C2.1a Lie algebras

Mathematical Institute, University of Oxford
Michaelmas Term 2012

Problem Sheet 6

Throughout this sheet we assume that all Lie algebras and all representations discussed are finite dimensional unless the contrary is explicitly stated, and we work over a field k *which is algebraically closed of characteristic zero.*

1. i) Show directly that if $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a surjective homomorphism of semisimple Lie algebras and $x = s + n$ is the Jordan decomposition of $x \in \mathfrak{g}_1$, then $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x) \in \mathfrak{g}_2$.

ii) Show that homomorphisms between semisimple Lie algebras are compatible with the Jordan decomposition, that is, if $\mathfrak{g}_1, \mathfrak{g}_2$ are semisimple Lie algebras, and $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism, then if $x = s + n$ is the Jordan decomposition of $x \in \mathfrak{g}_1$, $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ in $\mathfrak{g}_2$. (*For this part you may assume the fact, stated in lectures, that if $x = s + n$ is the Jordan decomposition of x and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, then $\rho(s)$ is semsimple and $\rho(n)$ is nilpotent.*)

2. i) Show that if V is a (finite dimensional) representation of $\mathfrak{sl}_2$ and $V = \bigoplus_{k\in\mathbb{Z}} V_k$ is the decomposition of V into (generalised) eigenspaces of h, then the number of irreducible constituents of V is equal to $\dim(V_0) + \dim(V_1)$.

ii) Show that if $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ is the Cartan decomposition of a semisimple Lie algebra $\mathfrak{g}$, and α, β and $\alpha + \beta$ are all in Φ, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. (*Hint: Use the representation theory of* $\mathfrak{sl}_2$.)

3. Use Weyl's theorem to give an alternative proof of the fact that any derivation of a semisimple Lie algebra $\mathfrak{g}$ is inner. (*Hint*: Suppose that δ is a derivation, show that $V = \mathsf{k} \oplus \mathfrak{g}$ has the structure of a $\mathfrak{g}$ representation via $x(a, y) = (0, a\delta(x) + [x, y])$, and consider a complement to the subrepresentation $\mathfrak{g}$.)

4. Let $\mathfrak{g}$ be a simple Lie algebra. Show that any nonzero trace form on $\mathfrak{g}$ is a multiple of the Killing form. (*Hint*: Show that the form can be used to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ as a $\mathfrak{g}$-representation.)

5. Use the previous question to show that the Killing form for $\mathfrak{sl}_n$ is given by:

$$\kappa(x, y) = 2n.\mathrm{tr}(xy), \quad x, y \in \mathfrak{sl}_n.$$

6. Let $\mathfrak{so}_4$ be the Lie algebra

$$\mathfrak{so}_4 = \{x \in \mathfrak{gl}_n(\mathbb{C}) : x^t S = -Sx\}$$

where

$$S = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

Show that the diagonal matrices $\mathfrak{h} \subset \mathfrak{so}_4(\mathbb{C})$ are a Cartan subalgebra and find the associated decomposition of $\mathfrak{so}_4(\mathbb{C})$.