The Partial Differential Equations of the Hull-Strominger System and its Associated Anomaly Flow

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Abstract

After introducing a couple of complex geometric preliminaries, the Hull–Strominger system and its associated Anomaly flow are introduced and specifically applied to the case of a complex torus and an Iwasawa manifold. In both cases, the existence of stationary points are investigated. Stationary points in the Anomaly flow represent solutions to the Hull–Strominger system. In the case of an Iwasawa manifold, the stability of a certain stationary point is further examined.

Contents

1 Introduction

The Hull-Strominger system [Hul86a; Str86] is of considerable interest both in physics, where it is the equations for supersymmetric compactifications of the heterotic string to a 4-dimensional spacetime, and in mathematics, where it is a non-Kähler generalization of a Calabi-Yau metric coupled to a Hermitian-Einstein connection.

Its formulation goes as follows. Suppose a compact complex threefold M together with a nowhere vanishing holomorphic $(3,0)$-form Ω. Consider a holomorphic vector bundle $E \to M$ with $c_1(E) = 0$. Let g be a Hermitian metric on the holomorphic tangent bundle $T^{1,0}(M)$ and denote by ω its associated fundamental 2-form. Most often, we implicitly identify a fundamental 2-form with its associated Hermitian metric. In addition, consider a Hermitian fiber metric H on E and write $R \in \Omega^2\left(\mathrm{End}\left(T^{1,0}(M)\right)\right)$ respectively $F \in \Omega^2(\mathrm{End}(E))$ for the curvatures of the Chern connections of g respectively H. Suppose further a *slope parameter* $\alpha' \in \mathbb{R}$. Then, the Hull-Strominger system is the following system of coupled partial differential equations for the metrics ω and H:

$$\mathrm{i}\partial\overline{\partial}\omega - \frac{\alpha'}{4}\left(\mathrm{Tr}(R \wedge R) - \mathrm{Tr}(F \wedge F)\right) = 0, \tag{1.1}$$

$$\mathrm{d}\left(\|\Omega\|_\omega \omega^2\right) = 0, \tag{1.2}$$

$$F \wedge \omega^2 = 0, \tag{1.3}$$

where the norm of Ω with respect to ω is defined by

$$\|\Omega\|_\omega^2 := \mathrm{i}\frac{\Omega \wedge \overline{\Omega}}{\omega^3}. \tag{1.4}$$

A generalization of the Hull-Strominger system to higher dimensions is discussed in [CHZ19].

If ω was fixed, the third equation (1.3) is the well-known Hermitian-Yang-Mills equation [Don87; UY86]. From the point of view of non-Kähler geometry and nonlinear partial differential equations, the novelty in the Hull-Strominger system lies in the first equation (1.1) and second equation (1.2). Equation (1.1) is called the *anomaly cancellation equation* or *Bianchi identity* and it appears in the Green-Schwarz cancellation mechanism in string theory. Observe that the Bianchi identity is quadratic in the curvature tensor. Equation (1.2) is called the *conformally balanced condition* which can be viewed as an analog of the Kähler condition for Ricci-flat metrics in this non-Kähler setting.

Many solutions to the Hull-Strominger system have been found both in the physics [Bec+06; CI10; DRS99] and in the mathematics literature [AG12; Fei16; FY15].

Due to the absence of an analog for the $\partial\overline{\partial}$-lemma, it is generally hard to produce an ansatz such that the conformally balanced condition is satisfied. This difficulty motivated the authors in [PPZ18b] to instead consider the so-called *Anomaly flow* of $(\omega(t), H(t))$ defined by

$$\partial_t \left(\|\Omega\|_\omega \omega^2\right) = \mathrm{i}\partial\bar{\partial}\omega - \frac{\alpha'}{4}\left(\mathrm{Tr}(R\wedge R) - \mathrm{Tr}(F\wedge F)\right) \tag{1.5}$$

$$H^{-1}\partial_t H = \Lambda_\omega F, \tag{1.6}$$

$$\omega(0) = \omega_0, \tag{1.7}$$

$$H(0) = H_0. \tag{1.8}$$

where the Hodge operator Λ_ω on $(1,1)$-forms ψ is defined by

$$\Lambda_\omega \psi := \frac{\omega^2 \wedge \psi}{\omega^3}. \tag{1.9}$$

The flow starts with some initial metric $H(0) = H_0$ on E and some initial metric $\omega(0) = \omega_0$ on $T^{1,0}(M)$ that is conformally balanced, i.e.

$$\mathrm{d}\left(\|\Omega\|_{\omega_0}\omega_0^2\right) = 0. \tag{1.10}$$

In equations (1.5) and (1.6), $R = R(t)$ and $F = F(t)$ denote the curvatures of the Chern connections of $\omega(t)$ and $H(t)$ on M and E respectively.

We will see that equation (1.10) will imply that $\omega(t)$ is conformally balanced for all times t. This implies that the conformally balanced condition does not need to be added to the flow equations. It is sufficient to determine whether the Anomaly flow exists for all times and whether it converges.

We start the exposition with section 2 in which we introduce the necessary complex geometric tools to introduce the Hull-Strominger system together with its associated Anomaly flow. The main literature for this section is from the books [Lee24; Huy05]. Notably, we introduce complex differential forms in section 2.5, connections in section 2.7 and the notion of curvature in section 2.8. We end section 2 with section 2.11 which is a computational preparation for section 3.

Section 3 is on the Hull-Strominger system and its associated anomaly flow. We first provide general observations in section 3.1 and then discuss the first application on a complex torus in section 3.2. The main result of section 3.2 is theorem 3.2.1 in which the Anomaly flow is solved. The main strategy is to provide a specific flow ansatz which translates into a soluble initial value problem. In particular, theorem 3.2.1 shows the non-existence of stationary points in the Anomaly flow.

In section 3.3 we repeat the analysis on an Iwasawa manifold. Theorem 3.3.3 shows that the Anomaly flow does not converge if $\alpha' \neq 8$. However, if the slope parameter is $\alpha' = 8$, theorem 3.3.4 demonstrates that only for a special type of initial metric on the Iwasawa manifold, a stationary point exists. In every other case, within the flow ansatz, the geometry on the holomorphic vector bundle $L \oplus L^*$ collapses either to L or to L^*. Finally, section 3.3.6 discusses the degree to which the stationary point from theorem 3.3.4 is unstable.

2 Complex Geometric Preliminaries

In this section, we introduce the complex geometric tools that will allow us to understand the Hull-Strominger system together with its associated Anomaly flow. We will consider the Anomaly flows in two different setups in sections 3.2 and 3.3.

The main literature for this section is from [Lee24; Huy05]. We develop the necessary complex geometry in an almost self-contained way and whenever proofs are omitted, exact literature references are provided. Einstein summation convention is implicitly implied whenever repeated indices appear in individual terms.

2.1 Complex Manifolds

We start with the definition of a complex manifold. A *topological manifold* is a second countable Hausdorff topological space with the property that every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ for some fixed $n \in \mathbb{N}$ called the *dimension* of the manifold. For a topological manifold M of dimension n, a *coordinate chart* for M is a pair (U, ϕ), where U is an open subset of M and ϕ is a homeomorphism $\phi\colon U \to \phi(U) \subseteq \mathbb{R}^n$. An *atlas* for M is a collection of charts whose domains cover M. Given two charts (U, ϕ) and (V, ψ) with overlapping domains, their *transition functions* are composite maps

$$\psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V), \tag{2.1}$$

$$\phi \circ \psi^{-1}\colon \psi(U \cap V) \to \phi(U \cap V). \tag{2.2}$$

Two charts are said to be *smoothly compatible* if their domains are disjoint or their transition functions are smooth as maps between open subsets of $\mathbb{R}^n$. A *smooth atlas* for M is an atlas with the property that any two charts in the atlas are smoothly compatible with each other. Finally, a *smooth structure* for M is a smooth atlas that is *maximal*, meaning that it is not properly contained in any larger smooth atlas. Saying that an atlas $\mathcal{A}$ is a maximal smooth atlas just means that every chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$. *Smooth manifolds* are topological manifolds endowed with a smooth structure.

We choose the following standard identification between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$:

$$\left(x^1, y^1, \dots, x^n, y^n\right) \leftrightarrow \left(x^1 + \mathrm{i}y^1, \dots, x^n + \mathrm{i}y^n\right). \tag{2.3}$$

Now suppose that M is a $2n$-dimensional topological manifold. If (U, ϕ) and (V, ψ) are two coordinate charts for M, we say they are *holomorphically compatible* if $U \cap V = \emptyset$ or both transition functions $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are holomorphic under our standard identification of $\phi(U \cap V)$ and $\psi(U \cap V)$ as open subsets of $\mathbb{C}^n$. A *holomorphic atlas* for M is an atlas with the property that any two charts in the atlas are holomorphically compatible with each other, and a *holomorphic structure* for M is a maximal holomorphic atlas. An *n-dimensional complex manifold* (or *holomorphic manifold*) is a topological manifold of dimension $2n$ endowed with a given holomorphic structure. Any one of the charts in the maximal holomorphic atlas is called a *holomorphic coordinate chart*, and the complex-valued coordinate functions $(z^1, \dots, z^n)$ are called *holomorphic coordinates*, where $z^j := x^j + \mathrm{i}y^j$ and $\bar{z}^j := x^j - \mathrm{i}y^j$.

Remark 2.1.1. Because holomorphic functions are smooth, a holomorphic atlas is also a smooth atlas and thus determines a unique smooth structure on M. Therefore, every complex manifold is also a smooth manifold in a canonical way. On the other hand, a given even-dimensional smooth manifold may have many different holomorphic structures that induce the given smooth structure, or it may have none at all. E.g., S^4 carries no holomorphic structure.

We finish this subsection with the definition of holomorphic maps between complex manifolds.

Definition 2.1.2. If M and N are complex manifolds, a map $f\colon M \to N$ is called **holomorphic** if for all $p \in M$ there exist holomorphic charts (U,ϕ) for M and (V,ψ) for N with $p \in U$, $f(p) \in V$ and $f(U) \subseteq V$ such that the coordinate representation of f, $\psi \circ f \circ \phi^{-1}\colon \phi(U) \to \psi(V)$, is a holomorphic map. If f bijective and holomorphic with holomorphic inverse f^{-1}, f is called a **biholomorphism**.

2.2 Complex Vector Bundles

We now introduce the notion of a complex vector bundle.

Definition 2.2.1. For a topological space M, a **complex vector bundle of rank k over M** is a topological space E together with a continuous subjective map $\pi\colon E \to M$ such that each **fiber** $E_p := \pi^{-1}(p)$ has the structure of a k-dimensional complex vector space. Furthermore, each $p \in M$ has a neighborhood U over which there exists a **local trivialization**, that is a homeomorphism $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^k$ that restricts to a complex linear isomorphism $E_q \to \{q\} \times \mathbb{C}^k$ for all $q \in U$. To summarize, the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow[\cong]{\phi} & U \times \mathbb{C}^k \\ & {\scriptstyle \pi|_{\pi^{-1}(U)}} \searrow & \downarrow {\scriptstyle \pi_1} \\ & & U \end{array} \tag{2.4}$$

shall commute, where $\pi_1\colon U \times \mathbb{C}^k \to U$ is the projection map onto the first factor.

If M and E are smooth manifolds, π is a smooth map, and local trivializations can be chosen to be diffeomorphisms, we call it a **smooth complex vector bundle**. Moreover, if M and E are complex manifolds, π is holomorphic, and the local trivializations can be chosen to be biholomorphisms, we speak of a **holomorphic vector bundle**. A local trivialization over all of M is called a **global trivialization** and if such a trivialization exists, the bundle is said to be a **trivial bundle**. A real or complex vector bundle of rank 1 is called a **line bundle**.

If we have two vector bundles $\pi\colon E \to M$ and $\pi'\colon E' \to M$ and a map $F\colon E \to E'$ such that $\pi' \circ F = \pi$ and such that $F|_{E_p}\colon E_p \to E'_p$ is a complex linear map, F is called a **bundle homomorphism**. If F is also a homomorphism, F is called a **bundle isomorphism** and E and E' are called **isomorphic**, denoted $E \cong E'$.

A fiberwise direct sum of vector spaces can be defined.

Definition 2.2.2. Suppose two holomorphic vector bundles $E \to M$ and $E' \to M$ of rank k and k', respectively, over the same base manifold M. Then their **Whitney sum** $E \oplus E'$ is the holomorphic vector bundle of rank $k + k'$ whose fiber at each $p \in M$ is the direct sum $E_p \oplus E'_p$.

Definition 2.2.3. Suppose $\pi \colon E \to M$ is a smooth vector bundle and $U \subseteq M$ is an open subset. A **local section of E over U** is a continuous map $\sigma \colon U \to E$ such that $\pi \circ \sigma = \mathrm{Id}_U$. The map σ is called a **global section of E** if $U = M$. We denote the space of smooth global sections of E by $\Gamma(E)$. For a holomorphic vector bundle $E \to M$, a local or global section is called a **holomorphic section** if it is holomorphic as a map between complex manifolds.

Given a k-tupel of local sections $(\sigma_1, \ldots, \sigma_k)$ over an open set $U \subseteq M$ whose values at each $p \in U$ form a basis for the fiber E_p, the k-tupel is called a **local frame for E**.

The existence of smooth sections is quite straightforward.

Remark 2.2.4. Every smooth vector bundle has a smooth **zero section** ξ, for which $\xi(p)$ is the zero element of E_p for all $p \in M$.

The following definition is important for later considerations.

Definition 2.2.5. Suppose M is a smooth manifold and $E \to M$ is a smooth complex vector bundle of rank m. A **Hermitian fiber metric** H on E is a choice of a Hermitian inner product H_p on each fiber E_p that is smoothly varying in the sense that for all smooth section σ and τ of E over an open subset $U \subseteq M$, $U \ni p \mapsto H_p(\sigma(p), \tau(p)) \in \mathbb{C}$ is a smooth function. A smooth complex vector bundle endowed with a Hermitian fiber metric is called a **Hermitian vector bundle**.

With a partition of unity argument, one can prove that any smooth complex vector bundle admits a Hermitian fiber metric [Huy05, Proposition 4.14]. One can define a natural Hermitian fiber metric on the dual of a given Hermitian vector bundle.

Theorem 2.2.6 ([Lee24, p. 218]). *Let $E \to M$ be a smooth complex vector bundle endowed with a Hermitian fiber metric H.*

1) *The metric H determines a smooth conjugate-linear bundle isomorphism $\hat{H} \colon E \to E^*$ by $\hat{H}(\sigma)(\tau) \coloneqq \langle \tau, \sigma \rangle_H$.*

2) *$\langle \phi, \psi \rangle_{H^*} \coloneqq \langle \hat{H}^{-1}(\psi), \hat{H}^{-1}(\phi) \rangle_H$ defines a Hermitian fiber metric H^* on E^*, called the dual metric*

2.3 Complex Vector Fields

Consider a complex manifold M and local holomorphic coordinates $(z^1, \ldots, z^n)$ on an open subset $U \subseteq M$. Denote the coordinate map by $\phi \colon U \to \mathbb{C}^n$ which can also be thought of a smooth coordinate map $U \to \mathbb{R}^{2n}$ with smooth coordinate functions $(x^1, y^1, \ldots, x^n, y^n)$

where $z^j = x^j + \mathrm{i}y^j$. These coordinates yield smooth coordinate vector fields $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\right)$, which act on a smooth function $f\colon U \to \mathbb{C}$ by

$$\left.\frac{\partial}{\partial x^j}\right|_p f := \frac{\partial f \circ \phi^{-1}}{\partial x^j} \text{ and } \left.\frac{\partial}{\partial y^j}\right|_p f := \frac{\partial f \circ \phi^{-1}}{\partial y^j}, \tag{2.5}$$

where the derivatives on the right-hand sides are ordinary partial derivatives on $\mathbb{R}^{2n}$.

We define a smooth local complex frame $\left\{\partial_j, \partial_{\bar{j}}\right\}$ for the complexfied tangent bundle $T_{\mathbb{C}}M$ by

$$\partial_j := \frac{\partial}{\partial z^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} - \mathrm{i}\frac{\partial}{\partial y^j}\right) \text{ and } \partial_{\bar{j}} := \frac{\partial}{\partial \bar{z}^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} + \mathrm{i}\frac{\partial}{\partial y^j}\right), \tag{2.6}$$

where $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^j}$ are the smooth vector fields on $U \subseteq M$. The vector fields in equation (2.6) are called *complex coordinate vector fields*, and the corresponding local frame is called a *complex coordinate frame*.

We call a section of $T_{\mathbb{C}}M$ a *complex vector field* and it can be written locally as a linear combination of coordinate vector fields with complex-valued coefficient functions, or as a sum of a real vector field plus i times another real vector field. A complex vector field $Z = X + \mathrm{i}Y$ acts on a smooth real-valued function f by

$$Zf := Xf + \mathrm{i}Yf, \tag{2.7}$$

and on complex-valued functions $f = u + \mathrm{i}v$ by the same formula, where

$$Xf := Xu + \mathrm{i}Xv \text{ and } Yf := Yu + \mathrm{i}Yv. \tag{2.8}$$

The Lie bracket operation can be extended to pairs of smooth complex vector fields by complex bilinearity.

$$[X_1 + \mathrm{i}Y_1, X_2 + \mathrm{i}Y_2] := [X_1, X_2] - [Y_1, Y_2] + \mathrm{i}\left([X_1, Y_2] + [Y_1, X_2]\right). \tag{2.9}$$

The formula $[fV, gW] = fg[V, W] + f(Vg)W - g(Wf)V$ also holds when the vector fields V and W and the functions f and g are complex.

Similarly, a section of the complexified cotangent bundle $T^*_{\mathbb{C}}M$ is called a *complex 1-form* or a *complex covector field*, and can be written locally as a linear combination of coordinate 1-forms with complex coefficients, or as a sum of real 1-forms plus i times another real 1-form.

2.4 Complex Structures

The tangent bundle $T\mathbb{C}^n$ has the canonical complex structure $J_{\mathbb{C}^n}\colon T\mathbb{C}^n \to T\mathbb{C}^n$ defined by

$$J_{\mathbb{C}^n}\frac{\partial}{\partial x^j} := \frac{\partial}{\partial y^j} \text{ and } J_{\mathbb{C}^n}\frac{\partial}{\partial y^j} := -\frac{\partial}{\partial x^j}. \tag{2.10}$$

The corresponding complexified tangent bundle $T_{\mathbb{C}}\mathbb{C}^n$ splits as $T_{\mathbb{C}}\mathbb{C}^n = T^{1,0}(\mathbb{C}^n) \oplus T^{0,1}(\mathbb{C}^n)$ where $T^{1,0}(\mathbb{C}^n)$ is spanned by $\partial_1, \dots, \partial_n$ and $T^{0,1}(\mathbb{C}^n)$ is spanned by $\partial_{\bar{1}}, \dots, \partial_{\bar{n}}$.

Proposition 2.4.1. *On every complex manifold M, there is a canonical complex structure $J_M : TM \to TM$.*

Proof. Let $p \in M$ and consider holomorphic coordinate chart (U, ϕ) on a neighborhood of p, and define $J_M\colon\ TM|_U \to TM|_U$ by

$$J_M \coloneqq D\phi^{-1} \circ J_{\mathbb{C}^n} \circ D\phi. \tag{2.11}$$

It remains to prove the independence of the definition of J_M on the holomorphic coordinate chart. Whenever two holomorphic charts (U, ϕ) and (V, ψ) overlap, the transition function $\psi \circ \phi^{-1}$ is a holomorphic map between subsets of $\mathbb{C}^n$ so that $D(\psi \circ \phi^{-1}) = D\psi \circ D\phi^{-1}$ commutes with $J_{\mathbb{C}^n}$ [Lee24, Lemma 1.54]. Finally, we find

$$\begin{aligned} D\psi^{-1} \circ J_{\mathbb{C}^n} \circ D\psi &= D\psi^{-1} \circ J_{\mathbb{C}^n} \circ (D\psi \circ D\phi^{-1}) \circ D\psi && (2.12)\\ &= D\psi^{-1} \circ (D\psi \circ D\phi^{-1}) \circ J_{\mathbb{C}^n} \circ D\psi = D\phi^{-1} \circ J_{\mathbb{C}^n} \circ D\phi. && (2.13) \end{aligned}$$

The fact that $J_M^2 = -\mathrm{Id}$ follows immediately from $J_{\mathbb{C}^n}^2 = -\mathrm{Id}$. □

On a complex manifold M, there are subbundles $T^{1,0}(M), T^{0,1}(M) \subseteq T_{\mathbb{C}}M$ whose fibers at each point are the i-eigenspace and $(-\mathrm{i})$-eigenspaces of the complexification of J_M, respectively. We then have the Whitney sum decomposition $T_{\mathbb{C}}M = T^{1,0}(M) \oplus T^{0,1}(M)$. Considering local holomorphic coordinates $z^j = x^j + \mathrm{i}y^j$, the complex vector fields $\{\partial_j\}$ form a local frame for $T^{1,0}(M)$. Similarly, the vector fields $\{\partial_{\bar{j}}\}$ form a local frame for $T^{0,1}(M)$. Since the two subbundles $T^{1,0}(M)$ and $T^{0,1}(M)$ are spanned locally by smooth vector fields, they are both smooth.

2.5 Complex Differential Forms

Let M be a complex manifold and denote the bundle of complex k-forms by $\bigwedge_{\mathbb{C}}^k M$ which is the complexification of $\bigwedge^k M$. Every smooth section of $\bigwedge_{\mathbb{C}}^k M$ can be written uniquely as a sum $\alpha + \mathrm{i}\beta$, where α and β are smooth real k-forms, and $\bigwedge_{\mathbb{C}}^k M$ has a natural conjugation operator given by $\overline{\alpha + \mathrm{i}\beta} = \alpha - \mathrm{i}\beta$. We call a complex differential form ω *real* if $\overline{\omega} = \omega$.

In the domain of any local holomorphic coordinates $(z^1, \dots, z^n)$, the 1-forms $(\mathrm{d}z^1, \dots, \mathrm{d}z^n, \mathrm{d}\bar{z}^1, \dots, \mathrm{d}\bar{z}^n)$ constitute a local frame for the complexified cotangent bundle and therefore

$$\left\{\mathrm{d}z^{j_1} \wedge \dots \wedge \mathrm{d}z^{j_p} \wedge \mathrm{d}\bar{z}^{l_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{l_q} \mid p + q = k,\ j_1 < \dots < j_p \text{ and } l_1 < \dots < l_q\right\} \tag{2.14}$$

delivers a smooth local frame for $\bigwedge_{\mathbb{C}}^k M$.

Suppose $p + q = k$. We call a complex k-form of *type* (p, q) or a (p, q)*-form* if in every local holomorphic coordinate chart $(z^1, \dots, z^n)$, the complex k-form can be expressed as a sum of terms whose summands have exactly p of the $\mathrm{d}z^j$ factors and q of the $\mathrm{d}\bar{z}^l$ factors. Denote by $\bigwedge^{p,q} M \subseteq \bigwedge_{\mathbb{C}}^k M$ the subset of (p, q)-forms. It is locally spanned by smooth sections of $\bigwedge_{\mathbb{C}}^k M$ and is thus a smooth subbundle of $\bigwedge_{\mathbb{C}}^k M$.

Any complex k-form is a sum of various types (p,q) so that $p+q=k$ and because

$$\bigwedge^{p,q} M \cap \bigwedge^{p',q'} M = \begin{cases} \bigwedge^{p,q} M & \text{if } p'=p \text{ and } q'=q, \\ 0 & \text{otherwise,} \end{cases} \tag{2.15}$$

there is the Whitney sum decomposition

$$\bigwedge_{\mathbb{C}}^{k} M = \bigoplus_{p+q=k} \bigwedge^{p,q} M. \tag{2.16}$$

For each (p,q) there exists a coordinate independent projection operator $\pi^{p,q} \colon \bigwedge_{\mathbb{C}}^{k} M \to \bigwedge^{p,q} M$. We denote $\Omega^k(M) \coloneqq \Gamma\left(\bigwedge_{\mathbb{C}}^{k} M\right)$ and $\Omega^{p,q}(M) \coloneqq \Gamma(\bigwedge^{p,q} M)$.

We can now define for $p,q \in \{0,\dots,n\}$ the Dolbeault operator $\overline{\partial} \colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ by $\overline{\partial} \coloneqq \pi^{p,q+1} \circ \mathrm{d}$ and its conjugate $\partial \colon \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ by $\partial \coloneqq \pi^{p+1,q} \circ \mathrm{d}$. More generally, the operators $\overline{\partial}$ and ∂ are extended onto $\Omega^k(M)$ by decomposing complex k-forms into types of type (p,q) with $p+q=k$ and applying $\overline{\partial}$ and ∂ on each term separately.

Considering holomorphic coordinates

$$\alpha \coloneqq \alpha_{j_1,\dots,j_p,l_1,\dots,l_q} \mathrm{d}z^{j_1} \wedge \cdots \wedge \mathrm{d}z^{j_p} \wedge \mathrm{d}\overline{z}^{l_1} \wedge \cdots \wedge \mathrm{d}\overline{z}^{l_q}, \tag{2.17}$$

the Dolbeault operators act as

$$\overline{\partial}\alpha = \left(\partial_{\overline{r}} \alpha_{j_1,\dots,j_p,l_1,\dots,l_q}\right) \mathrm{d}\overline{z}^{r} \wedge \mathrm{d}z^{j_1} \wedge \cdots \wedge \mathrm{d}z^{j_p} \wedge \mathrm{d}\overline{z}^{l_1} \wedge \cdots \wedge \mathrm{d}\overline{z}^{l_q}, \tag{2.18}$$

$$\partial\alpha = \left(\partial_{r} \alpha_{j_1,\dots,j_p,l_1,\dots,l_q}\right) \mathrm{d}z^{r} \wedge \mathrm{d}z^{j_1} \wedge \cdots \wedge \mathrm{d}z^{j_p} \wedge \mathrm{d}\overline{z}^{l_1} \wedge \cdots \wedge \mathrm{d}\overline{z}^{l_q}. \tag{2.19}$$

The following proposition is a direct implication of equation (2.18).

Proposition 2.5.1. *Suppose M is a complex manifold and f is a smooth function on M. Then f is holomorphic if and only if $\overline{\partial} f = 0$.*

Here are some useful properties of Dolbeault operators.

Proposition 2.5.2. *Let M be a complex manifold and α a complex differential form on M. Then the following identities hold:*

1) $\mathrm{d} = \partial + \overline{\partial}$,

2) $\overline{\partial\alpha} = \overline{\partial}\,\overline{\alpha}$,

3) $\partial\partial = 0 = \overline{\partial}\,\overline{\partial}$,

4) $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

Proof. We decompose α into types and apply the Dolbeault operators to each type separately. In this way, we can assume α to be of some type (p,q) to begin with. Point 1) is essentially a local calculation in local holomorphic coordinates. Using, equations (2.18) and (2.19), one immediately finds $\mathrm{d} = \partial + \overline{\partial}$. Point 2) also follows from the coordinate equations (2.18) and (2.19) since essentially $\mathrm{d}z^j$ and $\mathrm{d}\bar{z}^j$ are conjugates of each other. For the last two points, 3) and 4), we make the following observation by using point 1):

$$\partial\partial\alpha + \left(\partial\bar{\partial} + \bar{\partial}\partial\right)\alpha + \bar{\partial}\bar{\partial}\alpha = \left(\partial + \bar{\partial}\right)^2 \alpha = \mathrm{d}^2\alpha = 0. \tag{2.20}$$

Now, $\partial\partial\alpha \in \Omega^{p+2,q}(M)$, $\left(\partial\bar{\partial} + \bar{\partial}\partial\right)\alpha \in \Omega^{p+1,q+1}(M)$ and $\bar{\partial}\bar{\partial}\alpha \in \Omega^{p,q+2}(M)$ and these spaces intersect pairwise only in the trivial form implying that indeed $\partial\partial\alpha = 0 = \bar{\partial}\bar{\partial}\alpha$ and $\left(\partial\bar{\partial} + \bar{\partial}\partial\right)\alpha = 0$. □

Definition 2.5.3. A **Hermitian metric** on a complex manifold M is a Riemannian metric g for which the complex structure is an orthogonal map. A complex manifold endowed with a Hermitian metric is called a **Hermitian manifold.** On a Hermitian manifold (M, g), the 2-form $\omega := g(J\cdot, \cdot)$ is called the **fundamental 2-form** of the Hermitian metric.

To determine the coordinate expression of the fundamental 2-form ω, we write $\omega = \omega_{j\bar{k}}\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k$ with the coefficient functions $\omega_{j\bar{k}}$ being determined as

$$\omega_{j\bar{k}} = \omega\left(\partial_j, \partial_{\bar{k}}\right) = g\left(J\partial_j, \partial_{\bar{k}}\right) = \mathrm{i}g\left(\partial_j, \partial_{\bar{k}}\right) = \mathrm{i}g_{j\bar{k}}, \tag{2.21}$$

where J is the complex structure and we defined the matrix $g_{j\bar{k}} := g\left(\partial_j, \partial_{\bar{k}}\right)$. We denote by $\left(g^{\bar{k}j}\right)$ the inverse of the matrix $\left(g_{j\bar{k}}\right)$.

2.6 Wedge Product of Endomorphism Valued Forms

Consider a smooth manifold M and a smooth complex vector bundle $E \to M$. For each $q \in \mathbb{N}_0$, we define the *bundle of E-valued q-forms* as $\bigwedge^q_{\mathbb{C}} M \otimes E$ and denote $\Omega^q(M, E) := \Gamma\left(\bigwedge^q_{\mathbb{C}} M \otimes E\right)$ and $\Omega^0(M, E) = \Gamma(E)$. Similarly, we call $\bigwedge^q_{\mathbb{C}} M \otimes \mathrm{End}(E)$ the bundle of *endomorphism valued q-forms* and denote $\Omega^q(M, \mathrm{End}(E)) := \Gamma\left(\bigwedge^q_{\mathbb{C}} M \otimes \mathrm{End}(E)\right)$.

Especially important are wedge products with endomorphism-valued forms. For $A \otimes \alpha \in \Omega^q(M, \mathrm{End}(E))$, $B \otimes \beta \in \Omega^{q'}(M, \mathrm{End}(E))$, and $\sigma \otimes \gamma \in \Omega^{q''}(M, E)$, we define

$$(A \otimes \alpha) \wedge (B \otimes \beta) := (A \circ B) \otimes (\alpha \wedge \beta) \in \Omega^{q+q'}(M, \mathrm{End(E)}) \tag{2.22}$$

$$(A \otimes \alpha) \wedge (\gamma \otimes \sigma) := (A\sigma) \otimes \alpha \wedge \gamma \in \Omega^{q+q''}(M, E) \tag{2.23}$$

and extend bilinearly.

To see how to compute these locally, let (s_j) be a local frame for E and (ϵ^k) the dual frame for E^*. Because of the canonical isomorphism $\mathrm{End}(E) \cong E \otimes E^*$, each section $\omega \in \Omega^q(M, \mathrm{End}(E))$ can be expressed locally in the form

$$\omega = s_j \otimes \epsilon^k \otimes \omega^j_k \tag{2.24}$$

for a uniquely determined matrix ω^j_k of ordinary q-forms. The tensor product $s_j \otimes \epsilon^k$ represents the endomorphism of E whose action on a basis element s_i is

$$(s_j \otimes \epsilon^k)(s_i) = \delta_i^k s_j, \tag{2.25}$$

so the wedge product defined above satisfies

$$\omega \wedge \eta = (s_j \otimes \epsilon^k \otimes \omega_k^j) \wedge (s_l \otimes \epsilon^m \otimes \eta_m^l) \tag{2.26}$$
$$= (\delta_l^k s_j \otimes \epsilon^m) \otimes (\omega_k^j \wedge \eta_m^l) \tag{2.27}$$
$$= (s_j \otimes \epsilon^m) \otimes (\omega_k^j \wedge \eta_m^k). \tag{2.28}$$

In other words, the matrix of forms representing $\omega \wedge \eta$ is the matrix product of the ones representing ω and η, with individual entries combined via the wedge product. (In an expression like ω_k^j, we always interpret the upper index as a row number and the lower index as a column number.)

2.7 Connections

It is important to introduce the notion of connections before delving into the definition of curvature.

Definition 2.7.1. Let E be a smooth complex vector bundle of rank m on a smooth manifold M. A **connection** on E is a map

$$\nabla\colon \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \tag{2.29}$$
$$(X, \sigma) \mapsto \nabla_X \sigma \tag{2.30}$$

that is $C^\infty(M)$-linear in X, $\mathbb{C}$-linear in σ and satisfies the Leibniz rule

$$\nabla(f\sigma) = f\nabla\sigma + \mathrm{d}f \otimes \sigma \tag{2.31}$$

for all complex-valued functions f. The expression $\nabla_X\sigma$ is called the **covariant derivative of σ in the direction of X.**

Remark 2.7.2. The value of $\nabla_X\sigma$ at a $p \in M$ depends only on $X(p)$ and the value σ in an arbitrary small neighborhood of p. Consequently, any connection ∇ on E determines a connection, still denoted by ∇, on the restriction of E to any open subset of M.

For any smooth section σ, the section $\nabla\sigma \in \Gamma(T^*M \otimes E)$ of the bundle $T^*M \otimes E$ is called the *total covariant derivative*. The bundle $T^*M \otimes E$ is canonically isomorphic to $\mathrm{Hom}(TM, E)$ and the section is defined by $(\nabla\sigma)(X) \coloneqq \nabla_X\sigma$.

We can extend connections by complex linearity to accept complex vector fields. If ∇ is a connection on E and $Z \coloneqq X + \mathrm{i}Y$ is a smooth complex vector field on M, we set

$$\nabla_Z\sigma \coloneqq \nabla_X\sigma + \mathrm{i}\nabla_Y\sigma. \tag{2.32}$$

This operation is $C^\infty(M,\mathbb{C})$-linear in Z, and it allows us to regard the total covariant derivative $\nabla\sigma$ as a section of $T_{\mathbb{C}}^*M \otimes E \cong \mathrm{Hom}(T_{\mathbb{C}}M, E)$.

2.7.1 Connection Forms

It is sufficient to work locally to understand the action of a given connection. To do so, we suppose a smooth local frame $(s_1, \ldots, s_m)$ for E and an open subset $U \subseteq M$. For each vector field X on U there exist smooth functions $\theta^k_j(X)$ such that

$$\nabla_X s_j = \theta^k_j(X) s_k \tag{2.33}$$

for all $j \in \{1, \ldots, m\}$. Using the Leibniz rule, we observe that equation (2.33) determines the covariant derivative of arbitrary sections. Due to the $\mathbb{C}^\infty(M)$-linearity of ∇ in X, $\left(\theta^k_j\right)$ defines a matrix of complex 1-forms, called *connection 1-forms* with respect to this frame. Connection 1-forms are smooth because they act on all smooth vector fields X as

$$\epsilon^k(\nabla_X s_j) = \theta^l_j(X) \underbrace{\epsilon^k(s_l)}_{=\delta^k_l} = \theta^k_j(X) \tag{2.34}$$

where (ϵ^j) is the frame for E^* dual to (s_j).

In terms of a smooth local frame and associated connection 1-forms, we can write

$$\nabla s_j = \theta^k_j \otimes s_k, \tag{2.35}$$

$$\nabla\left(\sigma^j s_j\right) = \mathrm{d}\sigma^j \otimes s_j + \sigma^j \theta^k_j \otimes s_k. \tag{2.36}$$

Conversely, given an arbitrary matrix of smooth complex 1-forms θ^k_j on the domain U of a smooth local frame for E, the formula

$$\nabla_X\left(\sigma^j s_j\right) = \left(X\sigma^j\right) s_j + \sigma^j \theta^k_j(X) s_k \tag{2.37}$$

determines a connection on E over U.

If we have another local frame $(\tilde{s}_k)$, then where they overlap, we can write

$$\tilde{s}_k = \tau^j_k s_j, \tag{2.38}$$

for a $\mathrm{GL}(m, \mathbb{C})$-valued transition function $\tau := \left(\tau^j_k\right)$. We derive the transformation law of the connection 1-forms by

$$\tilde{\theta}^p_k \otimes \tilde{s}_p = \nabla \tilde{s}_k = \nabla\left(\tau^j_k s_j\right) = \mathrm{d}\tau^j_k \otimes s_j + \tau^j_k \theta^l_j \otimes s_l \tag{2.39}$$

$$= \left(\left(\tau^{-1}\right)^p_j \mathrm{d}\tau^j_k + \left(\tau^{-1}\right)^p_l \theta^l_j \tau^j_k\right) \otimes \tilde{s}_p, \tag{2.40}$$

where we have used $s_j = \left(\tau^{-1}\right)^l_j \tilde{s}_l$, and thus

$$\tilde{\theta}^p_k = \left(\tau^{-1}\right)^p_j \mathrm{d}\tau^j_k + \left(\tau^{-1}\right)^p_l \theta^l_j \tau^j_k \tag{2.41}$$

or in matrix notation

$$\tilde{\theta} = \tau^{-1}\mathrm{d}\tau + \tau^{-1}\theta\tau. \tag{2.42}$$

The matrices $\mathrm{d}\tau$ and θ are matrices of complex 1-forms and the order of matrix multiplication is, as usual, important.

Example 2.7.3. Given $E \to M$ and $E' \to M$ are holomorphic vector bundles of ranks k and k' respectively and holomorphic local frames $(s_1, \dots, s_k)$ for E and $(s'_1, \dots, s'_{k'})$ for E', we get a local frame $(s_1, \dots, s_k, s'_1, \dots, s'_{k'})$ for $E \oplus E'$. If τ and τ' are transition functions for overlapping local frames for E and E', respectively, then the transition function for $E \oplus E'$ is the $\mathrm{GL}(k+k', \mathbb{C})$-valued matrix function $\begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}$, which is holomorphic. Thus, by the chart lemma [Lee24, Lemma 3.4], $E \oplus E'$ is a holomorphic vector bundle of rank $k+k'$.

Metric compatibility of a connection is just half the story of being a *Chern connection* (which we will introduce later).

Definition 2.7.4. Suppose E is endowed with a Hermitian fiber metric $\langle \cdot, \cdot \rangle$. A connection ∇ on E is **compatible with the metric**, or a **metric connection** if the following indetify holds

$$X\langle \sigma, \tau \rangle = \langle \nabla_X \sigma, \tau \rangle + \langle \sigma, \nabla_X \tau \rangle . \tag{2.43}$$

Applying compatibility with the metric to complex vector fields Z yields

$$\nabla_Z \langle \sigma, \tau \rangle = \langle \nabla_Z \sigma, \tau \rangle + \langle \sigma, \nabla_{\overline{Z}} \tau \rangle \tag{2.44}$$

because of the conjugate linearity of the Hermitian inner product in the second argument.

Compatibility of a connection with a Hermitian fiber metric has consequences for the corresponding connection 1-forms.

Proposition 2.7.5. *Suppose $E \to M$ is a smooth complex vector bundle with Hermitian fiber metric and ∇ is a metric connection on E. The matrix of connection 1-forms with respect to any local orthonormal frame is skew-Hermitian:*

$$\theta_j^k = -\overline{\theta}_k^j. \tag{2.45}$$

Proof. Let (s_j) be a local orthonormal frame for E, and let θ_j^k be the corresponding connection 1-forms. For every local complex vector field Z, compatibility with the metric implies

$$\theta_j^k(Z) + \overline{\theta}_k^j(Z) = \theta_j^l(Z)\langle s_l, s_k \rangle + \overline{\theta_k^l(\overline{Z})}\langle s_j, s_l \rangle = \langle \nabla_Z s_j, s_k \rangle + \langle s_j, \nabla_{\overline{Z}} s_k \rangle = Z(\underbrace{\langle s_j, s_k \rangle}_{=\delta_{jk}}) = 0. \tag{2.46}$$

□

2.7.2 Compatibility with the Holomorphic Structure

An important technical theorem is the following

Theorem 2.7.6 (The $\overline{\partial}$-Poincaré Lemma, [Lee24, Theorem 4.13]). *Suppose $q \geq 1$ and ω is a smooth (p,q)-form on a complex manifold M that satisfies $\overline{\partial}\omega = 0$. Then locally, i.e., in a neighborhood of each point, there exists a smooth $(p,q-1)$-form η such that $\overline{\partial}\eta = \omega$.*

For each pair of nonnegative integers p,q, one defines the bundle of E-valued (p,q)-forms as the tensor product bundle $\bigwedge^{p,q} M \otimes E$, and $\Omega^{p,q}(M,E) := \Gamma(\bigwedge^{p,q} M \otimes E)$.

Proposition 2.7.7. *Suppose M is a complex manifold and $E \to M$ is a holomorphic vector bundle. There are operators $\overline{\partial}_E : \Omega^{p,q}(M,E) \to \Omega^{p,q+1}(M,E)$ satisfying the following properties.*

1. *For $\sigma \in \Omega^{0,0}(M,E) = \Gamma(E)$, $\overline{\partial}_E\sigma = 0 \leftrightarrow \sigma$ is a holomorphic section.*
2. *For all $\alpha \in \Omega^{p,q}(M)$ and $\beta \in \Omega^{p',q'}(M,E)$ one has*
$$\overline{\partial}_E(\alpha \wedge \beta) = \overline{\alpha} \wedge \beta + (-1)^{p+q}\alpha \wedge \left(\overline{\partial}_E\beta\right). \tag{2.47}$$
3. *For all $\gamma \in \Omega^{p,q}(M,E^*)$ and $\beta \in \Omega^{p',q'}(M,E)$ one has*
$$\overline{\partial}(\gamma \wedge \beta) = \overline{\partial}_{E^*}\gamma \wedge \beta + (-1)^{p+q}\gamma \wedge \overline{\partial}_E\beta. \tag{2.48}$$
4. *$\overline{\partial}_E \circ \overline{\partial}_E = 0$.*
5. *If $\alpha \in \Omega^{p,q}(M,E)$ satisfies $\overline{\partial}_E\alpha = 0$ then in a neighborhood of each point there exists $\beta \in \Omega^{p,q-1}(M,E)$ such that $\overline{\partial}_E\beta = \alpha$.*

Proof. Suppose $\sigma \in \Omega^{p,q}(M,E)$. In any open subset $U \subseteq M$ over which there is a holomorphic local frame (s_j) for E, we can write $\sigma|_U = \sigma^j \otimes s_j$ with scalar-valued forms σ^j. We define $\overline{\partial}_E\sigma$ by setting
$$\overline{\partial}_E\sigma\Big|_U := \left(\overline{\partial}\sigma^j\right) \otimes s_j. \tag{2.49}$$
We need to check whether this definition is independent of the choice of a holomorphic local frame. If $(\tilde{s}_k)$ is another holomorphic frame, then, where the domains overlap, we can write
$$\tilde{s}_k = \tau_k^j s_j \tag{2.50}$$
for some holomorphic functions τ_k^j. Then $\tilde{\sigma}^k\tau_k^j s_j = \tilde{\sigma}^k\tilde{s}_k = \sigma = \sigma^j s_j$ yielding $\tau_k^j\tilde{\sigma}^k = \sigma^j$. Because $\overline{\partial}\tau_k^j = 0$ (due to holomorphicity), we have
$$(\overline{\partial}\sigma^j) \otimes s_j = \overline{\partial}(\tau_k^j\tilde{\sigma}^k) \otimes s_j = (\overline{\partial}\tilde{\sigma}^k) \otimes (\tau_k^j s_j) = (\overline{\partial}\tilde{\sigma}^k) \otimes \tilde{s}_k. \tag{2.51}$$

This proves that $\overline{\partial}_E$ is well-defined.

1. Suppose $\sigma \in \Gamma(E)$. In terms of any local holomorphic frame (s_j), we can write $\sigma = f^j s_j$ for some complex-valued functions f^j. If σ is holomorphic, then each f^j is holomorphic and equation (2.49) shows that $\overline{\partial}_E\sigma = 0$.

Conversely, if $0 = \overline{\partial}_E \sigma = (\overline{\partial} f^j) \otimes s_j$ and since the sections s_j are linearly independent at each point, this shows that

$$\overline{\partial} f^j = 0 \tag{2.52}$$

for each j and thus σ is holomorphic by proposition 2.5.1.

2. We write locally $\beta = \beta^j \otimes s_j$ and compute

$$\overline{\partial}_E(\alpha \wedge \beta) = \overline{\partial}_E((\alpha \wedge \beta^j) \otimes s_j) = \overline{\partial}(\alpha \wedge \beta^j) \otimes s_j \tag{2.53}$$

$$= (\overline{\partial}\alpha \wedge \beta^j + (-1)^{p+q} \alpha \wedge \overline{\partial}\beta^j) \otimes s_j = (\overline{\partial}\alpha \wedge \beta^j) \otimes s_j + (-1)^{p+q} (\alpha \wedge \overline{\partial}\beta^j) \otimes s_j \tag{2.54}$$

$$= \overline{\partial}\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \overline{\partial}_E \beta. \tag{2.55}$$

3. Write $\beta \in \Omega^{p',q'}(M, E)$ locally as $\beta = \beta^j \otimes s_j$ and write $\gamma = \gamma_k \otimes \epsilon^k$, where (ϵ^k) denotes the local holomorphic frame for E^* dual to (s_j), so that

$$\gamma \wedge \beta = \left(\gamma_k \otimes \epsilon^k\right) \wedge (\beta^j \otimes s_j) = \epsilon^k (s_j) (\gamma_k \wedge \beta^j) = \gamma_j \wedge \beta^j. \tag{2.56}$$

We compute

$$\overline{\partial}_{E^*}\gamma \wedge \beta + (-1)^{p+q} \gamma \wedge \overline{\partial}_E \beta = \left(\overline{\partial}\gamma_k \otimes \epsilon^k\right) \wedge (\beta^j \otimes s_j) + (-1)^{p+q} \left(\gamma_k \otimes \epsilon^k\right) \wedge \left(\overline{\partial}\beta^j \otimes s_j\right) \tag{2.57}$$

$$= \overline{\partial}\gamma_j \wedge \beta^j + (-1)^{p+q} \gamma_j \wedge \overline{\partial}\beta^j = \overline{\partial}(\gamma_j \wedge \beta^j) = \overline{\partial}(\gamma \wedge \beta). \tag{2.58}$$

4. Let $\sigma \in \Omega^{p,q}(M, E)$ and suppose (s_j) is a holomorphic local frame for E on $U \subseteq M$. Writing $\sigma = \sigma^j \otimes s_j$, we conclude with equation (2.49) that

$$\overline{\partial}_E \left(\overline{\partial}_E \sigma\right) = \overline{\partial}_E \left(\left(\overline{\partial}\sigma^j\right) \otimes s_j\right) = \left(\overline{\partial}\,\overline{\partial}\sigma^j\right) \otimes s_j = 0 \tag{2.59}$$

5. Suppose $\alpha \in \Omega^{p,q}(M, E)$ satisfies $\overline{\partial}_E \alpha = 0$. In terms of a holomorphic local frame, we can write $\alpha = \alpha^j \otimes s_j$ for some scalar-valued (p,q)-forms α^j satisfying $0 = \overline{\partial}_E \alpha = (\overline{\partial}\alpha^j) \otimes s_j$. The pointwise linear independence of the s_j's implies $\overline{\partial}\alpha^j = 0$ for all j and the $\overline{\partial}$-Poincaré lemma (see theorem 2.7.6) yields $(p, q-1)$-forms β^j in a neighborhood of each point such that $\overline{\partial}\beta^j = \alpha^j$ and thus

$$\overline{\partial}_E (\beta^j \otimes s_j) = \overline{\partial}\beta^j \otimes s_j = \alpha^j \otimes s_j = \alpha. \tag{2.60}$$

□

Suppose M is a complex manifold and $E \to M$ is a holomorphic vector bundle. Using the decomposition $T^*_{\mathbb{C}} M = \bigwedge^1_{\mathbb{C}} M = \bigwedge^{1,0} M \oplus \bigwedge^{0,1} M$, we can decompose a connection ∇ on E as $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$, where $\nabla^{(1,0)}\sigma \in \Gamma\left(\bigwedge^{1,0} M \otimes E\right)$ and $\nabla^{(0,1)}\sigma \in \Gamma\left(\bigwedge^{0,1} M \otimes E\right)$ for $\sigma \in \Gamma(E)$.

Definition 2.7.8. A connection ∇ on a holomorphic vector bundle $E \to M$ is **compatible with the holomorphic structure** if $\nabla^{(0,1)} = \overline{\partial}_E$.

There are a few equivalent definitions of being compatible with a holomorphic structure.

Proposition 2.7.9 (Compatibility with the Holomorphic Structure). *Let $E \to M$ be a holomorphic vector bundle and ∇ a connection on E. The following statements are equivalent.*

1) *The connection ∇ is compatible with the holomorphic structure, i.e. $\nabla^{(0,1)} = \overline{\partial}_E$.*
2) *Whenever σ is a holomorphic local section of E and $\overline{Z}$ is a smooth local section of $T^{(0,1)}(M)$, we have $\nabla_{\overline{Z}}\sigma = 0$.*
3) *For each holomorphic local frame (s_j), we have $\nabla s_j = \theta_j^k \otimes s_k$ where the 1-forms θ_j^k are all of type $(1,0)$.*

Proof. Take any open set over which there is a holomorphic local frame (s_j). Let θ_j^k be the connection 1-forms with respect to this frame. Taking the projection of both sides of

$$\nabla\left(\sigma^j s_j\right) = \mathrm{d}\sigma^j \otimes s_j + \sigma^j \theta_j^k s_k \tag{2.61}$$

onto $\bigwedge^{0,1} M \otimes E$, we have

$$\nabla^{(0,1)}\left(\sigma^j s_j\right) = \overline{\partial}\sigma^j \otimes s_j + \sigma^j \left(\theta_j^k\right)^{(0,1)} \otimes s_k. \tag{2.62}$$

On the other hand, $\overline{\partial}_E\Big|_U = \overline{\partial}\sigma^j \otimes s_j$ locally, showing that $\overline{\partial}_E\left(\sigma^j s_j\right) = \overline{\partial}\sigma^j \otimes s_j$ and this implies

$$\nabla^{(0,1)}\left(\sigma^j s_j\right) = \overline{\partial}_E\left(\sigma^j s_j\right) + \sigma^j \left(\theta_j^k\right)^{(0,1)} \otimes s_k. \tag{2.63}$$

The equivalence *1)⇔3)* is immediately given by the observation that $\nabla^{(0,1)} = \overline{\partial}_E$ if and only if $\left(\theta_j^k\right)^{(0,1)} = 0$.

To prove *3)⇒2)*, we take a holomorphic local section σ of E and a smooth local section $\overline{Z}$ of $T^{0,1}(M)$. Decomposing $\sigma = \sigma^j s_j$ with holomorphic component functions σ^j, we observe

$$\nabla_{\overline{Z}}\sigma = \overline{Z}(\sigma^j)s_j + \theta_j^k(\overline{Z})s_k = 0 \tag{2.64}$$

because $\overline{Z}\left(\sigma^j\right) = 0$ and $\theta_j^k(\overline{Z}) = 0$. The former is because σ^j are holomorphic and the latter since θ_j^k are of type $(1,0)$ by assumption.

The reverse implication *2)⇒3)* is demonstrated by taking any local section $\overline{Z}$ of $T^{0,1}(M)$ and calculating

$$\theta_j^k(\overline{Z})s_k = \nabla_{\overline{Z}}s_j \overset{2)}{=} 0 \tag{2.65}$$

to conclude that θ_j^k vanishes on $T^{0,1}(M)$ and is thus of type $(1,0)$. □

2.7.3 Chern Connection

We now prove the existence and uniqueness on Hermitian holomorphic vector bundles.

Theorem 2.7.10 (Chern Connection Theorem). *On every Hermitian holomorphic vector bundle, there exists a unique connection, called the Chern connection, that is compatible with the metric and the holomorphic structure.*

Proof. To prove uniqueness, suppose ∇ is a Chern connection on $E \to M$ and let (s_j) be a holomorphic local frame for E over an open subset $U \subseteq M$. We write

$$\nabla s_j = \theta_j^k \otimes s_k \tag{2.66}$$

and note that the connection 1-forms θ_j^k are all of type $(1,0)$ according to proposition 2.7.9.

Writing $H_{jk} := \langle s_j, s_k \rangle$ and using compatibility with the metric implies for every local section Z of $T^{1,0}(M)$

$$Z(H_{jk}) = \left\langle \nabla_Z s_j, s_k \right\rangle + \left\langle s_j, \nabla_{\overline{Z}} s_k \right\rangle = \theta_j^l(Z) H_{lk}, \tag{2.67}$$

where we have used $\nabla_Z s_j = \theta_j^l(Z) s_l$ and $\nabla_{\overline{Z}} s_k = 0$ (the latter results from proposition 2.7.9). We can use the positive definiteness of the matrix (H_{jk}) to invert it and denote the inverse by (H^{jk}). We multiply equation (2.67) by H^{km} and obtain $\theta_j^m(Z) = H^{km} Z(H_{jk})$. Since this holds for every section Z of $T^{1,0}(M)$, we get

$$\theta_j^m = H^{km} \partial H_{jk}. \tag{2.68}$$

This proves that ∇ is uniquely determined by the Hermitian fiber metric H.

To prove existence, we use equation (2.68) as the definition of the connection 1-forms of ∇ in terms of each holomorphic local frame. These forms are of type $(1,0)$ by definition and thus the resulting connection ∇ is compatible with the holomorphic structure (see proposition 2.7.9). We start from the middle of equation (2.67), receive the right-hand side in which we insert equation (2.68) and thus obtain the left-hand side of equation (2.67) so that this finally proves the compatibility with the metric. □

In a natural way, we can also define the notion of a dual connection.

Theorem 2.7.11 ([Lee24, p. 219]). *Let $E \to M$ be a smooth complex vector bundle, and let ∇ be a connection on E. Define a map $\nabla^* : \Gamma(T_{\mathbb{C}}M) \times \Gamma(E^*) \to \Gamma(E^*)$ by $\left(\nabla_X^* \phi\right)(\sigma) := X(\phi(\sigma)) - \phi(\nabla_X \sigma)$ for all $X \in \Gamma(T_{\mathbb{C}}M)$, $\phi \in \Gamma(E^*)$ and $\sigma \in \Gamma(E)$.*

1) ∇^* is a connection on E^*, called the dual connection.

2) Suppose E is endowed with a Hermitian fiber metric H, and let H^* be the dual metric on E^*. Then if ∇ is a metric connection then

$$\nabla_X^* \left(\hat{H}(\sigma) \right) = \hat{H}\left(\nabla_{\overline{X}} \sigma\right) \tag{2.69}$$

and ∇^* is also a metric connection.

3) If E is a holomorphic Hermitian vector bundle and ∇ is its Chern connection, then ∇^* is the Chern connection of E^*.

2.8 Curvature

Having set up the notion of a connection on a vector bundle, we now provide the following

Definition 2.8.1. Suppose $M \to E$ is a smooth complex vector bundle and ∇ is a connection on E. Define the **curvature of** ∇ as the map $F\colon \Gamma(T_{\mathbb{C}}M) \times \Gamma(T_{\mathbb{C}}M) \times \Gamma(E) \to \Gamma(E)$ by

$$F(X,Y)\sigma := \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma. \tag{2.70}$$

A connection ∇ is called **flat** if $F = 0$.

One can prove quite quickly that F is $C^\infty(M,\mathbb{C})$-linear in all of its three arguments and antisymmetric in its first two arguments. Furthermore, $E \otimes E^* \cong E^* \otimes E$ is canonically isomorphic to $\mathrm{End}(E)$ so that finally $F \in \Omega^2(M, \mathrm{End}(E))$.

For connections ∇_1 on E and ∇_2 on E', there is a natural connection ∇ on $E \oplus E'$ such that $\nabla(s_1 \oplus s_2) = \nabla_1(s_1) \oplus \nabla_2(s_2)$, where s_1 is a section of E and s_2 a section of E'. In this case, $F_\nabla = F_{\nabla_1} \oplus F_{\nabla_2}$.

In the study of the Hull-Strominger system on a complex torus or an Iwasawa manifold, we will work with line bundles which is why we need

Proposition 2.8.2 ([Lee24, Proposition 7.20]). *Let $L \to M$ be a Hermitian holomorphic line bundle and let F_L be the curvature of its Chern connection. When L^* is endowed with the dual connection, the curvature of its Chern connection is given by $F_{L^*} = -F_L$.*

2.8.1 Curvature Forms

Now take a smooth local frame (s_j) for E and let θ^k_j be the matrix of connection 1-forms. The curvature F is completely determined by its action on each basis section s_j for arbitrary $X, Y \in \Gamma(T_{\mathbb{C}}M)$. We calculate

$$F(X,Y)s_j = \nabla_X\left(\theta^k_j(X)s_k\right) - \nabla_Y\left(\theta^k_j(X)s_k\right) - \theta^k_j([X,Y])s_k = \left(\mathrm{d}\theta^l_j + \theta^l_k \wedge \theta^k_j\right)(X,Y)s_l \tag{2.71}$$

where in the second equality sign, we used the invariant formula

$$\mathrm{d}\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \tag{2.72}$$

for a 1-form α. Thus, with respect to this frame, we define the *curvature 2-forms* F^l_j of ∇ by

$$F^l_j := \mathrm{d}\theta^l_j + \theta^l_k \wedge \theta^k_j. \tag{2.73}$$

Thus, F is represented locally by the matrix of 2-forms $\left(F^l_j\right)$. Interpreting θ^l_j and F^l_j as the local expressions for $\mathrm{End}(E)$-valued forms, we can use the wedge product of endomorphism-valued forms defined by equation (2.28) to write

$$F = \mathrm{d}\theta + \theta \wedge \theta. \tag{2.74}$$

Some remarks are in order about equation (2.74): Since θ is a matrix composed of 1-forms, $\theta \wedge \theta$ is in general not trivial. Moreover, θ is an endomorphism-valued form only in the domain of a specified frame. However, F is a globally defined $\mathrm{End}(E)$-valued 2-form. The fact that θ is not a local matrix representation for a globally defined $\mathrm{End}(E)$-valued 1-form is reflected by the transformation equation (2.42). If it were, the transformation law of θ under a change of frames would have been simply $\tilde{\theta} = \tau^{-1}\mathrm{d}\tau$.

We prove now that all curvature forms of a Chern connection are of type $(1,1)$.

Proposition 2.8.3. *Suppose M is a complex manifold and $E \to M$ is a Hermitian holomorphic vector bundle. With the Chern connection on E, we have $F \in \Omega^{1,1}(M, \mathrm{End}(E))$.*

Proof. In terms of a holomorphic local frame (s_j), we can write

$$F_j^m = \overline{\partial}\theta_j^m + \partial\theta_j^m + \theta_k^m \wedge \theta_j^k \tag{2.75}$$

having used $\mathrm{d} = \partial + \overline{\partial}$ (see proposition 2.5.2). According to proposition 2.7.9, θ_j^m is of type $(1,0)$ so that the first term $\overline{\partial}\theta$ in equation (2.75) is of type $(1,1)$ and the remaining two terms are of type $(2,0)$. So it remains to prove that the remaining two terms add up to zero. Inserting equation (2.68) into the last two terms of equation (2.75) yields

$$\partial\theta_j^m + \theta_k^m \wedge \theta_j^k = \partial H^{km} \wedge \partial H_{jk} + H^{pm} H^{qk} \partial H_{kp} \wedge \partial H_{jq}. \tag{2.76}$$

Differentiating $H^{kp}H_{pl} = \delta_l^k$ and multiplying the result by H^{lm} gives

$$\partial H^{km} = -H^{lm} H^{kp} \partial H_{pl} \tag{2.77}$$

and substituting this into the right-hand side of equation (2.76) provides

$$\partial\theta_j^m + \theta_k^m \wedge \theta_j^k = -H^{pm} H^{qk} \partial H_{kp} \wedge \partial H_{jq} + H^{pm} H^{qk} \partial H_{kp} \wedge \partial H_{jq} = 0 \tag{2.78}$$

implying that $F_j^m = \overline{\partial}\theta_j^m$ is indeed of type $(1,1)$. □

The connection and curvature forms on a dual bundle can be given in terms of the original vector bundle as described in

Theorem 2.8.4 ([Lee24, p. 219]). *Let $E \to M$ be a smooth complex vector bundle, let ∇ be a connection on E, and ∇^* the dual connection on E^*. Suppose (s_j) is a holomorphic local frame for E, and θ_k^j and F_k^j are its connection and curvature forms, respectively. Let (ϵ^j) be the dual frame for E^* defined by $\epsilon^j(s_k) := \delta_k^j$, and let $\theta^{*}{}_j^k$ and $F^{*}{}_j^k$ be the connection and curvature forms for ∇^* satisfying*

$$\nabla_X^* \epsilon^j = \theta^{*}{}_j^k(X)\epsilon^k \text{ and } F^*(X,Y)\epsilon^j = F^{*}{}_j^k(X,Y)\epsilon^k. \tag{2.79}$$

(A summation over repeated indices is also implicitly implied here.)

Then

$$\theta^{*}{}_j^k = -\theta_k^j \text{ and } F^{*}{}_j^k = -F_k^j. \tag{2.80}$$

2.9 The First Real Chern Class

Suppose a connection ∇ on a smooth complex vector bundle $E \to M$. Since $F \in \Omega^2(M, \mathrm{End}(E))$ and the trace of an endomorphism is independent of the choice of a basis, we can define the *first Chern form of* ∇ by

$$c_1(\nabla) := \frac{\mathrm{i}}{2\pi}\mathrm{Tr}(F) \tag{2.81}$$

which is a global scalar 2-form.

Theorem 2.9.1. *For any connection on a smooth complex vector bundle, the first Chern form is closed, and its de Rham cohomology class is independent of the choice of connection.*

Proof. Let $E \to M$ be a smooth complex vector bundle and ∇ a connection on E. In terms of a smooth local frame, we have

$$c_1(\nabla) = \frac{\mathrm{i}}{2\pi}F^j_j = \frac{\mathrm{i}}{2\pi}\mathrm{d}\theta^j_j, \tag{2.82}$$

where we have used that $\theta^j_l \wedge \theta^l_j = 0$. With equation (2.82), we therefore see that $c_1(\nabla)$ is locally exact and thus closed.

Now suppose $\tilde{\nabla}$ is another connection on E. We define the difference tensor $D\colon \Gamma(T_{\mathbb{C}}M)\times\Gamma(E)$ by $D(X)\sigma := \tilde{\nabla}_X\sigma - \nabla_X\sigma$. One can see by direct computation that D is $C^\infty(M)$-linear in both arguments which makes it a section of $\Omega^1(M, \mathrm{End}(E))$. This implies that its trace is a globally defined scalar 1-form and

$$c_1(\tilde{\nabla}) - c_1(\nabla) = \frac{\mathrm{i}}{2\pi}(\mathrm{Tr}(\mathrm{d}\tilde{\theta}) - \mathrm{Tr}(\mathrm{d}\theta)) = \frac{\mathrm{i}}{2\pi}\mathrm{Tr}(\mathrm{d}D) = \frac{\mathrm{i}}{2\pi}\mathrm{d}\mathrm{Tr}(D). \tag{2.83}$$

That is, $c_1(\tilde{\nabla})$ and $c_1(\nabla)$ differ by an exact form and thus represent the same de Rham cohomology class. □

The Chern form becomes real when we deal with metric connections, in particular when we work with Chern connections, as demonstrated in

Proposition 2.9.2. *Suppose $E \to M$ is a smooth complex vector bundle with a Hermitian fiber metric. If ∇ is a metric connection on E, then $c_1(\nabla)$ is a real 2-form.*

Proof. Let ∇ be a metric connection on E. In a neighborhood of each point, we may choose an orthonormal local frame (s_j) and denote by θ^k_j the corresponding connection 1-forms. Proposition 2.7.5 shows that

$$\theta^k_j + \overline{\theta}^j_k = 0 \tag{2.84}$$

Taking $k = j$ and summing over j yields that the 1-form θ^j_j is purely imaginary and so is $F^j_j = \mathrm{d}\theta^j_j$. This implies that

$$c_1(\nabla) = \frac{\mathrm{i}}{2\pi}F^j_j \tag{2.85}$$

is indeed real. □

Let $E \to M$ be a smooth complex vector bundle. We can always choose a Hermitian fiber metric on E and a connection ∇ compatible with it, so that $c_1(\nabla)$ is represented by a real 2-form. Thus, the cohomology class determined by $c_1(\nabla)$ lies in $H^2_{\mathrm{dR}}(M;\mathbb{R})$ where $H^2_{\mathrm{dR}}(M;\mathbb{R})$ is considered as the subspace of $H^2_{\mathrm{dR}}(M;\mathbb{C})$ consisting of cohomology classes that are invariant under conjugation.

We define the *first Chern class of* E, denoted by $c_1(E) \in H^2_{\mathrm{dR}}(M;\mathbb{R})$, to be the cohomology class of $c_1(\nabla)$, where ∇ is any connection on E.

2.10 Chern Connection and Curvature on $L \oplus L^*$

Let $L \to M$ be a Hermitian holomorphic line bundle and ∇ its Chern connection. Given a local holomorphic frame s for L (i.e., $s \neq 0$ holomorphic local section) over an open $U \subseteq M$, let θ be the corresponding connection 1-form. The fiber metric is completely determined in U by the strictly positive function

$$h := |s|^2 = \langle s, s \rangle. \tag{2.86}$$

It implies

$$\theta = h^{-1}\partial h = \partial(\log(h)). \tag{2.87}$$

Its curvature is the globally defined 2-form F_L whose expression in terms of each holomorphic local frame s is $F_L = \mathrm{d}\theta = \overline{\partial}\partial(\log(h)) = \overline{\partial}\partial\left(\log\left(|s|^2\right)\right)$. Thus, the Chern form for this connection has the local expression

$$c_1(\nabla) = \frac{\mathrm{i}}{2\pi}\overline{\partial}\partial(\log(h)) = \frac{\mathrm{i}}{2\pi}\overline{\partial}\partial\left(\log\left(|s|^2\right)\right). \tag{2.88}$$

On the Whitney sum $L \oplus L^*$ the curvature $F_{L\oplus L^*}$ becomes

$$F_{L\oplus L^*} = \begin{pmatrix} F_L & 0 \\ 0 & -F_L \end{pmatrix} \tag{2.89}$$

because $F_{L^*} = -F_L$. The dual connection ∇^* on L^* is automatically the Chern connection (see theorem 2.7.11).

Remark 2.10.1. Equation (2.89) implies directly $c_1(L \oplus L^*) = 0$ which is one of the necessary conditions in the formulation of the Hull-Strominger system if we use $L \oplus L^*$ as the holomorphic vector bundle over a complex threefold M.

2.11 Computational Preparation

Given a holomorphic vector bundle $E \to M$ and a connection ∇ on E, we can locally decompose the corresponding curvature as

$$F = F\left(\partial_i, \partial_{\overline{j}}\right) \otimes \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j + \frac{1}{2}F\left(\partial_i, \partial_j\right) \otimes \mathrm{d}z^i \wedge \mathrm{d}z^j + \frac{1}{2}F\left(\partial_{\overline{i}}, \partial_{\overline{j}}\right) \otimes \mathrm{d}\overline{z}^i \wedge \mathrm{d}\overline{z}^j \tag{2.90}$$

with (z^i) being local holomorphic coordinates on M. Further assuming ∇ to be the Chern connection on E, proposition 2.8.3 tells us that they must be of type $(1,1)$ which implies $F\left(\partial_i, \partial_j\right) = 0 = F\left(\partial_{\overline{i}}, \partial_{\overline{j}}\right)$ for all i and j, reducing equation (2.90) to

$$F = F\left(\partial_i, \partial_{\overline{j}}\right) \otimes \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j. \tag{2.91}$$

Taking the wedge of the curvature F with itself amounts to

$$F \wedge F = F\left(\partial_i, \partial_{\bar{j}}\right) F\left(\partial_k, \partial_{\bar{l}}\right) \otimes \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \wedge \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^l. \tag{2.92}$$

Going back to the definition of curvature, namely $F \in \Omega^2(M, \mathrm{End}(E))$, a straightforward observation is that we can think of F as being a matrix composed of 2-forms, or more precisely of $(1,1)$-forms since we are working with a Chern connection. This provides the computationally feasible observation that

$$(F \wedge F)^i_j = F^i_k \wedge F^k_j \tag{2.93}$$

Therefore, $F \wedge F$ is computed by standard matrix multiplication, but instead of using ordinary multiplication of complex numbers, we take the wedge of 2-forms and do the standard summation as is part of ordinary matrix multiplication.

We thus receive

$$\mathrm{Tr}(F \wedge F) = \mathrm{Tr}\left(F\left(\partial_i, \partial_{\bar{j}}\right) F\left(\partial_k, \partial_{\bar{l}}\right)\right) \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \wedge \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^l = F^i_j \wedge F^j_i. \tag{2.94}$$

3 The Hull-Strominger System

The Hull-Strominger system is a system of coupled partial differential equations and originates from supergravity [Hul86b; Str86] and it first appeared in the mathematics literature in [LY05]. There is a conjectural relation between the Hull-Strominger system and conformal field theory which arises in a certain physical limit in compactifications of the heterotic string theory.

Yau proposed to study this system of partial differential equations as a natural generalization of the Calabi problem for non-Kählerian complex manifolds [Yau05] and due to its relation to Reid's fantasy on the moduli space of projective Calabi-Yau threefolds [Rei87].

Solutions to the Hull-Strominger system are found by polystable holomorphic vector bundles and Kähler Ricci flat metrics when the dimension of the complex base manifold is 1 or 2 [Gar16]. In the case of a three-dimensional complex base manifold, arguing on the existence and uniqueness for the Hull-Strominger system is still an open problem. Under mild assumptions, the existence of solutions to the Hull-Strominger system has been conjectured by Yau [Yau10].

3.1 General Observations

Assuming (M,ω) to be a compact Kähler threefold with $c_1(M)=0$ and taking $E := T^{1,0}(M)$ and $H = g$, the anomaly condition is automatically satisfied. Furthermore, when $\alpha' = 0$, the Bianchi identity (1.1) is reduced to the Type IIB equation with no source [Pho23]. The main obstruction in proving the existence of solutions to the system of equations (i.e. when assuming $\dim_{\mathbb{C}}(M) = 3$), is the Bianchi identity in equation (1.1) that couples the fundamental 2-form ω of the conformally balanced Hermitian metric g on the complex manifold M with the curvatures of the Chern connections of g and H respectively.

If ω were time independent, the flow in H is the well-known Donaldson heat flow. Thus, the Anomaly flow has the interesting peculiarity that ω flows as well and couples to the Donaldson heat flow. Using Hamilton's version of the Nash-Moser implicit function theorem, the short-time existence for the Anomaly flow is proven in [PPZ18b].

An interesting way of finding solutions to the Hull-Strominger system in equations (1.1) to (1.3) is to search for stationary points of the Anomaly flow formulated in equations (1.5) to (1.8). The reason for its name is that the stationary points of the Anomaly flow obviously satisfy the Green-Schwarz anomaly cancellation equation (1.1). Given ω_0 in equation (1.7) being the fundamental 2-form of an initial Hermitian metric on $T^{1,0}(M)$ and H_0 in equation (1.8) an initial Hermitian fiber metric for E, we study the Anomaly flow of the pair of time-dependent metrics $(\omega(t), H(t))$ satisfying equations (1.5) and (1.6) simultaneously. The initial metric ω_0 shall satisfy the conformally balanced condition (1.2) and it is indeed possible to construct conformally balanced metrics [TW16].

Using $\mathrm{d}\partial\overline{\partial}\omega = \mathrm{d}^2\overline{\partial}\omega = 0$ and the fact that $\mathrm{Tr}(R \wedge R)$ and $\mathrm{Tr}(F \wedge F)$ are closed representatives of the Chern classes $c_2\left(T^{1,0}(M)\right)$ and $c_2(E)$ of the bundles $T^{1,0}(M)$ and E respectively [PPZ18b], we find with equation (1.5),

$$\partial_t \mathrm{d}\left(\|\Omega\|_\omega \omega^2\right) = \mathrm{d}\partial_t\left(\|\Omega\|_\omega \omega^2\right) = 0. \tag{3.1}$$

That is, $\omega(t)$ satisfies the conformally balanced condition for all times t since ω_0 does and this implies that the conformally balanced condition does not need to be added to the flow equations in order to ensure that its stationary points are indeed solutions to the Hull-Strominger system. Actually, to find solutions to the Hull-Strominger system, it is sufficient to determine whether the Anomaly flow exists for all times and whether it converges.

The solution to the Hull-Strominger system Fu and Yau constructed on a T^2-fibration over a $K3$ surface [FY08; FY07] can be recast into flowing metrics that satisfy the Anomaly flow and the associated stationary point recaptures the solution found by Fu and Yau [PPZ18a].

For future reference, we denote $\omega = \mathrm{i} g_{j\bar{k}} \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k$ for the local decomposition of the fundamental 2-form on M, and then

$$\omega^3 = -\mathrm{i} g_{j\bar{k}} g_{l\bar{m}} g_{p\bar{q}} \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k \wedge \mathrm{d}z^l \wedge \mathrm{d}\bar{z}^m \wedge \mathrm{d}z^p \wedge \mathrm{d}\bar{z}^q \tag{3.2}$$

$$= \mathrm{i} g_{\sigma_1\sigma_1'} g_{\sigma_2\sigma_2'} g_{\sigma_3\sigma_3'} \mathrm{d}z^{\sigma_1} \wedge \mathrm{d}z^{\sigma_2} \wedge \mathrm{d}z^{\sigma_3} \wedge \mathrm{d}\bar{z}^{\sigma_1'} \wedge \mathrm{d}\bar{z}^{\sigma_2'} \wedge \mathrm{d}\bar{z}^{\sigma_3'} \tag{3.3}$$

$$= \mathrm{i}\,\mathrm{sign}(\sigma)\mathrm{sign}(\sigma') g_{\sigma_1\sigma_1'} g_{\sigma_2\sigma_2'} g_{\sigma_3\sigma_3'} \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \wedge \mathrm{d}\bar{z}^1 \wedge \mathrm{d}\bar{z}^2 \wedge \mathrm{d}\bar{z}^3 \tag{3.4}$$

$$= \mathrm{i}3!\det(g) \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \wedge \mathrm{d}\bar{z}^1 \wedge \mathrm{d}\bar{z}^2 \wedge \mathrm{d}\bar{z}^3. \tag{3.5}$$

In the notation, we understand $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\sigma' = (\sigma_1', \sigma_2', \sigma_3')$ as individual elements of the symmetric group $\mathcal{S}_3$ over which a summation is implied.

Working locally with $\Omega = f \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3$ for some nowhere vanishing function f, we find by combining equation (1.4) with equation (3.5),

$$\|\Omega\|_\omega^2 = \frac{|f|^2}{6\det(g)}. \tag{3.6}$$

3.2 On a Complex Torus

Suppose a complex vector space V with $\dim_{\mathbb{C}}(V) = n$, considered as an abelian complex Lie group. A lattice in V is a discrete additive subgroup $\Lambda \subseteq V$ generated by $2n$ vectors $v_1, \ldots, v_{2n}$ that are linearly independent over $\mathbb{R}$. In [Lee24, Corollary 1.17] it is shown that V/Λ is an n-dimensional complex Lie group, called a *complex torus*. When $n = 0$, it is just a single point. When $n > 0$, the real linear isomorphism $A\colon \mathbb{R}^{2n} \to V$ given by $A(x^1, \ldots, x^{2n}) = x^j v_j$ descends to a diffeomorphism from $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ to V/Λ. Since $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is diffeomorphic to the $2n$-torus $S^1 \times \cdots \times S^1$, so is V/Λ. Thus, complex tori defined by different lattices are all diffeomorphic to each other. They are typically not biholomorphic.

3.2.1 Anomaly Flow Ansatz on a Complex Torus

Here, we consider a complex torus $M \coloneqq \mathbb{C}^3/\Lambda$ with Λ being some lattice in $\mathbb{C}^3$. For the torus M, we make the Anomaly flow ansatz

$$H(t) \coloneqq \begin{pmatrix} a(t)h_0 & 0 \\ 0 & a(t)^{-1}h_0^{-1} \end{pmatrix} \text{ and } \omega(t) \coloneqq b(t)\omega_0 \tag{3.7}$$

with

$$\omega_0 \coloneqq \mathrm{i} g_{j\bar{k}}(0) \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k \text{ and } H_0 \coloneqq \begin{pmatrix} h_0 & 0 \\ 0 & h_0^{-1} \end{pmatrix}, \tag{3.8}$$

where ω_0 is the fundamental 2-form of some initial metric on M that is conformally balanced and h_0 is the initial metric on the line bundle L. Thus, $H_0 \coloneqq H(0)$ constitutes the initial metric on $L \oplus L^*$. We further need $a(0) = 1 = b(0)$. The time-dependent functions a and b must necessarily be positive for all times to ensure we are dealing with Hermitian metrics for all $t \geq 0$. With the definition of the first Chern class and equation (2.89), we of course have $c_1(L \oplus L^*) = 0$ automatically.

3.2.2 Necessary Conditions on Curvature Form on L

We assume the curvature on $L \oplus L^*$ to be coming from the Chern connection of H and having the form

$$F(t) \coloneqq \begin{pmatrix} f(t)\omega(t) & 0 \\ 0 & -f(t)\omega(t) \end{pmatrix} \tag{3.9}$$

for some function f that solely depends on the flow parameter $t \geq 0$. The assumption necessitates

$$\begin{pmatrix} f(t)\omega(t) & 0 \\ 0 & -f(t)\omega(t) \end{pmatrix} = F(t) = \overline{\partial}\left(\partial H(t) \cdot H(t)^{-1}\right) = \begin{pmatrix} \overline{\partial}(h_0^{-1}\partial h_0) & 0 \\ 0 & -\overline{\partial}(h_0^{-1}\partial h_0) \end{pmatrix} \tag{3.10}$$

which implies $f = b^{-1}$ such that $F(t) = F(0)$ is actually time independent, and $\omega_0 = \overline{\partial}(h_0^{-1}\partial h_0)$, i.e., the assumption necessitates this particular interrelation between the initial metrics. This interrelation is achieved by, for example, assuming the connection 1-form on L to be $-\mathrm{i} g_{j\bar{k}} \bar{z}^k \mathrm{d}z^j$.

3.2.3 Solving the ODEs Associated to the Anomaly Flow

We first define the nowhere vanishing holomorphic $(3,0)$-form $\Omega = \tilde{f}\mathrm{d}z^1, \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3$ such that $|\tilde{f}|$ is constant and

$$\|\Omega\|_{\omega(t)} = \frac{\lambda}{\sqrt{\det(g(t))}} = \lambda b(t)^{-3/2} \tag{3.11}$$

for some $\lambda > 0$. For simplicity and to obtain equation (3.11), we implicitly assume the metric ω to be diagonal.

The conformally balanced condition

$$\mathrm{d}\left(\|\Omega\|_{\omega(t)}\omega(t)^2\right) = 0 \tag{3.12}$$

is of course satisfied for all times t since we assume ω to be position independent.

The Anomaly flow equations translate to a system of coupled ordinary differential equations in the functions a and b which is soluble.

Theorem 3.2.1. *The PDEs of the Anomaly flow are solved by*

$$\omega(t) = \left(\frac{\alpha'}{2\lambda}t+1\right)^2 \omega_0 \text{ and} \tag{3.13}$$

$$H(t) = \begin{pmatrix} \exp\left(\frac{2\lambda}{\alpha'}\left(1-\frac{1}{\frac{\alpha'}{2\lambda}t+1}\right)\right)h_0 & 0 \\ 0 & \exp\left(-\frac{2\lambda}{\alpha'}\left(1-\frac{1}{\frac{\alpha'}{2\lambda}t+1}\right)\right)h_0^{-1} \end{pmatrix}. \tag{3.14}$$

Proof. We have

$$\partial_t \log(a(t)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H(t)^{-1}\partial_t H(t) = \frac{\omega(t)^2 \wedge F(t)}{\omega(t)^3} = \frac{1}{b(t)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.15}$$

and

$$\frac{\lambda}{2}b'(t)b(t)^{-1/2}\omega_0^2 = \partial_t(\|\Omega\|_{\omega(t)}\omega(t)^2) = \mathrm{i}\partial\underbrace{\overline{\partial}\omega(t)}_{=0} - \frac{\alpha'}{4}(\mathrm{Tr}(\underbrace{R(t)}_{=0}\wedge R(t)) - \underbrace{\mathrm{Tr}(F(t)\wedge F(t))}_{=2\omega_0^2}) = \frac{\alpha'}{2}\omega_0^2 \tag{3.16}$$

We thus obtain the following coupled initial value problem.

$$\begin{cases} b'(t) = \frac{\alpha'}{\lambda}b(t)^{1/2} & \text{with } b(0)=1, \\ \partial_t \log(a(t)) = b(t)^{-1} & \text{with } a(0)=1. \end{cases} \tag{3.17}$$

The first equation in equation (3.17) implies

$$b(t) = \left(\frac{\alpha'}{2\lambda}t+1\right)^2 \tag{3.18}$$

and inserting equation (3.18) into the second equation of equation (3.17) delivers

$$a(t) = \exp\left(\frac{2\lambda}{\alpha'}\left(1-\frac{1}{\frac{\alpha'}{2\lambda}t+1}\right)\right). \tag{3.19}$$

Inserting equations (3.18) and (3.19) into the ansatz in equation (3.7), provides equations (3.13) and (3.14). □

Theorem 3.2.1 implies that the infinite-time behaviour of $t \mapsto \omega(t)$ is divergent although we find for $t \mapsto H(t)$,

$$H_\infty := \lim_{t\to\infty} H(t) = \begin{cases} \begin{pmatrix} e^{2\lambda/\alpha'}h_0 & 0 \\ 0 & e^{-2\lambda/\alpha'}h_0^{-1} \end{pmatrix} & \text{provided } \alpha' > 0, \\ \text{divergent with blow up time } T := -\frac{\alpha'}{2\lambda} & \text{provided } \alpha' < 0 \end{cases} \tag{3.20}$$

That is, although the flow has no fixed point, and thus, there is no solution to the Hull-Strominger system in this torus case, H converges only in the physical case $\alpha' > 0$.

3.3 On an Iwasawa Manifold

Inspired by [Láz+25, Section 4.2], we consider the subgroup $H \leq \mathrm{GL}(3,\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.21}$$

for $z^1, z^2, z^3 \in \mathbb{C}$. It is a complex Lie group that is biholomorphic to $\mathbb{C}^3$ with multiplication given by

$$(z^1, z^2, z^3) \cdot (w^1, w^2, w^3) := (z^1 + w^1, z^2 + w^2, z^3 + w^3 + z^1 w^2). \tag{3.22}$$

We consider the *standard Iwasawa manifold* which is the left coset space $M := H/\Gamma$, where $\Gamma \leq H$ is the discrete subgroup consisting of matrices in which z^1, z^2, z^3 are *Gaussian integers*, i.e., complex numbers of the form $m + in$ for $m, n \in \mathbb{Z}$. The Iwasawa manifold is a complex manifold [Lee24, Corollary 1.17] with $\dim_{\mathbb{C}}(M) = 3$. The subgroup Γ is cocompact, that is, M is compact. The variables z^i define complex coordinates on H and hence a complex structure J on M.

3.3.1 A Left-Invariant Basis and Anomaly Flow Ansatz

We define a global (left-invariant) basis for $\Omega^{1,0}(M)$ by $\alpha_1 := \mathrm{d}z^1$, $\alpha_2 := \mathrm{d}z^2$ and $\alpha_3 := -\mathrm{d}z^3 + z^1 \mathrm{d}z^2$ which satisfy $\mathrm{d}\alpha_1 = 0 = \mathrm{d}\alpha_2$ and $\mathrm{d}\alpha_3 = \alpha_1 \wedge \alpha_2$. We also define the nowhere vanishing holomorphic $(3,0)$-form $\Omega := \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = -\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3$.

Now, consider the Anomaly flow ansatz

$$\begin{aligned} \omega(t) &:= \frac{\mathrm{i}}{2} \sum_{i=1}^{3} b_i(t) \alpha_i \wedge \overline{\alpha}_i \\ &= \frac{\mathrm{i}}{2} \Big(b_1(t) \mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 + \left(b_2(t) + b_3(t) \left| z^1 \right|^2 \right) \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2 + b_3(t) \mathrm{d}z^3 \wedge \mathrm{d}\overline{z}^3 \\ &\qquad - b_3(t) \overline{z}^1 \mathrm{d}z^3 \wedge \mathrm{d}\overline{z}^2 - b_3(t) z^1 \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^3 \Big) \end{aligned} \tag{3.23}$$

such that

$$\left(g_{i\overline{j}}(t) \right) = \begin{pmatrix} \frac{b_1(t)}{2} & 0 & 0 \\ 0 & \frac{b_2(t) + b_3(t) |z^1|^2}{2} & -\frac{b_3(t)}{2} z^1 \\ 0 & -\frac{b_3(t)}{2} \overline{z}^1 & \frac{b_3(t)}{2} \end{pmatrix} \tag{3.24}$$

with $c_1 := b_1(0) > 0$, $c_2 := b_2(0) > 0$, $b_3(0) = 1$ and $b_i > 0$ being positive flow parameter-dependent functions and $\omega_0 := \omega(0)$ is the initial metric that is assumed to be conformally balanced. In particular,

$$\det\left(g_{i\overline{j}}(t) \right) = b_1(t) b_2(t) b_3(t)/8. \tag{3.25}$$

As in the torus case in section 3.2,

$$\|\Omega\|_{\omega(t)} = \frac{\lambda}{\sqrt{b_1(t)b_2(t)b_3(t)}} \tag{3.26}$$

for some $\lambda > 0$.

Analogous to section 3.2, we make the Anomaly flow ansatz

$$H(t) := \begin{pmatrix} a(t)h_0 & 0 \\ 0 & a(t)^{-1}h_0^{-1} \end{pmatrix} \tag{3.27}$$

with $a > 0$, $a(0) = 1$ and h_0 being the initial Hermitian fiber metric on the line bundle L and thus H the Hermitian fiber metric on $L \oplus L^*$.

3.3.2 Curvature of Iwasawa Manifold and Conformally Balanced Condition

Lemma 3.3.1. *The time-dependent Chern connection of the metric in equation (3.23) is flat.*

Proof. The associated matrix of connection 1-forms is computed rather quickly via

$$\theta = \partial g \cdot g^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha_1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.28}$$

so that the connection is flat since $R = \bar{\partial}\theta = 0$. □

It is a general fact that $\omega(t)$ is conformally balanced for all times t since ω_0 is [PPZ18b]. However, it is a straightforward calculation to prove the conformally balanced condition for $\omega(t)$ for all t by hand:

Lemma 3.3.2. *The metric $\omega(t)$ is conformally balanced for all $t \geq 0$.*

Proof. Squaring the fundamental 2-form delivers

$$\begin{aligned} \omega(t)^2 = -\frac{1}{2}(&b_1(t)b_2(t)\alpha_1 \wedge \overline{\alpha}_1 \wedge \alpha_2 \wedge \overline{\alpha}_2 + b_2(t)b_3(t)\alpha_2 \wedge \overline{\alpha}_2 \wedge \alpha_3 \wedge \overline{\alpha}_3 \\ &+ b_3(t)b_1(t)\alpha_3 \wedge \overline{\alpha}_3 \wedge \alpha_1 \wedge \overline{\alpha}_1). \end{aligned} \tag{3.29}$$

The conformally balanced condition is satisfied for all times since

$$\begin{aligned} \mathrm{d}\left(\|\Omega\|_{\omega(t)}\omega(t)^2\right) = -\frac{1}{2}\|\Omega\|_{\omega(t)} b_3(t)b_1(t)(\partial + \bar{\partial})\Big(&|z^1|^2 \mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2 \wedge \mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 \\ -z^1 \mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^3 \wedge \mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 - \bar{z}^1 \mathrm{d}z^3 \wedge \mathrm{d}\bar{z}^2 \wedge \mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1&\Big) = 0. \end{aligned} \tag{3.30}$$

□

3.3.3 Necessary Condition on Curvature Form on L

We define a Chern connection on the rank-2 holomorphic vector bundle $L \oplus L^* \to M$, so that it has the curvature

$$\bar{\partial}\left(\partial H(t) \cdot H(t)^{-1}\right) = F(t) = \frac{\mathrm{i}}{4}(\alpha_1 \wedge \bar{\alpha}_1 - \alpha_2 \wedge \bar{\alpha}_2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{3.31}$$

This implies that a necessary requirement for the initial Hermitian fiber metrics is

$$\bar{\partial}\left(h_0^{-1}\partial h_0\right) = \frac{\mathrm{i}}{4}\left(\alpha_1 \wedge \bar{\alpha}_1 - \alpha_2 \wedge \bar{\alpha}_2\right) \tag{3.32}$$

and this is how we choose the initial metric. This is for instance guaranteed by taking the connection 1-form on L as $\frac{\mathrm{i}}{4}\left(\bar{z}^2\alpha_2 - \bar{z}^1\alpha_1\right)$.

3.3.4 Solving the Anomaly Flow Equations

We now use the ansatz in equations (3.23) and (3.27) to solve the Anomaly flow partial differential equation in equations (1.5) to (1.8):

Theorem 3.3.3. *The Anomaly flow is solved by*

$$\omega(t) = \frac{\mathrm{i}}{2}\left(\sqrt{\frac{c_1}{c_2}}\left(\frac{1}{\lambda} - \frac{\alpha'}{8\lambda}\right)t + c_1\right)\alpha_1 \wedge \bar{\alpha}_1 + \frac{\mathrm{i}}{2}\left(\sqrt{\frac{c_2}{c_1}}\left(\frac{1}{\lambda} - \frac{\alpha'}{8\lambda}\right)t + c_2\right)\alpha_2 \wedge \bar{\alpha}_2 + \frac{\mathrm{i}}{2}\alpha_3 \wedge \bar{\alpha}_3, \tag{3.33}$$

and

$$H(t) = \begin{pmatrix} h_0\left(\frac{1}{\sqrt{c_1c_2}}\left(\frac{1}{\lambda} - \frac{\alpha'}{8\lambda}\right)t + 1\right)^{\frac{3/2}{\frac{1}{\lambda}-\frac{\alpha'}{8\lambda}}\left(\sqrt{\frac{c_1}{c_2}}+\sqrt{\frac{c_2}{c_1}}\right)} & 0 \\ 0 & h_0^{-1}\left(\frac{1}{\sqrt{c_1c_2}}\left(\frac{1}{\lambda} - \frac{\alpha'}{8\lambda}\right)t + 1\right)^{-\frac{3/2}{\frac{1}{\lambda}-\frac{\alpha'}{8\lambda}}\left(\sqrt{\frac{c_1}{c_2}}+\sqrt{\frac{c_2}{c_1}}\right)}\end{pmatrix}, \tag{3.34}$$

whereas the formula for H holds only in the case $\alpha' \neq 8$ and $c_1 \neq c_2$.

Proof. We have

$$\mathrm{Tr}(F \wedge F) = \frac{1}{4}\alpha_1 \wedge \bar{\alpha}_1 \wedge \alpha_2 \wedge \bar{\alpha}_2. \tag{3.35}$$

Furthermore,

$$\partial\bar{\partial}\omega = \partial\bar{\partial}\left(\frac{\mathrm{i}}{2}\left|z^1\right|^2 b_3(t)\mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2\right) = \frac{\mathrm{i}}{2}b_3(t)\alpha_1 \wedge \bar{\alpha}_1 \wedge \alpha_2 \wedge \bar{\alpha}_2. \tag{3.36}$$

The anomaly flow now delivers

$$
\begin{aligned}
&-\frac{\lambda}{2}\Bigg(\partial_t\sqrt{\frac{b_1(t)b_2(t)}{b_3(t)}}\alpha_1\wedge\overline{\alpha}_1\wedge\alpha_2\wedge\overline{\alpha}_2+\partial_t\sqrt{\frac{b_2(t)b_3(t)}{b_1(t)}}\alpha_2\wedge\overline{\alpha}_2\wedge\alpha_3\wedge\overline{\alpha}_3\\
&\qquad+\partial_t\sqrt{\frac{b_3(t)b_1(t)}{b_2(t)}}\alpha_3\wedge\overline{\alpha}_3\wedge\alpha_1\wedge\overline{\alpha}_1\Bigg)=\partial_t\left(\|\Omega\|_{\omega(t)}\omega(t)^2\right)\\
&\qquad=\mathrm{i}\partial\overline{\partial}\omega(t)-\frac{\alpha'}{4}(\mathrm{Tr}(R(t)\wedge R(t))-\mathrm{Tr}(F(t)\wedge F(t)))\\
&\qquad=\left(\frac{\alpha'}{16}-\frac{b_3(t)}{2}\right)\alpha_1\wedge\overline{\alpha}_1\wedge\alpha_2\wedge\overline{\alpha}_2
\end{aligned}
\tag{3.37}
$$

and using

$$
\omega(t)^3=-\frac{3}{4}\mathrm{i}b_1(t)b_2(t)b_3(t)\alpha_1\wedge\overline{\alpha}_1\wedge\alpha_2\wedge\overline{\alpha}_2\wedge\alpha_3\wedge\overline{\alpha}_3, \tag{3.38}
$$

we find

$$
\partial_t\log(a(t))\begin{pmatrix}1&0\\0&-1\end{pmatrix}=H(t)^{-1}\partial_tH(t)=\frac{\omega(t)^2\wedge F(t)}{\omega(t)^3}=\frac{3}{2}\left(\frac{1}{b_1(t)}-\frac{1}{b_2(t)}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{3.39}
$$

The first Anomaly flow equation (3.37) gives the initial value problem

$$
\begin{cases}
\partial_t\sqrt{\frac{b_1(t)b_2(t)}{b_3(t)}} &=\frac{b_3(t)}{\lambda}-\frac{\alpha'}{8\lambda},\\
\partial_t\sqrt{\frac{b_2(t)b_3(t)}{b_1(t)}} &=0,\\
\partial_t\sqrt{\frac{b_3(t)b_1(t)}{b_2(t)}} &=0,\\
b_1(0) &=c_1,\\
b_2(0) &=c_2,\\
b_3(0) &=1,
\end{cases}
\tag{3.40}
$$

and the second Anomaly flow equation (3.39) provides

$$
\begin{cases}
\partial_t\log(a(t)) &=\frac{3}{2}\left(\frac{1}{b_1(t)}-\frac{1}{b_2(t)}\right),\\
a(0) &=1.
\end{cases}
\tag{3.41}
$$

The second and third equations in equation (3.40) are easily soluble:

$$
\sqrt{\frac{b_2(t)b_3(t)}{b_1(t)}}=\sqrt{\frac{c_2}{c_1}}\text{ and }\sqrt{\frac{b_3(t)b_1(t)}{b_2(t)}}=\sqrt{\frac{c_1}{c_2}}. \tag{3.42}
$$

Multiplying these two equations with each other gives $b_3\equiv 1$. Inserting this into the integrated third line of equation (3.40) results in $b_2=\frac{c_2}{c_1}b_1$ and together with the first line in equation (3.40) provides

$$b_1(t) = \sqrt{\frac{c_1}{c_2}}\left(\frac{1}{\lambda} - \frac{\alpha'}{8\lambda}\right)t + c_1 \text{ and } b_2(t) = \sqrt{\frac{c_2}{c_1}}\left(\frac{1}{\lambda} - \frac{\alpha'}{8\lambda}\right)t + c_2. \tag{3.43}$$

Inserting b_1, b_2 and b_3 into the ansatz in equation (3.23) for ω, delivers equation (3.33).

We now integrate the differential equation (3.41) and find

$$\log(a(t)) = \frac{3/2}{\frac{1}{\lambda} - \frac{\alpha'}{8\lambda}}\left(\sqrt{\frac{c_1}{c_2}} + \sqrt{\frac{c_2}{c_1}}\right)\log\left(\frac{1}{\sqrt{c_1c_2}}\left(\frac{1}{\lambda} - \frac{\alpha'}{8\lambda}\right)t + 1\right), \tag{3.44}$$

so that

$$a(t) = \left(\frac{1}{\sqrt{c_1c_2}}\left(\frac{1}{\lambda} - \frac{\alpha'}{8\lambda}\right)t + 1\right)^{\frac{3/2}{\frac{1}{\lambda} - \frac{\alpha'}{8\lambda}}\left(\sqrt{\frac{c_1}{c_2}} + \sqrt{\frac{c_2}{c_1}}\right)}. \tag{3.45}$$

Inserting equation (3.45) into the ansatz for H in equation (3.27) finishes the proof. □

Theorem 3.3.3 shows that the infinite-time behaviour of ω and H is divergent if $\alpha' \neq 8$. It proves the non-existence of a stationary point and thus the non-existence of solutions to the Hull-Strominger system in the case $\alpha' \neq 8$.

It should be mentioned that equation (3.33) defines a Hermitian metric on $T^{1,0}(M)$ as long as $\alpha' \leq 8$. If $\alpha' > 8$, there exists a finite time $T > 0$ for which $\omega(T)$ is not the fundamental 2-form of any Hermitian fiber metric on $T^{1,0}(M)$.

3.3.5 The Special Slope Parameter $\alpha' = 8$

In the case $c_1 = c_2$, we get $b_1 = b_2$ and together with equation (3.39), we find

$$H(t) = H(0) \text{ for all times } t \geq 0. \tag{3.46}$$

However, also in this case, the infinite-time behavior of ω remains divergent if $\alpha' \neq 8$.

Clearly, the initial value problem in equation (3.40) demonstrates that the case $\alpha' = 8$ is special.

Theorem 3.3.4. *Suppose* $\alpha' = 8$.

1) *If* $c_1 = c_2$, *a stationary point exists in the Anomaly flow and is given by* $(\omega(0), H(0))$.

2) *If* $c_1 < c_2$, *the geometry of* $L \oplus L^*$ *collapses to the line bundle* L.

3) *If* $c_1 > c_2$, *the geometry of* $L \oplus L^*$ *collapses to the dual bundle* L^*.

Proof. Since $\alpha' = 8$, we obtain the constant solution $b_1 \equiv c_1$, $b_2 \equiv c_2$ and $b_3 \equiv 1$ and thus

$$\omega(t) = \omega_0 = \frac{\mathrm{i}}{2}(c_1\alpha_1 \wedge \overline{\alpha}_1 + c_2\alpha_2 \wedge \overline{\alpha}_2 + \alpha_3 \wedge \overline{\alpha}_3), \tag{3.47}$$

$$H(t) = \begin{pmatrix} h_0 e^{\frac{3}{2}\left(\frac{1}{c_1} - \frac{1}{c_2}\right)t} & 0 \\ 0 & h_0^{-1} e^{-\frac{3}{2}\left(\frac{1}{c_1} - \frac{1}{c_2}\right)t} \end{pmatrix}. \tag{3.48}$$

by equations (3.40) and (3.41). Hence, the infinite time behavior of ω exists trivially. The infinite time behaviour of H exists only if $c_1 = c_2$, and in this case, also $H(t) = H(0)$ for all times $t \geq 0$. We thus have

$$H_\infty := \lim_{t\to\infty} H(t) = \begin{cases} \begin{pmatrix} \infty & 0 \\ 0 & 0 \end{pmatrix} & \text{if } c_1 < c_2, \\ H(0) & \text{if } c_1 = c_2, \\ \begin{pmatrix} 0 & 0 \\ 0 & \infty \end{pmatrix} & \text{if } c_1 > c_2. \end{cases} \tag{3.49}$$

This shows that the geometry collapses to the line bundle L if $c_1 < c_2$ or to its dual L^* in the case $c_1 > c_2$ at infinite times. Only at $c_1 = c_2$, the metric H is the initial metric on the Whitney sum $L \oplus L^*$. □

The existence of a solution to the Hull-Strominger system on the Iwasawa manifold was proven in [Láz+25].

3.3.6 Stability Analysis for the Stationary Point

Rescaling the metric H in equation (3.48) to

$$H(t) = \begin{pmatrix} h_0 & 0 \\ 0 & h_0^{-1} e^{-3\left(\frac{1}{c_1} - \frac{1}{c_2}\right)t} \end{pmatrix} \tag{3.50}$$

and assuming $c_1 < c_2$ (in the case $c_1 > c_2$ we can rescale this H by multiplying equation (3.48) with $e^{\frac{3}{2}\left(\frac{1}{c_1} - \frac{1}{c_2}\right)t}$), we find

$$H_\infty = \lim_{t\to\infty} H(t) = \begin{pmatrix} h_0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.51}$$

This collapses the discussion to considering the line bundle L over the Iwasawa manifold M. Together with the limiting metric $\omega_\infty := \lim_{t\to\infty} \omega(t) = \omega(0)$ one might ask, whether this pair $(\omega_\infty, H_\infty)$ poses a solution to the Hull-Strominger system.

After a small calculation, one sees rather quickly that the Bianchi identity (1.1) of the Hull-Strominger system is not satisfied. Indeed, the associated curvature reads

$$F_\infty := \frac{\mathrm{i}}{4}(\alpha_1 \wedge \overline{\alpha}_1 - \alpha_2 \wedge \overline{\alpha}_2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.52}$$

and thus

$$\mathrm{Tr}(F_\infty \wedge F_\infty) = \frac{1}{8} \alpha_1 \wedge \overline{\alpha}_1 \wedge \alpha_2 \wedge \overline{\alpha}_2. \tag{3.53}$$

Also,

$$\mathrm{i}\partial\overline{\partial}\omega_\infty = -\frac{1}{2}\partial\overline{\partial}(\alpha_3 \wedge \overline{\alpha}_3) = -\frac{1}{2}\alpha_1 \wedge \overline{\alpha}_1 \wedge \alpha_2 \wedge \overline{\alpha}_2. \tag{3.54}$$

Having $\alpha' = 8$, we find

$$i\partial\overline{\partial}\omega_\infty - \frac{\alpha'}{4}(\mathrm{Tr}(\underbrace{R_\infty}_{=0} \wedge R_\infty) - \mathrm{Tr}(F_\infty \wedge F_\infty)) = -\frac{1}{4}\alpha_1 \wedge \overline{\alpha}_1 \wedge \alpha_2 \wedge \overline{\alpha}_2 \neq 0 \tag{3.55}$$

and thus the Bianchi identity (1.1) is not satisfied. This renders the case $\alpha' = 8$ and $c_1 = c_2$, for which a solution to the Hull-Strominger system exists, as highly unstable.

4 Conclusion

We first developed the necessary complex geometric tools in a self-contained way to introduce the Hull-Strominger system together with its associated Anomaly flow. We studied the Anomaly flow on a torus and on an Iwasawa manifold separately. More concretely, we gave specific individual flow ansatzes, translated the Anomaly flow into individual soluble coupled initial value problems and proved the existence or non-existence of stationary points. The existence of stationary points is equivalent to the existence of solutions to the Hull-Strominger system on the particular setup under inspection. In this procedure, we gave necessary conditions for the connection forms of the Chern connection on the Whitney sum $L \oplus L^*$ to receive the curvature F that we would like to work with (see equations (3.10) and (3.31)).

It is clearly the case that the Anomaly flow we discussed on the torus does not have a stationary point and thus the Hull-Strominger system does not exhibit a solution on the torus. On the Iwasawa manifold, there were three cases we were able to differentiate which are summarized in equation (3.49). Only when $\alpha' = 8$ and only in the case $c_1 = c_2$ we can guarantee the existence of a stationary point in the Anomaly flow on the Iwasawa manifold. This stationary point is actually given by the initial metrics (ω_0, H_0). All other values of α' deliver divergent infinite time behaviours for the Hermitian fiber metric $H(t)$ on $L \oplus L^*$ (see equation (3.34)) and the fundamental 2-form $\omega(t)$ on the Iwasawa manifold (see equation (3.33)).

The stationary point on the Iwasawa manifold, for $\alpha' = 8$ and $c_1 = c_2$, is identified to be an unstable one in the following sense. Assuming $\alpha' = 8$ but $c_1 \neq c_2$ does not only imply the non-existence of solutions to the Hull-Strominger system. After rescaling $H(t)$ as in equation (3.50), its infinite time behaviour is convergent (see equation (3.51)) but the limit does not qualify as a solution to the Hull-Strominger system.

In [FY08], Fu and Yau develop a solution to the Hull-Strominger system. They consider a T^2-bundle (M, ω, Ω) over a complex surface (S, ω_S, Ω_S) with a non-vanishing holomorphic 2-form Ω_S. The surface S must be a finite quotient of a $K3$ surface, a complex torus or a Kodaira surface due to the classification of complex surfaces by Enriques and Kodaira. Fu and Yau rule out the existence of solutions to the Hull-Strominger system on T^2-bundles over Kodaira surfaces. They further argue that due to duality from M-theory, a supersymmetric solution to the Hull-Strominger system is not expected when the base manifold is a complex torus (Iwasawa manifold included). Thus, they prove the existence of a solution to the Hull-Strominger system on T^2-bundles over $K3$ surfaces.

Phong, Picard and Zhang [PPZ18a] take the Fu-Yau ansatz from [FY08; FY07] and assume certain components to be time dependent. In this way, they can write down the Anomaly flow and rediscover the Fu-Yau solution as a stationary point. More precisely, they consider a Goldstein-Prokushkin fibration over a Calabi-Yau surface and consider a stable vector bundle. After a mild cohomological assumption, they prove for given initial data that the Anomaly flow exists for all times and converges to a solution of the Hull-Strominger system.

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