

EXPONENTIAL ODD-DISTANCE SETS UNDER THE MANHATTAN METRIC

ALBERTO ESPUNY DÍAZ, EMMA HOGAN, FREDDIE ILLINGWORTH, LUKAS MICHEL,
JULIEN PORTIER, AND JUN YAN

ABSTRACT. We construct a set of 2^n points in \mathbb{R}^n such that all pairwise Manhattan distances are odd integers, which improves the recent linear lower bound of Golovanov, Kupavskii and Sagdeev. In contrast to the Euclidean and maximum metrics, this shows that the odd-distance set problem behaves very differently to the equilateral set problem under the Manhattan metric. Moreover, all coordinates of the points in our construction are integers or half-integers, and we show that our construction is optimal under this additional restriction.

1. INTRODUCTION

An *equilateral set* in a metric space is a set of points in which all pairwise distances are the same. Determining the largest equilateral set in a metric space has been well-studied and the case of \mathbb{R}^n equipped with the ℓ_p -norm¹ $\|\cdot\|_p$ has received significant attention. Denoting the size of the largest equilateral set in \mathbb{R}^n with the ℓ_p -norm by $e_p(n)$, the following table gives a summary of the best bounds known for $p \in \{1, 2, \infty\}$.

	Lower bound		Upper bound	
$e_1(n)$	$2n$	cross polytope	$cn \log n$	Alon and Pudlák [1]
$e_2(n)$	$n + 1$	unit simplex	$n + 1$	folklore
$e_\infty(n)$	2^n	unit hypercube	2^n	Petty [5]

TABLE 1. The size of the largest equilateral sets in $(\mathbb{R}^n, \|\cdot\|_p)$.

An old conjecture of Kusner [4] states that $e_p(n) = n + 1$ for all integers $1 < p < \infty$ and $e_1(n) = 2n$. For progress on Kusner's conjecture, see for example [1, 6, 7].

Graham, Rothschild and Straus [3] introduced the following relaxation of an equilateral set. An *odd-distance set* in a metric space is a set of points in which all pairwise distances are odd integers. We denote the size of the largest odd-distance set in \mathbb{R}^n equipped with the ℓ_p -norm by $\text{odd}_p(n)$. In a normed space, any equilateral set can be scaled to a set where all pairwise distances are 1, and so $\text{odd}_p(n) \geq e_p(n)$. The following table gives a summary of the best bounds known for $\text{odd}_p(n)$. The bounds for $\text{odd}_2(n)$ were proved in the original paper of Graham, Rothschild and Straus [3], who in fact determined the exact value of $\text{odd}_2(n)$ (it is $n + 1$ except when $n \equiv -2 \pmod{16}$). The other bounds are due to a recent paper of Golovanov, Kupavskii and Sagdeev [2].

Date: 24th October 2024.

A. Espuny Díaz was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through project no. 513704762. E. Hogan's research was supported by EPSRC grant EP/W524311/1. F. Illingworth's research was supported by EPSRC grant EP/V521917/1 and the Heilbronn Institute for Mathematical Research. J. Yan was supported by the Warwick Mathematics Institute CDT and funding from the EPSRC grant EP/W523793/1.

¹A point $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ has ℓ_p -norm $\|\mathbf{v}\|_p := (\sum_i |v_i|^p)^{1/p}$ for $1 \leq p < \infty$ and has ℓ_∞ -norm $\|\mathbf{v}\|_\infty := \max_i |v_i|$.

	Lower bound		Upper bound	
$\text{odd}_1(n)$	$(\frac{7}{3} - o(1))n$	[2]	$(4 + o(1))n! n \log n$	[2]
$\text{odd}_2(n)$	$n + 1$	unit simplex	$n + 2$	[3]
$\text{odd}_\infty(n)$	2^n	unit hypercube	2^n	[2]

TABLE 2. The size of the largest odd-distance sets in $(\mathbb{R}^n, \|\cdot\|_p)$.

From these results we see that relaxing from the equilateral to the odd-distance problem makes very little difference for the ℓ_2 and ℓ_∞ -norms. The case of the ℓ_1 -norm is less clear: the lower bound of Golovanov, Kupavskii and Sagdeev [2] for $\text{odd}_1(n)$ is larger than Kusner's conjectured upper bound for $e_1(n)$, but only by a constant multiplicative factor, and it is still smaller than the upper bound proven by Alon and Pudlák [1].

In this note, we significantly improve the lower bound for $\text{odd}_1(n)$, showing that there are odd-distance sets of size 2^n under the ℓ_1 -norm. Together with the upper bound for $e_1(n)$ of Alon and Pudlák [1], this confirms in a strong sense that the equilateral and odd-distance problems behave very differently for the ℓ_1 -norm.

Theorem 1. *For all positive integers n , $\text{odd}_1(n) \geq 2^n$. Moreover, there is an ℓ_1 -odd-distance set $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^n$ of size 2^n .*

It is not possible to replace the half-integer lattice $(\frac{1}{2} \cdot \mathbb{Z})^n$ by the integer lattice \mathbb{Z}^n in the statement of Theorem 1. Indeed, there is not even an odd-distance set of size 3 in \mathbb{Z}^n : every integer x satisfies $|x| \equiv x \pmod{2}$ and so, for any integer points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^n$, we have that $\|\mathbf{a} - \mathbf{b}\|_1 + \|\mathbf{b} - \mathbf{c}\|_1 \equiv \|\mathbf{a} - \mathbf{c}\|_1 \pmod{2}$.

We further show that Theorem 1 is best possible in the case that all coordinates are half-integers.

Proposition 2. *Let $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^n$ be an ℓ_1 -odd-distance set. Then, $|\mathcal{P}| \leq 2^n$.*

As a last result, we also note that, when considering odd-distance sets in the ℓ_1 -norm, we may restrict the coordinates to be rational numbers whose denominator is a power of 2.

Proposition 3. *Let $\mathcal{T} \subseteq \mathbb{Q}$ be the set of rational numbers whose denominators are powers of 2. For any ℓ_1 -odd-distance set $\mathcal{P} \subseteq \mathbb{R}^n$, there exists an ℓ_1 -odd-distance set $\mathcal{Q} \subseteq \mathcal{T}^n$ of the same size as \mathcal{P} .*

2. PROOFS

To prove the lower bound on $\text{odd}_1(n)$, we will inductively construct an odd-distance set of size 2^n . We will achieve this by splitting in two a coordinate of each point of an odd-distance set in \mathbb{R}^n . By doing this in two distinct ways, we replace each point with two points in \mathbb{R}^{n+1} . This construction relies on the following simple lemma.

Lemma 4. *For every $x \in \frac{1}{2} \cdot \mathbb{Z}$, there exists an odd-distance set $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^2$ of size 2 such that the coordinates of each point sum to x and all coordinates are at least $x/2 - 1/2$ and at most $x/2 + 1/2$.*

Proof. If $x \in \mathbb{Z}$, let

$$\mathcal{P} = \left\{ \left(\frac{x}{2} + \frac{1}{2}, \frac{x}{2} - \frac{1}{2} \right), \left(\frac{x}{2}, \frac{x}{2} \right) \right\}.$$

Otherwise, if $x \in \mathbb{Z} + \frac{1}{2}$, let

$$\mathcal{P} = \left\{ \left(\frac{x}{2} + \frac{1}{4}, \frac{x}{2} - \frac{1}{4} \right), \left(\frac{x}{2} - \frac{1}{4}, \frac{x}{2} + \frac{1}{4} \right) \right\}.$$

Clearly, in both cases \mathcal{P} satisfies the claim. \square

We may now prove our main result. Our strategy is to start with an odd-distance set $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^n$ and first space out the points in \mathcal{P} in the first dimension. We achieve this by translating the first coordinate of each point by some even integer while maintaining their order in the first dimension and the odd-distance property. Then, we replace each first coordinate by the odd-distance set in $(\frac{1}{2} \cdot \mathbb{Z})^2$ given by Lemma 4. Because the first dimension was sufficiently spaced out in advance, this guarantees that the relative order of the points in the two new dimensions will be the same as their relative order in the old dimension. As a result, the new set will be an odd-distance set in $(\frac{1}{2} \cdot \mathbb{Z})^{n+1}$ of size $2|\mathcal{P}|$.

Theorem 1. *For all positive integers n , $\text{odd}_1(n) \geq 2^n$. Moreover, there is an ℓ_1 -odd-distance set $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^n$ of size 2^n .*

Proof. We will prove, by induction on n , that $(\frac{1}{2} \cdot \mathbb{Z})^n$ contains an ℓ_1 -odd-distance set of size 2^n for all $n \geq 1$. For $n = 1$, the set $\mathcal{P} = \{0, 1\}$ suffices. Now, suppose \mathcal{P} is an odd-distance set in $(\frac{1}{2} \cdot \mathbb{Z})^n$ of size 2^n . We label the points of \mathcal{P} as $\mathbf{p}_1, \dots, \mathbf{p}_{2^n}$ so that the first coordinate is increasing. That is, for all $i \in [2^n]$ we may write $\mathbf{p}_i = (a_i, \mathbf{v}_i)$, where $a_1 \leq a_2 \leq \dots \leq a_{2^n}$ and the vectors \mathbf{v}_i are in $(\frac{1}{2} \cdot \mathbb{Z})^{n-1}$.

For each $i \in [2^n]$, let $b_i := a_i + 2i$ and $\mathbf{q}_i := (b_i, \mathbf{v}_i)$. Note that $\mathcal{Q} := \{\mathbf{q}_1, \dots, \mathbf{q}_{2^n}\}$ is also an ℓ_1 -odd-distance set in $(\frac{1}{2} \cdot \mathbb{Z})^n$. Indeed, for $1 \leq i < j \leq 2^n$,

$$\begin{aligned} \|\mathbf{q}_j - \mathbf{q}_i\|_1 &= (b_j - b_i) + \|\mathbf{v}_j - \mathbf{v}_i\|_1 \\ &= (a_j - a_i) + \|\mathbf{v}_j - \mathbf{v}_i\|_1 + 2(j - i) \\ &= \|\mathbf{p}_j - \mathbf{p}_i\|_1 + 2(j - i), \end{aligned}$$

which is odd.

By Lemma 4, for each $i \in [2^n]$ there is an odd-distance set $C_i = \{\mathbf{c}_i^{(1)}, \mathbf{c}_i^{(2)}\} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^2$ where the coordinates of each point sum to b_i and all coordinates are between $b_i/2 - 1/2$ and $b_i/2 + 1/2$. For each $i \in [2^n]$, let $\mathbf{q}_i^{(1)} := (\mathbf{c}_i^{(1)}, \mathbf{v}_i)$ and $\mathbf{q}_i^{(2)} := (\mathbf{c}_i^{(2)}, \mathbf{v}_i)$. Finally, let $\mathcal{Q}' := \{\mathbf{q}_1^{(1)}, \mathbf{q}_1^{(2)}, \dots, \mathbf{q}_{2^n}^{(1)}, \mathbf{q}_{2^n}^{(2)}\}$. Clearly, since all the b_i are distinct, \mathcal{Q}' is a subset of $(\frac{1}{2} \cdot \mathbb{Z})^{n+1}$ of size 2^{n+1} . It remains to show that it is an ℓ_1 -odd-distance set.

First note that, for all $i \in [2^n]$,

$$\|\mathbf{q}_i^{(1)} - \mathbf{q}_i^{(2)}\|_1 = \|\mathbf{c}_i^{(1)} - \mathbf{c}_i^{(2)}\|_1 + \|\mathbf{v}_i - \mathbf{v}_i\|_1 = \|\mathbf{c}_i^{(1)} - \mathbf{c}_i^{(2)}\|_1,$$

which is odd as C_i is an odd-distance set. Now fix $1 \leq i < j \leq 2^n$ and consider the distance between $\mathbf{q}_i^{(r)}$ and $\mathbf{q}_j^{(s)}$, where $r, s \in \{1, 2\}$. Recall, from our choice of the b_i , that $b_j \geq b_i + 2$. Since the coordinates of $\mathbf{c}_i^{(r)}$ are at most $b_i/2 + 1/2$ and the coordinates of $\mathbf{c}_j^{(s)}$ are at least $b_j/2 - 1/2$, both coordinates of $\mathbf{c}_i^{(r)}$ are bounded from above by both coordinates of $\mathbf{c}_j^{(s)}$.

Since the sum of the coordinates of $\mathbf{c}_i^{(r)}$ is b_i and the sum of the coordinates of $\mathbf{c}_j^{(s)}$ is b_j , it follows that

$$\|\mathbf{c}_j^{(s)} - \mathbf{c}_i^{(r)}\|_1 = b_j - b_i.$$

Thus,

$$\|\mathbf{q}_j^{(s)} - \mathbf{q}_i^{(r)}\|_1 = \|\mathbf{c}_j^{(s)} - \mathbf{c}_i^{(r)}\|_1 + \|\mathbf{v}_j - \mathbf{v}_i\|_1 = (b_j - b_i) + \|\mathbf{v}_j - \mathbf{v}_i\|_1 = \|\mathbf{q}_j - \mathbf{q}_i\|_1,$$

which is odd since Q is an odd-distance set, as required. \square

If we restrict a point set to having half-integer coordinates, we show that our construction is optimal. We prove this with a simple pigeonhole argument.

Proposition 2. *Let $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^n$ be an ℓ_1 -odd-distance set. Then, $|\mathcal{P}| \leq 2^n$.*

Proof. Let $\varphi: \mathcal{P} \rightarrow \{0, 1\}^n$ be defined by setting $\varphi(\mathbf{p}) := (\mathbb{1}_{\{p_i \in \mathbb{Z} + 1/2\}})_{i \in [n]}$.

First, we note that there cannot be two points $\mathbf{p}, \mathbf{q} \in \mathcal{P}$ with $\|\varphi(\mathbf{p})\|_1 \not\equiv \|\varphi(\mathbf{q})\|_1 \pmod{2}$. Indeed, in such a case, there would be an odd number of coordinates $i \in [n]$ such that $|p_i - q_i| \in \mathbb{Z} + 1/2$, which implies that $\|\mathbf{p} - \mathbf{q}\|_1$ is not an integer. In particular, $|\varphi(\mathcal{P})| \leq 2^{n-1}$.

Secondly, we claim that there cannot be three points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{P}$ with $\varphi(\mathbf{a}) = \varphi(\mathbf{b}) = \varphi(\mathbf{c})$. Indeed, otherwise $a_i - b_i$, $b_i - c_i$, and $c_i - a_i$ are integers for all $i \in [n]$, and so

$$\begin{aligned} \|\mathbf{a} - \mathbf{b}\|_1 &= \sum_{i=1}^n |a_i - b_i| \equiv \sum_{i=1}^n (a_i - b_i) \pmod{2}, \\ \|\mathbf{b} - \mathbf{c}\|_1 &= \sum_{i=1}^n |b_i - c_i| \equiv \sum_{i=1}^n (b_i - c_i) \pmod{2}, \\ \|\mathbf{a} - \mathbf{c}\|_1 &= \sum_{i=1}^n |a_i - c_i| \equiv \sum_{i=1}^n (a_i - c_i) \pmod{2}. \end{aligned}$$

Adding the first two expressions and comparing with the last one, we conclude that

$$\|\mathbf{a} - \mathbf{b}\|_1 + \|\mathbf{b} - \mathbf{c}\|_1 \equiv \|\mathbf{a} - \mathbf{c}\|_1 \pmod{2},$$

which is impossible, as the sum of two odd numbers is even. Thus, $|\mathcal{P}| \leq 2 \cdot |\varphi(\mathcal{P})| \leq 2^n$. \square

Finally, we show that we may assume all points in ℓ_1 -odd-distance sets \mathcal{P} have rational coordinates. Indeed, the pairwise distances between the points of \mathcal{P} can be expressed by a system of linear equations on the coordinates of the points of \mathcal{P} . Solutions to this system of linear equations can be characterised by a set of free variables whose choice determines all other variables. If we now replace all coordinates corresponding to free variables by nearby rational numbers, this determines all remaining coordinates, and those coordinates will also be rational since the coefficients of the linear equations were rational. This gives a point set with rational coefficients, and if the order of the coordinates did not change, then the distances between the points remain the same. So, the point set will be an odd-distance set.

Lemma 5. *For any ℓ_1 -odd-distance set $\mathcal{P} \subseteq \mathbb{R}^n$, there exists an odd-distance set $Q \subseteq \mathbb{Q}^n$ of the same size as \mathcal{P} .*

Proof. By iteratively translating coordinates as in the proof of [Theorem 1](#), we may assume that $|p_i - q_i| \geq 2$ for all distinct points $\mathbf{p}, \mathbf{q} \in \mathcal{P}$ and all $i \in [n]$. Let $s_i^{\mathbf{p}, \mathbf{q}} := 1$ if $p_i > q_i$ and $s_i^{\mathbf{p}, \mathbf{q}} := -1$ otherwise. Then,

$$\|\mathbf{p} - \mathbf{q}\|_1 = \sum_{i=1}^n s_i^{\mathbf{p}, \mathbf{q}} (p_i - q_i).$$

Write $d_{\mathbf{p}, \mathbf{q}} := \|\mathbf{p} - \mathbf{q}\|_1$ and introduce a variable $x_{\mathbf{p}, i}$ for all $\mathbf{p} \in \mathcal{P}$ and all $i \in [n]$. This means that the system of linear equations

$$\sum_{i=1}^n s_i^{\mathbf{p}, \mathbf{q}} (x_{\mathbf{p}, i} - x_{\mathbf{q}, i}) = d_{\mathbf{p}, \mathbf{q}}$$

for all distinct $\mathbf{p}, \mathbf{q} \in \mathcal{P}$ has a solution given by $x_{\mathbf{p}, i} = p_i$ for all $\mathbf{p} \in \mathcal{P}$ and all $i \in [n]$.

Since all coefficients and all $d_{\mathbf{p}, \mathbf{q}}$ are integers, the reduced row echelon form of this system of linear equations only has rational coefficients. Thus, in the reduced row echelon form, there exists a set of indices $\mathcal{F} \subseteq \mathcal{P} \times [n]$ such that x_f is a free variable for all $f \in \mathcal{F}$ while x_g is a dependent variable for all $g \notin \mathcal{F}$, with $x_g = a_g + \sum_{f \in \mathcal{F}} b_{f, g} x_f$ for some $a_g, b_{f, g} \in \mathbb{Q}$.

Let $C := \max_{f \in \mathcal{F}, g \notin \mathcal{F}} |b_{f, g}|$. For each $(\mathbf{p}, i) \in \mathcal{F}$, choose $y_{\mathbf{p}, i} \in \mathbb{Q}$ so that $|y_{\mathbf{p}, i} - p_i| < 1/(C|\mathcal{P}|n)$. For each $(\mathbf{p}, i) \notin \mathcal{F}$ let $y_{\mathbf{p}, i} = a_{\mathbf{p}, i} + \sum_{f \in \mathcal{F}} b_{f, (\mathbf{p}, i)} y_f$, and note that $y_{\mathbf{p}, i} \in \mathbb{Q}$ and

$$\begin{aligned} |y_{\mathbf{p}, i} - p_i| &= \left| \left(a_{\mathbf{p}, i} + \sum_{(\mathbf{q}, j) \in \mathcal{F}} b_{(\mathbf{q}, j), (\mathbf{p}, i)} y_{\mathbf{q}, j} \right) - \left(a_{\mathbf{p}, i} + \sum_{(\mathbf{q}, j) \in \mathcal{F}} b_{(\mathbf{q}, j), (\mathbf{p}, i)} q_j \right) \right| \\ &\leq \sum_{(\mathbf{q}, j) \in \mathcal{F}} |b_{(\mathbf{q}, j), (\mathbf{p}, i)} (y_{\mathbf{q}, j} - q_j)| < \frac{|\mathcal{F}|C}{C|\mathcal{P}|n} \leq 1, \end{aligned}$$

where the first equality used the fact that $x_{\mathbf{p}, i} = p_i$ is a solution to the system of equations.

For each $\mathbf{p} \in \mathcal{P}$, let $\mathbf{p}' \in \mathbb{Q}^n$ be the point with $p'_i = y_{\mathbf{p}, i}$. Consider two distinct points $\mathbf{p}, \mathbf{q} \in \mathcal{P}$. For each i , if $p_i < q_i$, then $p_i + 2 \leq q_i$ by assumption, and since $|p'_i - p_i| < 1$ and $|q'_i - q_i| < 1$, this implies that $p'_i < q'_i$. Similarly, if $p_i > q_i$, then $p'_i > q'_i$. Therefore,

$$\|\mathbf{p}' - \mathbf{q}'\|_1 = \sum_{i=1}^n s_i^{\mathbf{p}, \mathbf{q}} (p'_i - q'_i) = d_{\mathbf{p}, \mathbf{q}},$$

where the last equality uses the fact that $x_{\mathbf{p}, i} = p'_i$ is a solution to the system of equations. In particular, the distance between \mathbf{p}' and \mathbf{q}' is odd, and so $\mathcal{Q} = \{\mathbf{p}' : \mathbf{p} \in \mathcal{P}\} \subseteq \mathbb{Q}^n$ is an odd-distance set of the same size as \mathcal{P} . \square

[Proposition 3](#) is now an easy consequence since, if we scale all coordinates of an odd-distance set by a fixed odd integer, the resulting set is still an odd-distance set.

Proposition 3. *Let $\mathcal{T} \subseteq \mathbb{Q}$ be the set of rational numbers whose denominators are powers of 2. For any ℓ_1 -odd-distance set $\mathcal{P} \subseteq \mathbb{R}^n$, there exists an ℓ_1 -odd-distance set $\mathcal{Q} \subseteq \mathcal{T}^n$ of the same size as \mathcal{P} .*

Proof. By [Lemma 5](#), there exists an odd-distance set $\mathcal{Q} \subseteq \mathbb{Q}^n$ of the same size as \mathcal{P} . Let C be the least common multiple of all odd integers that divide the denominator of any coordinate of any point of \mathcal{Q} . Clearly, C is an odd integer, and $\mathcal{Q}' := \{C\mathbf{q} : \mathbf{q} \in \mathcal{Q}\} \subseteq \mathcal{T}^n$. Moreover, for all distinct points $C\mathbf{p}, C\mathbf{q} \in \mathcal{Q}'$ we have $\|C\mathbf{p} - C\mathbf{q}\|_1 = C\|\mathbf{p} - \mathbf{q}\|_1$, which is an odd integer as a product of two odd integers, and so \mathcal{Q}' is an odd-distance set. \square

ACKNOWLEDGEMENT

The research leading to the results in this note began during the *Early Career Researchers in Combinatorics* (ECRiC) workshop, 15-19 July 2024, funded by the International Centre for Mathematical Sciences, Edinburgh.

REFERENCES

- [1] N. Alon and P. Pudlák, Equilateral sets in l_p^n . *Geom. Funct. Anal.* 13.3 (2003), 467–482, DOI: [10.1007/s00039-003-0418-7](https://doi.org/10.1007/s00039-003-0418-7).
- [2] A. Golovanov, A. Kupavskii and A. Sagdeev, Odd-distance and right-equidistant sets in the maximum and Manhattan metrics. *Eur. J. Comb.* 107 (2023), Paper No. 103 603, DOI: [10.1016/j.ejc.2022.103603](https://doi.org/10.1016/j.ejc.2022.103603).
- [3] R. L. Graham, B. L. Rothschild and E. G. Straus, Are there $n + 2$ points in E^n with odd integral distances? *Am. Math. Mon.* 81 (1974), 21–25, DOI: [10.2307/2318906](https://doi.org/10.2307/2318906).
- [4] R. K. Guy, An olla-podrida of open problems, often oddly posed. *Am. Math. Mon.* 90.3 (1983), 196–200, DOI: [10.2307/2975549](https://doi.org/10.2307/2975549).
- [5] C. M. Petty, Equilateral sets in Minkowski spaces. *Proc. Am. Math. Soc.* 29.2 (1971), 369–374, DOI: [10.2307/2038145](https://doi.org/10.2307/2038145).
- [6] C. Smyth, Equilateral sets in ℓ_p^d . *Thirty essays on geometric graph theory*, 483–487, Berlin: Springer, ISBN 978-1-4614-0109-4; 978-1-4614-0110-0 (2013), DOI: [10.1007/978-1-4614-0110-0_25](https://doi.org/10.1007/978-1-4614-0110-0_25).
- [7] K. J. Swanepoel, A problem of Kusner on equilateral sets. *Arch. Math.* 83.2 (2004), 164–170, DOI: [10.1007/s00013-003-4840-8](https://doi.org/10.1007/s00013-003-4840-8).

(Espuny Díaz) INSTITUT FÜR INFORMATIK, UNIVERSITÄT HEIDELBERG, 69120 HEIDELBERG, GERMANY.

Email address: `espuny-diaz@informatik.uni-heidelberg.de`

(Hogan, Michel) MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, UNITED KINGDOM.

Email address: `{hogan,michel}@maths.ox.ac.uk`

(Illingworth) DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON, UNITED KINGDOM.

Email address: `f.illingworth@ucl.ac.uk`

(Portier) DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS (DPMMS), UNIVERSITY OF CAMBRIDGE, UNITED KINGDOM.

Email address: `jp899@cam.ac.uk`

(Yan) MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, UNITED KINGDOM.

Email address: `jun.yan@warwick.ac.uk`