LIE ALGEBRAS: BASIC THEORY

1. BASIC NOTIONS

1.1. **Background.** *In this section I use some material, like multivariable analysis, which is not necessary for the main body of the course, but if you know it (and hopefully if you don't but are willing to think imprecisely at some points) it will help to put the course in context. For those worried about such things, fear not, it is non-examinable.*

In mathematics, group actions give a way of encoding the symmetries of a space or physical system. Formally these are defined as follows: an action of a group G on a space[1] X is a map $a\colon G \times X \to X$, written $(g.x) \mapsto a(g,x)$ or more commonly $(g,x) \mapsto g.x$ which satisfies the properties

(1) $e.x = x$, for all $x \in X$, where $e \in G$ is the identity;
(2) $(g_1 g_2).x = g_1.(g_2.x)$ for all $g_1, g_2 \in G$ and $x \in X$.

Natural examples of actions are that of the group of rigid motions SO_3 on the unit sphere $\{x \in \mathbb{R}^3 : ||x|| = 1\}$, or the general linear group $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$.

Whenever a group acts on a space X, there is a resulting linear action (a representation) on the vector space of functions on X. Indeed if $\mathrm{Fun}(X)$ denotes the vector space of real-valued functions on X, then the formula

$$g(f)(x) = f(g^{-1}.x), \quad (g \in G, f \in \mathrm{Fun}(X), x \in X),$$

defines a representation of G on $\mathrm{Fun}(X)$. If X and G have more structure. *e.g.* that of a topological space or smooth manifold, then this action may also preserve the subspaces of say continuous, or differentiable functions. Lie algebras arise as the "infinitesimal version" of group actions, which loosely speaking means they are what we get by trying to differentiate group actions.

Example 1.1. Take for example the natural action of the circle S^1 by rotations on the plane $\mathbb{R}^2$. This action can be written explicitly using matrices:

$$g(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

where we have smoothly parametrized the circle S^1 using the trigonometric functions. Note that for this parametrization, $g(t)^{-1} = g(-t)$. The induced action on $\mathrm{Fun}(\mathbb{R}^2)$ restricts to an action on $\mathcal{C}^\infty(\mathbb{R}^2)$ the space of smooth (*i.e.* infinitely differentiable) functions on $\mathbb{R}^2$. Using our parametrization, it makes sense to differentiate this action at the identity element (*i.e.* at $t = 0$) to get an operation $\nu\colon \mathcal{C}^\infty(\mathbb{R}^2) \to \mathcal{C}^\infty(\mathbb{R}^2)$, given by

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[1] I'm being deliberately vague here about what a "space" is, X could just be a set, but it could also have a more geometric nature, such as a topological space or submanifold of $\mathbb{R}^n$.

$$\begin{aligned} f \mapsto & \frac{d}{dt}\big(f(g(-t).\left(\begin{array}{c} x \\ y \end{array}\right))\big)_{|t=0} \\ & = y\frac{df}{dx} - x\frac{df}{dy}. \end{aligned}$$

It is immediate from the product rule for differentiation, that the operator ν constructed in the above example obeys the "Leibniz rule":

$$\nu(f.g) = \nu(f).g + f.\nu(g).$$

An operator on smooth functions which satisfies this property is called a *derivation*. It's not hard to see that any such operator on $\mathcal{C}^\infty(\mathbb{R}^2)$ will be of the form $a(x,y)\frac{d}{dx} + b(x,y)\frac{d}{dy}$ where $a, b \in \mathcal{C}^\infty(\mathbb{R}^2)$. Thus, heuristically for now, we think of the infinitesimal version of a group action as the collection of derivations on smooth functions we obtain by "differentiating the group action at the identity element". (For the circle the collection of vector fields we get are just the scalar multiples of the vector field ν, but for actions of larger group we would attach a larger space of derivations). It turns out this set of derivations forms a vector space, but it also has a kind of "product" which is a sort of infinitesimal remnant of the group multiplication[2]. Let's set this up a little more formally.

Definition 1.2. A *vector field* on $X = \mathbb{R}^n$ (or, with a bit more work, any manifold) is a (smooth) function $\nu\colon \mathbb{R}^n \to \mathbb{R}^n$, which one can think of as giving the infinitesimal direction of a flow (*e.g.* of a fluid, or an electric field say). The set of vector fields forms a vector space which we denote by Θ_X. Such fields can be made to act on functions $f\colon X \to \mathbb{R}$ by differentiation. If $\nu = (a_1, a_2, \dots, a_n)$ in standard coordinates, then set

$$\nu(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}.$$

This formula gives an action of Θ_X on the space of smooth functions $\mathcal{C}^\infty(X)$, since if $f \in \mathcal{C}^\infty(X)$, then so is $\nu(f)$. This action is linear, and interacts with multiplication of functions according to the Leibniz rule, that is, if ν is a vector field, and $f_1, f_2 \in \mathcal{C}^\infty(X)$ then

$$\nu(f_1 f_2) = \nu(f_1).f_2 + f_1.\nu(f_2),$$

in other words, vector fields act as derivations on smooth functions in the above sense. Note we can talk about vector fields and derivations interchangeably, since the derivation given by a vector field completely determines the vector field, and any derivation comes from a vector field (this is an exercise that is worth checking).

Note that if we compose two derivations $\nu_1 \circ \nu_2$ we again get an operator on functions, but it is not given by a vector field, since it involves second order differential operators. However, it is easy to check using the symmetry of mixed partial derivatives that if ν_1, ν_2 are derivations, then $[\nu_1, \nu_2] = \nu_1 \circ \nu_2 - \nu_2 \circ \nu_1$ is again a derivation. Thus the space Θ_X of vector fields on X is equipped with a natural product[3] $[.,.]$ which is called a *Lie bracket*. The derivatives of a group action give subalgebras of the algebra Θ_X.

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[2] To be a bit more precise, it comes from the conjugation action of the group on itself.

[3] This is in the weakest sense, in that it is a bilinear map $\Theta_X \times \Theta_X \to \Theta_X$. It is not even associative – the axiom it does satisfy is discussed shortly.

Example 1.3. Consider the action of $\mathrm{SO}_3(\mathbb{R})$ on $\mathbb{R}^3$. Using the fact that every element of $\mathrm{SO}_3(\mathbb{R})$ is a rotation about some axis through the origin it is not too hard to find the space of vector fields on $\mathbb{R}^3$ which can be associated to this action, and check that it forms a Lie algebra. Indeed as we saw before, the action of the circle fixing the z-axis gives the derivation $-y\frac{d}{dx} + x\frac{d}{dy}$, and the derivation obtained from rotation about any other axis will be obtained by an orthogonal change of coordinates. It can be shown that these form the 3-dimensional space $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{x\frac{d}{dy} - y\frac{d}{dx}, y\frac{d}{dz} - z\frac{d}{dy}, z\frac{d}{dx} - x\frac{d}{dz}\}$, and moreover it is then not hard to check that $\mathfrak{g}$ is closed under the bracket operations $[\cdot,\cdot]$. (This also gives a non-trivial example of a 3-dimensional Lie algebra).

1.2. **Definitions and Examples.**

The definition of a Lie algebra is an abstraction of the above example of the product on vector fields. It is purely algebraic, so it makes sense over any field k.

Definition 1.4. A Lie algebra over a field k is a pair $(\mathfrak{g}, [.,.]_{\mathfrak{g}})$ consisting of a k-vector space $\mathfrak{g}$, along with a bilinear form $[.,.]_{\mathfrak{g}}\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ taking values in $\mathfrak{g}$ known as a Lie bracket, which satisfies the following axioms:

1. $[.,.]_{\mathfrak{g}}$ is alternating, *i.e.* $[x,x]_{\mathfrak{g}} = 0$ for all $x \in \mathfrak{g}$.
2. The Lie bracket satisfies the *Jacobi Identity*: that is, for all $x, y, z \in \mathfrak{g}$ we have:

$$[x,[y,z]_{\mathfrak{g}}]_{\mathfrak{g}} + [y,[z,x]_{\mathfrak{g}}]_{\mathfrak{g}} + [z,[x,y]_{\mathfrak{g}}]_{\mathfrak{g}} = 0.$$

Remark 1.5. It is easy to check directly from the definition that the Lie bracket we put on the space of vector fields Θ_X satisfies the above conditions.

Note that by considering the bracket $[x+y, x+y]_{\mathfrak{g}}$ it is easy to see that the alternating condition implies that for all $x, y \in L$ we have $[x,y]_{\mathfrak{g}} = -[y,x]_{\mathfrak{g}}$, that is $[.,.]_{\mathfrak{g}}$ is skew-symmetric. If $\mathrm{char}(\mathsf{k}) \neq 2$, the alternating condition is equivalent to skew-symmetry. Note that a Lie algebra is an algebra with a product which is neither commutative nor associative, and moreover it does not have an identity element[4]. We will normally simply write $[.,.]$ and reserve use the decorated bracket only for emphasis or where there is the potential for confusion.

Example 1.6.

1. If V is any vector space then setting the Lie bracket $[.,.]$ to be zero we get a (not very interesting) Lie algebra. Such Lie algebras are called *abelian* Lie algebras.
2. Generalising the example of vector fields a bit, if A is a k-algebra and $\delta\colon A \to A$ is a k-linear map, then we say δ is a k-*derivation* if it satisfies the Leibniz rule, that is, if:

$$\delta(a.b) = \delta(a).b + a.\delta(b), \quad \forall a, b \in A.$$

It is easy to see by a direct calculation that if $\mathrm{Der}_{\mathsf{k}}(A)$ denotes the k-vector space of k-derivations on A, then $\mathrm{Der}_{\mathsf{k}}(A)$ is a Lie algebra under the commutator product, that is:

$$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1.$$

Indeed the alternating property is immediate, so the only thing to check is the Jacobi identity, which is an easy computation.

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[4] This makes them sound awful. However, as we will see this is not the way to think about them!

(3) For a more down-to-earth example, take $\mathfrak{g} = \mathfrak{gl}_n$ the k-vector space of $n \times n$ matrices with entries in k. It is easy to check that this is a Lie algebra for the commutator product:

$$[X, Y] = X.Y - Y.X.$$

Slightly more abstractly, if V is a vector space, then we will write $\mathfrak{gl}(V)$ for the Lie algebra $\mathrm{End}(V)$ equipped with the commutator product as for matrices.

(4) If $\mathfrak{g}$ is a Lie algebra and $N < \mathfrak{g}$ is a k-subspace of $\mathfrak{g}$ on which the restriction of the Lie bracket takes values in N, so that it induces a bilinear form $[.,.]_N \colon N \times N \to N$, then $(N, [.,.]_N)$ is clearly a Lie algebra, and we say N is a (Lie) *subalgebra* of $\mathfrak{g}$.

(5) Let $\mathfrak{sl}_n = \{X \in \mathfrak{gl}_n : \mathrm{tr}(X) = 0\}$ be the space of $n \times n$ matrices with trace zero. It is easy to check that $\mathfrak{sl}_n$ is a Lie subalgebra of $\mathfrak{gl}_n$ (even though it is *not* a subalgebra of the associative algebra $\mathrm{End}(V)$). More generally we say any Lie subalgebra of $\mathfrak{gl}(V)$ for a vector space V is a *linear Lie algebra*.

(6) If A is an associative k-algebra, then if $a \in A$ let $\delta_a \colon A \to A$ be the linear map given by

$$\delta_a(b) = a.b - b.a, \quad b \in A.$$

One can check that δ_a is a derivation on A, and that this is equivalent to the statement that $(A, [.,.]_A)$ is a Lie algebra, where $[.,.]_A$ is the commutator bracket on A, that is $[a, b]_A = a.b - b.a$. Thus any associative algebra can be given the structure of a Lie algebra. (This is a generalisation of the case of $n \times n$ matrices).

Remark 1.7. One could begin to try and classify all (say finite-dimensional) Lie algebras. In very low dimension this is actually possible. For dimension 1 clearly there is a unique (up to isomorphism) Lie algebra since the alternating condition demands that the bracket is zero. In dimension two, one can again have an abelian Lie algebra, but there is another possibility: if $\mathfrak{g}$ has a basis $\{e, f\}$ then we may set $[e, f] = f$, and this completely determines the Lie algebra structure. All two-dimensional Lie algebras which are not abelian are isomorphic to this one (check this). It is also possible to classify three-dimensional Lie algebras, but it becomes rapidly intractable to do this in general as the dimension increases. In this course we will focus on a particular class of Lie algebras, known as *semisimple Lie algebras*, for which an elegant classification theorem is known.