LIE ALGEBRAS: LECTURE 9.

1. SIMPLE AND SEMISIMPLE LIE ALGEBRAS

Definition 1.1. We say that a Lie algebra is *simple* if it is non-Abelian and has no nontrivial proper ideal. We now show that this notion is closed related to our notion of a semisimple Lie algebra.

Proposition 1.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and let I be an ideal of $\mathfrak{g}$. Then $\mathfrak{g} = I \oplus I^{\perp}$.*

Proof. Since $\mathfrak{g}$ is semisimple, the Killing form is nondegenerate, hence (see for example notes on the course webpage) we have

$$\dim(I) + \dim(I^{\perp}) = \dim(\mathfrak{g}). \tag{1.1}$$

Now consider $I \cap I^{\perp}$. The Killing form of $\mathfrak{g}$ vanishes identically on $I \cap I^{\perp}$ by definition, and since it is an ideal, the Killing form of $I \cap I^{\perp}$ is just the restriction of the Killing form of $\mathfrak{g}$. It follows from Cartan's Criterion that $I \cap I^{\perp}$ is solvable, and hence since $\mathfrak{g}$ is semisimple we must have $I \cap I^{\perp} = 0$. But then by Equation (1.1) we must have $\mathfrak{g} = I \oplus I^{\perp}$ as required (note that this is a direct sum of Lie algebras, since $[I, I^{\perp}] \subset I \cap I^{\perp}$). □

Proposition 1.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra.*

1. *Any ideal and any quotient of $\mathfrak{g}$ is semisimple.*
2. *Then there exist ideals $\mathfrak{g}_1, \mathfrak{g}_2, \dots \mathfrak{g}_k \subseteq \mathfrak{g}$ which are simple Lie algebras and for which the natural map:*

$$\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_k \to \mathfrak{g},$$

is an isomorphism. Moreover, any simple ideal $\mathfrak{a} \in \mathfrak{g}$ is equal to some $\mathfrak{g}_i$ $(1 \leq i \leq k)$. In particular the decomposition above is unique up to reordering, and $\mathfrak{g} = D\mathfrak{g}$.

Proof. For the first part, if I is an ideal of $\mathfrak{g}$, by the previous Proposition we have $\mathfrak{g} = I \oplus I^{\perp}$, so that the Killing form of $\mathfrak{g}$ restricted to I is nondegenerate. Since this is just the Killing form of I, Cartan's criterion shows that I is semisimple. Moreover, clearly $\mathfrak{g}/I \cong I^{\perp}$ so that any quotient of $\mathfrak{g}$ is isomorphic to an ideal of $\mathfrak{g}$ and hence is also semisimple.

For the second part we use induction on the dimension of $\mathfrak{g}$. Let $\mathfrak{a}$ be a minimal non-zero ideal in $\mathfrak{g}$. If $\mathfrak{a} = \mathfrak{g}$ then $\mathfrak{g}$ is simple, so we are done. Otherwise, we have $\dim(\mathfrak{a}) < \dim(\mathfrak{g})$. Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$, and by induction $\mathfrak{a}^{\perp}$ is a direct sum of simple ideals, and hence clearly $\mathfrak{g}$ is also.

To show the moreover part, suppose that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_k$ is a decomposition as above and $\mathfrak{a}$ is a simple ideal of $\mathfrak{g}$. Now as $\mathfrak{z}(\mathfrak{g}) = \{0\}$, we must have $0 \neq [\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$, and hence by simplicity of $\mathfrak{a}$ it follows that $[\mathfrak{g}, \mathfrak{a}] = \mathfrak{a}$. But then we have

$$\mathfrak{a} = [\mathfrak{g}, \mathfrak{a}] = [\bigoplus_{i=1}^{k} \mathfrak{g}_i, \mathfrak{a}] = [\mathfrak{g}_1, \mathfrak{a}] \oplus [\mathfrak{g}_2, \mathfrak{a}] \oplus \dots \oplus [\mathfrak{g}_k, \mathfrak{a}],$$

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(the ideals $[\mathfrak{g}_i, \mathfrak{a}]$ are contained in $\mathfrak{g}_i$ so the last sum remains direct). But $\mathfrak{a}$ is simple, so direct sum decomposition must have exactly one nonzero summand and we have $\mathfrak{a} = [\mathfrak{g}_i, \mathfrak{a}]$ for some i ($1 \leq i \leq k$). Finally, using the simplicity of $\mathfrak{g}_i$ we see that $\mathfrak{a} = [\mathfrak{g}_i, \mathfrak{a}] = \mathfrak{g}_i$ as required. To see that $\mathfrak{g} = D\mathfrak{g}$ note that it is now enough to check it for simple Lie algebras, where it is clear[1].

□

2. The Jordan Decomposition

If V is a vector space and $x \in \mathrm{End}(V)$, then we have the natural direct sum decomposition of V into the generalized eigenspaces of x. This can be viewed as giving a decomposition of the endomorphism x in a semisimple (or diagonalisable) and nilpotent part, as the next Lemmas show.

Lemma 2.1. *If $x, y \in End(V)$ are commuting linear maps then if both are nilpotent, so is $x + y$, and similarly if both are semisimple, so is $x + y$.*

Proof. For semisimple linear maps this follows from the fact that if s is a semisimple linear map, its restriction to any invariant subspace is again semisimple. For nilpotent linear maps it follows because

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

so that if n is large enough (*e.g.* $n \geq 2\dim(V)$) each of these terms will be zero (since x and y are nilpotent). □

Proposition 2.2. *Let V be a finite dimensional vector space $x \in End(V)$. Then we may write $x = x_s + x_n$ where x_s where x_s is semisimple and x_n is nilpotent, and x_s and x_n commute, i.e. $[x_s, x_n] = 0$. Moreover, this decomposition is unique, and if U is a subspace of V preserved by x, it is also preserved by x_s, x_n*

Proof. Let $V = \bigoplus_{\lambda \in \mathsf{k}} V_\lambda$ be the generalised eigenspace decomposition of V, and let $p_\lambda \colon V \to V_\lambda$ be the projection with with kernel $\bigoplus_{\mu \neq \lambda} V_\mu$. If we set x_s to be $\sum_\lambda \lambda.p_\lambda$, clearly x_s and x commute, and their difference $x_n = x - x_s$ is nilpotent. This establishes the existence of the Jordan decomposition.

To see the uniqueness, suppose that $x = s + n$ is another such decomposition. Now since s commutes with x, it must preserve the generalised eigenspaces of x, and so, since x_s is just a scalar on each V_λ, clearly s commutes with x_s. It follows s and n both commute with x_s and x_n. But then by Lemma 2.1 $x_s - s$ and $n - x_n$ are semisimple and nilpotent respectively. Since $s + n = x_s + x_n$ they are equal, and the only endomorphism which is both semisimple and nilpotent is zero, thus $s = x_s$ and $n = x_n$ as required.

Finally, to see that x_s and x_n preserve any subspace U which is preserved by x, note that if $U = \bigoplus_{\lambda \in \mathsf{k}} U_\lambda$ is the decomposition of U into generalised eigenspaces of x, then clearly $U_\lambda \subseteq V_\lambda$, ($\forall \lambda \in \mathsf{k}$) and since x_s is a scalar on V_λ it certainly preserves U_λ, and hence all of U. As $x_n = x - x_s$ clearly x_n also preserves U.

□

[1] This is one reason for insisting simple Lie algebras are nonabelian.

Lemma 2.3. *Let V be a vector space and $x \in End(V)$. If x is semisimple then*

$$ad(x)\colon End(V) \to End(V)$$

is also semisimple, and similarly if x is nilpotent. In particular, if $x = x_s + x_n$ is the Jordan decomposition of x, then $ad(x) = ad(x_s) + ad(x_n)$ is the Jordan decomposition of $ad(x)$. In other words, $ad(x)_s = ad(x_s)$ and $ad(x)_n = ad(x_n)$.

Proof. If x is nilpotent, then $\mathrm{ad}(x) = \lambda_x - \rho_x$ where λ_x and ρ_x denote left and right multiplication by x. Since λ_x and $-\rho_x$ are clearly nilpotent if x is, and evidently commute, $\mathrm{ad}(x)$ is nilpotent by Lemma 2.1. □

We now return to Lie algebras. The above linear algebra allows us to define an "abstract" Jordan decomposition for the elements of any Lie algebra (over an algebraically closed field).

Definition 2.4. Suppose that $\mathfrak{g}$ is a Lie algebra and $x \in \mathfrak{g}$. The endomorphism $\mathrm{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ has a unique Jordan decomposition $\mathrm{ad}(x) = \mathrm{ad}(x)_s + \mathrm{ad}(x)_n$ in $\mathfrak{gl}(\mathfrak{g})$. If we may write $x = s + n$ where $s, n \in \mathfrak{g}$ are such that $\mathrm{ad}(s) = \mathrm{ad}(x)_s$ and $\mathrm{ad}(n) = \mathrm{ad}(x)_n$, we say the Lie algebra elements s, n are an *abstract Jordan decomposition* of x.

For an arbitrary Lie algebra it is not automatic that the elements s, n exist or indeed are well-defined. For example, if $\mathfrak{g}$ has a nontrivial centre $\mathfrak{z}(\mathfrak{g})$, then the adjoint representation is not faithful. Note however that at least if $\mathfrak{g} = \mathfrak{gl}(V)$ for some vector space V, then Lemma 2.3 shows that the abstract Jordan decomposition for an element $x \in \mathfrak{gl}(V)$ is just the naive one (*i.e.* the one for x thought of as a linear map from V to itself). (Although of course for $\mathfrak{gl}(V)$, the adjoint representation has a kernel, and hence the abstract Jordan decomposition in this case is not unique.)

3. The abstract Jordan decomposition

Unless explicitly stated to the contrary, in this section we work over a field k *which is algebraically closed of characteristic zero.*

3.1. **Derivations of semisimple Lie algebras.** Let $\mathfrak{g}$ be a Lie algebra. Then $\mathrm{Der}_{\mathsf{k}}(\mathfrak{g})$ the Lie algebra of k-derivations of $\mathfrak{g}$ is a Lie algebra, which we may view as a subalgebra of the Lie algebra $\mathfrak{gl}(\mathfrak{g})$. The map $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is in fact a Lie algebra homomorphism from $\mathfrak{g}$ into $\mathrm{Der}_{\mathsf{k}}(\mathfrak{g})$. Its image is denoted $\mathrm{Inn}_{\mathsf{k}}(\mathfrak{g})$. We will show that Jordan decompositions exist and are unique for semisimple Lie algebras by showing that they always exist for the *a priori* larger Lie algebra $\mathrm{Der}_{\mathsf{k}}(\mathfrak{g})$, and also that for semisimple Lie algebras $\mathrm{Der}_{\mathsf{k}}(\mathfrak{g}) = \mathrm{Inn}_{\mathsf{k}}(\mathfrak{g})$.