Solution of the Age-Structured Model with Uniform Age Distribution. In the age-structured model, if the initial distribution of ages is uniform, given by $n_i(a) = \frac{\hat{n}_i}{t_i}$, then the stem cell population is given by

$$N_0(t, a) = \frac{\hat{n}_0}{t_0} (2a_3)^n$$  \hspace{1cm} \text{for } (n-1)t_0 < t - a \leq nt_0. \hspace{1cm} [34]$$

If the cell cycle times satisfy $t_1/t_0 = p/q$, where $p$ and $q$ are integers, the general solution for the semidifferentiated cells, at the points where $t - a = qnt_1$, is given by

$$N_1(a + qnt_1, a) = \frac{\hat{n}_1}{t_1} (2b_3)^q n + \frac{2a_2\hat{n}_0/t_0}{(2a_3)^p - (2b_3)^q} \left( (2a_3)^{p-1} + \sum_{k=1}^{q-1} (2b_3)^{q-k}(2a_3)^{\lfloor kp/q \rfloor} \right) [(2a_3)^m - (2b_3)^m], \hspace{1cm} [35]$$

where $\lfloor \cdot \rfloor$ denotes integer part.

Relating the Age-Structured and Continuous Models. We relate the age-structured and continuous models for the case in which all cells start with age zero, and the age-structured solution is given by 6-8. To find the total stem, semidifferentiated and fully differentiated cell populations at a given time in the age-structured model we integrate the age distribution function over all possible ages. For the stem cell population, integrating 6 gives

$$\hat{N}_0(t) = \hat{n}_0 \int_0^t \sum_{n=0}^{\infty} \delta(t - nt_0 - a)(2a_3)^n \ da$$

$$= \hat{n}_0 \sum_{n=0}^{\infty} (2a_3)^n \left[ \int_0^{t_0} \delta(t - nt_0 - a) \ da \right]$$

$$= \hat{n}_0 (2a_3)^{\lfloor t/t_0 \rfloor}. \hspace{1cm} [36]$$

The last equality follows since the only $\delta$-function which gives a non-zero integral is that satisfying $nt_0 < t < (n+1)t_0$, which picks out the single value $n = \lfloor t/t_0 \rfloor$ from the sum. If $t$ is much greater than $t_0$, so that $\lfloor t/t_0 \rfloor \approx t/t_0$, we have

$$\hat{N}_0(t) \approx \hat{n}_0 (2a_3)^{t/t_0}. \hspace{1cm} [37]$$

For the semidifferentiated cell population, integrating 7 gives

$$\hat{N}_1(t) = \hat{n}_1 \sum_{m=0}^{\infty} (2b_3)^m \left[ \int_0^{t_1} \delta(t - a - nt_0) \ da \right]$$

$$+ 2a_2\hat{n}_0 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (2a_3)^{n-1}(2b_3)^m \left[ \int_0^{t_1} \delta(t - a - nt_0 - mt_1) \ da \right]. \hspace{1cm} [38]$$
The integral of the first \( \delta \)-function picks out the value \( m = \lfloor t/t_1 \rfloor \), while that of the second picks out the value \( m = \lfloor t/t_1 - nt_0/t_1 \rfloor \), giving

\[
\hat{N}_1(t) = \hat{n}_1(2b_3)^{\lfloor t/t_1 \rfloor} + 2a_2\hat{n}_0 \sum_{n=1}^{\lfloor t/t_0 \rfloor} (2a_3)^{n-1}(2b_3)^{\lfloor t/t_1 - nt_0/t_1 \rfloor}. \tag{39}
\]

The sum can be evaluated exactly at the times \( t = rt_0t_1 \), where \( r, t_0, \) and \( t_1 \) are integers, giving the approximation

\[
\hat{N}_1(t) \approx \hat{A}(2a_3)^{t/t_0} + \left( \hat{n}_1 - \hat{A} \right)(2b_3)^{t/t_1}, \tag{40}
\]

where

\[
\hat{A} = \frac{2a_2\hat{n}_0}{(2a_3)^{t_1} - (2b_3)^{t_0}}, \quad \text{and} \quad f_1 = \sum_{n=1}^{t_1} (2a_3)^{t_1-n}(2b_3)^{(n-1)t_0/t_1}. \tag{41}
\]

For the fully differentiated cell population, integrating 8 gives

\[
\hat{N}_2(t) = \tilde{n}_2 \sum_{p=0}^{\infty} (1-c)^p \left[ \int_0^{t_2} \delta(t-a-p t_2) \, da \right] + 2b_2\tilde{n}_1 \sum_{m=1}^{\lfloor t/t_1 \rfloor} \sum_{p=0}^{\infty} (2b_3)^{m-1}(1-c)^p \left[ \int_0^{t_2} \delta(t-a-m t_1 - pt_2) \, da \right]
+ 2a_2\tilde{n}_0 (2b_2) \sum_{n=1}^{\lfloor t/t_0-t_1/t_0 \rfloor} \sum_{m=1}^{\lfloor t/t_0-t_1-t_0/t_1 \rfloor} (2a_3)^{n-1}(2b_3)^{m-1}(1-c)^{t/t_2-nt_0/t_2-mt_1/t_2}. \tag{42}
\]

The first \( \delta \)-function picks out the value \( p = \lfloor t/t_2 \rfloor \), the second picks out the value \( p = \lfloor t/t_2 - mt_1/t_2 \rfloor \), and the third picks out the value \( p = \lfloor t/t_2 - nt_0/t_2 - mt_1/t_2 \rfloor \), giving

\[
\hat{N}_2(t) = \tilde{n}_2 (1-c)^{\lfloor t/t_2 \rfloor} + 2b_2\tilde{n}_1 \sum_{m=1}^{\lfloor t/t_1 \rfloor} (2b_3)^{m-1}(1-c)^{\lfloor t/t_2 - mt_1/t_2 \rfloor}
+ 2a_2\tilde{n}_0 (2b_2) \sum_{n=1}^{\lfloor t/t_0-t_1/t_0 \rfloor} \sum_{m=1}^{\lfloor t/t_0-t_1-t_0/t_1 \rfloor} (2a_3)^{n-1}(2b_3)^{m-1}(1-c)^{t/t_2-nt_0/t_2-mt_1/t_2}. \tag{43}
\]

Estimating \( t/t_0 - t_1/t_0 \approx t/t_0 \) for large times in 43, and choosing \( t_2 = t_1 \), the resulting sums can again be evaluated exactly at times \( t = rt_0t_1 \), where \( r, t_0, \) and \( t_1 \) are integers, giving the approximation

\[
\hat{N}_2(t) \approx \hat{B}(2a_3)^{t/t_0} + \hat{C}(2b_3)^{t/t_1} + (\tilde{n}_2 - \hat{B} - \hat{C})(1-c)^{t/t_2}, \tag{44}
\]

where

\[
\hat{B} = \frac{2a_2\tilde{n}_0 (2b_2)}{2b_3 - (1-c)} \left( \frac{f_1}{(2a_3)^{t_1} - (2b_3)^{t_0}} - \frac{f_2}{(2a_3)^{t_1} - (1-c)^{t_0}} \right), \quad \hat{C} = \frac{2b_2(\tilde{n}_1 - \hat{A})}{2b_3 - (1-c)}, \tag{45}
\]

and

\[
f_2 = \sum_{n=1}^{t_1} (2a_3)^{t_1-n}(1-c)^{(n-1)t_0/t_1}. \tag{46}
\]