LIE ALGEBRAS: LECTURE 2.

1. HOMOMORPHISMS AND IDEALS

We have already introduced the notion of a subalgebra of a Lie algebra in the examples above, but there are other standard constructions familiar from rings which make sense for Lie algebras. A *homomorphism* of Lie algebras $(\mathfrak{g}, [.,.]_{\mathfrak{g}})$ and $(\mathfrak{g}', [.,.]_{\mathfrak{g}'})$ is a k-linear map $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ which respects the Lie brackets, that is:

$$\phi([a,b]_{\mathfrak{g}}) = [\phi(a), \phi(b)]_{\mathfrak{g}'} \quad \forall a, b \in \mathfrak{g}.$$

An *ideal* in a Lie algebra $\mathfrak{g}$ is a subspace I such that for all $a \in \mathfrak{g}$ and $x \in I$ we have $[a,x]_{\mathfrak{g}} \in I$. It is easy to check that if $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism, then $\ker(\phi) = \{a \in \mathfrak{g} : \phi(a) = 0\}$ is an ideal of $\mathfrak{g}$. Conversely, if I is an ideal of $\mathfrak{g}$ then it is easy to check that the quotient space $\mathfrak{g}/I$ inherits the structure of a Lie algebra, and the canonical quotient map $q\colon \mathfrak{g} \to \mathfrak{g}/I$ is a Lie algebra homomorphism with kernel I.

Remark 1.1. Note that because the Lie bracket is skew-symmetric, we do not need to consider notions of left, right and two-sided ideals, as they will all coincide. If a nontrivial Lie algebra has no nontrivial ideals we say it is *simple*.

Just as for groups and rings, we have the normal stable of isomorphism theorems, and the proofs are identical.

Theorem 1.2. (1) *Let $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ be a homomorphism of Lie algebras. The subspace $\phi(\mathfrak{g}) = im(\phi)$ is a subalgebra of $\mathfrak{g}'$ and ϕ induces an isomorphism $\bar{\phi}\colon \mathfrak{g}/ker(\phi) \to im(\phi)$.*

(2) *If $I \subset J \subset \mathfrak{g}$ are ideals of $\mathfrak{g}$ then we have:*

$$(\mathfrak{g}/J)/(I/J) \cong \mathfrak{g}/J$$

(3) *If I, J are ideals of $\mathfrak{g}$ then we have*

$$(I + J)/J \cong I/(I \cap J).$$

2. REPRESENTATIONS OF LIE ALGEBRAS

Just as for finite groups (or indeed groups in general) one way of studying Lie algebras is to try and understand how they can act on other objects. For Lie algebras, we will use actions on linear spaces, or in other words, "representations". Formally we make the following definition.

Definition 2.1. A *representation* of a Lie algebra $\mathfrak{g}$ is a vector space V equipped with a homomorphism of Lie algebras $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. In other words, ρ is a linear map such that

$$\rho([x,y]) = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x)$$

where $\circ$ denotes composition of linear maps. We may also refer to a representation of $\mathfrak{g}$ as a $\mathfrak{g}$-module. A representation is *faithful* if $\ker(\rho) = 0$. When there is no

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danger of confusion we will normally suppress ρ in our notation, and write $x(v)$ rather than $\rho(x)(v)$, for $x \in \mathfrak{g}, v \in V$.

We will study representation of various classes of Lie algebras in this course, but for the moment we will just give some basic examples.

Example 2.2. (1) If $\mathfrak{g} = \mathfrak{gl}(V)$ for V a vector space, then the identity map $\mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{gl}(V)$ on V, which is known as the vector representation. Clearly any subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ also inherits V as a representation, where then the map ρ is just the inclusion map.

(2) Given an arbitrary Lie algebra $\mathfrak{g}$, there is a natural representation ad of $\mathfrak{g}$ on $\mathfrak{g}$ itself known as the adjoint representation. The homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(\mathfrak{g})$ is given by

$$\text{ad}(x)(y) = [x, y], \quad \forall x, y \in \mathfrak{g}.$$

Indeed the fact that this is a representation is just a rephrasing[1] of the Jacobi identity. Note that while the vector representation is clearly faithful, in general the adjoint representation is not. Indeed the kernel is known as the *centre* of $\mathfrak{g}$:

$$\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, y] = 0, \forall y \in \mathfrak{g}\}.$$

Note that if $x \in \mathfrak{z}(\mathfrak{g})$ then for any representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ the endomorphism $\rho(x)$ commutes with all the elements $\rho(y) \in \text{End}(V)$ for all $y \in \mathfrak{g}$.

(3) If $\mathfrak{g}$ is any Lie algebra, then the zero map $\mathfrak{g} \to \mathfrak{gl}_1$ is a Lie algebra homomorphism. The corresponding representation is called the *trivial representation*. It is the Lie algebra analogue of the trivial representation for a group (which send every group element to the identity).

(4) If (V, ρ) is a representation of $\mathfrak{g}$, we say that a subspace $U < V$ is a *subrepresentation* if $\phi(x)(U) \subseteq U$ for all $x \in \mathfrak{g}$. It follows immediately that ϕ restricts to give a homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(U)$, hence $(U, \phi_{|U})$ is again a representation of $\mathfrak{g}$. Note also that if $\{V_i : i \in I\}$ are a collection of invariant subspaces, their sum $\sum_{i \in I} V_i$ is clearly also invariant, and so again a subrepresentation.

There are a number of standard ways of constructing new representations from old, all of which have their analogue for group representations. For example, recall that if V is a k-vector space, and U is a subspace, then we may form the quotient vector space V/U. If $\phi\colon V \to V$ is an endomorphism of V which preserves U, that is if $\phi(U) \subseteq U$, then there is an induced map $\bar{\phi}\colon V/U \to V/U$. Applying this to representations of a Lie algebra $\mathfrak{g}$ we see that if V is a representation of $\mathfrak{g}$ and U is a subrepresentation we may always form the *quotient representation* V/U. Next, if (V, ρ) and (W, σ) are representations of $\mathfrak{g}$, then clearly $V \oplus W$ the direct sum of V and W becomes a $\mathfrak{g}$-representation via the obvious homomorphism $\rho \oplus \sigma$. More interestingly, the vector space $\text{Hom}(V, W)$ of linear maps from V to W has the structure of a $\mathfrak{g}$-representation via

$$x(\phi) = \sigma(x) \circ \phi - \phi \circ \rho(x), \quad \forall x \in \mathfrak{g}, \phi \in \text{Hom}(V, W).$$

[1] Check this! It's also (for some people) a useful way of remembering what the Jacobi identity says.

It is straight-forward to check that this gives a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(\mathrm{Hom}(V,W))$, since[2]:

$$\begin{aligned}
x(y(\phi)) - y(x(\phi)) &= x(\sigma(y)\phi - \phi\rho(y)) - y(\sigma(x)\phi - \phi\rho(x)) \\
&= x(\sigma(y)\circ\phi) - x(\phi\rho(y)) - y(\sigma(x)\phi) + y(\phi\rho(x)) \\
&= \sigma(x)\sigma(y)\phi - \sigma(y)\phi\rho(x) - \sigma(x)\phi\rho(y) + \phi\rho(y)\rho(x) \\
&\quad - \sigma(y)\sigma(x)\phi + \sigma(x)\phi\rho(y) + \sigma(y)\phi\rho(x) - \phi\rho(x)\rho(y) \\
&= (\sigma(x)\sigma(y) - \sigma(y)\sigma(x))\phi - \phi(\rho(x)\rho(y) - \rho(y)\rho(x)) \\
&= \sigma[x,y]\phi - \phi\rho([x,y]) \\
&= [x,y](\phi).
\end{aligned}$$

An important special case of this is where $W = \mathsf{k}$ is the trivial representation (as above, so that the map $\sigma\colon \mathfrak{g} \to \mathfrak{gl}(\mathsf{k})$ is the zero map). This allows us to give $V^* = \mathrm{Hom}(V,\mathsf{k})$, the dual space of V, a natural structure of $\mathfrak{g}$-representation where (since $\sigma = 0$) the action of $x \in \mathfrak{g}$ on $f \in V^*$ is given by $\rho^*\colon \mathfrak{g} \to \mathfrak{gl}(V^*)$ where

$$\rho^*(x)(f) = -f \circ (\rho(x)) \quad (f \in V^*).$$

If $\alpha\colon V \to V$ is any linear map, recall that the transpose map $\alpha^t\colon V^* \to V^*$ is defined by $\alpha^t(f) = f \circ \alpha$, thus our definition of the action of $x \in \mathfrak{g}$ on V^* is just $-\rho(x)^t$. This makes it clear that the action of $\mathfrak{g}$ on V^* is compatible with the standard constructions on dual spaces, *e.g.* if U is a subrepresentation of V, the U^0 the annihilator of U will be a subrepresentation of V^*, and moreover, the natural isomorphism of V with V^{**} is an isomorphism of $\mathfrak{g}$-representations.

We end this section with some terminology which will be useful later.

Definition 2.3. A representation is said to be *irreducible* if it has no proper non-zero subrepresentations, and it is said to be *completely reducible* if it is isomorphic to a direct sum of irreducible representations.

Example 2.4. Giving a representation of $\mathfrak{gl}_1$ is equivalent to giving a vector space equipped with a linear map. Indeed as a vector space $\mathfrak{gl}_1 = \mathsf{k}$, hence if (V,ρ) is a representation of $\mathfrak{gl}_1$ we obtain a linear endomorphism of V by taking $\rho(1)$. Since every other element of $\mathfrak{gl}_1$ is a scalar multiple of 1 this completely determines the representation, and this correspondence is clearly reversible.

If we assume k is algebraically closed, then you know the classification of linear endomorphisms is given by the Jordan canonical form. From this you can see that the only irreducible representations of $\mathfrak{gl}_1$ are the one-dimensional ones, while indecomposable representations correspond to linear maps with a single Jordan block.

[2] This is basically the same calculation as the one which shows the commutator satisfies the Jacobi identity, but it's a bit complicated to sort out how to say it that way, in the same way that the action of a group G on $\mathrm{Hom}(V,W)$ for G-representations V and W is basically by conjugation, but it's slightly tricky to say that precisely.