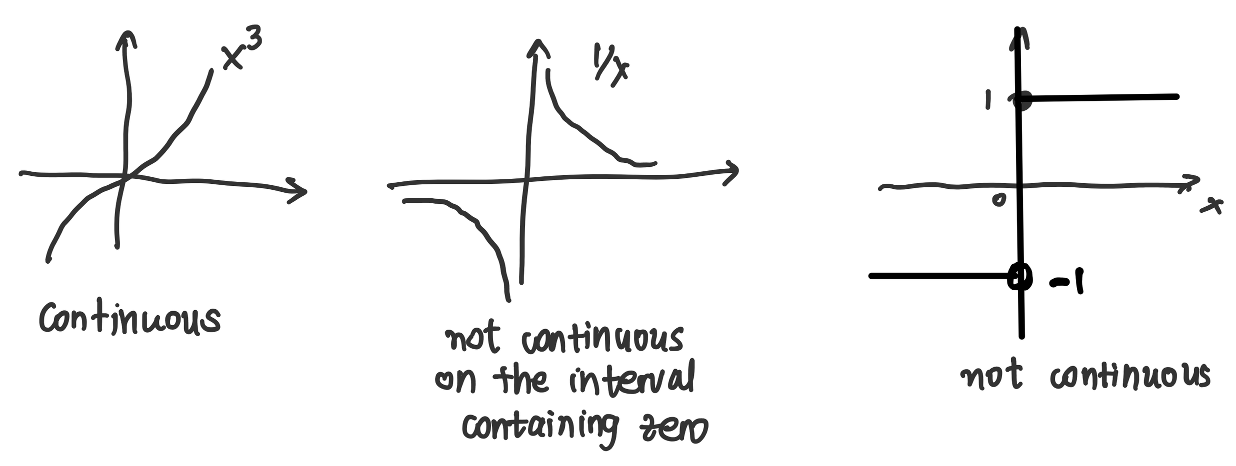
Limits

Sunday, July 12, 2020 10:56 PM

Intro to limits and continuity

A function is continuous on an interval if its graph has no jumps or holes in that interval.

$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$



Continuous — not continuous on the interval containing zero — not continuous

A function is continuous at a point if nearby values of the independent variable give nearby values of the function.

Limit

We write $\lim_{x \to c} f(x) = L$ if the values of $f(x)$ approach L as x approaches c.

Limits of a continuous function

If a function $f(x)$ is continuous at $x = c$, the limit is the value of $f(x)$ there

$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

Example Use algebra to deduce what the limit is

$$\lim_{x \to 4} \frac{x^2-16}{x-4} = \lim_{x \to 4} \frac{\cancel{(x-4)}(x+4)}{\cancel{x-4}} = \lim_{x \to 4} x+4 = 8$$

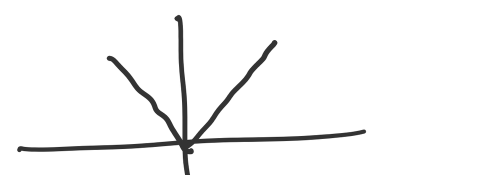
When does a limit not exist?

Eg $\lim_{x \to 2} \frac{|x-2|}{x-2} \Rightarrow \lim_{x \to 2^-} \frac{|x-2|}{x-2} = -1$

$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = 1$

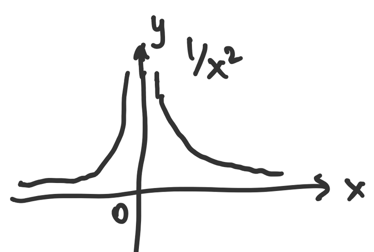
Left and right limits are different so the limit does not exist.

Reminder $y = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$



$\frac{|x-2|}{x-2} = \begin{cases} \frac{-(x-2)}{x-2} = -1, & x < 2 \\ \frac{x-2}{x-2} = 1, & x > 2 \end{cases}$

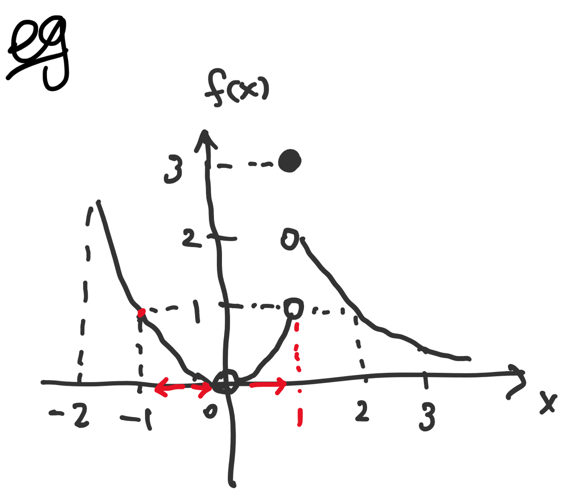
eg $\lim_{x \to 0} \frac{1}{x^2}$



DNE

because as $x \to 0$, $f(x) = \frac{1}{x^2}$ gets arbitrarily large and so it cannot approach a finite number L.

eg



a) $\lim_{x \to -1^+} f(x) = 1$

b) $\lim_{x \to 0^-} f(x) = 0$

c) $\lim_{x \to 0} f(x) = 0$

d) $\lim_{x \to 1^-} f(x) = 1$

e) $\lim_{x \to 1} f(x) = \text{DNE}$

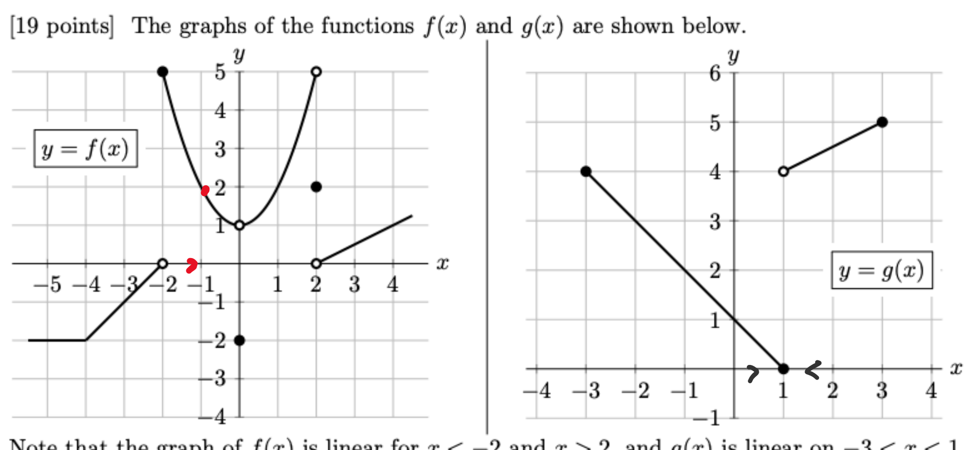
$\lim_{x \to 1^+} f(x) = 2$

Theorem 1.2: Properties of Limits

Assuming all the limits on the right-hand side exist:

1. If b is a constant, then $\lim_{x \to c} (bf(x)) = b\left(\lim_{x \to c} f(x)\right)$.
2. $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.
3. $\lim_{x \to c} (f(x)g(x)) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$.
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided $\lim_{x \to c} g(x) \neq 0$.
5. For any constant k, $\lim_{x \to c} k = k$.
6. $\lim_{x \to c} x = c$.

1. [19 points] The graphs of the functions $f(x)$ and $g(x)$ are shown below.



Note that the graph of $f(x)$ is linear for $x < -2$ and $x > 2$, and $g(x)$ is linear on $-3 < x < 1$ and $1 < x < 3$.

For each of the following parts, find the given limit. If any of the quantities do not exist (including the case of limits that diverge to ∞ or $-\infty$), write DNE. If the limit cannot be found based on the information given, write NOT ENOUGH INFO. *You do not need to show any work.*

a. [2 points] Find $\lim_{x \to -1} f(x)$.

$\lim_{x \to -1} f(x) =$ 2

b. [2 points] Find $\lim_{t \to 2^-} 2(f(t) - 3)$.

$\lim_{t \to 2^-} 2(f(t) - 3) =$ 4

c. [2 points] Find $\lim_{x \to 1} f(x)g(x)$.

$\lim_{x \to 1} f(x)g(x) =$ DNE

d. [2 points] Find $\lim_{x \to \infty} f(e^{-x})$.

$g^{-1}(2) = -1$

$\lim_{x \to \infty} f(e^{-x}) =$ 1

e. [2 points] Find $\lim_{x \to 2^+} g^{-1}(x)$.

$\lim_{x \to 2^+} g^{-1}(x) =$ -1

f. [2 points] Find $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$.

$f'(3)$

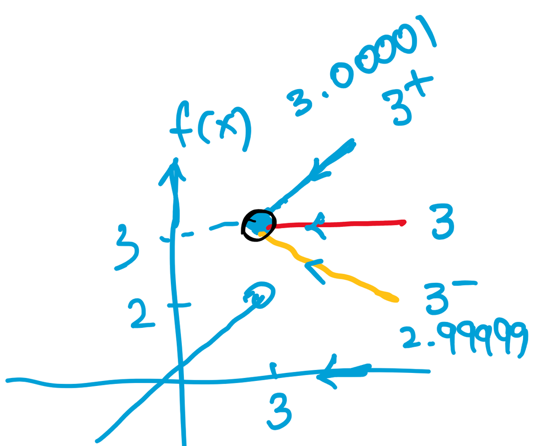
$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} =$ 1/2

c) $\lim_{x \to 1} f(x)g(x)$

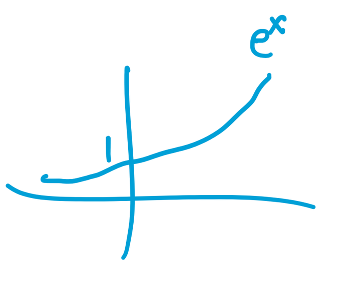
$= \left(\lim_{x \to 1} f(x)\right)\left(\lim_{x \to 1} g(x)\right)$

$=$ DNE

$f(x)$ 3.00001 3+



e^x



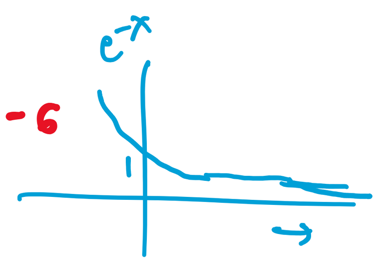
$\lim_{t \to 2^-} 2(f(t) - 3)$

$= \lim_{t \to 2^-} 2f(t) - 6$

$= 2(5) - 6$

$= 4$

e^{-x}



$\lim_{x \to 3^+} f(x) =$

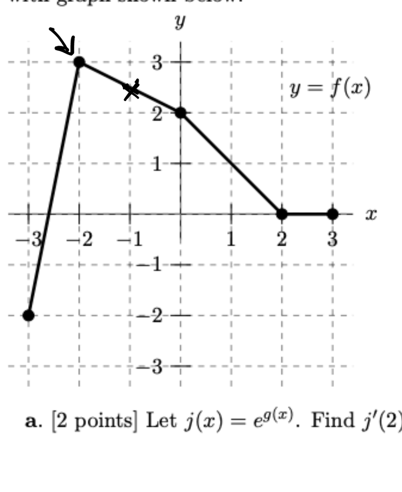
$\lim_{x \to \infty} e^{-x} = 0^+$

$\lim_{x \to \infty} f(e^{-x}) = \lim_{x \to 0^+} f(x)$

$= 1$

2. [12 points]

Let f be the piecewise linear function with graph shown below.



The table below gives several values of a differentiable function g and its derivative g'.
Assume that both $g(x)$ and $g'(x)$ are invertible.

x	-2	-1	0	2	5
$g(x)$	21	11	5	-1	-3
$g'(x)$	-12	-8	-4	-2	-0.4

You are not required to show your work on this problem. However, limited partial credit may be awarded based on work shown.

For each of parts **a.-f.** below, find the value of the given quantity. If there is not enough information provided to find the value, write "NOT ENOUGH INFO". If the value does not exist, write "DOES NOT EXIST".

a. [2 points] Let $j(x) = e^{g(x)}$. Find $j'(2)$.

$j'(x) = e^{g(x)} g'(x)$

$j'(2) = e^{g(2)} g'(2)$

$= e^{-1}(-2)$

$= -2/e$

Answer: $-2/3$

b. [2 points] Let $k(x) = f(x)f(x+2)$. Find $k'(-1)$.

$k'(x) = f'(x)f(x+2) + f(x)f'(x+2)$, $k'(-1) = f'(-1)f(1) + f(-1)f'(1)$

$= -\frac{1}{2}(1) + \frac{5}{2}(-1)$

$= -3$

$f(-1+2) = f(1)$

Answer:

c. [2 points] Let $h(x) = 3f(x) + g(x)$. Find $h'(-2)$.

$h'(x) = 3f'(x) + g'(x)$

$h'(-2) = 3f'(-2) + g'(-2)$

Answer: DNE

d. [2 points] Find $(g^{-1})'(2)$.

Answer: not enough info

e. [2 points] Let $m(x) = g(f(g(x)))$. Find $m'(2)$.

$g'(2.5)$

$m'(x) = g'(f(g(x))) \cdot f'(g(x)) \cdot g'(x)$

$m'(2) = g'(f(\underset{-1}{g(2)})) f'(g(2)) \cdot g'(2)$

Answer: not enough info

f. [2 points] Let $\ell(x) = \frac{f(x)}{g(2x)}$. Find $\ell'(-1)$.

quotient rule.

Answer: ≈ 0.1122

Integration

Tuesday, July 14, 2020 10:42 AM

During last class we saw limits and differentiation. This time we will cover Integration methods. In particular we will use integration by parts, by substitution. Next time I'll quickly cover partial fractions at the beginning of class.

Integration by substitution

How does it work? It's based on reverse chain rule.

Assume you have an integral of the form $\int f(g(x))g'(x)\,dx$

If F is the antiderivative of f then $F' = f$ then by the chain rule you have

$$\frac{d}{dx}\left(F(g(x))\right) = f(g(x))g'(x)$$

Thus $\int f(g(x))g'(x)\,dx = F(g(x)) + C$

So for the substitution use $\boxed{u = g(x)}$ $\Rightarrow$ $\frac{du}{dx} = g'(x)$

$$\int f(u)\frac{du}{dx}\,dx = F(u) + C$$

Since $F' = f$ $\Rightarrow$ $\underline{\int f(u)\,du = F(u) + C}$

SUBSTITUTION

Examples

① $\int t\cos(t^2)\,dt$ $\quad u = t^2 \Rightarrow du = 2t\,dt \Rightarrow t\,dt = \frac{1}{2}du$

$= \int \frac{1}{2}\cos(u)\,du = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(t^2) + C$

② $\int x(x^2+3)^2\,dx$ $\quad u = x^2+3 \Rightarrow du = 2x\,dx$

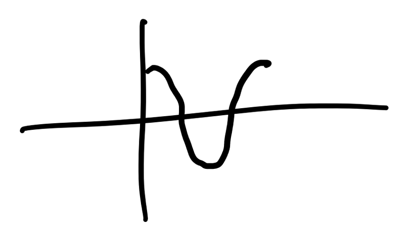
$\int \frac{1}{2}u^2\,du = \frac{1}{2}\frac{u^3}{3} + C = \frac{1}{6}u^3 + C = \frac{1}{6}(x^2+3)^3 + C$

Check $\frac{d}{dx}\left(\frac{1}{6}(x^2+3)^3 + C\right) = \frac{1}{\cancel{2}}(x^2+3)^2(\cancel{2}x) = x(x^2+3)^2$

③ $\int_0^{\pi/2} e^{-\cos\theta}\sin\theta\,d\theta$ $\quad u = -\cos\theta$, $du = \sin(\theta)\,d\theta$ $\quad \theta = 0 \Rightarrow u = -\cos 0 = -1$; $\theta = \frac{\pi}{2} \Rightarrow u = -\cos\frac{\pi}{2} = 0$

$= \int_{-1}^{0} e^u\,du$

$= \left[e^u\right]_{-1}^{0} = e^0 - e^{-1} = 1 - \frac{1}{e}$



Integration by parts:

This method is based on the product rule: $\frac{d}{dx}(uv) = uv' + u'v$

Rearrange this to write $uv' = \frac{d}{dx}(uv) - u'v$

Integrate both sides $\boxed{\int uv'\,dx = uv - \int u'v\,dx}$

How do you choose u and v':

- whatever you let v' be, you need to be able to get v
- It's good if u' is simpler than u
- It's good if v is simpler than v'

BY PARTS

Example $\int \ln(x)\,dx$ $\quad u = \ln(x)$, $\frac{dv}{dx} = 1$, $\frac{du}{dx} = \frac{1}{x}$, $v = x$

When you have $\ln(x)$ or something similar choose $u = \ln(x)$

$= x\ln(x) - \int \frac{1}{x}(x)\,dx$

$= x\ln(x) - \int 1\,dx$

$= x\ln(x) - x + C$

e.g. $\int x^2 e^x\,dx$ $\quad u = x^2$, $\frac{dv}{dx} = e^x$, $\frac{du}{dx} = 2x$, $v = e^x$

$= x^2e^x - \int 2xe^x\,dx$ $\quad u = 2x$, $\frac{dv}{dx} = e^x$, $\frac{du}{dx} = 2$, $v = e^x$

$= x^2e^x - \left[2xe^x - \int 2e^x\,dx\right]$

$= x^2e^x - 2xe^x + 2e^x + C$

Exercise $\int \frac{1}{2+2\sqrt{x}}\,dx$ $\quad 2(1+\sqrt{x})$ $\quad \boxed{u = \sqrt{x}}$ $\Rightarrow$ $du = \frac{1}{2}x^{-1/2}\,dx$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$

$2\sqrt{x}\,du = dx$

$2u\,du = dx$

$\int \frac{\cancel{2}u\,du}{\cancel{2}(1+u)} = \int \frac{u}{1+u}\,du$

$= \int \frac{1+u-1}{1+u}\,du$

$= \int \frac{(1+u)-1}{1+u}\,du$

$= \int \left(\frac{1+u}{1+u} - \frac{1}{1+u}\right)du$

$= \int \left(1 - \frac{1}{1+u}\right)du$

$= u - \ln(1+u) + C$

$= \sqrt{x} - \ln(1+\sqrt{x}) + C$

HW1 (hints)

x	-1	0	1	2	3	4
$z(x)$	2				π	3
$z'(x)$			5	π	12	

④ $s(x) = z(x)z(x+2)$ where $\boxed{s'(1) = 17\pi}$

$s'(x) = z(x)z'(x+2) + z'(x)z(x+2)$

$\underbrace{s'(1)}_{17\pi} = \underbrace{z(1)}\,\underbrace{z'(3)}_{12} + \underbrace{z'(1)}_{5}\,\underbrace{z(3)}_{\pi}$

$17\pi = 12z(1) + 5\pi$

$12\pi = 12z(1)$

$\boxed{z(1) = \pi}$

$$u(x) = \begin{cases} z'(-1)\frac{x^2+1}{2^x} & \text{for } x<0 \\ 3 & \text{for } x=0 \\ e^{z(x)} & \text{for } x>0 \end{cases}$$

$\lim_{x\to 0^-} u(x) = \underbrace{z'(-1) = 3}$ by continuity

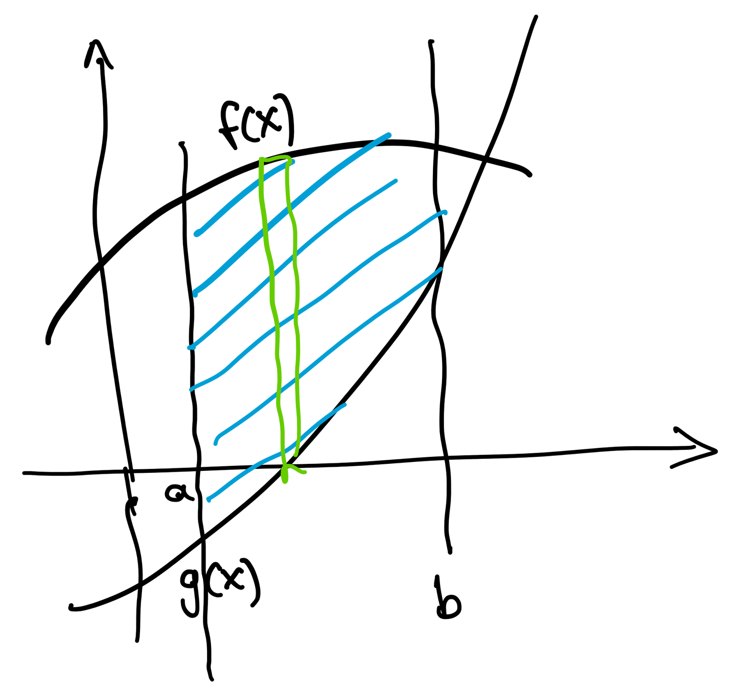
$\lim_{x\to 0^+} u(x) = \underbrace{e^{z(0)} = 3}$

$z(0) = \ln(3)$

Areas and Volumes

Friday, July 17, 2020 9:29 AM

Areas between curves

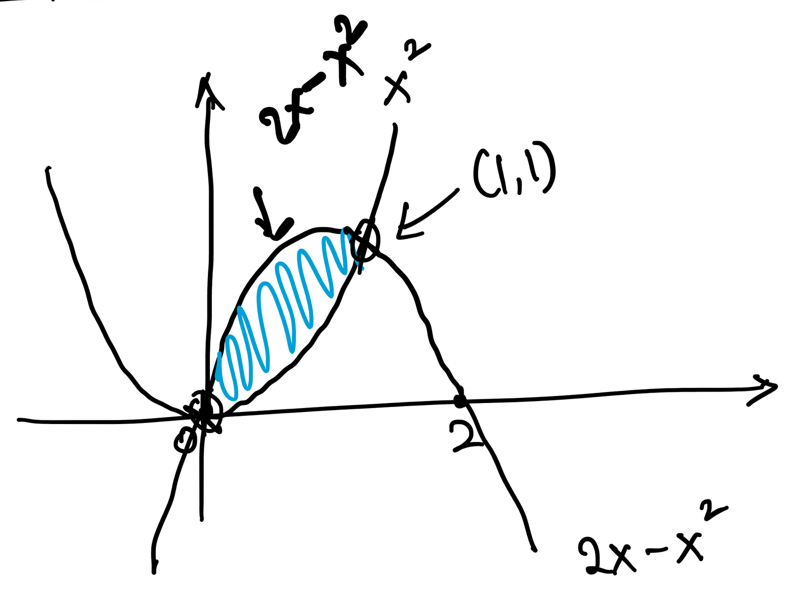


The area of a region bounded by the curves $y=f(x)$ and $y=g(x)$ and the lines $x=a$ and $x=b$, where f and g are continuous, with $f(x) \geq g(x)$ for all x in $[a,b]$ is

$$\text{area} = \int_a^b [f(x)-g(x)]\,dx$$

$$\left(= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right)$$

Example Find the area enclosed by the parabolas $y=x^2$ and $y=2x-x^2$
$= x(2-x)$



intersection

$x^2 = 2x - x^2$

$\cancel{2}x^2 = \cancel{2}x$

$x^2 - x = 0$

$x(x-1) = 0$

$x=0 \quad x=1$

$x=0 \Rightarrow y=0 \qquad (0,0)$

$x=1 \Rightarrow y=1 \qquad (1,1)$

"top - bottom"

$$\text{area} = \int_0^1 (2x-x^2) - x^2\,dx$$

$$= \int_0^1 (2x - 2x^2)\,dx$$

$$= \left[x^2 - \frac{2}{3}x^3\right]_0^1$$

$$= 1 - \frac{2}{3}$$

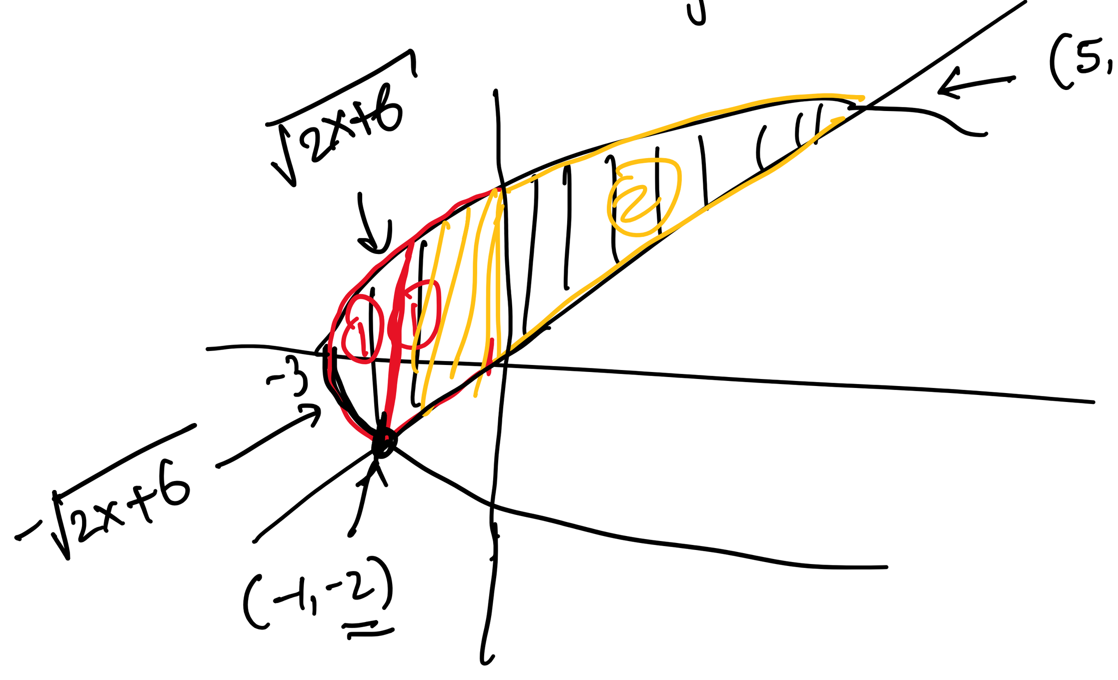
$$= \frac{1}{3} \quad //$$

Examples

$y = x-1$

$y^2 = 2x+6$

$x-1=y \Rightarrow x=y+1$



$y^2 = 2x+6$

$\Rightarrow x = \frac{y^2-6}{2}$

" right - left "

$$\text{area} = \int_{-2}^{4} (y+1) - \left(\frac{y^2-6}{2}\right)\,dy$$

$$= \int_{-2}^{4} y+1 - \frac{y^2}{2} + 3\,dy$$

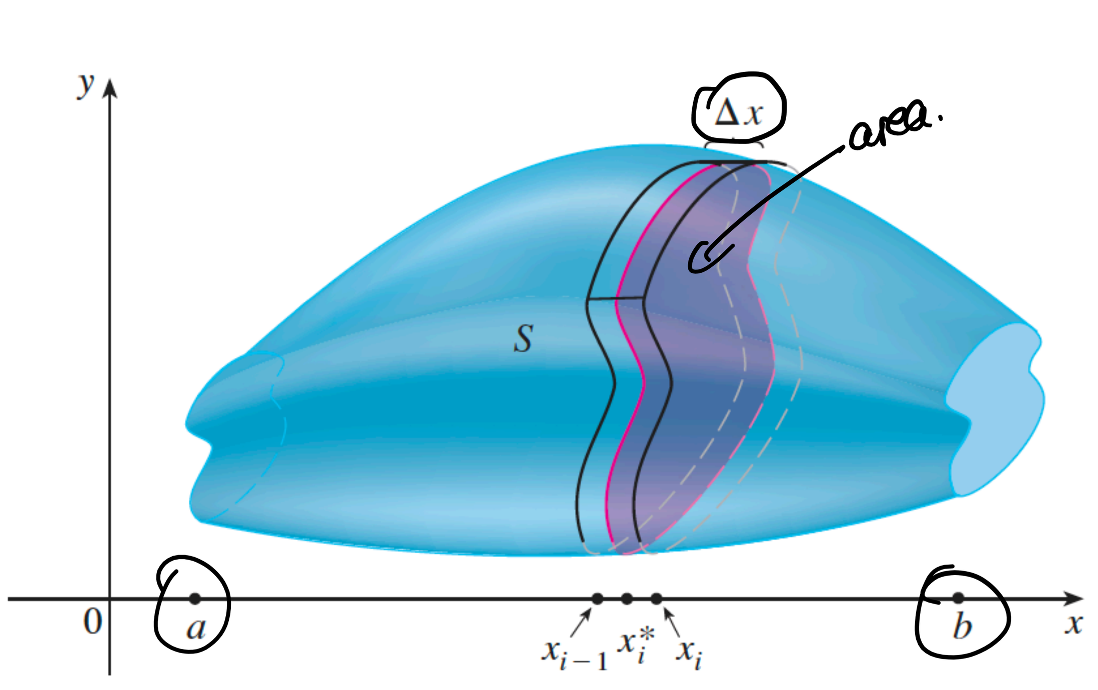
$$= \int_{-2}^{4} y + 4 - \frac{y^2}{2}\,dy$$

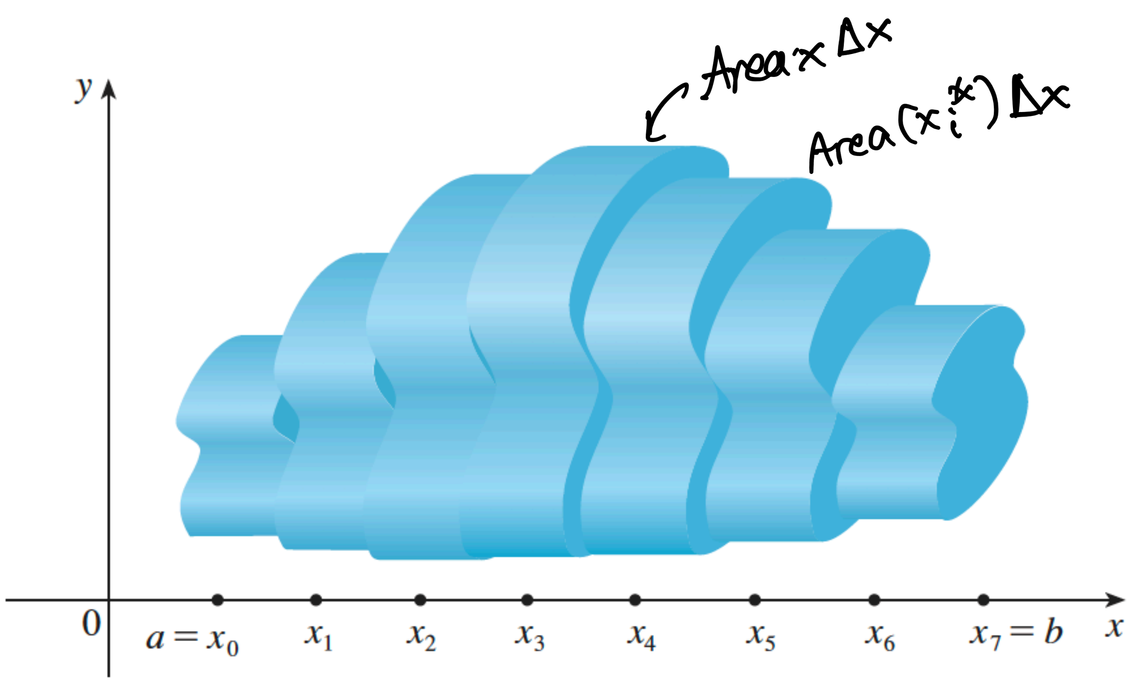
$$= \left[\frac{y^2}{2} + 4y - \frac{y^3}{6}\right]_{-2}^{4}$$

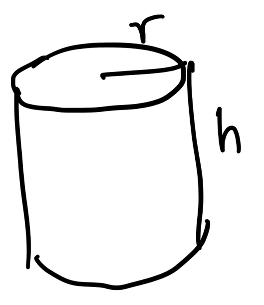
$$= \frac{4^2}{2} + 4(4) - \frac{4^3}{6} - \left(\frac{(-2)^2}{2} + 4(-2) - \frac{(-2)^3}{6}\right)$$

$$= 18 \quad //$$

Volumes







$\pi r^2 h = \text{Area} \times h$

πr^2 = area

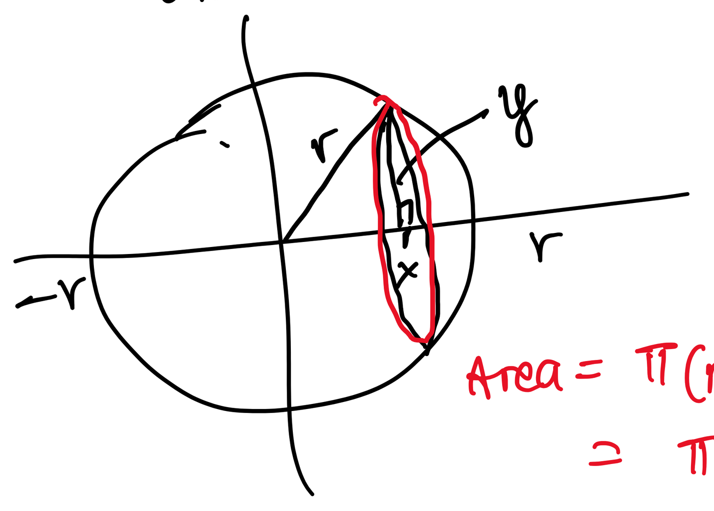
Def^n of volume

Let S be the solid that lies between $x=a$ and $x=b$. If the cross-sectional area of S perpendicular to the x-axis is $A(x)$ and (A is a continuous function) volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

Example

Show that the volume of a sphere of radius r is given by $V = \frac{4\pi r^3}{3}$.



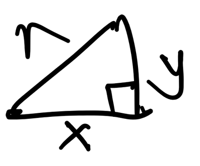
Area $= \pi(\text{radius})^2$

$= \pi y^2$

$= \pi\left(\sqrt{r^2-x^2}\right)^2$

$= \pi(r^2-x^2)$

Pythagorean theorem: $y = \sqrt{r^2-x^2}$



$$\text{Volume} = \int_{-r}^{r} \pi(r^2-x^2)\,dx$$

$$= \left[\pi r^2 x - \frac{\pi x^3}{3}\right]_{-r}^{r}$$

$$= \pi r^3 - \frac{\pi r^3}{3} - \left(\pi r^2(-r) - \frac{\pi(-r)^3}{3}\right)$$

$$= \frac{2\pi r^3}{3} - \left(-\frac{2\pi r^3}{3}\right)$$

$$= \frac{4\pi r^3}{3}$$

Partial fractions

Wednesday, July 15, 2020 6:41 PM

The integrand of some rational functions can be obtained by splitting the integrand into partial fractions.

e.g Find $\boxed{\int \frac{1}{(x-3)(x-7)}dx}$

Write $\frac{1}{(x-3)(x-7)} = \frac{A}{x-3} + \frac{B}{x-7}$ where A and B are constants to be found.

$= \frac{A(x-7)+B(x-3)}{(x-3)(x-7)}$

Get identity $1 = A(x-7) + B(x-3)$ ☆

Two ways:

① Eliminate B and solve for A:

let $x=3 \Rightarrow 1 = A(3-7) \Rightarrow 1 = -4A \Rightarrow A = -\frac{1}{4}$

② Eliminate A and solve for B:

let $x=7 \Rightarrow 1 = B(7-3) \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$

OR Equate coefficients (☆) $1 = (A+B)x - 7A - 3B$

const: $1 = -7A - 3B$

x: $0 = A + B \Rightarrow A = -B$

$1 = -7(-B) - 3B = 4B \Rightarrow B = 1/4$

$A = -1/4$

$$\int \frac{1}{(x-3)(x-7)}dx = \int \left(\frac{-1/4}{x-3} + \frac{1/4}{x-7} \right) dx$$

$$= -\frac{1}{4}\ln|x-3| + \frac{1}{4}\ln|x-7| + C$$

$$= \frac{1}{4}\ln\left|\frac{x-7}{x-3}\right| + C$$

$\boxed{\ln|A| - \ln|B| = \ln\left|\frac{A}{B}\right|}$

$\ln|A-B| \neq \frac{\ln|A|}{\ln|B|}$

e.g $\int \frac{x}{(x-1)^2(x-2)}dx = \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} \right) dx$

Multiply through by $(x-1)^2(x-2)$:

$$x = \frac{A(x-1)(x-2) + B(x-2) + C(x-1)^2}{(x-1)^2(x-2)}$$

$(x-1)^2(x-2)$

let $x=2 \Rightarrow 2 = C(2-1)^2 \Rightarrow 2 = C$

let $x=1 \Rightarrow 1 = B(1-2) \Rightarrow B = -1$

let $x=0 \Rightarrow 0 = A(-1)(-2) + \underset{-1}{B}(-2) + \underset{2}{C}(1)$ $\quad (0-1)^2 = (-1)^2 = 1$

$\Rightarrow 0 = 2A + 2 + 2$

$2A = -4$

$A = -2$

$$\int \left(\frac{-2}{x-1} - \frac{1}{(x-1)^2} + \frac{2}{x-2} \right) dx = -2\ln|x-1| + (x-1)^{-1} + 2\ln|x-2| + C$$

$$= 2\ln\left|\frac{x-2}{x-1}\right| + \frac{1}{x-1} + C$$

$\int -(x-1)^{-2}dx$

Examples from Jamboard

(40) $\int \frac{1}{(x+7)(x-2)}dx = -\frac{1}{9}\ln|x+7| + \frac{1}{9}\ln|x-2| + C$

(44) $\int \frac{1}{3P - 3P^2}dP = \int \frac{1}{3P(1-P)}dP = \int \frac{A}{3P} + \frac{B}{1-P}\ dP$ where A & B are constants to be found

$1 = A(1-P) + B(3P)$

let $P=1 \Rightarrow 1 = 3B \Rightarrow B = 1/3$

let $P=0 \Rightarrow 1 = A$

$\int \left(\frac{1}{3P} + \frac{1}{3}\frac{1}{1-P} \right) dP = \frac{1}{3}\ln|P| - \frac{1}{3}\ln|1-P| + C$

check answer

$\frac{d}{dP}\left(\frac{1}{3}\ln|P| - \frac{1}{3}\ln|1-P| + C \right)$

$= \frac{1}{3}\frac{1}{P} - \frac{1}{3}\frac{1}{1-P}\underset{\text{chain rule}}{(-1)} = \frac{1}{3P} + \frac{1}{3}\frac{1}{1-P}$ ✓

(45) $\int \frac{3x+1}{x^2-3x+2}dx = \int \frac{3x+1}{(x-2)(x-1)}dx$

$A = 7$

$B = -4$

$\frac{A}{x-2} + \frac{B}{x-1} = \frac{3x+1}{(x-2)(x-1)}$

Final answer

$7\ln|x-2| - 4\ln|x-1| + C$

Parametric equations

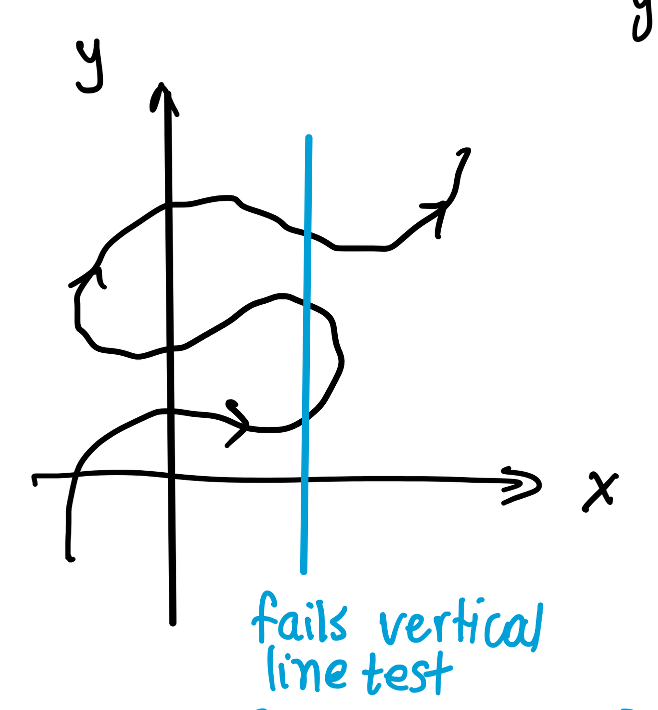
Saturday, July 18, 2020   11:00 PM

So far you have described curves by giving y as a function of x $(y=f(x))$ or by implicitly defining y as a function of x $(f(x,y)=0)$.

Some curves are best handled when x and y are both given in terms of a third variable t (we call this the parameter)

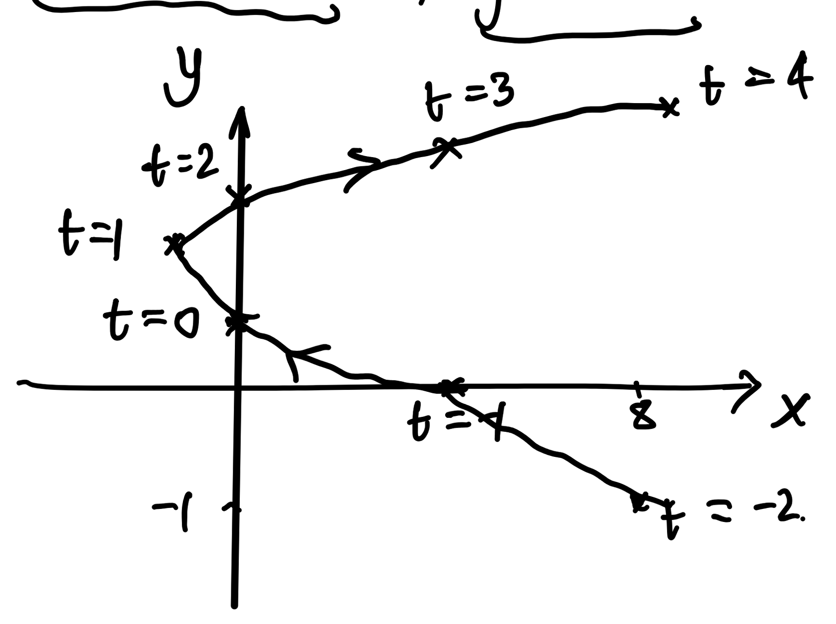
Parametric equations: $\boxed{x=f(t) \quad , \; y=g(t)}$

For each value of t you get an x and y and you can plot this



Example Sketch and identify the curve given by $\underbrace{x=t^2-2t}$, $\underbrace{y=t+1}$

t	x	y
−2	8	−1
−1	3	0
0	0	1
1	−1	2
2	0	3
3	3	4
4	8	5



$\underline{x=f(y)}$

$x=t^2-2t \qquad y=t+1 \Rightarrow \boxed{t=y-1}$

$x=(y-1)^2-2(y-1)$

$\;\;=y^2-2y+1-2y+2$

$\boxed{x=y^2-4y+3}$

Speed and velocity

Recall $x=f(t)$, $y=g(t)$

The instantaneous speed of a moving particle is defined to be

$$\left\{\text{speed}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\right]$$

where $\frac{dx}{dt}$ represents the instantaneous velocity in the x-direction

$\frac{dy}{dt}$ 〃 y-direction

velocity vector

$v=\frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}$

unit vectors

Example

As a particle moves in the xy-plane with $x=2t^3-9t^2+12t$ and $y=3t^4-16t^3+18t^2$, where t is time, find:

(a) At what times is the particle stopped

(b) At what times is the particle moving parallel to the x- or y-axis

(c) The speed of the particle at time t.

Solution: (a) Both $\frac{dx}{dt}=0$ and $\frac{dy}{dt}=0$ for particle to be stopped.

$x=2t^3-9t^2+12t \Rightarrow \frac{dx}{dt}=6t^2-18t+12=6(t-2)(t-1)$

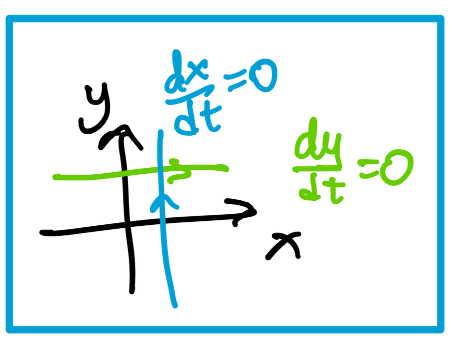
$y=3t^4-16t^3+18t^2 \Rightarrow \frac{dy}{dt}=12t^3-48t^2+36t=12t(t^2-4t+3)$

$=12t(t-3)(t-1)$

$\frac{dx}{dt}=0 \Rightarrow t=2, t=1$

$\frac{dy}{dt}=0 \Rightarrow t=0,3,1$ } stopped at $t=1$

(b)



$\frac{dx}{dt}=0$ when $t=2,1$

$\frac{dy}{dt}=0$ when $t=0,\cancel{1},3$

Particle is parallel to the x-axis if $\frac{dy}{dt}=0$ and $\frac{dx}{dt}\neq 0$ so when $\boxed{t=0,3}$

Particle is parallel to the y-axis if $\frac{dx}{dt}=0$ and $\frac{dy}{dt}\neq 0$ so when $\boxed{t=2}$

(c) speed $=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{(6t^2-18t+12)^2+(12t^3-48t^2+36t)^2}$

Slope & concavity of parametric curves

Slope from chain rule is given by $\boxed{\frac{dy}{dx}=\frac{dy/dt}{dx/dt}}$

$y=h(x)$ and $x=f(t)$, $y=g(t)$

$\underline{y(t)=y(x(t))} \Rightarrow \frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$

Rearranging $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Concavity (you need the 2nd derivative, $\frac{d^2y}{dx^2}$).

If you are given $\underline{w=\frac{dy}{dx}}$, then $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dw}{dx}=\frac{dw/dt}{dx/dt}$

$w=\frac{dy}{dx}$

$w=\frac{dy}{dx}$

Concavity: $\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

Polar coordinates

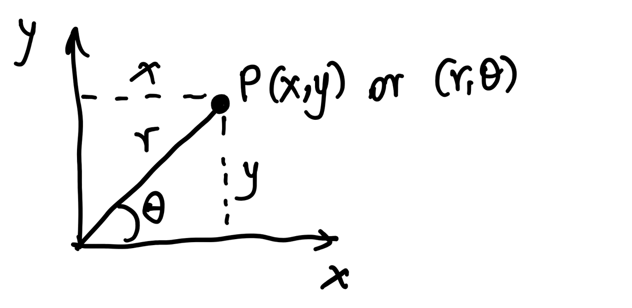
Tuesday, July 21, 2020 9:11 PM

Polar coordinates are an alternative way of describing a point P in a two-dimensional space.

You need two measurements to describe the position of this point.

a) the distance from the pole (usually the origin O), r

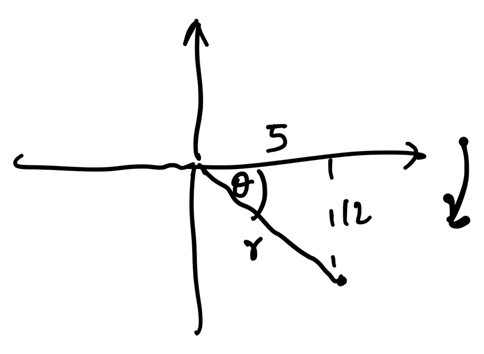
b) the angle measured anticlockwise from the initial line (usually the x-axis), θ



To convert between Cartesian coordinates and polar coordinates.

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}} \longleftrightarrow \boxed{\begin{aligned} r^2 &= x^2+y^2 \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}}$$

e.g Find the polar coordinates of the point $(x,y) = (5,-12)$



$r = \sqrt{12^2+5^2} = 13$

$\theta = -\arctan\left(\frac{12}{5}\right) = -1.176$

$(r,\theta) = (13, -1.176)$.

e.g. Find the Cartesian coords of $(r,\theta) = \left(10, \frac{4\pi}{3}\right)$

$x = r\cos\theta = 10\cos\left(\frac{4\pi}{3}\right) = -5$

$y = r\sin\theta = 10\sin\left(\frac{4\pi}{3}\right) = -5\sqrt{3}$

$(x,y) = (-5, -5\sqrt{3})$

Polar equations of curves are usually given by $r = f(\theta)$. For example $r = 1+2\cos\theta$, $r = 3$, $r = 2\sin\theta$, etc.

e.g Find the Cartesian equation of $r = 2+\cos 2\theta$ Use identity $\cos 2\theta = 2\cos^2\theta - 1$

$r = 2+\cos 2\theta$

$r = 2 + 2\cos^2\theta - 1$

$r = 1 + \underbrace{2\cos^2\theta}$

Use $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2+y^2$

Multiply both sides by r^2

$r^3 = r^2 + \underbrace{2r^2\cos^2\theta}_{x^2}$ use $x = r\cos\theta$

$(x^2+y^2)^{3/2}$ $\quad x^2+y^2$

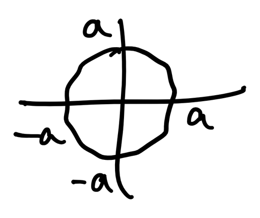
$(x^2+y^2)^{3/2} = x^2+y^2+2x^2$

$$\boxed{(x^2+y^2)^{3/2} = 3x^2+y^2}$$

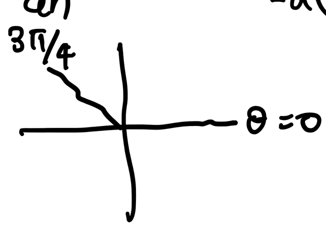
Sketching polar curves

Standard curves:

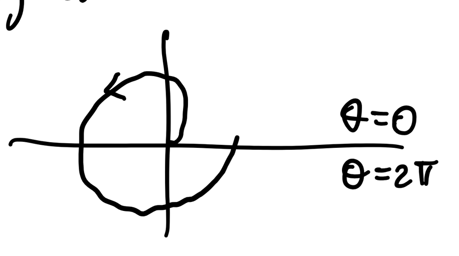
- $r = a$ is a circle of radius a centered at the origin



- $\theta = \alpha$ is a half-line through O and that makes an angle α w/ the x-axis. e.g. $\theta = \frac{3\pi}{4}$



- $r = a\theta$. This is a spiral starting at O

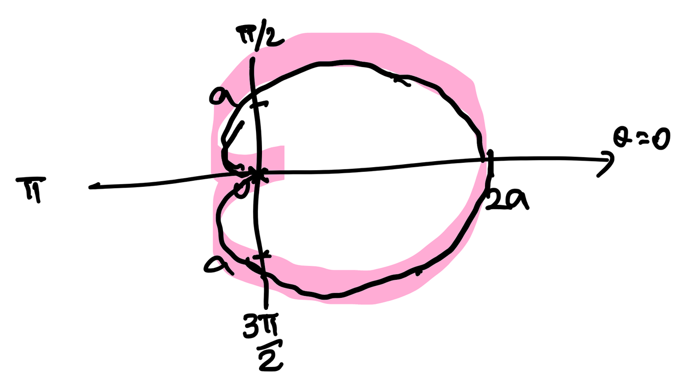


Note

You can sketch curves by drawing up a table of values of r for particular values of θ. It's common to use values of θ that result in positive r.

e.g Sketch $r = (1+\cos\theta)a$

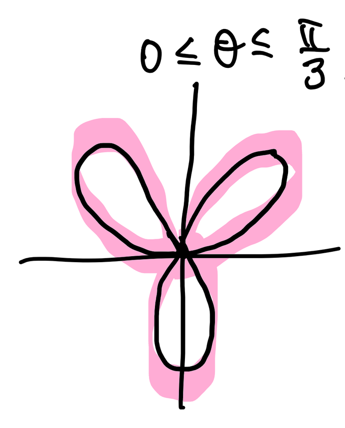
θ	0	$\pi/2$	π	$3\pi/2$	2π
r	$2a$	a	0	a	$2a$



Cardioid.

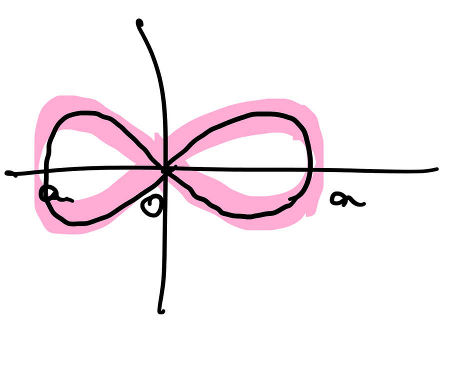
Jamboards

b) $r = a\sin 3\theta$, $0 \le \theta \le \frac{\pi}{3}$, $\frac{2\pi}{3} \le \theta \le \pi$, $\frac{4\pi}{3} \le \theta \le \frac{5\pi}{3}$ give positive r



symmetric

c) $r^2 = a^2\cos 2\theta$ $\quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$, $\frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}$.

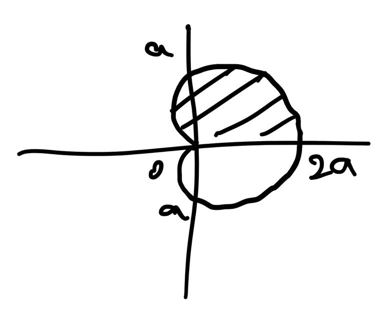


Area enclosed by polar curve

The area of a sector bounded by a polar curve and the half-lines $\theta = \alpha$ and $\theta = \beta$, where θ is in radians, is given by

$$\left[\text{Area} = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta = \frac{1}{2}\int_\alpha^\beta (f(\theta))^2\,d\theta\right]$$ (recall curve is given by $r = f(\theta)$).

e.g Find the area enclosed by $r = a(1+\cos\theta)$



$$\text{Area} = 2\left(\frac{1}{2}\int_0^\pi (a(1+\cos\theta))^2\,d\theta\right)$$

Tangents of polar curves.

If you are given a curve $r = f(\theta)$ then you can use

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

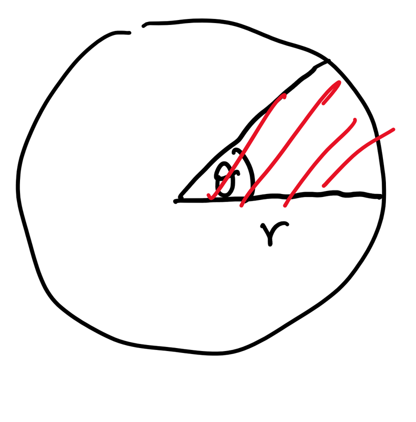
Parametric eqns: $\boxed{\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}}$

When $\frac{dy}{d\theta} = 0$, the tangent to the curve is horizontal

When $\frac{dx}{d\theta} = 0$, the tangent to the curve is vertical.

Deriving the area in polar coordinates

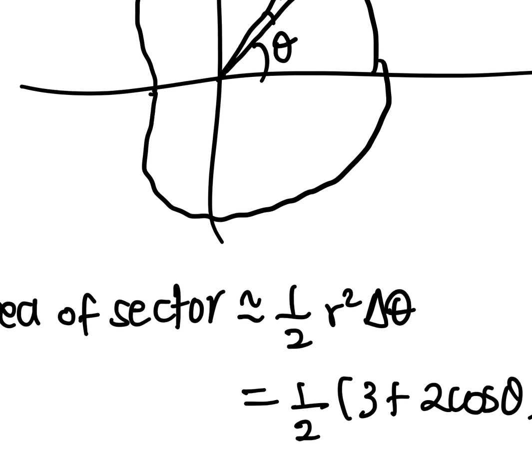
$$\text{Area} = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$



area of sector $= \frac{\theta}{\cancel{2\pi}}\,\cancel{\pi}r^2 = \frac{1}{2}r^2\theta$

circle.

Example



Area of sector $\approx \frac{1}{2}r^2\Delta\theta$

$= \frac{1}{2}(3+2\cos\theta)^2\Delta\theta$

Area of the whole region is

$\sum \frac{1}{2}(3+2\cos\theta)^2\,\Delta\theta$

As $n \to \infty$ and $\Delta\theta \to 0$

$$\text{Area} = \int_0^{2\pi} \frac{1}{2}(3+2\cos\theta)^2\,d\theta$$

Example

$$\frac{1}{2}\int_0^{\pi/3} r_1^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} r^2\,d\theta$$

More on polar coordinates

Thursday, July 23, 2020 3:18 PM

Arc length in polar coordinates

We can calculate the arclength of the curve $r = f(\theta)$ by expressing x and y in terms of θ as a parameter

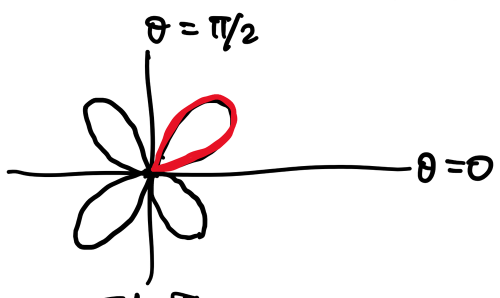
$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

and using the formula for arclength in parametric equations

$$\boxed{\text{arclength} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta}$$

Example Find the arclength of one petal of the "rose" curve $\boxed{r = 3\sin 2\theta} \leftarrow f(\theta)$ $\quad f'(\theta) = 6\cos 2\theta$

for $0 \le \theta \le \frac{\pi}{2}$



$$x = r\cos\theta = 3\sin 2\theta\cos\theta$$
$$y = r\sin\theta = 3\sin 2\theta\sin\theta$$

$$\text{arclength} = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{\pi/2} \sqrt{(6\cos 2\theta\cos\theta - 3\sin 2\theta\sin\theta)^2 + (6\cos 2\theta\sin\theta + 3\sin 2\theta\cos\theta)^2}\, d\theta$$

$$\frac{dx}{d\theta} = 6\cos 2\theta\cos\theta - 3\sin 2\theta\sin\theta$$

$$\frac{dy}{d\theta} = 6\cos 2\theta\sin\theta + 3\sin 2\theta\cos\theta$$

OR $\displaystyle\int_0^{\pi/2} \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta$

$\displaystyle = \int_0^{\pi/2} \sqrt{(6\cos 2\theta)^2 + (3\sin 2\theta)^2}\, d\theta$

The calculations can be simplified if we use instead

$$\text{arclength} = \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta$$

Proof

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \quad \text{using product rule}$$
$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\text{arclength} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2$$

$$= (f'(\theta))^2\cos^2\theta - \cancel{2f'(\theta)f(\theta)\cos\theta\sin\theta} + (f(\theta))^2\sin^2\theta$$
$$+ (f'(\theta))^2\sin^2\theta + \cancel{2f'(\theta)f(\theta)\sin\theta\cos\theta} + (f(\theta))^2\cos^2\theta$$

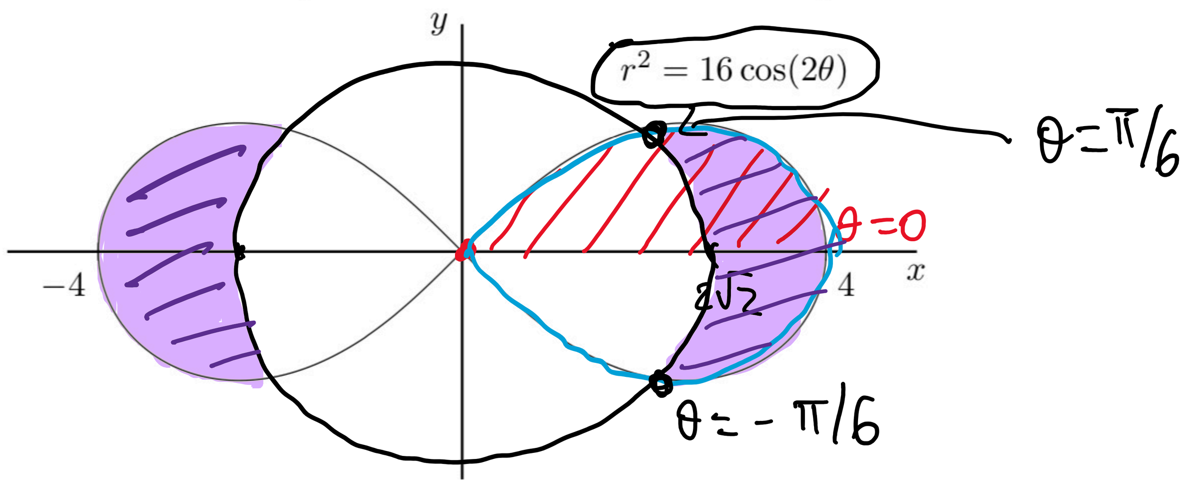
using $\cos^2\theta + \sin^2\theta = 1$

$$= (f'(\theta))^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + (f(\theta))^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}$$

$$= (f'(\theta))^2 + (f(\theta))^2 \quad ✓$$

Math 116 / Exam 2 (March 20, 2017) DO NOT WRITE YOUR NAME ON THIS EXAM *page 3*

2. [12 points] Chancelor was doodling in his coloring book one Sunday afternoon when he drew an infinity symbol, or lemniscate. The picture he drew is the polar curve $r^2 = 16\cos(2\theta)$, which is shown on the axes below. (The axes are measured in inches.)



Example
$$y^2 = 4x$$
$$2y\frac{dy}{dx} = 4$$

a. [4 points] Chancelor decides to color the inside of the lemniscate red. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total area, in square inches, that he has to fill in with red.

$$\text{Area} = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

$r^2 = 16\cos 2\theta = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}$

$\theta = \pi/4$

$$\text{area} = 4\left(\frac{1}{2}\int_0^{\pi/4} 16\cos 2\theta\, d\theta\right) \text{ square inches}$$

b. [4 points] He decides he wants to outline the right half (the portion to the right of the y-axis) of the lemniscate in blue. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total length, in inches, of the outline he must draw in blue.

$$\text{arclength} = \int_\alpha^\beta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

$$f(\theta) = \sqrt{16\cos 2\theta}$$

$$2r\frac{dr}{d\theta} = 16(-2\sin 2\theta)$$

$$\frac{dr}{d\theta} = -\frac{\overset{16}{\cancel{32}}\sin 2\theta}{\cancel{2}r} = \frac{-16\sin 2\theta}{\sqrt{16\cos 2\theta}} = -\frac{4\sin 2\theta}{\sqrt{\cos 2\theta}}$$

either use symmetry or

$$\int_{-\pi/4}^{\pi/4} \sqrt{16\cos 2\theta + \left(-\frac{4\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2}\, d\theta \quad \text{inches}$$

c. [4 points] Chancelor draws another picture of the same lemniscate, but this time also draws a picture of the circle $r = 2\sqrt{2}$. He would like to color the area that is inside the lemniscate but outside the circle purple. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total area, in square inches, that he must fill in with purple.

$16\cos 2\theta = (2\sqrt{2})^2$ intersection

$16\cos 2\theta = 8$

$\cos 2\theta = \frac{1}{2}$

$2\theta = -\frac{\pi}{3}, \frac{\pi}{3} \Rightarrow \theta = -\frac{\pi}{6}, \frac{\pi}{6}$

$$2\left(\frac{1}{2}\int_{-\pi/6}^{\pi/6} (16\cos 2\theta - 8)\, d\theta\right) \text{ square inches.}$$

by symmetry

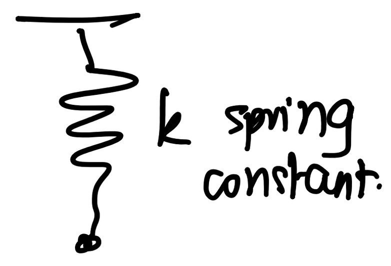
Intro to differential equations

Thursday, July 23, 2020 3:21 PM

A differential equation is an equation that states how a rate of change (a "differential") in one variable is related to the other variables.

eg The amount of stretch in the spring is directly related to the position of a particle, x. We can write this as a differential equation for the velocity

$$\frac{dv}{dt} = -kx \qquad \text{Hooke's law}$$



eg Suppose we are interested in how fast an employee learns a new task. One theory claims that the more the employee already knows of the task, the slower he/she learns. In other words, if $y\%$ is the percentage of the task that the employee has already mastered and $\frac{dy}{dt}$ is the rate at which the employee learns then $\frac{dy}{dt}$ decreases as y increases

$$\boxed{\frac{dy}{dt} = 100 - y}$$

A formula for the solution

Let's suppose $y = 100 + Ce^{-t}$ is a solution. How do you check that?

$$LHS = \frac{dy}{dt} = -Ce^{-t}$$

$$RHS = 100 - y = \cancel{100} - (\cancel{100} + Ce^{-t}) = -Ce^{-t}$$

$\therefore$ LHS = RHS

The $y = 100 + Ce^{-t}$ satisfies the differential equation & must be a solution

a. [4 points] December is a busy time for cookie bakers and cookie eaters. Suppose that there is so much baking going on that cookies are added to the cookie supply of Ann Arbor at a rate of 10 pounds per minute. At the same time, 2% of the cookies are eaten every minute. Write a differential equation for the number of pounds C of cookies in Ann Arbor at time t, in minutes.

pounds of cookies per minute

pounds of cookies per minute

$$\frac{dC}{dt} = 10 - 0.02C$$

pounds of cookies per minute

Example Wild rabbits were introduced to Australia in 1859. The behavior of the rabbit population P in Australia at a time t years after 1859 was modeled by the differential equation

$$P' = P + e^{-t} \qquad \leftarrow \frac{dP}{dt} = P + e^{-t}$$

Q For what value of B is

$$P = 3e^{t} + Be^{-t}$$

a solution to the differential equation?

S

$$LHS = \frac{dP}{dt} = 3e^{t} - Be^{-t}$$

$$RHS = P + e^{-t} = \underbrace{\boxed{3e^{t} + Be^{-t}} + e^{-t}}_{e^{-t}(B+1)} = 3e^{t} + (B+1)e^{-t}$$

Since it's a solution, we must have LHS = RHS

$$3e^{t} - Be^{-t} = 3e^{t} + (B+1)e^{-t}$$

Comparing coefficients

$$-B = B + 1$$

$$2B = -1$$

$$\boxed{B = -\frac{1}{2}}$$

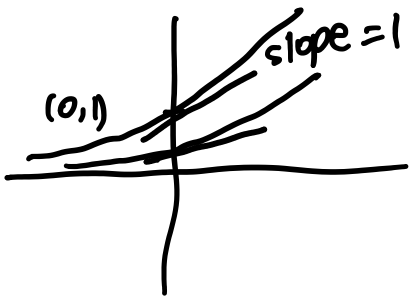
Slope fields

Saturday, July 25, 2020 1:29 PM

Slope fields help us visualize differential equations. Let's take for example

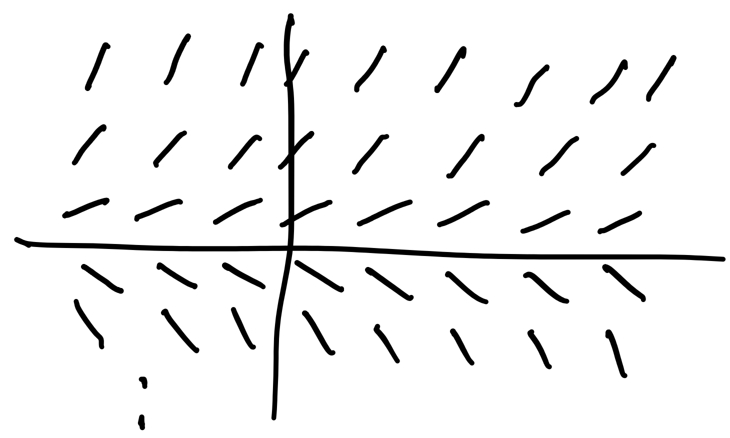
$$\frac{dy}{dx} = y.$$

This implies that any solution to this differential eqn has the property that the slope at any point is the y-coordinate at that point.



<u>Note</u> slope is constant on a horizontal line where y is constant.

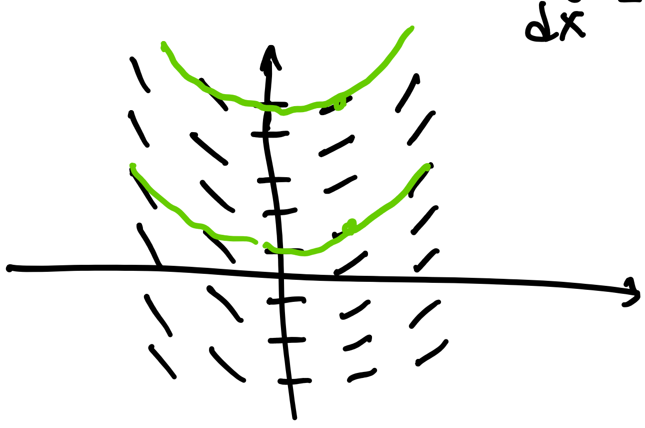
Slope field :



reflection

The higher the y-value for $y>0$ then the steeper the slope field line is

How does it behave if $\frac{dy}{dx} = x$?



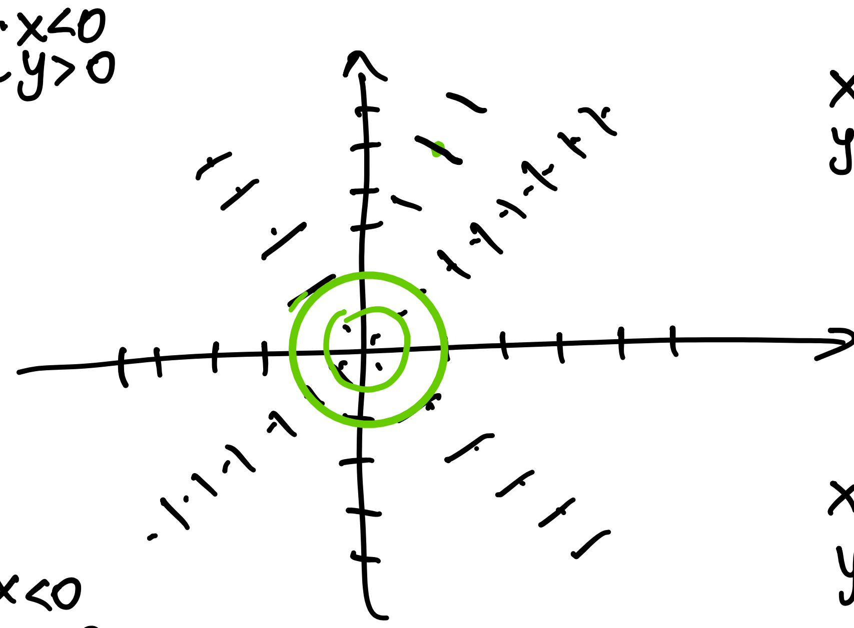
if you connect the slope fields they should give you a parabola

$$\int \frac{dy}{dx}\,dx = \int x\,dx$$

$$y = \frac{x^2}{2} + C \quad \text{parabolas.}$$

<u>Example</u> $\frac{dy}{dx} = -\frac{x}{y}$

$\frac{dy}{dx} > 0$. $\left[\begin{array}{l} x<0 \\ y>0 \end{array}\right.$



$\left.\begin{array}{l} x>0 \\ y>0 \end{array}\right] \frac{dy}{dx} < 0$

$\left.\begin{array}{l} x>0 \\ y<0 \end{array}\right] \frac{dy}{dx} > 0$

$\frac{dy}{dx} < 0$ $\left[\begin{array}{l} x<0 \\ y<0 \end{array}\right.$

Solution curves: $x^2 + y^2 = C$, where C is a constant.

Check using implicit differentiating.

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \checkmark$$

Separation of variables

Saturday, July 25, 2020 1:29 PM

Consider the same example $\frac{dy}{dx} = -\frac{x}{y}$.

How do you obtain that $x^2+y^2=C$ is the solution?

The method of separation of variables works by putting all the x-values on one side of the equation and all the y-values on the other

METHOD

Step 1 Separate the x's with the y's.

$$y\,dy = -x\,dx$$

Step 2 Integrate each side separately

NB
If you are given an initial condition of the form $y(A)=B$ you can use it to find the constant of integration.

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$y^2 = -x^2 + 2C \qquad \text{let } k=2C.$$

$$x^2+y^2=k \quad \leftarrow \text{circles.}$$

Note A differential equation is called separable, if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

1. Determine which of the following differential equations are separable. Do not solve the equations. Y = yes, N = no

(a) $y' = y$ Y
(b) $y' = x + y$ N
(c) $y' = xy$ Y
(d) $y' = \sin(x + y)$ N
(e) $y' - xy = 0$ Y
(f) $y' = y/x$ Y
(g) $y' = \ln(xy)$ N
(h) $y' = (\sin x)(\cos y)$ Y
(i) $y' = (\sin x)(\cos(xy))$ N
(j) $y' = x/y$
(k) $y' = 2x$
(l) $y' = (x + y)/(x + 2y)$

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{dx} = x+y$$

$$\frac{dy}{dx} - y = x$$

$$dy - y\,dx = x\,dx$$

$$\frac{dy}{dx} = \ln(xy)$$

$$e^{dy/dx} = xy$$

Example

$$B^2 + 2B\frac{dB}{dt} = 2500, \qquad B(0)=0$$

$$2B\frac{dB}{dt} = 2500 - B^2$$

$$\frac{dB}{dt} = \frac{2500-B^2}{2B} \leftarrow \text{divide by } \left(\frac{2500-B^2}{2B}\right)$$

$$\frac{2B}{2500-B^2}\frac{dB}{dt} = 1$$

$$\int \frac{2B}{2500-B^2}\,dB = \int dt$$

partial fractions or u-subst

Equilibrium solutions and their stability

Tuesday, July 28, 2020 8:56 PM

An equilibrium solution is constant for all values of the independent variable. (The graph is a horizontal line).

★ Equilibrium solutions are found using $\frac{dy}{dx} = 0$ and solving for y. ★

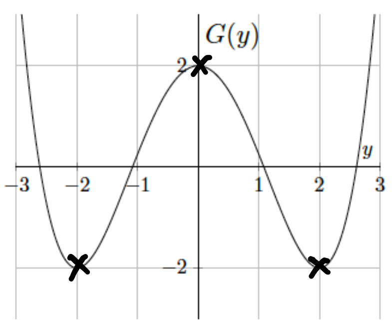
Stability

An eqm solⁿ is <u>stable</u> if a small change in the initial conditions gives a solution that tends towards the equilibrium as the independent variable goes to ∞.

An eqm solⁿ is <u>unstable</u> if a small change in the initial conditions gives a solution that tends away from the equilibrium as the independent variable goes to ∞.

[11 points] The graph of $G(y)$ is shown below. Suppose that $G'(y) = g(y)$. Consider the differential equation $\frac{dy}{dt} = g(y)$.

derivative = 0



← y-axis

Eqm solⁿ $\frac{dy}{dt} = 0$

then solve for y

$\frac{dy}{dt} = g(y) = 0$

$G'(y) = g(y)$

Note again that $\frac{dy}{dt} = g(y)$ and the given graph depicts $G(y)$ **not** $g(y)$.

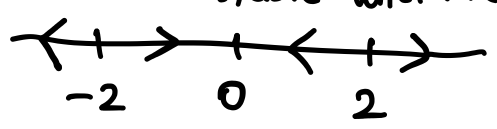
a. [6 points] The differential equation has 3 equilibrium solutions. Find the 3 solutions and indicate whether they are stable or unstable by circling the correct answer.

Equilibrium solution 1: -2 Stable Unstable

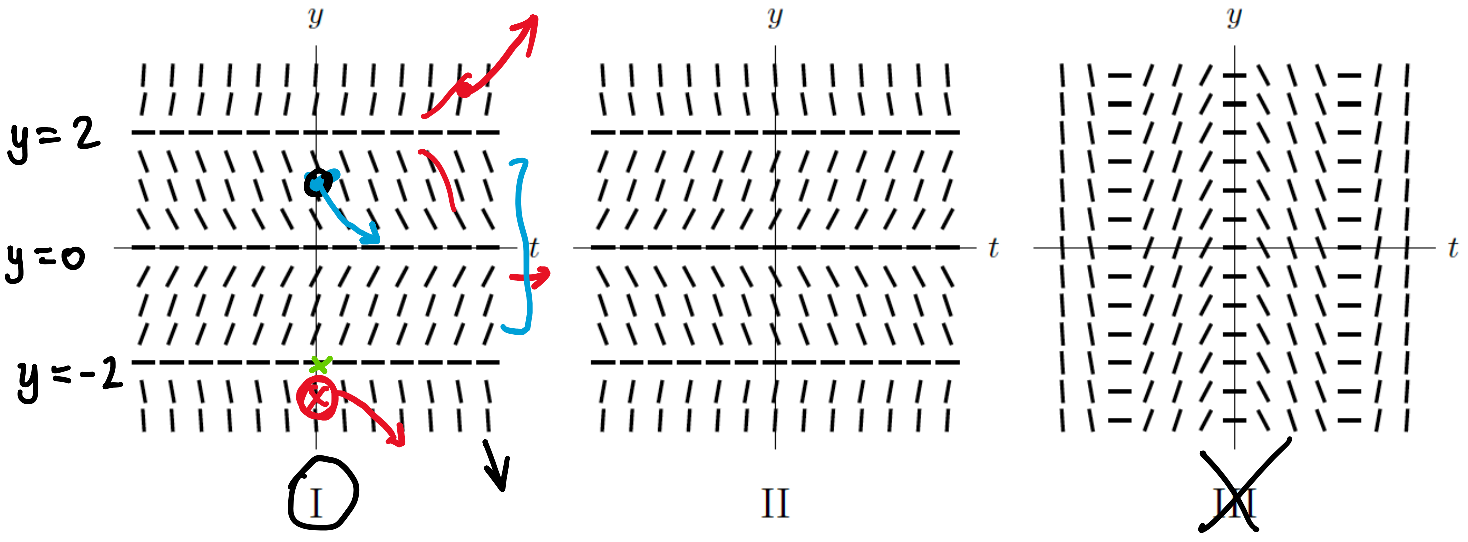
Equilibrium solution 2: 0 Stable Unstable

Equilibrium solution 3: 2 Stable Unstable

Example



b. [2 points] Circle the graph that could be the slope field of the above differential equation.



c. [3 points] Suppose $y_1(t), y_2(t)$ and $y_3(t)$ are all solutions of the differential equation with different initial conditions as indicated below:

- $y_1(t)$ solves the differential equation with initial condition $y(0) = -2$.
- $y_2(t)$ solves the differential equation with initial condition $y(0) = 1.5$.
- $y_3(t)$ solves the differential equation with initial condition $y(0) = -2.1$.

$y(t)$

$y(0) = -2$

At $t = 0$ $y = -2$

Compute the following limits:

$\lim_{t\to\infty} y_1(t) = -2$ $\lim_{t\to\infty} y_2(t) = 0$ $\lim_{t\to\infty} y_3(t) = -\infty$

Example a) Consider a differential equation

$$\frac{dy}{dx} = (x-y)(y-2)$$

What are the equilibrium solutions? $\boxed{y=2}$

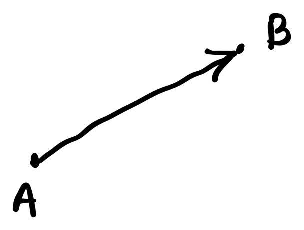
b) Use inequalities to describe the regions in the slope field where the solution curves are increasing.

$\frac{dy}{dx} > 0$ $x - y > 0$ and $y - 2 > 0$ $\Rightarrow$ $y < x$ and $y > 2$.

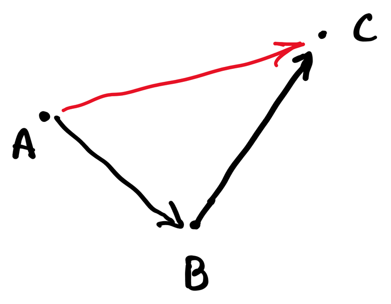
<u>OR</u> $x - y < 0$ and $y - 2 < 0$ etc.

Vectors and vector functions
Friday, July 31, 2020 9:00 AM

The term vector is used to describe a quantity that has both magnitude and direction

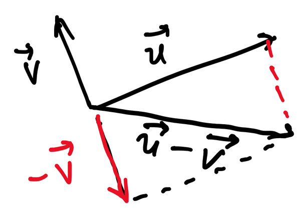


the directed line segment is the vector
$\vec{v} = \vec{AB}$

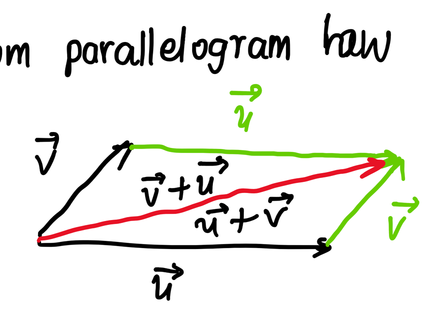


$\vec{AC} = \vec{AB} + \vec{BC}$

You can subtract vectors.



from parallelogram law

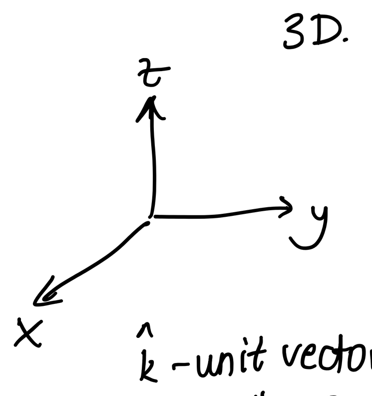


Vector-valued function — It's a function whose domain is a set of real numbers and whose range is a set of vectors.

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k} = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$

where $f(t), g(t), h(t)$ are the components of the vector $\vec{r}(t)$ and t is the independent variable.

3D.



$\hat{k}$ - unit vector in the z-direction
$\hat{j}$ - unit vector in the y-direction
$\hat{i}$ - unit vector in the x-direction

Example $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$. $f(t) = t^3, g(t) = \ln(3-t), h(t) = \sqrt{t}$

$3 - t > 0 \Rightarrow t < 3$
$t \geq 0 \Rightarrow$ $t \in [0,3)$

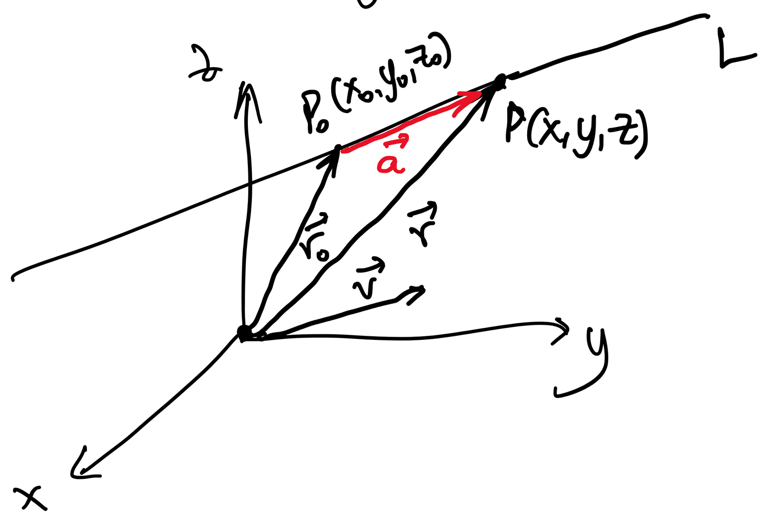
t is an element of the interval $[0,3)$

Equation of a line — As in 2D space, a line in 3D is determined when we know

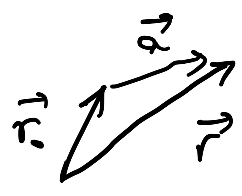
- a point $P_0(x_0, y_0, z_0)$ on L
- the direction of L (its slope).

In 3D the direction of a line is described by a vector, so we let $\vec{v}$ be a vector parallel to L.

Let $P(x,y,z)$ be a point on L and let $\vec{r_0}$ and $\vec{r}$ be the position vectors of P_0 and P.



If $\vec{a}$ is the vector for $\vec{P_0P}$ then $\vec{r} = \vec{r_0} + \vec{a}$



But $\vec{a}$ and $\vec{v}$ are parallel and so $\vec{a} = t\vec{v}$
($\vec{a}$ is a scalar multiple of $\vec{v}$)

Use $\vec{r} = \vec{r_0} + \vec{a}$ and $\vec{a} = t\vec{v}$ to write

$\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$ vector equation for a line

If $\vec{r} = \langle x, y, z \rangle$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ and $\vec{v} = \langle a, b, c \rangle$ then

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the direction vector (a, b, c).

EXAMPLE 1
(a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
(b) Find two other points on the line.

a) $\vec{r} = \vec{r_0} + t\vec{v}$ where $\vec{r_0} = \langle 5, 1, 3 \rangle$ and $\vec{v} = \langle 1, 4, -2 \rangle$

$\vec{r} = \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle$
$= \langle 5+t, 1+4t, 3-2t \rangle$

b) Choose a parameter $t = 1$ $x = 6, y = 5, z = 1$ so $(6, 5, 1)$ is a point on L
and similarly $t = -1$ $x = 4, y = -3, z = 5$ so $(4, -3, 5)$ is another point on L.

Showing if two lines intersect

Example $\vec{r_1} = (3\hat{i} + 8\hat{j} - 2\hat{k}) + t(2\hat{i} - \hat{j} + 3\hat{k}) = \begin{pmatrix} 3 \\ 8 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$\vec{r_2} = (7\hat{i} + 4\hat{j} + 3\hat{k}) + s(2\hat{i} + \hat{j} + 4\hat{k}) = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

At the intersection point $\begin{pmatrix} 3+2t \\ 8-t \\ -2+3t \end{pmatrix} = \begin{pmatrix} 7+2s \\ 4+s \\ 3+4s \end{pmatrix}$ $\begin{matrix} \leftarrow x \\ \leftarrow y \\ \leftarrow z \end{matrix}$

Equate the x and y-components:

$3+2t = 7+2s \rightarrow 3+\cancel{2t} = 7+2s$
$8-t = 4+s \rightarrow 16-\cancel{2t} = 8+2s$ +

$19 = 15+4s$
$4 = 4s$
$s = 1$

$8 - t = 4 + 1$
$t = 3$

Check that the z-components are also equal

$-2+3t = -2+9 = 7$ ✓
$3+4s = 3+4 = 7$ ✓

If the z-coordinate does not agree then the lines do not intersect!

Intersection point has position vector $\begin{pmatrix} 3+2t \\ 8-t \\ -2+3t \end{pmatrix}$ with $t = 3 \Rightarrow \vec{r} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$

Motion in space: velocity and acceleration

Friday, July 31, 2020 9:50 AM

Let $\vec{r}(t)$ be the position vector at time t then the velocity vector is given by

$$\vec{v}(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \vec{r}'(t)$$

The speed of a particle at time t is the magnitude of the velocity vector $|\vec{v}(t)|$

NB

$$|\vec{v}(t)| = |\vec{r}'(t)| = \text{rate of change of distance wrt time.}$$

$$\vec{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \qquad = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The acceleration of a particle is $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$

EXAMPLE 3 A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\boxed{\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}}$. Its acceleration is $\mathbf{a}(t) = 4t\,\mathbf{i} + 6t\,\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.

Since $\vec{a}(t) = \vec{v}'(t) \Rightarrow \vec{v}(t) = \int \vec{a}(t)\,dt = \int (4t\,\hat{i} + 6t\,\hat{j} + \hat{k})\,dt$

$$= 2t^2\,\hat{i} + 3t^2\,\hat{j} + t\hat{k} + \vec{C}$$

Using $\vec{v}(0) = \hat{i} - \hat{j} + \hat{k} = \vec{C}$

$$\Rightarrow \vec{v}(t) = (2t^2+1)\hat{i} + (3t^2-1)\hat{j} + (t+1)\hat{k}$$

Since $\vec{v}(t) = \vec{r}'(t) \Rightarrow \vec{r}(t) = \int \vec{v}(t)\,dt$

$$= \left(\frac{2}{3}t^3 + t\right)\hat{i} + (t^3 - t)\hat{j} + \left(\frac{1}{2}t^2 + t\right)\hat{k} + \vec{D}$$

Using $\vec{r}(0) = \langle 1, 0, 0 \rangle = \vec{D}$.

Example Let $x(t) = 10t$, $y(t) = 20t$, $z(t) = 30t - 5t^2$, $t \geq 0$

A toy is hit by a ball at the coordinate $(20, 40, 40)$

Is the ball moving upward or downward when it hits the toy?

$$(20, 40, 40) = (10t, 20t, 30t - 5t^2) \to t = 2.$$

$$\vec{v}(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle 10, 20, 30 - 10t \rangle$$

At $t = 2$ $\quad \vec{v}(2) = \langle 10, 20, 10 \rangle$

moving upward since $z'(2) > 0$.

Partial derivatives

Sunday, August 2, 2020 9:19 PM

Suppose f is a function of two variables x and y. If only x varies and y is constant, say at $y=b$ then we're considering a function of one variable x, i.e. $g(x) = f(x,b)$
If g has a derivative at a, then we call it the partial derivative of f wrt x at (a,b) and denote it by $f_x(a,b) = \frac{\partial f}{\partial x}(a,b)$

Thus $f_x(a,b) = g'(a)$ where $g(x) = f(x,b)$

By definition a derivative is $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h} = g'(a)$

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Similarly, the partial derivative of f wrt y at (a,b) (keep constant $x=a$)

$$\left(\frac{\partial f}{\partial y}(a,b)\right) f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Rule for finding the partial derivatives

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ WRT x
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ wrt y

Example If $f(x,y) = x^3 + x^2y^3 - 2y^2$ find $f_x(2,1)$ and $f_y(2,1)$

$f_x(x,y) = 3x^2 + 2xy^3$

$f(x,y) = x^3 + x^2y^3 - 2y^2$ "constants"

$f_x(x,y)$ keep y as a constant and treat x as a variable.

$f_x(x,y) = 3x^2 + 2xy^3 \cancel{+0}$

$f_y(x,y) = 3x^2y^2 - 4y$

$f_x(2,1) = 3(2)^2 + 2(2)(1)^3 = 12 + 4 = 16$

x y

$f_y(2,1) = 3(2)^2(1)^2 - 4(1) = 8$

$f(T,H)$

Table 1 Heat index I as a function of temperature and humidity

Relative humidity (%)

Actual temperature (°F) T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Find the partial derivative when $H = 70\%$

$\boxed{f_H(96,70)}$ ← keep 1st variable constant & vary the second.

Use limit definition of a derivative

$$f_H(96,70) = \lim_{h\to 0} \frac{f(96,70+h) - f(96,70)}{h}$$

$f(T,H)$

take $H = \pm 5$

used $H = -5$

$$f_H(96,70) \approx \frac{f(96,65) - f(96,70)}{-5} = \frac{121-125}{-5} = 0.8$$

$H=5$

$$f_H(96,70) \approx \frac{f(96,75) - f(96,70)}{5} = \frac{130-125}{5} = 1$$

average the two

$$f_H(96,70) \approx \frac{0.8+1}{2} = 0.9$$

Interpretation When the temperature is 96°F and the relative humidity is 70%, the heat index increases by about 0.9°F for every percent that the relative humidity rises.

NB $D_x f = f_x = \frac{\partial f}{\partial x}$

$\frac{df}{dx}$ if f is a function of a single variable

$\frac{d^2f}{dx^2}, \frac{d^3f}{dx^3}, \ldots$

Higher derivatives

$(f_x)_x = \frac{\partial^2 f}{\partial x^2}$ $\quad (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$

$(f_y)_y = \frac{\partial^2 f}{\partial y^2}$ $\quad (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$ ← first differentiate wrt y and then wrt x

e.g. $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x = 3x^2 + 2xy^3$

$f_y = 3x^2y^2 - 4y$

$f_{xx} = 6x + 2y^3$

$f_{yy} = 6x^2y - 4$

$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left[3x^2 + 2xy^3\right] = 0 + 6xy^2$

$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3x^2y^2 - 4y\right) = 6xy^2$

Theorem Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D then $f_{xy}(a,b) = f_{yx}(a,b)$

$\left(\frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial^2 f}{\partial x \partial y}\right)$

4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notations for Partial Derivatives If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Tangent planes

Sunday, August 2, 2020 9:57 PM

Suppose a surface S has equation $z = f(x,y)$ and let $P(x_0, y_0, z_0)$ be a point on S (see figure). Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at point P. Then the tangent plane to the surface S at P is the plane that contains both tangent lines T_1 and T_2.

Eqn of a plane through the point $P(x_0, y_0, z_0)$ is of the form

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

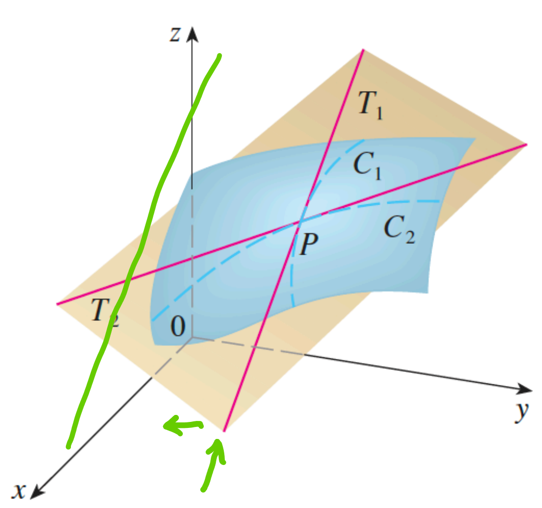


FIGURE 1
The tangent plane contains the tangent lines T_1 and T_2.

Divide through by C: $\frac{A}{C}(x-x_0) + \frac{B}{C}(y-y_0) + z - z_0 = 0$

$$z - z_0 = -\frac{A}{C}(x-x_0) - \frac{B}{C}(y-y_0)$$

let $a = -\frac{A}{C}$ and $b = -\frac{B}{C}$

$$\Rightarrow \quad z - z_0 = a(x-x_0) + b(y-y_0) \qquad (*)$$

If $(*)$ represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be tangent to T_1.

If we substitute $y = y_0$ into $(*)$ we obtain

$$z - z_0 = a(x - x_0)$$ ← point slope formula of a line with slope a.

The slope of this tangent line is $f_x(x_0, y_0)$ and thus $a = f_x(x_0, y_0)$

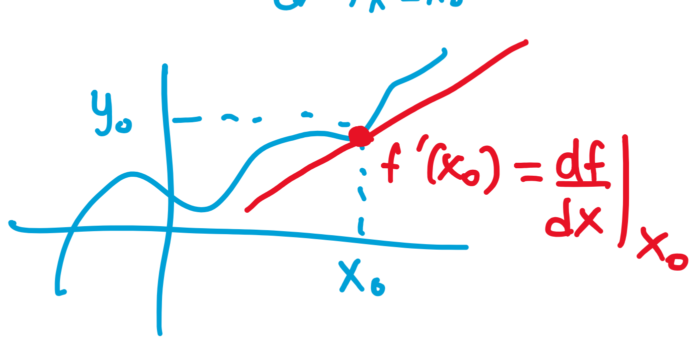
Similarly $b = f_y(x_0, y_0)$.

Thus, $(*)$ becomes

$$\boxed{z - z_0 = \underbrace{f_x(x_0,y_0)}_{\text{evaluate } \frac{\partial f}{\partial x} \text{ at } (x_0,y_0)} \cdot (x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)}$$

equation of a tangent plane through (x_0, y_0, z_0)

$$y - y_0 = \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0)$$



<u>Example</u> Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $\underbrace{(1,1,3)}_{(x_0,\, y_0,\, z_0)}$

Let $\underline{z = f(x,y)} = 2x^2 + y^2$

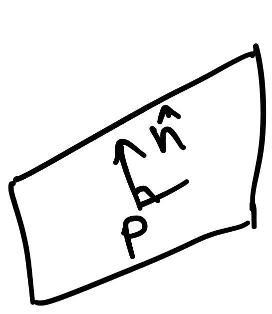
$\frac{\partial f}{\partial x} = f_x = 4x$ $\qquad \frac{\partial f}{\partial x}(x_0, y_0) = 4(1) = 4$

$\frac{\partial f}{\partial y} = f_y = 2y$ $\qquad \frac{\partial f}{\partial y}(x_0, y_0) = 2(1) = 2$

$$\underset{z_0}{\underset{\uparrow}{z}} - 3 = \underset{f_x(x_0,y_0)}{\underset{\uparrow}{4}}(x - \overset{x_0}{\overset{\downarrow}{1}}) + \underset{f_y(x_0,y_0)}{\underset{\uparrow}{2}}(y - \overset{y_0}{\overset{\downarrow}{1}})$$ where $(x_0, y_0, z_0) = (1,1,3)$

$$z - 3 = 4x - 4 + 2y - 2$$

$$\boxed{z = 4x + 2y - 3}$$ tangent plane



$$x \cdot \hat{n} = p \cdot \hat{n}$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\0\end{pmatrix} = 1(2) + 2(-1) + 3(0)$$

$$= 2 - 2 + 0$$

$$= 0$$

Lagrange multipliers

Tuesday, August 4, 2020 11:50 PM

(Not in Exam)

This is an example of where you can use partial derivatives.

Maximum and minimum values

Theorem If $f(x,y)$ has a local minimum or maximum at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

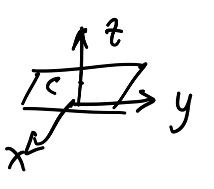
Notation: this can be written as $\boxed{\nabla f(a,b)=\vec{0}}$ where $\underset{\text{grad}}{\nabla f} = \text{grad } f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$

(nabla : ∇)

Note From the previous class we saw that the tangent plane at the point (a,b,c)

$$z - c = \underbrace{f_x(a,b)}_{0}\cdot(x-a) + \underbrace{f_y(a,b)}_{0}\cdot(y-b)$$

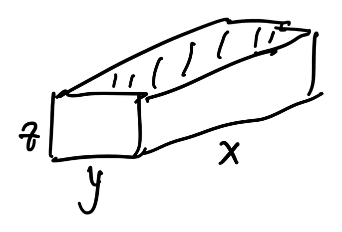
$$\Rightarrow \quad z = c$$



The geometric interpretation is that if the graph of f has a tangent plane at a local minimum or maximum then the tangent plane is horizontal.

A point (a,b) is called a critical point of f if $\boxed{f_x(a,b)=0}$ and $\boxed{f_y(a,b)=0}$ or if one of f_x or f_y does not exist.

Example A rectangular box without a lid is made of $12\,m^2$ of cardboard. Find the maximum volume of this box.



Sol^n $\quad V = xyz$

Constraint : $\quad 2xz + 2yz + xy = 12$

$$\Rightarrow \quad z(2x+2y) + xy = 12$$

$$\Rightarrow \quad \boxed{z = \frac{12-xy}{2(x+y)}}$$

Objective function: $V = xyz$

$$= xy\left(\frac{12-xy}{2(x+y)}\right)$$

$$= \frac{12xy - x^2y^2}{2(x+y)} \quad \leftarrow$$

Critical points: $\frac{\partial V}{\partial x} = 0$ and $\frac{\partial V}{\partial y} = 0$

$$\frac{\partial V}{\partial x} = \frac{\cancel{2}(x+y)\left[12y - 2xy^2\right] - \cancel{2}\left[12xy - x^2y^2\right]}{2^{\cancel{2}}(x+y)^2} = \frac{\cancel{12xy} + 12y^2 - 2x^2y^2 - 2xy^3 - \cancel{12xy} + x^2y^2}{2(x+y)^2}$$

$$= \frac{12y^2 - x^2y^2 - 2xy^3}{2(x+y)^2}$$

$$= \frac{y^2(12 - x^2 - 2xy)}{2(x+y)^2}$$

$$\frac{\partial V}{\partial y} = \frac{12x^2 - x^2y^2 - 2yx^3}{2(x+y)^2} = \frac{x^2(12 - y^2 - 2yx)}{2(x+y)^2}$$

$x = 0 = y \Rightarrow V = 0$

$12 - y^2 - 2yx = 0$ and $12 - x^2 - 2xy = 0 \quad \Rightarrow \quad x = y$

using $x = y \Rightarrow$

$$12 - x^2 - 2x^2 = 0$$
$$12 - 3x^2 = 0$$
$$x^2 = 4$$
$$\boxed{x = 2} \Rightarrow y = 2 \quad \Rightarrow \quad z = \frac{12 - (2)(2)}{2(2+2)} = 1$$

$x = 2,\ y = 2,\ z = 1 \quad \Rightarrow \quad V = 2(2)(1) = 4\ m^3$

To find the absolute maximum and minimum values of a continuous function on a closed, bounded set D:

1. Find the values of f at the critical points of f in D
2. Find the values of f on the boundary of D.
3. The largest of the values from 1 and 2 is the maximum and the smallest is the minimum

Lagrange multipliers

This is used for maximizing or minimizing an objective function $f(x,y,z)$ subject to a constraint of the form $g(x,y,z) = k$.

Step 1 Find all values of x, y, z and λ such that

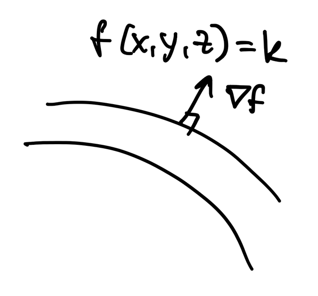
$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and $\quad g(x,y,z) = k$

λ: Lagrange multiplier

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$$

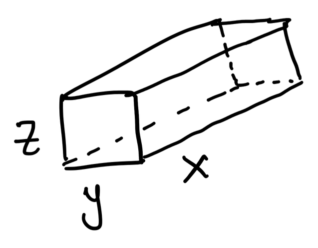
Aside.



Step 2 Evaluate f at all points (x,y,z) that result from step 1

The largest of these values is the maximum of f.

Example



$V = xyz$

Constraint : $\quad 2xz + 2yz + xy = 12 \quad$ (recall the lid is missing)

$$f = xyz$$
$$g = 2xz + 2yz + xy$$

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} \qquad \nabla g = \begin{pmatrix} 2z+y \\ 2z+x \\ 2x+2y \end{pmatrix}$$

$$\nabla f = \lambda \nabla g \quad \Rightarrow \quad \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} 2z+y \\ 2z+x \\ 2x+2y \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix}$$

multiply

$(\times x)$	$yz = \lambda(2z+y)$	$xyz = \lambda(2xz+xy)$	
$(\times y)$	$xz = \lambda(2z+x)$	$\rightarrow xyz = \lambda(2yz+xy)$	$\boxed{\lambda(2xz+xy) = \lambda(2yz+xy)}$
$(\times z)$	$xy = \lambda(2x+2y)$	$\rightarrow \boxed{xyz = \lambda(2xz+2yz)}$	

$\lambda = 0 \Rightarrow xy = xz = yz = 0$

this cannot be true because the constraint is not satisfied.

$$2xz + xy = 2yz + xy \quad \Rightarrow \quad x = y$$

$x = y \quad \Rightarrow \quad 2yz + xy = 2xz + 2yz \quad \Rightarrow \quad \cancel{2yz} + y^2 = \cancel{2yz} + 2yz \quad \Rightarrow \quad y^2 - 2yz = 0$

$$y(y - 2z) = 0$$
$$z = \frac{y}{2}$$

Using the constraint $\quad 2xz + 2yz + xy = 12$

$$\cancel{2}y\left(\frac{y}{\cancel{2}}\right) + \cancel{2}y\left(\frac{y}{\cancel{2}}\right) + y^2 = 12$$

$$y^2 + y^2 + y^2 = 12 \Rightarrow 3y^2 = 12$$

$$y = 2 \quad \Rightarrow \quad x = 2 \quad \Rightarrow \quad z = 1 \qquad V = 4\ m^3$$

Example Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$

sketching

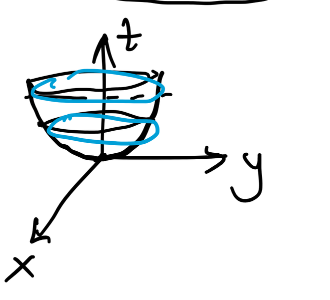
$x^2 + 2y^2 = z \quad$ where k is a constant

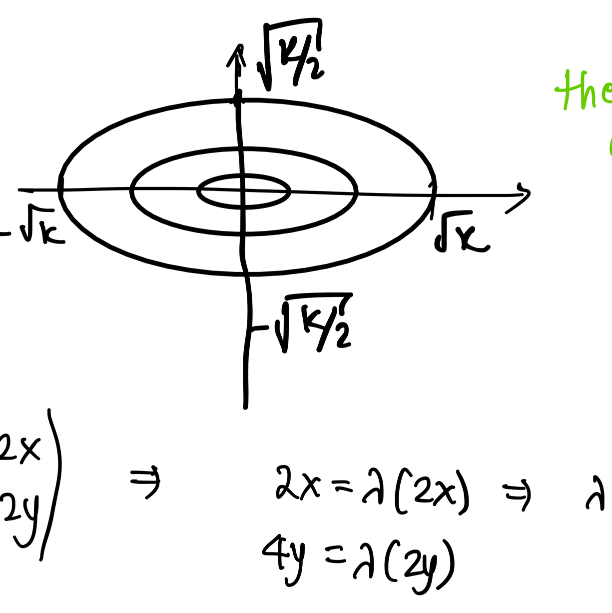
Setting $z = \text{constant} = k$

$$x^2 + 2y^2 = k \qquad \left(\text{Ellipse } \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\right)$$

$$\frac{x^2}{k} + \frac{2y^2}{k} = 1$$

$$\left(\frac{x}{\sqrt{k}}\right)^2 + \left(\frac{y}{\left(\sqrt{k/2}\right)}\right)^2 = 1$$





the cross-sections are ellipses!

$$\left.\begin{matrix} \nabla f = \lambda \nabla g \\ \text{and } x^2 + y^2 = 1 \end{matrix}\right] \Rightarrow \quad \begin{pmatrix} 2x \\ 4y \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} 2x = \lambda(2x) \\ 4y = \lambda(2y) \end{matrix} \quad \Rightarrow \quad \lambda = 1 \text{ or } x = 0$$

$$\underset{0}{x^2} + y^2 = 1$$
$$y = \pm 1$$

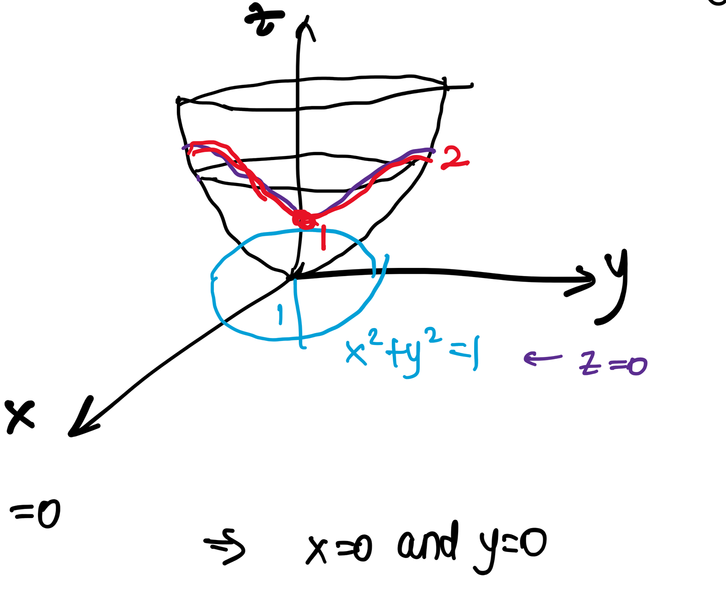
If $\lambda = 1 \Rightarrow 4y = 2y \Rightarrow y = 0$

then $x = \pm 1$ from constraint

recall $\boxed{f = x^2 + 2y^2}$

max $\quad f(0, \pm 1) = 0 + 2(\pm 1)^2 = 2$

min $\quad f(\pm 1, 0) = (\pm 1)^2 + 2(0)^2 = 1$



If constraint $\quad x^2 + y^2 \le 1$

Extra step $\quad$ find critical points of f

$$\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix} \qquad \begin{matrix} f_x = 2x = 0 \\ f_y = 4y = 0 \end{matrix} \quad \Rightarrow \quad x = 0 \text{ and } y = 0$$

$f(0,0) = 0 \quad$ min

$f(0, \pm 1) = 2 \quad$ max