LIE ALGEBRAS: LECTURE 7.

1. THE CARTAN DECOMPOSITION

In this lecture we work over an algebraically closed field k *of characteristic zero.*

Our study of the representation theory of nilpotent Lie algebras can now be used to study the structure of an arbitrary Lie algebra. Indeed, if $\mathfrak{g}$ is any Lie algebra, we have shown that it contains a Cartan subalgebra $\mathfrak{h}$, and the restriction of the adjoint action makes $\mathfrak{g}$ into an $\mathfrak{h}$-representation. As such it decomposes into a direct sum

$$\mathfrak{g} = \bigoplus_{\lambda \in (\mathfrak{h}/D\mathfrak{h})^*} \mathfrak{g}_\lambda.$$

The next Lemma establishes some basic properties of this decomposition.

Lemma 1.1. *Let* $\mathfrak{g}, \mathfrak{h}$ *be as above. Then* $\mathfrak{h} = \mathfrak{g}_0$*. Moreover, if* λ, μ *are one-dimensional representations, then*

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}.$$

Proof. Clearly $\mathfrak{h} \subseteq \mathfrak{g}_0$, since $\mathfrak{h}$ is nilpotent. Consider $\mathfrak{g}_0/\mathfrak{h}$ as an $\mathfrak{h}$-representation. By Lie's theorem it contains a one-dimensional submodule L, and since $\mathfrak{h}$ acts nilpotently on $\mathfrak{g}_0$ be definition, $\mathfrak{h}$ acts as zero on L. But then the preimage of L in $\mathfrak{g}_0$ normalizes $\mathfrak{h}$ which contradicts the assumption that $\mathfrak{h}$ is a Cartan subalgebra.

The second part follows from the identity:

$$(\mathrm{ad}(x) - (\lambda(x)+\mu(x))1)^n[y,z] = \sum_{i=0}^{n} \binom{n}{i} [(\mathrm{ad}(x) - \lambda(x).1)^i(y), (\mathrm{ad}(x) - \mu(x).1)^{n-i}(z)].$$

which is straight-forward to establish by induction on n. □

Definition 1.2. By the previous Lemma, if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ then $\mathfrak{g}$ decomposes into a direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \neq 0} \mathfrak{g}_\lambda.$$

This is known as the *Cartan decomposition* of $\mathfrak{g}$. The set Φ of non-zero $\lambda \in (\mathfrak{g}/D\mathfrak{g})^*$ for which the subspace $\mathfrak{g}_\lambda$ is non-zero is called the set of *roots* of $\mathfrak{g}$, and the subspaces $\mathfrak{g}_\lambda$ are known as the *root spaces* of $\mathfrak{g}$.

Although we will not prove it in this course, any two Cartan subalgebras of $\mathfrak{g}$ are conjugate by an automorphism[1] of $\mathfrak{g}$, that is, given any two Cartan subalgebras $\mathfrak{h}_1, \mathfrak{h}_2$ there is an isomorphism $\alpha\colon \mathfrak{g} \to \mathfrak{g}$ such that $\alpha(\mathfrak{h}_1) = \mathfrak{h}_2$, so that the decomposition of $\mathfrak{g}$ is unique up to automorphisms.

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[1]In fact, they are even conjugate by what is known as an *inner automorphism*.

2. The Killing form

Any Lie algebra has a natural symmetric bilinear form which will play an important role in the rest of the course. See Appendix 2 for a brief review of the basic theory of symmetric bilinear forms[2].

Definition 2.1. If $\mathfrak{g}$ is a Lie algebra we may define a bilinear form $\kappa\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ on $\mathfrak{g}$, known as the *Killing form* by setting

$$\kappa(x,y) = \text{tr}(\text{ad}(x)\text{ad}(y)).$$

Clearly κ is a symmetric bilinear form (*i.e.* $\kappa(x,y)=\kappa(yx)$) because $\text{tr}(ab)=\text{tr}(ba)$, but it also satisfies another important property: it is *invariant*, that is:

$$\kappa([x,y],z) = \kappa(x,[y,z]).$$

To see this, note that

$$\begin{aligned}\kappa([x,y],z) &= \text{tr}(\text{ad}([x,y])\text{ad}(z))\\ &= \text{tr}(\text{ad}(x)\text{ad}(y)\text{ad}(z)) - \text{tr}(\text{ad}(y)\text{ad}(x)\text{ad}(z))\\ &= \text{tr}(\text{ad}(x)\text{ad}(y)\text{ad}(z)) - \text{tr}(\text{ad}(x)\text{ad}(z)\text{ad}(y))\\ &= \text{tr}(\text{ad}(x),\text{ad}([y,z]))\\ &= \kappa(x,[y,z]).\end{aligned}$$

where in the second line we used the fact that $\text{tr}(ab)=\text{tr}(ba)$. Note that if $\mathfrak{a}\subseteq\mathfrak{g}$ is a subalgebra, the Killing form of $\mathfrak{a}$ is not necessarily equal to the restriction of that of $\mathfrak{g}$. We will write $\kappa^{\mathfrak{g}}$ when it is not clear from context which Lie algebra is concerned.

If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, then in fact the Killing form is unambiguous, as the following Lemma shows.

Lemma 2.2. *Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. The Killing form $\kappa^{\mathfrak{a}}$ of $\mathfrak{a}$ is given by the restriction of the Killing form $\kappa^{\mathfrak{g}}$ on $\mathfrak{g}$, that is:*

$$\kappa^{\mathfrak{g}}_{|\mathfrak{a}} = \kappa^{\mathfrak{a}}.$$

Proof. Then if $a\in\mathfrak{a}$ we have $\text{ad}(a)(\mathfrak{g})\subseteq\mathfrak{a}$, thus the same will be true for the composition $\text{ad}(a_1)\text{ad}(a_2)$ for any $a_1,a_2\in\mathfrak{a}$. Thus if we pick a vector space complement W to $\mathfrak{a}$ in $\mathfrak{g}$, the matrix of $\text{ad}(a_1)\text{ad}(a_2)$ with respect to a basis compatible with the subspaces $\mathfrak{a}$ and W will be of the form

$$\begin{pmatrix} A & B\\ 0 & 0.\end{pmatrix}$$

where $A\in\text{End}(\mathfrak{a})$ and $B\in\text{Hom}_{\mathsf{k}}(\mathfrak{a},W)$. Then clearly $\text{tr}(\text{ad}(a_1)\text{ad}(a_2))=\text{tr}(A)$. Since A is clearly given by $\text{ad}(a_1)_{|\mathfrak{a}}\text{ad}(a_2)_{|\mathfrak{a}}$, we are done. □

The Killing form also allows us to produce ideals: If $\mathfrak{a}$ denotes a subspace of $\mathfrak{g}$, then we will write $\mathfrak{a}^{\perp}$ for the subspace

$$\{x\in\mathfrak{g} : \kappa(x,y)=0, \forall y\in\mathfrak{a}\}.$$

Lemma 2.3. *Let $\mathfrak{g}$ be Lie algebra and let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. Then $\mathfrak{a}^{\perp}$ is also an ideal of $\mathfrak{g}$.*

[2]Part A Algebra focused more on positive definite and Hermitian forms, but there is a perfectly good theory of general symmetric bilinear forms.

Proof. Suppose that $x \in \mathfrak{g}$ and $z \in \mathfrak{a}^\perp$. We need to show that $[x,z] \in \mathfrak{a}^\perp$. But if $y \in \mathfrak{a}$ we have

$$\kappa([x,z],y) = -\kappa([z,x],y) = -\kappa(z,[x,y]) = 0,$$

since $[x,y] \in \mathfrak{a}$ since $\mathfrak{a}$ is an ideal. Hence $[x,z] \in \mathfrak{a}^\perp$ as required. □

3. Cartan's Criterion

We now wish to show how the Killing form yields a criterion for determining whether a Lie algebra is solvable or not. For this we need a couple of technical preliminaries.

Lemma 3.1. *Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra, so that the Cartan decomposition of $\mathfrak{g}$ is $\mathfrak{h} \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}_\lambda$. Let α, β be roots. Then there is an $r \in \mathbb{Q}$ such that the restriction of β to $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is equal to $r\alpha$.*

Proof. The set of roots Φ is finite, thus there are positive integers p, q such that $\beta + t\alpha \in \Phi$ for all integers t with $-p \leq t \leq q$ but $\beta - (p+1)\alpha \notin \Phi$ and $\beta + (q+1)\alpha \notin \Phi$. Let $M = \bigoplus_{-p \leq t \leq q} \mathfrak{g}_{\beta + t\alpha}$. If $z \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is of the form $[x,y]$ where $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ then since $\text{ad}(x)(\mathfrak{g}_{\beta+q\alpha}) \subseteq \mathfrak{g}_{\beta+(q+1)\alpha} = \{0\}$, we see that $\text{ad}(x)$ and $\text{ad}(y)$ preserve M. Thus the action of $\text{ad}(z)$ on M is the commutator of the action of $\text{ad}(x)$ and $\text{ad}(y)$ on M, and so $\text{tr}(\text{ad}(z), M) = 0$. On the other hand, we may also compute the trace of $\text{ad}(z)$ on M directly:

$$\begin{aligned} 0 &= \text{tr}(\text{ad}(z), M) \\ &= \sum_{-p \leq t \leq q} \text{tr}(\text{ad}(z), \mathfrak{g}_{\beta+t\alpha}) \\ &= \sum_{-p \leq t \leq q} (\beta(z) + t\alpha(z)) \dim(\mathfrak{g}_{\beta+t\alpha}). \end{aligned}$$

since any $h \in \mathfrak{h}$ acts on a root space $\mathfrak{g}_\lambda$ with unique eigenvalue $\lambda(h)$. Rearranging the above equation gives $\beta(z) = r\alpha(z)$ for some $r \in \mathbb{Q}$ as required (where the denominator is a sum of dimensions of subspaces which are not all zero, and hence is nonzero). □