

LIE ALGEBRAS: LECTURE 12.

1. \mathfrak{h} AND INNER PRODUCT SPACES:

In this section we work over an algebraically closed field k of characteristic zero.

Recall that since $\kappa|_{\mathfrak{h}}$ is non-degenerate, it gives an isomorphism $\theta: \mathfrak{h}^* \rightarrow \mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$, we write t_λ for $\theta(\lambda)$, so that $\kappa(t_\lambda, h) = \lambda(h)$, ($\forall \lambda \in \mathfrak{h}^*, h \in \mathfrak{h}$). Recall that $\kappa(t_\alpha, t_\alpha) = \alpha(t_\alpha) \neq 0$, and $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$ is the unique element in $\mathfrak{h}_\alpha = k \cdot t_\alpha$ with $\alpha(h_\alpha) = 2$. We begin with a simple lemma.

Lemma 1.1. *Let $\alpha \in \Phi$ be a root. Then we have $\kappa(t_\alpha, t_\alpha) \cdot \kappa(h_\alpha, h_\alpha) = 4$. Moreover, if $\alpha, \beta \in \Phi$ then $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$ and $\kappa(t_\alpha, t_\beta) \in \mathbb{Q}$.*

Proof. The first equality follows immediately from the fact that $h_\alpha = 2t_\alpha / \kappa(t_\alpha, t_\alpha)$. The fact that $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$ follows from calculating $\kappa(h_\alpha, h_\beta)$ using our expression for the Killing form on \mathfrak{h} in terms of the Cartan decomposition. Indeed now that we know that root spaces are one-dimensional, this becomes:

$$\kappa(h_1, h_2) = \sum_{\gamma \in \Phi} \gamma(h_1)\gamma(h_2).$$

Applying this to $h_1 = h_\alpha$ and $h_2 = h_\beta$, and using the fact that $\gamma(h_\alpha)$ and $\gamma(h_\beta)$ are integers for any $\gamma \in \Phi$, we immediately see that $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$ as claimed. Finally, the first part of the lemma now immediately gives $\kappa(t_\alpha, t_\beta) \in \mathbb{Q}$. \square

Let $(-, -)$ denote the bilinear form on \mathfrak{h}^* which is obtained by identifying \mathfrak{h}^* with \mathfrak{h} : that is

$$(\lambda, \mu) = \kappa(t_\lambda, t_\mu).$$

Clearly it is a nondegenerate symmetric bilinear form, and via the previous Lemma, $(\alpha, \beta) = \kappa(t_\alpha, t_\beta) \in \mathbb{Q}$ for all $\alpha, \beta \in \Phi$.

Lemma 1.2. *The \mathbb{Q} -span of the roots Φ is a \mathbb{Q} -vector space of dimension $\dim_k(\mathfrak{h}^*)$.*

Proof. We know that Φ spans \mathfrak{h}^* , so we may pick a subset $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ which forms a k -basis of \mathfrak{h}^* . To prove the Lemma it is enough to show that every $\beta \in \Phi$ lies in the \mathbb{Q} -span of the $\{\alpha_i : 1 \leq i \leq l\}$. But now if we write $\beta = \sum_{j=1}^l c_j \alpha_j$ for $c_j \in k$, then we see that $(\alpha_i, \beta) = \sum_{j=1}^l (\alpha_i, \alpha_j) c_j$. But the matrix $C = (\alpha_i, \alpha_j)_{i,j}$ is invertible since $(-, -)$ is nondegenerate¹, and its entries are in \mathbb{Q} hence so are those of C^{-1} . But then we have $(c_i) = C^{-1}((\alpha_i, \beta))$, and the objects on the right-hand side all have \mathbb{Q} -entries, so we are done. \square

Let $\mathfrak{h}_{\mathbb{Q}}^*$ denote the \mathbb{Q} -span of the roots. Although you are perhaps more used to inner product spaces over \mathbb{R} or \mathbb{C} , the definition of a positive definite symmetric bilinear form makes perfectly good sense over \mathbb{Q} . We now show that $(-, -)$ is such an inner product on $\mathfrak{h}_{\mathbb{Q}}^*$.

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¹One way to see this is to note that C is the matrix of the linear map κ yields from \mathfrak{h}^* to \mathfrak{h} , with respect to the basis $\{\alpha_1, \dots, \alpha_l\}$ and its dual basis.

Lemma 1.3. *The form $(-, -)$ is positive definite on $\mathfrak{h}_{\mathbb{Q}}^*$.*

Proof. Note first that $(-, -)$ is \mathbb{Q} -valued on $\mathfrak{h}_{\mathbb{Q}}^*$, since $(\alpha, \beta) \in \mathbb{Q}$ for all roots α, β . Moreover $(-, -)$ is clearly a nondegenerate symmetric bilinear form on $\mathfrak{h}_{\mathbb{Q}}^*$, since it already is on \mathfrak{h}^* , thus we only need to show that it is positive definite, *i.e.* that if $\lambda \in \mathfrak{h}_{\mathbb{Q}}^*$ then $(\lambda, \lambda) \geq 0$ with equality if and only if $\lambda = 0$.

Using the formula for κ on \mathfrak{h} given by the root space decomposition we see that, for $\lambda \in \mathfrak{h}_{\mathbb{Q}}^*$,

$$(\lambda, \lambda) = \kappa(t_{\lambda}, t_{\lambda}) = \sum_{\alpha \in \Phi} \alpha(t_{\lambda})^2 = \sum_{\alpha \in \Phi} (\alpha, \lambda)^2 \geq 0.$$

It follows that $(\lambda, \lambda) \in \mathbb{Q}_{\geq 0}$, and $(\lambda, \lambda) = 0$ if and only if $(\alpha, \lambda) = 0$ for all $\alpha \in \Phi$. But then since Φ spans \mathfrak{h}^* and $(-, -)$ is nondegenerate on \mathfrak{h}^* this implies $\lambda = 0$ as required. □

Remark 1.4. Note that the Cartan decomposition has given us a lot of information about the Lie algebra \mathfrak{g} , much of which we capture just remembering the set of roots Φ in $\mathfrak{h}_{\mathbb{Q}}^*$ viewed as an \mathbb{Q} -inner product space (via the restriction of the Killing form). In fact, we will see that by examining this data more closely we can completely classify them and hence all semisimple Lie algebras up to isomorphism.

2. REPRESENTATIONS OF SEMISIMPLE LIE ALGEBRAS AND WEYL'S THEOREM

In this section we will assume, unless explicitly stated otherwise, that our field is algebraically closed of characteristic zero.

In this section we study the representations of a semisimple Lie algebra. Our goal is to show, just as for representations of a finite group over \mathbb{C} , that every representation is a direct sum of irreducibles. If (V, ρ) and (W, σ) are representations of \mathfrak{g} , then a linear map $\phi: V \rightarrow W$ is a *homomorphism* of representations if

$$\phi(\rho(x)(v)) = \sigma(x)(\phi(v)) \quad \forall x \in \mathfrak{g}, v \in V.$$

We will write $\text{Hom}_{\mathfrak{g}}(V, W)$ for the space of homomorphisms from V to W . To make the notation less cluttered, where there is no danger for confusion we will often suppress the notation for the map $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, so that, for example, the condition for ϕ to be a homomorphism will be written simply as $\phi(x(v)) = x(\phi(v))$.

Definition 2.1. If \mathfrak{g} is any Lie algebra, the one-dimensional representation² given by the zero map $\mathfrak{g} \rightarrow \mathfrak{gl}_1(k)$ is called the *trivial* representation of \mathfrak{g} . If (V, ρ) is any representation of \mathfrak{g} , then we set

$$V^{\mathfrak{g}} = \{v \in V : x(v) = 0, \forall x \in \mathfrak{g}\}.$$

The subrepresentation $V^{\mathfrak{g}}$ is known as the *invariants*, or invariant vectors, in V .

Note that if V, W are \mathfrak{g} -representations then the invariant vectors in $\text{Hom}(V, W)$ are exactly the \mathfrak{g} -homomorphisms, that is, we have

$$\text{Hom}(V, W)^{\mathfrak{g}} = \text{Hom}_{\mathfrak{g}}(V, W).$$

The following Lemma is proved in the exercises (the proof is identical to the corresponding results for representations of groups). For it we suppose that our field k is algebraically closed (but not necessarily of characteristic zero).

²Really I should say isomorphism class of one-dimensional representations.

Lemma 2.2. (*Schur's Lemma*). Let (V, ρ) be an irreducible representation of a Lie algebra \mathfrak{g} . Then if $\phi: V \rightarrow V$ is a homomorphism of representations, then $\phi = \lambda \cdot \text{id}_V$ for some $\lambda \in \mathfrak{k}$.

The next result holds when \mathfrak{k} is algebraically closed of characteristic zero. Recall that in the representation theory of finite groups, when the field is of characteristic zero, every representation is a direct sum of irreducible representations – representations of this form are said to be *completely reducible*. For semisimple Lie algebras, the same result holds (at least when we assume \mathfrak{k} is algebraically closed), but the proof is somewhat more subtle.

Theorem 2.3. (*Weyl's Theorem*). Let V be a finite dimensional representation of a semisimple Lie algebra over an algebraically closed field \mathfrak{k} of characteristic zero. Then V is completely reducible.

We will prove the theorem by induction on the dimension of V (the case when V is 1-dimensional being immediate). If V is irreducible, then we are done immediately. Otherwise, we may find a proper subrepresentation U of V , and hence we get an exact sequence

$$0 \longrightarrow U \xrightarrow{i} V \xrightarrow{q} V/U \longrightarrow 0$$

Now suppose there is a right-inverse to the quotient map q , i.e. a map $s: V/U \rightarrow V$ of \mathfrak{g} -representations such that $q \circ s = \text{id}_{V/U}$. Then its image, T say, is a complement to U . Indeed clearly $\dim(V) = \dim(U) + \dim(T)$, and if $v \in U \cap T$ we have $v = s(w)$ for some $w \in V/U$ so that $0 = q(v) = q(s(w)) = w$, and hence $v = s(0) = 0$. Thus $V = U \oplus T$ is a direct sum of subrepresentations, and as $\dim(U) < \dim(V)$ and $\dim(T) = \dim(V/U) < \dim(V)$, by induction T and U are a direct sum of irreducible representations and the proof is complete.

Thus we see that the proof of Weyl's theorem reduces to showing that if $q: V \rightarrow W$ is a surjective map of \mathfrak{g} -representations, then q has a right inverse $s: W \rightarrow V$. (Such a map s is also often called a *section* of q , and we say that s *splits* the map q .)

Thus to complete the proof of Weyl's theorem, we must show that every surjective map of representations splits. In order to do this, we construct certain endomorphisms of a representation known as Casimir operators. For these we will need the following definition.

2.1. Casimir operators. Let \mathfrak{g} as before be a semisimple Lie algebra and let (V, ρ) be a representation of \mathfrak{g} . We may define a symmetric bilinear form on \mathfrak{g} generalizing the Killing form by setting

$$t_V(x, y) = \text{tr}_V(\rho(x)\rho(y)).$$

This is known as a *trace form* on \mathfrak{g} , and the Killing form is just the trace form associated to the adjoint representation. Exactly the same argument which showed that the Killing form is invariant shows that any trace form is invariant.