{m+1}(\psi_1, ..., \psi_m)$, the expression of P, and more exactly of $\alpha_{0,m}, \alpha_{1,m}, \beta_{2,m}, \beta_{3,m}$, is an analytic map of its variables, that vanish at order 0 and 1. From Theorem 5.2.5, when we add the variables Y_m we can solve the equation with estimates

$$\|\psi_m\|_{C^{l,\alpha}_\nu} + \|Y_m\|_{C^{l+1,\alpha}_{\nu+1}} \leq C \left(\|(\alpha_{0,m}, \beta_{2,m}, \beta_{3,m})\|_{C^{k,\alpha}_\nu} + \|\alpha_{1,m}\|_{C^{k-1,\alpha}_{\nu-1}} \right)$$

Now using the analyticity of the expressions of $\alpha_{0,m}, \alpha_{1,m}, \beta_{2,m}, \beta_{3,m}$ and the discussion following Proposition 3.4.6, there must exist a series $\sum_{m \geq 2} C_m \epsilon^m$ with positive convergence radius such that the inequality

$$\|\alpha_{0,m}, \beta_{2,m}, \beta_{3,m})\|_{C^{k,\alpha}_\nu} + \|\alpha_{1,m}\|_{C^{k-1,\alpha}_{\nu-1}} \leq Q \sum_{m \geq 2} C_m \|\psi_1\|_{C^{l,\alpha}_{-1}} \|\psi_2\|_{C^{l,\alpha}_\nu} ... \|\psi_m\|_{C^{l,\alpha}_\nu}$$

holds for some $Q > 0$ independent of m. Using the induction and equation (5.19), it follows that we have [11, §8.2, p. 44]:

$$\|\psi_{m+1}\|_{C^{l,\alpha}_\nu} + \|Y_{m+1}\|_{C^{l+1,\alpha}_{\nu+1}} \leq CQ \left(\sum_{k=2}^{m+1} C_k \left(\frac{b}{c}\right)^{k-1} \right) A_{m+1}$$

Since the series $\sum C_k x^k$ converges, choosing c big enough allows us to obtain $B(\epsilon) \ll A(\epsilon)$, which imply that B has positive radius of convergence. Hence the expansion Ψ_ϵ and Y_ϵ converge in the $C^{l,\alpha}_\nu$ and $C^{l+1,\alpha}_{\nu+1}$ norms. Since they satisfy the equations of Proposition 5.2.4, we conclude that, for $\epsilon > 0$ small enough, we must have $Y_\epsilon = 0$.

Thus, the expansions $\operatorname{Re}\Omega_\epsilon = \operatorname{Re}\Omega_0 + \epsilon\rho + \epsilon^2\rho'$, $h_\epsilon = 1 + \epsilon^2 h'$ and $\epsilon\theta_\epsilon = \epsilon\theta + \epsilon^2\gamma'$, where (ρ, θ) is the solution of the linearized problem given by Theorem 5.2.3 and (ρ', h', γ') are the solutions we constructed above that converge in $C^{l,\alpha}_\nu$-norm, define a genuine solution of the Apostolov-Salamon equations. However, as it was the case for the Kodaira-Spencer argument in Chapter 4, it does not follow from the construction that this solution is smooth, since the radius of convergence of the series may depend on l. In order to prove that (ρ', h', γ') is smooth, we want to show that the tuple is solution of an elliptic problem.

Let us write again $\psi_\epsilon = \psi_0 + \epsilon\psi + \epsilon^2\psi'$ for the tuple $(\Omega_\epsilon, h_\epsilon, \theta_\epsilon)$, with $\psi' = (\rho', h', \gamma')$. The Apostolov-Salamon equations $A(\psi_\epsilon) = 0$ can be written as $A = L + P$, where L is the linearization at (ω_0, Ω_0) and P the non-linear part. If we expand this equation, we have

$$A(\psi_\epsilon) = \epsilon L(\psi) + \epsilon^2 L(\psi) + P(\psi_\epsilon) = 0 \tag{5.20}$$

and the non-linear part $P(\psi_\epsilon)$ can be written in the form

$$P(\psi_\epsilon) = \epsilon^2 Q(\psi) + \epsilon^3 R(\psi, \psi')$$

where $Q(\psi)$ is an expression involving the solution of the linearized problem, which is smooth by Theorem 5.2.3, and $R(\psi, \psi')$ is an non-linear differential operator in ψ'. Since moreover we have $L(\psi) = 0$ by construction, equation (5.20) can be written as

$$L(\psi') + \epsilon R(\psi, \psi') = -Q(\psi) \tag{5.21}$$

As we noted in §5.2.3, L is not an elliptic operator, but with the variables added by Proposition 5.2.4, it is then elliptic. Here we already showed that these variables have to vanish, but we can add the variables $X' = 0$ in (5.21), so that (ψ', X') is solution of an equation of the form

$$\tilde{L}(\psi', X') + \epsilon\tilde{R}(\psi, \psi', X') = -Q(\psi) \tag{5.22}$$

where $\tilde{L}$ is elliptic. Hence, for ϵ small enough, $(\psi', 0)$ is solution of a non-linear elliptic problem, which implies that ψ' must be smooth.

Remark 5.2.5. In the discussion above, ψ' implicitly depends on ϵ, but we are only proving that for ϵ fixed, the tuple ψ' is smooth.

The metrics associated with the torsion-free G_2-structure φ_ϵ constructed above on the total space M have the form

$$g_{\varphi_\epsilon} = h_\epsilon^{\frac{1}{2}} g_\epsilon + \epsilon^2 h_\epsilon^{-1} \theta_\epsilon^2$$

and are asymptotic to metrics of the form $g_0 + \epsilon^2 \theta$, where θ is the connection form given by theorem 5.2.3, up to terms of order $O(r^{-\min\{1,-\mu\}})$, where $\mu < 0$ is the rate of the AC Calabi-Yau manifold (B, ω_0, Ω_0), with similar decay for all the derivatives. Such metrics are called Asymptotically Locally Conical (ALC).

It remains to prove that the restricted holonomy group is not a proper subgroup of G_2. Since $c_1(M) \neq 0$ and B is simply-connected, then the fundamental group of M is finite, and thus we may assume that M is simply-connected. As we noted at the end of §2.3.2, it suffices to prove that there are no parallel 1-forms. This fact is proven by extending the results of Chapter 3 from AC to ALC manifolds, as given in [13]. We refer the reader to the article [11, p. 41] for details.

Chapter 6

Example(s)

By the construction of Chapter 5, one can construct ALC metrics with restricted holonomy G_2 from a circle bundle $M^7 \to B^6$ over an AC Calabi-Yau manifold (B, ω_0, Ω_0). Remember that, for this construction to work, there are few topological conditions to be satisfied by this data. First, B must be asymptotic to a cone $C(\Sigma)$, where the universal cover of Σ is not isometric to the round 5-sphere. Then, the circle bundle M must be non-trivial, but we want the topological condition $c_1(M) \cup [\omega_0] = 0 \in H^4(B)$ to be satisfied. This condition may be difficult to check in practice, but if $H^4(B) = 0$, it trivially holds. Hence, the construction of ALC G_2-metrics is reduced to constructing AC Calabi-Yau manifolds satisfying the above topological conditions. Such manifolds often arise a resolutions or smoothing of Calabi-Yau cones. We will not try here to say anything about the general theory of singular varieties and their resolutions, but we want to point out that the structure of AC Calabi-Yau manifolds of crepant resolutions of Calabi-Yau cones is fully understood [11, Theorem 9.1]. Hence, it suffices to construct Calabi-Yau cones. In general, it is hard to decide whether a cone $C(\Sigma)$ admits a Calabi-Yau structure, but for cones $C(\Sigma) \subset \mathbf{C}^N$, a necessary condition is known, called K-stability. We refer to [11, §9] for a more precise description of some examples of AC Calabi-Yau manifolds constructed in this way.

In this chapter, we want to describe one example, constructed by Candelas-De La Ossa in [6]. Since this example admits a cohomogeneity one action of $SU(2) \times SU(2)$, the computations reduce to an ODE in one dimension, and everything can be made explicit.

6.1 Sasaki-Einstein structure on $S^2 \times S^3$

6.1.1 Crepant resolution

The projective space $\mathbf{CP}^1 = \{[z_1 : z_2], (z_1, z_2) \neq (0,0)\}$ has a natural complex structure given by the two charts

$$z \in \mathbf{C} \mapsto [z : 1] \in U_1 = \mathbf{CP}^1 \backslash \{[1:0]\}, \quad z \in \mathbf{C} \mapsto [1 : z] \in U_2 = \mathbf{CP}^1 \backslash \{[0:1]\}$$

where the transition map $z \in \mathbf{C}^* \mapsto \frac{1}{z} \in \mathbf{C}^*$ is indeed holomorphic. Each point $p \in \mathbf{CP}^1$ is identified with a 1-dimensional subspace $p \subset \mathbf{C}^2$. The *tautological line bundle* $\mathcal{O}(-1) \to \mathbf{CP}^1$ is the line bundle whose fiber at p is p itself, seen as a line in $\mathbf{C}^2$. In particular, we have a natural holomorphic map $\mathcal{O}(-1) \to \mathbf{C}^2$, which defines a biholomorphism $\mathcal{O}(-1)\backslash\mathbf{CP}^1 \simeq \mathbf{C}^2\backslash\{0\}$. We have explicit trivializations of $\mathcal{O}(-1)$ given by

$$U_1 \times \mathbf{C} \to \mathcal{O}(-1),\ ([z:1],t) \to (tz,t) \in \mathbf{C}^2$$

and similarly for U_2. Therefore the transition defining this bundle is

$$U_1 \cap U_2 \times \mathbf{C} \to U_1 \cap U_2 \times \mathbf{C},\ ([z:1],t) \mapsto ([z:1],z^{-1}t)$$

which justifies the notation $\mathcal{O}(-1)$. In general, the line bundle $\mathcal{O}(n)$ is defined by the transition function $([1:z],t) \mapsto ([1:z],z^n t)$, and every line bundle over $\mathbf{CP}^1$ is isomorphic to exactly one of the $\mathcal{O}(n)$.

Now consider the rank 2 vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$. Denote by B the total space of this bundle. Since two vectors (w^1, w^2) and (w^3, w^4) in $\mathbf{C}^2$ are on the same line if and only if $w^1w^4 - w^2w^3 = 0$, the image of the natural map

$$\pi : \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{C}^2 \oplus \mathbf{C}^2 = \mathbf{C}^4$$

is the hypersurface $C = \{w^1w^4 - w^2w^3 = 0\}$. This hypersurface has an isolated singularity at 0, and $\pi^{-1}(0) = \mathbf{CP}^1$. Moreover, π defines a biholomorphism $B\backslash\pi^{-1}(0) \simeq C\backslash\{0\}$. We say that (B,π) is a resolution of C at 0.

Consider $\mathbf{C}^4$ as the set of matrices $W = \begin{pmatrix} w^1 & w^3 \\ w^2 & w^4 \end{pmatrix}$ with the quadratic form

$$q(W) = \det W = w^1w^4 - w^2w^3$$

We have an action of $SU(2) \times SU(2)$ on $\mathbf{C}^4$ by

$$(L,R) \cdot W = LWR^\dagger \tag{6.1}$$

This action leaves invariant q, so that in particular the hypersuface C is invariant under the $SU(2) \times SU(2)$ action. In addition to the quadratic form q, the action of $SU(2) \times SU(2)$ also leaves invariant the norm defined by

$$|W|^2 = \mathrm{tr}(W^\dagger W)$$

Let Σ be the intersection of C with the unit ball for this norm. A particular element of Σ is

$$Z_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The stabilizer of Z_0 under $SU(2) \times SU(2)$ is $U(1)$ acting by

$$e^{i\theta} \in U(1) \longmapsto \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \right) \in SU(2) \times SU(2) \tag{6.2}$$

and Σ is the orbit of Z_0 under $SU(2) \times SU(2)$. Hence as a smooth manifold we have $\Sigma = SU(2) \times SU(2)/U(1)$. Topologically, it is diffeomorphic to $S^2 \times S^3$. Indeed, the projection $SU(2) \times SU(2)/U(1) \to SU(2)/U(1)$, $[(L, R)] \mapsto [L]$ defines a principal $SU(2)$-bundle over $SU(2)/U(1) = S^2$, and all such bundles are trivial.

Remark 6.1.1. If we introduce another quadratic form on $\mathbf{C}^4$ defined by

$$Q(z^1, z^2, z^3, z^4) = (z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2$$

then Q is equivalent to q under the linear coordinate change

$$w^1 = z^1 - iz^4, \quad w^2 = iz^2 + z^3, \quad w^3 = iz^2 - z^3, \quad w^4 = z^1 + iz^4$$

Hence we can also define C as the hypersurface $\{(z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 = 0\}$. Written in this way, the action of $SU(2) \times SU(2)$ is the double cover of the action of $SO(4)$ on $\mathbf{C}^4 = \mathbf{R}^4 \otimes \mathbf{C}$.

6.1.2 Homogeneous Einstein metrics on $S^2 \times S^3$

Let G be a compact Lie group and (Σ, g) a Riemannian manifold. Σ is called G-homogeneous if G acts transitively on Σ by isometries. In this case, if $x \in \Sigma$ and ϕ denotes the map

$$\phi : G \longrightarrow \Sigma, \ h \longmapsto h \cdot x$$

and $K \subset G$ is the compact subgroup of G that stabilizes x, then ϕ induces a diffeomorphism $G/K \to \Sigma$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a decomposition of the Lie algebra of G, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is a complement of $\mathfrak{k}$ invariant by $\mathrm{Ad}_G(K)$. Since we assumed the groups to be compact, such a decomposition always exists. Then the tangent map $d\phi_e : \mathfrak{g} \to T_x\Sigma$ induces an identification $T_x\Sigma \simeq \mathfrak{p}$, and under this identification the metric g of Σ gives an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$. Since K acts by isometries on Σ and fixes x, then K acts by isometries on $T_x\Sigma$. Moreover, with the identification $T_x\Sigma \simeq \mathfrak{p}$, the action of K is identified with the adjoint action of K. Indeed, if $v \in T_x\Sigma$ is written in a unique way $v = \frac{d}{dt} e^{tX} \cdot x$ for $X \in \mathfrak{p}$, and $k \in K$, we have

$$kv = k\frac{d}{dt} e^{tX} \cdot x = \frac{d}{dt} k e^{tX} k^{-1} \cdot x = \frac{d}{dt} e^{t\,\mathrm{Ad}(k)X} \cdot x$$

Therefore, the inner product on $\mathfrak{p}$ induced by the metric g is $\mathrm{Ad}_G(K)$-invariant. Conversely, any $\mathrm{Ad}_G(K)$-invariant inner product on $\mathfrak{p}$ determines an homogeneous metric on G/K.

In the case of interest to us, we have $G = SU(2) \times SU(2)$ and $K = U(1)$ embedded in $SU(2) \times SU(2)$ as in (6.2). Recall that the Lie algebra $\mathfrak{su}(2)$ has three generators

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

These generators satisfy the commutation relations $[\sigma_i, \sigma_j] = \sigma_k$ for all circular permutations (ijk) of (123). Hence the Lie algebra of $SU(2) \times SU(2)$ has two sets of generators $\{\sigma_i^L\}$ and $\{\sigma_i^R\}$ for $i = 1, 2, 3$, that satisfy the commutation relations

$$[\sigma_i^L, \sigma_j^R] = 0, \quad [\sigma_i^L, \sigma_j^L] = \sigma_k^L, \quad [\sigma_i^R, \sigma_j^R] = \sigma_k^R \tag{6.3}$$

for all circular permutations (ijk) of (123). The Lie algebra of $U(1) \subset SU(2) \times SU(2)$ is generated by $\sigma_3^L - \sigma_3^L$ and has a complement $\mathfrak{p}$, invariant by the adjoint action of $U(1)$, spanned by $\{\sigma_1^L, \sigma_2^L, \sigma_1^R, \sigma_2^R, \sigma_3^L + \sigma_3^R\} = \{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5\}$. In order to find inner products on $\mathfrak{p}$ that are invariant under the adjoint action of $U(1)$, we seek metrics with respect to which the endomorphism $T = \mathrm{ad}(\sigma_3^L - \sigma_3^R)$ of $\mathfrak{p}$ is antisymmetric. In the basis $\{\xi_i\}$, T is written

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In particular, if we write $\{\xi^i\}$ the dual basis of $\{\xi_i\}$, the the inner products of the form

$$\Lambda_L^2(\xi^1 \otimes \xi^1 + \xi^2 \otimes \xi^2) + \Lambda_R^2(\xi^3 \otimes \xi^3 + \xi^4 \otimes \xi^4) + \lambda^2 \xi^5 \otimes \xi^5 \tag{6.4}$$

are invariant under the adjoint action of $U(1)$, where Λ_L, Λ_R and λ are positive constants.

The advantage of homogeneous metrics is that the Ricci curvature has a simple expression. If we choose an orthonormal basis

$$X_1 = \Lambda_L^{-1}\sigma_1^L, \quad X_2 = \Lambda_L^{-1}\sigma_2^L, \quad X_3 = \Lambda_R^{-1}\sigma_1^R,$$
$$X_4 = \Lambda_R^{-1}\sigma_2^R, \quad X_5 = \lambda^{-1}(\sigma_3^L + \sigma_3^R)$$

then for all $X \in \mathfrak{p}$, the Ricci tensor is given by

$$\mathrm{Ric}(X, X) = -\frac{1}{2}\sum_i |[X, X_i]_{\mathfrak{p}}|^2 - \frac{1}{2}B(X, X) + \frac{1}{4}\sum_{i,j}\langle [X_i, X_j]_{\mathfrak{p}}, X\rangle^2$$

where $[\cdot, \cdot]_{\mathfrak{p}}$ denotes the projection of the Lie bracket onto $\mathfrak{p}$ according to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and $B(X, Y) = \mathrm{tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$ is the Cartan form of $\mathfrak{g}$.

Remark 6.1.2. We picked up this expression in [4, Corollary 7.38]. In the expression given by Besse, there is one more term, but in our case where $\mathfrak{g}$ is the Lie algebra of $SU(2) \times SU(2)$, the adjoint representation is trace-free, and hence the additional term vanishes.

Using this expression, we can explicitly compute the Ricci tensor of the metric (6.4). in particular we obtain:

$$\begin{aligned}
\mathrm{Ric}(X_1, X_1) = \mathrm{Ric}(X_2, X_2) &= \frac{1}{\Lambda_L^2}\left(1 - \frac{1}{8}\frac{\lambda^2}{\Lambda_L^2}\right), \\
\mathrm{Ric}(X_3, X_3) = \mathrm{Ric}(X_4, X_4) &= \frac{1}{\Lambda_R^2}\left(1 - \frac{1}{8}\frac{\lambda^2}{\Lambda_R^2}\right), \\
\mathrm{Ric}(X_5, X_5) &= \frac{\lambda^2}{8}\left(\frac{1}{\Lambda_L^4} + \frac{1}{\Lambda_R^4}\right)
\end{aligned}$$

and the homogeneous metric (6.4) on Σ satisfies $\mathrm{Ric}_g = 4g$ if and only if

$$4 = \frac{1}{\Lambda_L^2}\left(1 - \frac{1}{8}\frac{\lambda^2}{\Lambda_L^2}\right) = \frac{1}{\Lambda_R^2}\left(1 - \frac{1}{8}\frac{\lambda^2}{\Lambda_R^2}\right) = \frac{\lambda^2}{8}\left(\frac{1}{\Lambda_L^4} + \frac{1}{\Lambda_R^4}\right) \tag{6.5}$$

Remark 6.1.3. This expression is coherent with [6, Equation 2.9]. In the article of Candelas-De La Ossa, the metric is explicitly given in terms of Euler angles, and they make a slightly different choice of constants. The constant that we denote Λ_L^2 and Λ_R^2 respectively correspond to Λ_1^{-1} and Λ_2^{-1} in the article, and the constant denoted λ in the article is $\frac{\lambda}{2}$ in our conventions.

A particular choice of constants that satisfy (6.5) is

$$\Lambda_L = \Lambda_R = \frac{\sqrt{6}}{6}, \qquad \lambda = \frac{2}{3}$$

Denote by g_Σ the induced metric on $\Sigma = SU(2) \times SU(2)/U(1)$. As proven by Candelas-De La Ossa in [6, §3], this metric is Sasaki-Einstein. We will show how to find an explicit Kähler potential for the conical metric $dr^2 + r^2 g_\Sigma$. Beforehand, it is useful to give a more global expression for g_Σ. Let $Z = LZ_0R^\dagger$ for $(L, R) \in SU(2) \times SU(2)$. According to [6, Equation 2.33], we may write

$$g_\Sigma = \frac{2}{3}\operatorname{tr}(dZ^\dagger dZ) - \frac{2}{9}|\operatorname{tr} Z^\dagger dZ|^2$$

6.1.3 Kähler structure

In order to prove that $C(\Sigma)$ is a Calabi-Yau cones for the homogeneous metric derived in the last part, it is sufficient to prove that the metric is Kähler, in virtue of Lemma 2.2.2. In general, a hermitian metric $g_{\alpha\overline{\beta}}$ over a complex manifold (B, J) is Kälher if and only there exists locally a real function F such that

$$g_{\alpha\overline{\beta}} = \partial_\alpha \partial_{\overline{\beta}} F$$

where implicitly we work in local complex coordinates z^α, and write $\partial_\alpha = \frac{\partial}{\partial z^\alpha}$ and $\partial_{\overline{\beta}} = \frac{\partial}{\partial \overline{z}^\beta}$. The function F is called a *local Kähler potential*. The Kähler potential is not unique, but two Kähler potentials F_1 and F_2 are locally related by $F_2 - F_1 = \log|f|^2$ for some local holomorphic function f.

For the cone $C(\Sigma)$, we may actually find a global Kähler potential. Recall that we parametrize $\mathbf{C}^4$ by the matrix $W = \begin{pmatrix} w^1 & w^3 \\ w^2 & w^4 \end{pmatrix}$ and we have a norm

$$r^2 = |W|^2 = \operatorname{tr}(W^\dagger W) = \sum_{j=1}^{4} |w_j|^2$$

We may also set $W = rZ$. The cone $C(\Sigma)$ is defined by the equation $\det W = 0$, and any matrix Z satisfying $\det Z = 0$ and $|Z| = 1$ can be written in the form $Z = LZ_0R^\dagger$ with $(L, R) \in SU(2) \times SU(2)$. Since we are working with metrics homogeneous under the action of $SU(2) \times SU(2)$ on $C(\Sigma)$, we look for a potential as a function of r^2, say $F(r^2)$. For such a function we have

$$\partial_\alpha \partial_{\overline{\beta}} F = \frac{\partial^2 r^2}{\partial w^\alpha \partial \overline{w}^\beta} F'(r^2) + \frac{\partial r^2}{\partial w^\alpha} \frac{\partial r^2}{\partial \overline{w}^\beta} F''(r^2)$$

where the indices refer to the coordinates $(w^1, ..., w^4)$ on $\mathbf{C}^4$. Since we have

$$\frac{\partial^2 r^2}{\partial w^\alpha \partial \overline{w}^\beta} = \delta_{\alpha\overline{\beta}}, \quad \frac{\partial r^2}{\partial w^\alpha} \frac{\partial r^2}{\partial \overline{w}^\beta} = \overline{w}^\alpha w^\beta$$

a hermitian metric corresponding to such a potential can be written using the matrix W in the following way [6, Equation 3.3]:

$$g = F'(r^2) \operatorname{tr}(dW^\dagger dW) + F''(r^2) |\operatorname{tr} W^\dagger dW|^2$$

Since we are looking for a metric on the cone $C(\Sigma)$, we let $W = rZ$ with $Z = LZ_0R^\dagger$, $(L, R) \in SU(2) \times SU(2)$. Since in particular $\operatorname{tr}(Z^\dagger Z) = 1$, we obtain:

$$\begin{aligned} \operatorname{tr}(dW^\dagger dW) &= dr^2 + r^2 \operatorname{tr}(dZ^\dagger dZ) + r dr \operatorname{tr}(Z^\dagger dZ + Z dZ^\dagger) \\ |\operatorname{tr} W^\dagger dW|^2 &= r^2 dr^2 + r^4 |\operatorname{tr} Z^\dagger dZ|^2 + r^3 dr \operatorname{tr}(Z^\dagger dZ + Z dZ^\dagger) \end{aligned}$$

An important fact is that the quantity $\operatorname{tr}(Z^\dagger dZ + Z dZ^\dagger)$ vanishes on Σ. Indeed, by homogeneity we may check this fact at $Z = Z_0$. If $\xi = \xi^L + \xi^R$ is in $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, then we have

$$\begin{aligned} \operatorname{tr}(Z_0^\dagger(\xi^L Z_0 + Z_0 \xi^R) + &Z_0(Z_0^\dagger (\xi^L)^\dagger + (\xi^R)^\dagger Z_0^\dagger)) \\ &= \operatorname{tr}(Z_0 Z_0^\dagger(\xi^L + (\xi^L)^\dagger) + Z_0^\dagger Z_0(\xi^R + (\xi^R)^\dagger)) \\ &= 0 \end{aligned}$$

Hence the potential $F(r^2)$ gives metrics of the form

$$g = F'(r^2)(dr^2 + r^2 \operatorname{tr}(dZ^\dagger dZ)) + r^2 F''(r^2)(dr^2 + r^2 |\operatorname{tr} Z^\dagger dZ|^2)$$

Choosing a simple potential of the form $F(r^2) = r^{2\gamma}$ for some $\gamma \in \mathbf{R}$ yields

$$g = \gamma^2 r^{2(\gamma-1)} dr^2 + r^{2\gamma}(\gamma \operatorname{tr}(dZ^\dagger dZ) + \gamma(\gamma - 1)|\operatorname{tr} Z^\dagger dZ|^2)$$

and with a choice of new radial coordinate $\rho = r^\gamma$, we obtain a conical metric

$$g = d\rho^2 + \rho^2(\gamma \operatorname{tr}(dZ^\dagger dZ) + \gamma(\gamma-1)|\operatorname{tr} Z^\dagger dZ|^2)$$

Hence, we see that for $\gamma = \frac{2}{3}$, we recover the metric g_Σ constructed in §6.1.2. By construction the conical metric $d\rho^2 + \rho^2 g_\Sigma$ is Kähler, and since g_Σ satisfies $\mathrm{Ric}_\Sigma = 4g_\Sigma$, the conical metric is Ricci-flat. This proves that (Σ, g_Σ) is a Sasaki-Einstein manifold. Moreover, the complex structure induced by the Calabi-Yau structure of the cone $C(\Sigma)$ is by construction the complex structure induced by the inclusion $C(\Sigma) \subset \mathbf{C}^4$.

Remark 6.1.4. We may as well choose a potential $F(r^2) = ar^{2\gamma}$ for some $a > 0$, and accordingly rescale the radial coordinate as $\rho = a^{\frac{1}{2}} r^\gamma$, which yields the same conical metric. In the original article [6], Candelas-De La Ossa choose a scaling $a = \frac{3}{2}$.

6.2 Calabi-Yau structure on the resolution

6.2.1 Cohomogeneity one Kähler structure

The complex structure on $B = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is given by the two charts

$$\begin{aligned} (z,u,v) \in \mathbf{C}^3 &\longmapsto ([z:1], (zu,u), (zv,v)) \in H_1 \subset \mathcal{O}(-1) \oplus \mathcal{O}(-1), \\ (z,u,v) \in \mathbf{C}^3 &\longmapsto ([1:z], (u,zu), (v,zv)) \in H_2 \subset \mathcal{O}(-1) \oplus \mathcal{O}(-1) \end{aligned} \tag{6.6}$$

In particular the gluing function is given by

$$(z,u,v) \in \mathbf{C}^* \times \mathbf{C}^2 \longmapsto (z^{-1}, zu, zv) \in \mathbf{C}^* \times \mathbf{C}^2$$

An important point is that the action of $SU(2) \times SU(2)$ on $C(\Sigma) \subset \mathbf{C}^4$ can be extended in a unique way into an action on the resolution $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Indeed, if we take $(L, R) \in SU(2) \times SU(2)$ written as

$$L = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad R = \begin{pmatrix} k & -\bar{l} \\ l & \bar{k} \end{pmatrix}$$

the action of $SU(2) \times SU(2)$ on an element $W \in C(\Sigma)$ written as $W = \begin{pmatrix} zu & zv \\ u & v \end{pmatrix}$ is given by:

$$LWR^\dagger = \begin{pmatrix} (\bar{k}u - lv)(az - \bar{b}) & (\bar{l}u + kv)(az - \bar{b}) \\ (\bar{k}u - lv)(bz + \bar{a}) & (\bar{l}u + kv)(bz + \bar{a}) \end{pmatrix}$$

which is in the fiber of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over the element $[az - \bar{b} : bz + \bar{a}] \in \mathbf{CP}^1$. Hence the action of $SU(2) \times SU(2)$ on $C(\Sigma)$ extends to B, where the action of $SU(2) \times SU(2)$ on $\mathbf{CP}^1$ is the action of the left factor by homographic maps. Note that under the usual identification $\mathbf{CP}^1 \simeq S^2$, this action of $SU(2)$ is just the double cover of the usual action of $SO(3)$, and in particular it is an isometric action for the usual metric on S^2.

Since the Calabi-Yau cone structure constructed on $C(\Sigma)$ is invariant under the action of $SU(2) \times SU(2)$, it is natural to look for a Calabi-Yau structure on the resolution B which is also invariant under this action. However, we cannot look at a Kähler potential which is only a function of r^2, otherwise we would get a conical metric as in §6.1.3, which does not extend as a metric on B, since we cannot find an extension to the sphere replacing the apex of the cone.

On $\mathbf{CP}^1$, the standard metric coming from the identification $\mathbf{CP}^1 \simeq S^2$ is Kähler, and the corresponding Kähler form is called the *Fubini-Study structure*. It can be generated via the potential

$$f(z) = 4 \log(1 + |z|^2)$$

Since the gluing function of $\mathbf{CP}^1$ is $z \mapsto z^{-1}$, this potential transforms as $f(z) \to f(z) - 4 \log |z|^2$. Since $\partial\overline{\partial} \log |z|^2 = 0$, the the quantity $\partial\overline{\partial} f$ is globally defined.

The idea for building an $SU(2) \times SU(2)$-invariant AC Calabi-Yau structure on B asymptotic to the structure constructed in §6.1.3 is find an interpolation between the conical metric and the Fubini-Study structure over $\mathbf{CP}^1$. In coordinates (z, u, v) as in (6.6), we may look for a potential of the form

$$K(z, u, v) = F(r^2) + 4a^2 \log(1 + |z|^2)$$

where here the expression of the radius is $r^2 = (1 + |z|^2)(|u|^2 + |v|^2)$. The corresponding metric has the form

$$g = F'(r^2) \operatorname{tr}(dW^\dagger dW) + F''(r^2)|\operatorname{tr} W^\dagger dW|^2 + 4a^2 \frac{|\overline{z}dz + zd\overline{z}|^2}{(1 + |z|^2)^2}$$

where induced metric on $\mathbf{CP}^1$ is the metric of a standard sphere of radius a. On a Kähler manifold, the Ricci form ρ is given by $\rho_{\alpha\overline{\beta}} = \partial_\alpha \partial_{\overline{\beta}} \log \det g$. Hence the condition for g as above to be Calabi-Yau reduces to an ODE for the function F. In [6, §3], Candelas-De La Ossa give an explicit solution of to this ODE, asymptotic to the function giving the Calabi-Yau cone structure on $C(\Sigma)$. We refer to the article for an explicit computation. We will denote (ω_0, Ω_0) this AC Calabi-Yau structure on B.

Hence, the resolution $B \to C(\Sigma)$ admits an explicit AC Calabi-Yau structure, asymptotic to $C(\Sigma)$, where Σ is endowed with a homogeneous Sasaki-Einstein structure. Since topologically $\Sigma \simeq S^2 \times S^3$, it is simply connected, and the assumption that it is not isometric to the round 5-sphere is clearly satisfied. In the next section, we describe the circle bundles over B, in order to find an explicit example of ALC manifold with full holonomy G_2 obtained from the construction of Foscolo-Haskins-Nordström.

6.2.2 Circle bundles over $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$

Recall that the complex manifold B has two charts $H_1, H_2 \simeq \mathbf{C}^3$ with gluing function $(z, u, v) \in \mathbf{C}^* \times \mathbf{C}^2 \to (z^{-1}, zu, zv) \in \mathbf{C}^* \times \mathbf{C}^2$. Since B is a holomorphic vector bundle over $\mathbf{CP}^1$, it has essentially the same line bundles as

$\mathbf{CP}^1$, parametrized by an integer n, for line bundles L_n associated with gluing functions $f_n(z, u, v) = z^n$. Choosing a hermitian metric on the line bundle L_n gives principal $U(1)$-bundle M_n over B, the topology of which does not depend on the choice of metric. Moreover, M_n and M_{-n} are related by a change of orientation.

We look for a non-trivial simply connected circle bundle. Since it is clear that the circle bundles over B have the same homotopy type as circle bundles over $\mathbf{CP}^1$, M_n has the same fundamental group as the $U(1)$-bundle associated to the line bundle $\mathcal{O}(n)$ over $\mathbf{CP}^1$. In particular, the circle bundle associated with the standard hermitian metric on $\mathcal{O}(-1)$ is just the Hopf fibration $S^3 \to S^2$. Hence, $M = M_{-1}$ is a simply connected circle bundle over B. It is clearly non-trivial, and since B retracts onto $\mathbf{CP}^1$, the condition $C_1(M) \cup [\omega_0] = 0$ is trivially satisfied, because $H^4(B) = 0$. Hence the construction of Foscolo-Haskins-Nordström insures the existence of a one-parameter family of complete ALC metrics g_ϵ on M with full holonomy G_2.

Actually, M is the only simply connected circle bundle over B, up to orientation. Indeed, the fundamental group of M_n can be easily computed via the Van Kampen theorem. Since M_n is trivial over the simply connected subsets $H_1, H_2 \subset B$, we have $\pi_1(M_{n|H_1}) \simeq \mathbf{Z} \simeq \pi_1(M_{n|H_2})$. Let γ_1, γ_2 be respectively generators of $\pi_1(M_{n|H_1})$ and $\pi_1(M_{n|H_2})$. There is a surjective map $F(\gamma_1, \gamma_2) \to \pi_1(M_n)$, where $F(\gamma_1, \gamma_2)$ is the free group generated by γ_1, γ_2. The fundamental group of the intersection $M_{n|H_1} \cap M_{n|H_2} \simeq \mathbf{C}^* \times \mathbf{C}^2 \times S^1$ has two generators σ (coming from the factor $\mathbf{C}^*$) and γ (coming from the factor S^1). Under the morphism of fundamental groups induced by the inclusion $M_{n|H_1} \cap M_{n|H_2} \subset M_{n|H_1}$, σ is mapped to the trivial loop, whereas γ is mapped to γ_1. On the other hand, since the transition function of M_n is given by $(z, u, v) \to z^n$, σ is mapped to γ_2^n under the morphism induced by the inclusion $M_{n|H_1} \cap M_{n|H_2} \subset M_{n|H_2}$, and γ is mapped to γ_2. Hence the fundamental group of M_n is

$$\pi_1(M_n) \simeq F(\gamma_1, \gamma_2)/\{\gamma_1 = \gamma_2,\ \gamma_2^n = 1\} \simeq \mathbf{Z}/n\mathbf{Z}$$

In the case where $n = 0$, $M_0 = B \times S^1$ is the trivial bundle, and since B is simply connected, we recover the fact that $\pi_1(M_0) = \mathbf{Z}$. For $n = \pm 1$, we obtain another proof of the fact that $M = M_{-1}$ is simply connected.

For $n \geq 2$, M_n is a non-trivial vector bundle over B, which satisfies $c_1(M_n) \cup [\omega_0] = 0 \in H^4(B)$, and the construction of Foscolo-Haskins-Nordström leads to a one-parameter family of complete ALC metrics with restricted holonomy G_2. However, since M_n is not simply connected, the full holonomy group may strictly contain G_2. For $n = \pm 1$, as we noted above, M is simply connected and hence the construction gives complete ALC metrics with full holonomy G_2. For the trivial bundle, the construction only gives trivial product metrics, for which the restricted holonomy group is contained in $SU(3)$.

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