LIE ALGEBRAS: LECTURE 11.

1. More on the Cartan decomposition

Proposition 1.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra, and $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ the associated Cartan decomposition.*

1. *If $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ and $h \in \mathfrak{h}$ then*
$$\kappa(h, [x, y]) = \alpha(h)\kappa(x, y).$$
2. *The roots $\alpha \in \Phi$ span $\mathfrak{h}^*$.*
3. *The subspace $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ is one-dimensional and $\alpha(\mathfrak{h}_\alpha) \neq 0$.*
4. *If $\alpha \in \Phi$, and $e_\alpha \in \mathfrak{g}_\alpha \backslash \{0\}$, there exist $f_\alpha \in \mathfrak{g}_\alpha$ and $h_\alpha \in \mathfrak{h}_\alpha$ so that the map $e \mapsto e_\alpha$, $f \mapsto f_\alpha$ and $h \mapsto h_\alpha$ gives an embedding $\mathfrak{sl}_2 \to \mathfrak{g}_\alpha \oplus \mathfrak{h}_\alpha \oplus \mathfrak{g}_{-\alpha}$. (Here e, f, h denote the standard basis of $\mathfrak{sl}_2$.)*

Proof. For (1) we have
$$\kappa(h, [x, y]) = \kappa([h, x], y) = \kappa(\alpha(h)x, y) = \alpha(h)\kappa(x, y),$$
as required.

For (2), suppose that $W = \text{span}\{\Phi\}$. If W is a proper subspace of $\mathfrak{h}^*$, then we may find an $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$. But then it follows from our formula for the Killing form in terms of the Cartan decomposition that $\kappa(h, x) = 0$ for all $x \in \mathfrak{h}$, which contradicts the nondegeneracy of the form $\kappa_{|\mathfrak{h}}$.

For (3), as in the remark above, since $\kappa_{|\mathfrak{h}}$ is nondegenerate it yields an isomorphism $\mathfrak{h}^* \to \mathfrak{h}$, given by $\lambda \mapsto t_\lambda$ where $(t_\lambda, h) = \lambda(h)$ for all $h \in \mathfrak{h}$. Since we know that Φ spans $\mathfrak{h}^*$, it follows that $\{t_\alpha : \alpha \in \Phi\}$ spans $\mathfrak{h}$. Suppose that $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$. Then by (1) we see that $[x, y] = \kappa(x, y)t_\alpha$, so that $\mathfrak{h}_\alpha \subseteq \text{span}\{t_\alpha\}$. Since κ is nondegenerate on $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ we may find $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ such that $\kappa(x, y) \neq 0$, hence $\mathfrak{h}_\alpha = \text{span}\{t_\alpha\}$ as required.

Next we wish to show that $\alpha(\mathfrak{h}_\alpha) \neq 0$. For this note that if $\alpha(\mathfrak{h}_\alpha) = 0$ then pick $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ so that $z = [x, y] \in \mathfrak{h}_\alpha$ is nonzero. Then $[z, x] = \alpha(z)x = 0 = -\alpha(z)y = [z, y]$, so that $\mathfrak{a} = \mathsf{k}\text{-span}\{x, y, z\}$ is a solvable subalgebra of $\mathfrak{g}$. In particular, by Lie's theorem we may find a basis of $\mathfrak{g}$ with respect to which the matrices of $\text{ad}(\mathfrak{a})$ act by upper triangular matrices, and so $\text{ad}(z) = \text{ad}([x, y])$ acts by a strictly upper triangular matrix, and hence is nilpotent. Since we also know $z \in \mathfrak{h}$ we have $\text{ad}(z)$ is semisimple, hence $\text{ad}(z)$ is both semisimple and nilpotent, which implies it is zero, contradicting $z \neq 0$.

Given $\alpha(\mathfrak{h}_\alpha) \neq 0$, it is clear that there is a unique $h_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(h_\alpha) = 2$, indeed $h_\alpha = \frac{2}{\alpha(t_\alpha)}t_\alpha$. Next if $e_\alpha \in \mathfrak{g}_\alpha$ is nonzero, then using the nondegeneracy of κ and part (1) we may find an $f_\alpha \in \mathfrak{g}_{-\alpha}$ so that $\kappa(e_\alpha, f_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}$, and hence using part $i)$ we see $[e_\alpha, f_\alpha] = h_\alpha$. It is now easy to check that $\{e_\alpha, f_\alpha, h_\alpha\}$ span an copy of $\mathfrak{sl}_2$ in $\mathfrak{g}$ which establishes (4).

□

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Remark 1.2. A triple of elements $\{e, f, h\}$ in a Lie algebra $\mathfrak{g}$ which obey the relations of the standard generators of $\mathfrak{sl}_2$ (that is, $[e, f] = h, [h, e] = 2e, [h, f] = 2f$) is called an $\mathfrak{sl}_2$-*triple*.

Lemma 1.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra with Cartan decomposition $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. Then*

- *The root spaces $\mathfrak{g}_\alpha$ are one-dimensional.*
- *If $\alpha \in \Phi$ and $c\alpha \in \Phi$ for some $c \in \mathbb{Z}$ then $c = \pm 1$.*

Proof. Choose a nonzero vector $e_\alpha \in \mathfrak{g}_\alpha$. Then as above we may find an element $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha \in \mathfrak{h}$ (since κ restricted to $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is nondegenerate). Consider the subspace:

$$M = \mathsf{k}.e_\alpha \oplus \mathsf{k}.t_\alpha \oplus \bigoplus_{p<0} \mathfrak{g}_{p\alpha}.$$

(this is a finite direct sum as $\mathfrak{g}$ is finite-dimensional). Then since $\mathrm{ad}(e_\alpha)(e_\alpha) = 0$, and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathsf{k}.h_\alpha$, and $[e_\alpha, h_\alpha] = 2e_\alpha$, it is easy to see that M is stable under $e_\alpha, e_{-\alpha}$ and h_α. We commute the trace of h_α on M in two ways: on the one hand, it a commutator and so has trace zero. On the other hand it acts semisimply on each of the direct sums defining M, so that

$$\begin{aligned} 0 = \mathrm{tr}(h_\alpha) &= \alpha(h_\alpha) + \sum_{p<0} \dim(\mathfrak{g}_{p\alpha}).p\alpha(h_\alpha) \\ &= \alpha(h_\alpha)(1 - \sum_{p>0} p.\dim(\mathfrak{g}_{-p\alpha}). \end{aligned}$$

Since we know that $\alpha(h_\alpha) \neq 0$, the only way the above equality can hold is if $\dim(\mathfrak{g}_{p\alpha}) = 0$ for $p > 1$ and $\dim((\mathfrak{g}_{-\alpha}) = 1$. Since $-\alpha \in \Phi$ if and only if $\alpha \in \Phi$, this completes the proof. □

Remark 1.4. It follows immediately from Proposition 1.1 part iv) and Lemma 1.3 part i) that for any $\alpha \in \Phi$, the direct sum $\mathfrak{g}_\alpha \oplus \mathfrak{h}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$. We will denote this subalgebra as $\mathfrak{sl}_\alpha$. (Note $\mathfrak{sl}_\alpha = \mathfrak{sl}_\beta$ if and only if $\alpha = \pm\beta$.)

We can refine somewhat the structure of the Cartan decomposition we have already obtained, using the same techniques. Suppose that α, β are two roots in $\mathfrak{g}$ such that $\beta \neq k\alpha$ for $k \in \mathbb{Z}$. Then we may consider the roots which have the form $\alpha + k\beta$. Clearly, since $\mathfrak{g}$ is finite dimensional, there are integers $p, q > 0$ such that $\alpha + k\beta \in \Phi$ for each k with $-p \leq k \leq q$, but neither $-(p+1)\alpha$ nor $(q+1)\alpha$ are not in Φ. This set of roots is called the *α-string* through β.

Proposition 1.5. *Let $\beta - p\alpha, \dots, \beta + q\alpha$ be the α-string through β. Then we have*

$$\beta(h_\alpha) = \kappa(h_\alpha, t_\beta) = \frac{2\kappa(t_\alpha, t_\beta)}{\kappa(t_\alpha, t_\alpha)} = p - q.$$

In particular $\beta - \beta(h_\alpha).\alpha \in \Phi$. Moreover, if $\alpha \in \Phi$ and $c \in \mathsf{k}$ has $c\alpha \in \Phi$ then $c \in \{\pm 1\}$.

Proof. We consider the subspace $M = \bigoplus_{-p \leq k \leq q} \mathfrak{g}_{\alpha + k\beta}$. Pick $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $0 \neq [e_\alpha, e_{-\alpha}] = h_\alpha$ and so that $\{e_\alpha, e_{-\alpha}, h_\alpha\}$ form the standard generators of $\mathfrak{sl}_2$ as above. It is clear that $e_\alpha, h_\alpha, e_{-\alpha}$ preserve M, so we $\text{tr}_{|M}(h_\alpha) = 0$, and so, using the fact root spaces are 1-dimensional, we have the identity:

$$\sum_{-p\leq k\leq q} (\beta + k\alpha)(h_\alpha) = 0,$$

and so

$$\big(q(q+1)/2 - p(p+1)/2)\alpha(h_\alpha) + (p+q+1)\beta(h_\alpha) = 0,$$

and so since $p + q + 1 \neq 0$ and $\alpha(h_\alpha) = 2$, we obtain:

$$\beta(h_\alpha) = p - q.$$

as required. Since $\beta + (p-q)\alpha$ is certainly in the α-string through β it follows that $\beta + \beta(h_\alpha).\alpha \in \Phi$.

For the second part, since we know from the previous lemma that if $c \in \mathbb{Z}$ then $c \in \{\pm 1\}$, it suffices to consider the case where $c \in \mathsf{k}\backslash\mathbb{Z}$. But then we may apply the first part of the lemma to $\beta = c\alpha$ to find that $2c = c.\alpha(h_\alpha) = p - q$, that is, $c = \frac{1}{2}(p-q)$. Since $c \notin \mathbb{Z}$, the difference $p - q$ must be odd, and the α-string through $\beta = \frac{(p-q)}{2}\alpha$ has the form:

$$\frac{-(p+q)}{2}\alpha, \ldots, \frac{(p-q)}{2}\alpha, \ldots, \frac{(p+q)}{2}\alpha,$$

which clearly then contains $\frac{1}{2}\alpha$ so that $\frac{1}{2}\alpha \in \Phi$. But then we get a contradiction as $\alpha = 2(\frac{1}{2}\alpha)$. □

Remark 1.6. In fact, if $\alpha, \beta \in \Phi$ and $\alpha+\beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Indeed let $\mathfrak{sl}_2$ act on $\mathfrak{g}$ via the triple $\{e_\alpha, h_\alpha, e_{-\alpha}\}$ as in the proof above. Then the $\mathfrak{sl}_2$-representation $M = \bigoplus_{-p\leq k\leq q} \mathfrak{g}_{\alpha+k\beta}$ is easily seen to be an irreducible representation (because each h_α weight space is one-dimensional and the eigenvalues all have the same parity as $\beta(h_\alpha)$. Then the explicit description of the irreducible representations of $\mathfrak{sl}_2$ worked out in the problem sheet shows that e_α is injective except on the subspace $\mathfrak{g}_{\beta+q\alpha}$, and so in particular on $\mathfrak{g}_\beta$, hence $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ is nonzero and hence all of $\mathfrak{g}_{\alpha+\beta}$.