Small families of partially shattering permutations

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Abstract

We say that a family of permutations t-shatters a set if it induces at least t distinct permutations on that set. What is the minimum number $f_k(n,t)$ of permutations of $\{1,\ldots,n\}$ that t-shatter all subsets of size k? For $t \leqslant 2$, $f_k(n,t) = \Theta(1)$. Spencer showed that $f_k(n,t) = \Theta(\log \log n)$ for $3 \leqslant t \leqslant k$ and $f_k(n,k!) = \Theta(\log n)$. In 1996, Füredi asked whether partial shattering with permutations must always fall into one of these three regimes. Johnson and Wickes recently settled the case $k = 3$ affirmatively and proved that $f_k(n,t) = \Theta(\log n)$ for $t > 2(k-1)!$.

We give a surprising negative answer to the question of Füredi by showing that a fourth regime exists for $k \geqslant 4$. We establish that $f_k(n,t) = \Theta(\sqrt{\log n})$ for certain values of t and prove that this is the only other regime when $k = 4$. We also show that $f_k(n,t) = \Theta(\log n)$ for $t > 2^{k-1}$. This greatly narrows the range of t for which the asymptotic behaviour of $f_k(n,t)$ is unknown.

1 Introduction

A family $\mathcal{P}$ of permutations of $[n] = \{1,\ldots,n\}$ *shatters* a set $X \subseteq [n]$ if the permutations of $\mathcal{P}$ induce every possible permutation on the elements of X. Shattering families of permutations were first studied by Spencer [12] who asked the following question.

> What is the smallest family of permutations of $[n]$ that shatters all subsets of a fixed size k?

Spencer [12] showed that such families have size $\Theta(\log n)$, with subsequent work improving the constant of the lower bound [6, 5, 9].

A natural refinement of this problem is to consider partial shattering. For $t \geqslant 1$, we say that a family $\mathcal{P}$ *t-shatters* a set X if $\mathcal{P}$ induces at least t distinct permutations on X. Let $f_k(n,t)$ be the minimum number of permutations of $[n]$ that t-shatter all subsets of size k. From above we know that $f_k(n,k!) = \Theta(\log n)$, and monotone permutations can be used to prove that $f_k(n,t) = t$ for $t \leqslant 2$. Moreover, an argument of Hajnal

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and Spencer [12] shows that $f_k(n,t) = \Theta(\log\log n)$ for $3 \leqslant t \leqslant k$.[1] Therefore, the asymptotic behaviour of $f_k(n,t)$ falls into at least three distinct regimes.

In 1996, Füredi [5] asked whether these might be the only possible regimes, even in a much more general version of partial shattering. Let $\mathcal{S}$ be a family of sets of permutations of $[k]$. Then, a family $\mathcal{P}$ of permutations of $[n]$ is *$\mathcal{S}$-mixing* if for every subset $X \subseteq [n]$ of size k, the set of permutations that $\mathcal{P}$ induces on X is a member of $\mathcal{S}$. Moreover, $\mathcal{S}$ is *monotone* if for all $S \in \mathcal{S}$ and $S \subseteq T$ we have $T \in \mathcal{S}$. If $\mathcal{S}$ is the family of sets with at least t permutations of $[k]$, then $\mathcal{S}$ is monotone and $\mathcal{S}$-mixing families are exactly those families that t-shatter all subsets of size k.

Even in this more general $\mathcal{S}$-mixing framework, the minimum size of an $\mathcal{S}$-mixing family in all previously known cases was in one of the three regimes $\Theta(1)$, $\Theta(\log\log n)$, and $\Theta(\log n)$. This prompted Füredi [5] to ask the following question.

Question 1.1 (Füredi, 1996)**.** *If $\mathcal{S}$ is a monotone family of sets of permutations of $[k]$, is the minimum size of an $\mathcal{S}$-mixing family either $\Theta(1)$, $\Theta(\log\log n)$, or $\Theta(\log n)$?*

Johnson and Wickes [7] recently made progress on this question for $f_k(n,t)$. They showed that $f_k(n,t) = \Theta(\log n)$ for $t > 2(k-1)!$. Together with the previously known asymptotics, this yields the following partial classification.

Theorem 1.2 (Johnson, Wickes, 2023)**.** *Let $k \geqslant 3$ be an integer. Then,*

$$f_k(n,t) = \begin{cases} t & \text{for } t \leqslant 2, \\ \Theta(\log\log n) & \text{for } 3 \leqslant t \leqslant k, \\ \Theta(\log n) & \text{for } 2(k-1)! < t \leqslant k!. \end{cases}$$

Moreover, Johnson and Wickes [7] settled the case $k = 3$ completely by additionally proving that $f_3(n,4) = \Theta(\log\log n)$. Given these results, they reiterated Füredi's question and asked specifically whether $f_k(n,t)$ must always fall into one of the three regimes $\Theta(1)$, $\Theta(\log\log n)$, and $\Theta(\log n)$ [1, 7].

We answer the questions of Füredi and of Johnson and Wickes negatively. For $k \geqslant 4$, we show that a fourth regime exists with $f_k(n,t) = \Theta(\sqrt{\log n})$. More generally, we improve the partial classification of the asymptotic behaviour of $f_k(n,t)$ as follows.

Theorem 1.3. *Let $k \geqslant 4$ be an integer. Then,*

$$f_k(n,t) = \begin{cases} t & \text{for } t \leqslant 2, \\ \Theta(\log\log n) & \text{for } 3 \leqslant t \leqslant 2^{\lceil \log_2 k \rceil}, \\ \Theta(\sqrt{\log n}) & \text{for } 2^{\lceil \log_2 k \rceil} < t \leqslant \min\{2k, 2^{\lceil \log_2 k \rceil} + 4\}, \\ \Theta(\log n) & \text{for } 2^{k-1} < t \leqslant k!. \end{cases}$$

For $k = 4$, this settles the asymptotic behaviour of $f_4(n,t)$ completely as all values of t are covered. However, if k is large, there is still an exponential gap between the regime $\Theta(\sqrt{\log n})$ and $\Theta(\log n)$.

[1] Spencer [12] considered the slightly different problem of determining the minimum number of permutations such that for every subset $X \subseteq [n]$ of size k and every element $x \in X$, there is a permutation where x is the largest element of X. Johnson and Wickes [7] observed that this transfers to $f_k(n,t)$.

Theorem 1.3 is based on a series of new lower and upper bounds on $f_k(n,t)$. First, we show that the lower bound $f_k(n,t) = \Omega(\log n)$ holds for a wider range of values of t.

Theorem 1.4. *Let $k \geqslant 3$ and $t > 2^{k-1}$. Then, $f_k(n,t) = \Omega(\log n)$.*

The main observation for this result is that for any small family $\mathcal{P}$ of permutations of $[n]$, we can construct two large subsets $A, B \subseteq [n]$ that are ordered in the following sense: for each permutation of $\mathcal{P}$ either all elements of A are smaller than all elements of B, or all elements of B are smaller than all elements of A. By recursively constructing ordered sets in A, we find a subset of size k that is only 2^{k-1}-shattered by $\mathcal{P}$.

For smaller values of t, we provide a new lower bound of the form $\Omega(\sqrt{\log n})$.

Theorem 1.5. *Let $k \geqslant 3$ and $t > 2^{\lceil \log_2 k \rceil}$. Then, $f_k(n,t) = \Omega(\sqrt{\log n})$.*

The proof of this result is inspired by the proof of Theorem 1.4. However, instead of only constructing ordered sets in A, we recursively construct ordered sets both in A and in B. We then use a Ramsey-theoretic argument about vertex-coloured binary trees to find a subset of size k that is only $2^{\lceil \log k \rceil}$-shattered.

We note that the lower bound on t in Theorem 1.5 cannot be replaced by anything smaller. Indeed, a careful analysis of the construction of Hajnal and Spencer [12] shows that $f_k(n,t) = \Theta(\log \log n)$ for $3 \leqslant t \leqslant 2^{\lceil \log_2 k \rceil}$. We provide an equivalent construction which proves this and which serves as a motivating example for what follows.

Theorem 1.6. *Let $k \geqslant 3$ and $t \leqslant 2^{\lceil \log_2 k \rceil}$. Then, $f_k(n,t) = \mathcal{O}(\log \log n)$.*

Our construction identifies $[n]$ with $[2]^d$ for $d = \log_2 n$. Then, we consider lexicographic permutations of $[2]^d$ which are premutations where the order of $x, y \in [2]^d$ only depends on the values x_i and y_i for the first position i with $x_i \neq y_i$. We show that an appropriate choice of these permutations ensures that all subsets of size k are $2^{\lceil \log_2 k \rceil}$-shattered.

Finally, for $k \geqslant 4$, we show that there exist $t > 2^{\lceil \log_2 k \rceil}$ with $f_k(n,t) = \mathcal{O}(\sqrt{\log n})$. Together with Theorem 1.5, this establishes the existence of a fourth regime for $f_k(n,t)$.

Theorem 1.7. *Let $k \geqslant 4$ and $t \leqslant \min\{2k, 2^{\lceil \log_2 k \rceil} + 4\}$. Then, $f_k(n,t) = \mathcal{O}(\sqrt{\log n})$.*

This result is proved similar to Theorem 1.6, but we identify $[n]$ with $[2^d]^d$ for $d = \sqrt{\log_2 n}$. Then, most subsets of size k are $2k$-shattered by lexicographic permutations, and the remaining subsets have a very specific structure. We exploit this structure and add a few more permutations which ensure that all subsets of size k are t-shattered.

We remark that partial shattering with permutations is quite different to partial shattering with sets. A family $\mathcal{F}$ of subsets of $[n]$ *shatters* a set $X \subseteq [n]$ if for every subset $Y \subseteq X$ there exists $F \in \mathcal{F}$ with $F \cap X = Y$, and $\mathcal{F}$ *t-shatters* X if for at least t distinct subsets $Y \subseteq X$ there exists $F \in \mathcal{F}$ with $F \cap X = Y$. The study of shattering families of sets dates back to the seminal works of Sauer [10], Shelah [11], and Vapnik and Chervonenkis [13].

As for permutations, Kleitman and Spencer [8] showed that the minimum number of subsets of $[n]$ that shatter all subsets of size k is $\Theta(\log n)$. However, in the case of partial shattering, the family $\{\varnothing, [n]\}$ 2-shatters all subsets of size k, and every family that

3-shatters all subsets of size k already needs $\Omega(\log n)$ sets.[2] Therefore, partial shattering with sets only has the two regimes $\Theta(1)$ and $\Theta(\log n)$.

The rest of the paper is structured as follows. In Section 2 we prove the lower bounds of Theorems 1.4 and 1.5. Afterwards, in Section 3, we prove the upper bounds of Theorems 1.6 and 1.7. We finish with some open problems in Section 4.

Notation. Throughout the paper, a permutation ρ of a set X is a total order of the elements of X. We denote this order by $<_\rho$. Note that if $Y \subseteq X$, then ρ induces a permutation on Y. If ρ is a permutation of $A \cup B$, we write $A <_\rho B$ if for all $a \in A$ and $b \in B$ we have $a <_\rho b$.

2 Lower bounds

To prove lower bounds for $f_k(n,t)$, we first need to define the concept of ordered sets. Let $\mathcal{P}$ be a family of permutations of a set X. Then, we say that a pair of disjoint subsets $A, B \subseteq X$ is *$\mathcal{P}$-ordered* if for each permutation $\rho \in \mathcal{P}$ either $A <_\rho B$ or $B <_\rho A$. The following result shows that any set contains large $\mathcal{P}$-ordered subsets.

Lemma 2.1. *Let X be a set and let $\mathcal{P}$ be a family of m permutations of X. Then, there exists a $\mathcal{P}$-ordered pair $A, B \subseteq X$ with $\min\{|A|,|B|\} \geqslant \lfloor |X|/2^{m+1} \rfloor$.*

Proof. Let $\mathcal{P} = \{\rho_1, \rho_2, \dots, \rho_m\}$ and define $\mathcal{P}_i = \{\rho_1, \rho_2, \dots, \rho_i\}$. We inductively construct a $\mathcal{P}_i$-ordered pair $A_i, B_i \subseteq X$ with $|A_i| = |B_i| \geqslant \lfloor |X|/2^{i+1} \rfloor$. For $i = 0$, we choose $A_0, B_0 \subseteq X$ as two disjoint subsets of size $\lfloor |X|/2 \rfloor$.

Now suppose we have constructed A_i and B_i. Let $\ell = |A_i| = |B_i| \geqslant \lfloor |X|/2^{i+1} \rfloor$ and consider the subset L of the ℓ largest elements of $A_i \cup B_i$ under $<_{\rho_{i+1}}$. Note that either $|L \cap A_i| \geqslant \ell/2$ or $|L \cap B_i| \geqslant \ell/2$. Without loss of generality, assume that $|L \cap A_i| \geqslant \ell/2$.

Set $A_{i+1} = L \cap A_i$ and $B_{i+1} = B_i \setminus L$. Since $b <_{\rho_{i+1}} a$ for all $a \in A_{i+1}$ and $b \in B_{i+1}$, the pair (A_{i+1}, B_{i+1}) is $\mathcal{P}_{i+1}$-ordered. We also have $|A_{i+1}| \geqslant \ell/2 \geqslant \lfloor |X|/2^{i+2} \rfloor$ and $|B_{i+1}| = \ell - |L \cap B_i| = \ell - (\ell - |L \cap A_i|) = |A_{i+1}|$, as required.

For $i = m$, the pair (A_m, B_m) is $\mathcal{P}$-ordered with $\min\{|A_m|, |B_m|\} \geqslant \lfloor |X|/2^{m+1} \rfloor$. □

Using $\mathcal{P}$-ordered pairs, we can prove Theorem 1.4. For this, we construct a sequence of nested $\mathcal{P}$-ordered pairs $A_i, B_i \subseteq A_{i-1}$ for $i \in [k]$, where $A_0 = [n]$. Then, if we pick an element $x_i \in B_i$ for each $i \in [k]$, the fact that all pairs are $\mathcal{P}$-ordered implies that $\{x_1, \dots, x_k\}$ is only 2^{k-1}-shattered by $\mathcal{P}$, as required.

Proof of Theorem 1.4. Let $t > 2^{k-1}$. We claim that $f_k(n,t) > (\log_2 n)/k - 1$. Indeed, suppose that $\mathcal{P}$ is a family of m permutations of $[n]$ with $m \leqslant (\log_2 n)/k - 1$, and so $n \geqslant 2^{k(m+1)}$. We show that there is a subset of size k that is not t-shattered by $\mathcal{P}$.

[2] If $n \geqslant k \cdot 2^{|\mathcal{F}|}$, one of the intersections $\bigcap_{F \in \mathcal{F}} A_F$ with $A_F \in \{F, [n] \setminus F\}$ must have size at least k, and any subset $X \subseteq \bigcap_{F \in \mathcal{F}} A_F$ of size k satisfies $F \cap X \in \{\varnothing, X\}$ for all $F \in \mathcal{F}$.

To do so, we inductively construct a sequence of $\mathcal{P}$-ordered pairs as follows. Start with $A_0 = [n]$. Then, for $i = 1, \dots, k$, apply Lemma 2.1 to A_{i-1} to obtain a $\mathcal{P}$-ordered pair $A_i, B_i \subseteq A_{i-1}$ with $\min\{|A_i|, |B_i|\} \geqslant \lfloor |A_{i-1}|/2^{m+1} \rfloor$. Note that this implies that $\min\{|A_i|, |B_i|\} \geqslant \lfloor n/2^{i(m+1)} \rfloor \geqslant \lfloor n/2^{k(m+1)} \rfloor \geqslant 1$, and so A_i and B_i are non-empty.

For each $i \in [k]$, pick $x_i \in B_i$. We claim that $\mathcal{P}$ does not t-shatter the set $\{x_1, \dots, x_k\}$. Indeed, let $i \in [k-1]$. Note that $x_j \in A_i$ for all $j > i$. Since (A_i, B_i) is $\mathcal{P}$-ordered, it follows that for each permutation $\rho \in \mathcal{P}$ we either have $x_i <_\rho \{x_{i+1}, \dots, x_k\}$ or $\{x_{i+1}, \dots, x_k\} <_\rho x_i$. This provides two choices for the position of x_i relative to $\{x_{i+1}, \dots, x_k\}$. Moreover, given such a choice for each $i \in [k-1]$, this uniquely determines the permutation that ρ induces on $\{x_1, \dots, x_k\}$. Hence, $\mathcal{P}$ induces at most 2^{k-1} permutations on $\{x_1, \dots, x_k\}$, and therefore does not t-shatter $\{x_1, \dots, x_k\}$. □

To prove Theorem 1.5, we will construct a complete binary tree of nested $\mathcal{P}$-ordered pairs. Namely, for each $\mathcal{P}$-ordered pair (A, B) that we obtain, we recursively construct new $\mathcal{P}$-ordered pairs in each of A and B.

This is useful for the following reason. Suppose that $k = 4$, (A, B) is a $\mathcal{P}$-ordered pair, and two $\mathcal{P}$-ordered pairs $A', B' \subseteq A$ and $A'', B'' \subseteq B$ in the tree are *synchronised* in the sense that for all $\rho \in \mathcal{P}$ we have $A' <_\rho B'$ if and only if $A'' <_\rho B''$. Pick $x_1 \in A'$, $x_2 \in B'$, $x_3 \in A''$, and $x_4 \in B''$. Then, since (A, B) is $\mathcal{P}$-ordered, we have $\{x_1, x_2\} <_\rho \{x_3, x_4\}$ or $\{x_3, x_4\} <_\rho \{x_1, x_2\}$ for all $\rho \in \mathcal{P}$, and since (A', B') and (A'', B'') are synchronised, we have $x_1 <_\rho x_2$ if and only if $x_3 <_\rho x_4$. This implies that $\mathcal{P}$ induces at most 4 permutations on $\{x_1, x_2, x_3, x_4\}$ which is less than $2^{k-1} = 8$.

To find synchronised $\mathcal{P}$-ordered pairs, we use a Ramsey-theoretic argument about vertex-coloured binary trees. For that, we need to introduce some terminology. A *binary tree* T rooted at a vertex r is a tree where r has degree 0 or 2 and every other vertex has degree 1 or 3. A leaf of T is a vertex of degree at most 1. The *layer* of T at height h is the set $N_h(r) \subseteq V(T)$ of those vertices at graph distance exactly h from r. For a vertex $v \in N_h(r)$, its children are its neighbours in $N_{h+1}(r)$, and if $v \neq r$ then its parent is its neighbour in $N_{h-1}(r)$. Clearly, $|N_h(r)| \leqslant 2^h$. If there exists some h such that $|N_h(r)| = 2^h$ and $|N_{h+1}(r)| = 0$, then T is a *complete binary tree* of height h.

Let S be a second binary tree with root q. A *subdivision* of S in T is an injective map $\varphi\colon V(S) \to V(T)$ such that for all vertices $v \in V(S) \setminus \{q\}$ with parent p, $\varphi(v)$ is contained in the subtree of T rooted at $\varphi(p)$. If the vertices of T are coloured (not necessarily properly), we say that the layers of the subdivision are *monochromatic* if $\varphi(N_h(q))$ is monochromatic for all h. We often identify $V(S)$ with $\varphi(V(S))$.

In our proof, T will be a complete binary tree whose vertices correspond to nested $\mathcal{P}$-ordered pairs, and we colour the vertices of T in such a way that monochromatic layers correspond to collections of synchronised $\mathcal{P}$-ordered pairs. Then, if we find a subdivision of a complete binary tree of height $\lceil \log_2 k \rceil$ in T with monochromatic layers, we can pick one element from each of its leaves to obtain a set of size k that is only $2^{\lceil \log_2 k \rceil}$-shattered by $\mathcal{P}$, as required.

To do this, we need to find such subdivisions in large vertex-coloured complete binary trees. For all integers $c \geqslant 1$ and $h \geqslant 0$, let $g(c, h)$ be the smallest integer such that every complete binary tree of height $g(c,h)$ whose vertices are coloured with c colours contains a subdivision of a complete binary tree of height h with monochromatic layers. We prove the following upper bound on $g(c,h)$.

Lemma 2.2. *For all integers $c \geqslant 1$ and $h \geqslant 0$ we have $g(c,h) \leqslant h\lceil \log_2(c^{h-1}+1)\rceil$.*

Proof. We proceed by induction on h. If $h = 0$, a complete binary tree of height h is a single vertex, and so $g(c,0) = 0$.

If $h \geqslant 1$, let $d = \lceil \log_2(c^{h-1}+1)\rceil$ and let T be a complete binary tree with root r and height $d + g(c,h-1)$ that is coloured with c colours. For each vertex $v \in N_d(r)$, the subtree of T rooted at v has height $g(c,h-1)$ and must therefore contain a subdivision S_v of a complete binary tree of height $h-1$ with monochromatic layers. Since such subdivisions admit at most c^{h-1} valid colourings and $|N_d(r)| = 2^d \geqslant c^{h-1}+1$, there must be two subdivisions S_u and S_v whose colourings coincide. Then, the subdivision formed by S_u, S_v, and the lowest common ancestor of u and v is a subdivision of a complete binary tree of height h with monochromatic layers.

Therefore, we get that

$$g(c,h) \leqslant d + g(c,h-1) \leqslant d + (h-1)\lceil \log_2(c^{h-2}+1)\rceil \leqslant h\lceil \log_2(c^{h-1}+1)\rceil. \qquad \square$$

Using the strategy outlined above, we now prove our lower bound for $t > 2^{\lceil \log_2 k\rceil}$.

Proof of Theorem 1.5. Let $h = \lceil \log_2 k\rceil$ and $t > 2^h$. We claim $f_k(n,t) > \sqrt{\log_2 n}/h - 1$. Indeed, suppose that $\mathcal{P}$ is a family of m permutations of $[n]$ with $m \leqslant \sqrt{\log_2 n}/h - 1$, and so using Lemma 2.2 we get $\log_2 n \geqslant h^2(m+1)^2 \geqslant g(2^m,h)(m+1)$. We show that there is a subset of size k that is not t-shattered by $\mathcal{P}$.

Consider a complete binary tree T of height $g(2^m,h)$. We associate to each vertex v of T a set $X_v \subseteq [n]$ and a $\mathcal{P}$-ordered pair $A_v, B_v \subseteq X_v$ as follows. For the root r of T, let $X_r = [n]$ and let $A_r, B_r \subseteq X_r$ be a $\mathcal{P}$-ordered pair given by Lemma 2.1. Then, for a vertex v of T with $\mathcal{P}$-ordered pair (A_v, B_v) and children v_1 and v_2, let $X_{v_1} = A_v$ and $X_{v_2} = B_v$, and let $A_{v_1}, B_{v_1} \subseteq X_{v_1}$ and $A_{v_2}, B_{v_2} \subseteq X_{v_2}$ be $\mathcal{P}$-ordered pairs given by Lemma 2.1. This implies that $|X_w| \geqslant \lfloor n/2^{g(2^m,h)(m+1)}\rfloor \geqslant 1$ for every $w \in V(T)$, and so X_w is non-empty.

Colour each vertex $v \in V(T)$ with a colour c_v as follows. For $\rho \in \mathcal{P}$, let

$$c_v(\rho) = \begin{cases} 1 & \text{if } A_v <_\rho B_v, \\ 0 & \text{if } B_v <_\rho A_v. \end{cases}$$

Since (A_v, B_v) is $\mathcal{P}$-ordered, this is well-defined. This colours each vertex of T with one of 2^m colours. By Lemma 2.2, T admits a subdivision of a complete binary tree S of height h with monochromatic layers. Let q be the root of S.

For each leaf ℓ of S, pick $y_\ell \in X_\ell$, and let $Y = \{y_\ell : \ell \in N_h^S(q)\}$ be the set of all of these elements. Note that $|Y| = 2^h \geqslant k$. We claim that Y is not t-shattered. Indeed, consider the colouring that a fixed permutation $\rho \in \mathcal{P}$ induces on the first h layers $N_{<h}^S(q)$ of S

by assigning $c_v(\rho)$ to each $v \in N^S_{<h}(q)$. Since each layer of S is monochromatic, this colours $N^S_{<h}(q)$ with one of at most 2^h distinct colourings (accross all $\rho \in \mathcal{P}$).

Now, for any two distinct leaves $\ell, \ell' \in N^S_h(q)$, the lowest common ancestor $v \in N^S_{<h}(q)$ of ℓ and ℓ' satisfies $y_\ell \in A_v$ and $y_{\ell'} \in B_v$ (or vice versa). Then, if $c_v(\rho) = 1$, we have $A_v <_\rho B_v$ and so $y_\ell <_\rho y_{\ell'}$, and if $c_v(\rho) = 0$ we have $B_v <_\rho A_v$ and so $y_{\ell'} <_\rho y_\ell$. Thus, the colours $c_v(\rho)$ for the vertices $v \in N^S_{<h}(q)$ uniquely determine the permutation that ρ induces on Y. Since there are at most 2^h distinct colourings of $N^S_{<h}(q)$, it follows that $\mathcal{P}$ induces at most 2^h permutations on Y, and therefore does not t-shatter Y. □

3 Upper bounds

To prove our upper bounds on $f_k(n,t)$, we use lexicographic permutations. Identify $[n]$ with a subset of $[b]^d$ where $b \geqslant 2$ and $d \geqslant \lceil \log_b n \rceil$ and define a *lex-permutation* ρ of $[b]^d$ to be a permutation that is constructed as follows. For each $i \in [d]$, pick a permutation ρ_i of $[b]$. Then, for distinct $x, y \in [b]^d$, set $x <_\rho y$ if and only if $x_i <_{\rho_i} y_i$ where $i = i(x,y) = \min\{i \in [d] : x_i \neq y_i\}$. That is, the relative order of x and y in ρ is entirely determined by the first position where x and y differ. It is easy to check that $<_\rho$ is a total order, and so ρ is a permutation.

We say that a family of permutations $\mathcal{P}$ of $[b]^d$ is *k-lex-shattering* if it is a family of lex-permutations such that the following holds. Let $\mathcal{I} \subseteq [d]$ be a subset of size at most k, and for each $i \in \mathcal{I}$ fix a permutation σ_i of a subset $Y_i \subseteq [b]$ of size at most k. Then, there exists a permutation $\rho \in \mathcal{P}$ such that ρ_i induces σ_i on Y_i for all $i \in \mathcal{I}$. We will later show that k-lex-shattering families $2^{\lceil \log_2 k \rceil}$-shatter every subset of size k. However, we first have to show that small k-lex-shattering families exist.

Lemma 3.1. *For $k \geqslant 1$, there exists a k-lex-shattering family of $[b]^d$ with size $\mathcal{O}(\log(bd))$.*

Proof. By Theorem 1.2, there is a family $\mathcal{Q}$ of permutations of $[bd]$ with size $\mathcal{O}(\log(bd))$ that shatters every subset of size k^2. For each $\tau \in \mathcal{Q}$, construct a lex-permutation ρ^τ such that ρ^τ_i is the permutation that τ induces on $(i-1)b + [b] = \{(i-1)b + 1, \dots, ib\}$. We claim that $\mathcal{P} = \{\rho^\tau : \tau \in \mathcal{Q}\}$ is k-lex-shattering.

Indeed, let $\mathcal{I} \subseteq [d]$ have size at most k, and for every $i \in \mathcal{I}$ let σ_i be a permutation of a subset $Y_i \subseteq [b]$ of size at most k. Since $\mathcal{Q}$ shatters the set $\bigcup_{i \in \mathcal{I}}((i-1)b + Y_i)$, there exists a permutation $\tau \in \mathcal{Q}$ which induces σ_i on $(i-1)b + Y_i$ for all $i \in \mathcal{I}$. This implies that ρ^τ_i induces σ_i on Y_i for all $i \in \mathcal{I}$. □

To determine the permutations that a k-lex-shattering family $\mathcal{P}$ induces on a subset X, consider the lexicographic structure of X. Let $i(X) = \inf\{i(x,y) : x, y \in X\}$ be the first position where two elements of X differ, and let $S(X) = \{x_{i(X)} : x \in X\}$ denote the *slice* of X at that position if $i(X) < \infty$. This partitions X into the subsets $X_s = \{x \in X : x_{i(X)} = s\}$ for $s \in S(X)$. Note that the sets X_s are pairwise $\mathcal{P}$-ordered and that their relative order in a permutation $\rho \in \mathcal{P}$ is determined by the permutation that $\rho_{i(X)}$ induces on $S(X)$. We will denote this permutation by $\rho(X)$.

If we partition a set X_s in the same way, this yields a partition of X_s into $\mathcal{P}$-ordered sets whose relative order is determined by $\rho(X_s)$. Continuing recursively like this, it follows that the order that ρ induces on X is determined by the permutations $\rho(Y)$ for $Y \subseteq X$. This is captured by the following lemma.

Lemma 3.2. *Let ρ be a lex-permutation of $[b]^d$ and let $X \subseteq [b]^d$. Then, the permutation that ρ induces on X is uniquely determined by the set of permutations $\rho(Y)$ for $Y \subseteq X$.*

Proof. Let $x, y \in X$ and $i = i(x, y)$. Then, $Y = \{x, y\} \subseteq X$ and $\rho(Y)$ is the permutation that ρ_i induces on $S(Y) = \{x_i, y_i\}$. This implies that $x <_\rho y$ if and only if $x_i <_{\rho_i} y_i$ which in turn is equivalent to $x_i <_{\rho(Y)} y_i$. So, the permutation that ρ induces on X is determined by the set of permutations $\rho(Y)$ for $Y \subseteq X$.

Conversely, suppose that ρ and τ are lex-permutations with $\rho(Y) \neq \tau(Y)$ for some $Y \subseteq X$. Let $i = i(Y)$. Since $\rho(Y) \neq \tau(Y)$, there must exist $x, y \in Y$ with $i(x, y) = i$ such that $x_i <_{\rho(Y)} y_i$ and $y_i <_{\tau(Y)} x_i$ (or vice versa). As above, this is equivalent to $x <_\rho y$ and $y <_\tau x$, and so ρ and τ induce different permutations on X. □

Next, we deduce a lower bound on the number of permutations that $\mathcal{P}$ induces on X. Let $\mathcal{I}(X) = \{i(Y) : Y \subseteq X \text{ and } i(Y) < \infty\}$. We rely on the following oberservation.

Lemma 3.3. *Let $\mathcal{P}$ be a k-lex-shattering family of permutations of $[b]^d$ and let $X \subseteq [b]^d$ be a subset of size k. For each $i \in \mathcal{I}(X)$, let $Y_i \subseteq X$ be such that $i(Y_i) = i$. Then, X is $(\prod_{i \in \mathcal{I}(X)} |S(Y_i)|!)$-shattered by $\mathcal{P}$.*

Proof. We claim that $|\mathcal{I}(X)| < k$. Indeed, write $<$ for the standard lexicographic order on $[b]^d$, so $x < y$ if and only if $x_i < y_i$ where $i = i(x, y)$. Note that if $x, y, z \in [b]^d$ satisfy $x < y < z$, then $i(x, z) = \min\{i(x, y), i(y, z)\}$. Therefore, if we write $X = \{x_1, \ldots, x_k\}$ with $x_1 < \cdots < x_k$, this implies that $\mathcal{I}(X) = \{i(x_\ell, x_{\ell+1}) : \ell \in [k-1]\}$, and so $|\mathcal{I}(X)| \leqslant k - 1$ as claimed.

Since $\mathcal{P}$ is k-lex-shattering, it follows that if σ_i is a permutation of $S(Y_i)$ for every $i \in \mathcal{I}(X)$, then there is a permutation $\rho \in \mathcal{P}$ with $\rho(Y_i) = \sigma_i$ for all $i \in \mathcal{I}(X)$. By Lemma 3.2, each choice of the permutations σ_i induces a different permutation on X, and so X is $(\prod_{i \in \mathcal{I}(X)} |S(Y_i)|!)$-shattered by $\mathcal{P}$. □

To maximise this product, consider again the decomposition of X into its lexicographic structure from above. Suppose that we only pick the largest set X_s whenever we recursively continue the decomposition, and let the sets obtained during this process be the sets Y_i. Then, we will show that $\prod_{i \in \mathcal{I}} |S(Y_i)|! \geqslant 2^{\lceil \log_2 k \rceil}$, and so every set of size k is $2^{\lceil \log_2 k \rceil}$-shattered by $\mathcal{P}$. This suffices to prove Theorem 1.6.

For Theorem 1.7, we will additionally infer some structural information about X whenever X is not $2k$-shattered by $\mathcal{P}$. Later, this information will allow us to add more permutations to $\mathcal{P}$ which ensure that X is t-shattered for some $t > 2^{\lceil \log_2 k \rceil}$. Overall, we obtain the following lemma.

Lemma 3.4. *Let $\mathcal{P}$ be a k-lex-shattering family of permutations of $[b]^d$, and let $h = \lceil \log_2 k \rceil$. Then, $\mathcal{P}$ is a family that 2^h-shatters all subsets of $[b]^d$ of size k. Moreover, if $X \subseteq [b]^d$ is a subset* *of size k that $\mathcal{P}$ does not $2k$-shatter, then the set $\mathcal{I}(X)$ has size h, all $Y \subseteq X$ with $i(Y) < \infty$ satisfy $|S(Y)| = 2$, and all $Y, Z \subseteq X$ with $i(Y) = i(Z) < \infty$ satisfy $S(Y) = S(Z)$.*

Proof. Let $X \subseteq [b]^d$ be a subset of size k and let $\mathcal{I} = \mathcal{I}(X)$.

Claim 3.5. *There exist sets $Y_i \subseteq X$ with $i(Y_i) = i$ for each $i \in \mathcal{I}$ such that $\prod_{i\in\mathcal{I}}|S(Y_i)| \geqslant k$.*

Proof. We prove by induction on j that for all $0 \leqslant j \leqslant d$ there exist sets $Y_i \subseteq X$ with $i(Y_i) = i$ for each $i \in \mathcal{I} \cap [j]$ and a subset $Z \subseteq X$ which satisfy $i(Z) > j$ and $|Z| \cdot \prod_{i\in\mathcal{I}\cap[j]}|S(Y_i)| \geqslant k$. Applying this with $j = d$ proves the claim.

For $j = 0$, let $Z = X$. Now suppose we have constructed such sets for $j - 1$. If $i(Z) > j$, the same sets work for j, where we choose any $Y_j \subseteq X$ with $i(Y_j) = j$ if $j \in \mathcal{I}$.

Otherwise, $i(Z) = j$. Let $Y_j = Z$, and for every $s \in S(Z)$ let $Z_s = \{z \in Z : z_j = s\}$. Pick $s \in S(Z)$ such that $|Z_s| \geqslant |Z|/|S(Z)|$. Note that $i(Z_s) > j$ and $|Z_s| \cdot \prod_{i\in\mathcal{I}\cap[j]}|S(Y_i)| \geqslant (|Z|/|S(Z)|) \cdot \prod_{i\in\mathcal{I}\cap[j-1]}|S(Y_i)| \cdot |S(Z)| \geqslant k$. So we have constructed the sets for j. ■

For all $i \in \mathcal{I}$, let $Y_i \subseteq X$ be a set with $i(Y_i) = i$ that maximises $|S(Y_i)|$. By Claim 3.5, $\prod_{i\in\mathcal{I}}|S(Y_i)| \geqslant k$, and by Lemma 3.3 $\mathcal{P}$ induces at least $\prod_{i\in\mathcal{I}}|S(Y_i)|!$ permutations on X. If $|S(Y_i)| \geqslant 3$ for some $i \in \mathcal{I}$, then $\prod_{i\in\mathcal{I}}|S(Y_i)|! \geqslant 2\prod_{i\in\mathcal{I}}|S(Y_i)| \geqslant 2k \geqslant 2^h$. Otherwise, $|S(Y_i)| = 2$ for all $i \in \mathcal{I}$. Then, since $2^{|\mathcal{I}|} = \prod_{i\in\mathcal{I}}|S(Y_i)| \geqslant k$, we have $|\mathcal{I}| \geqslant h$ and so $\prod_{i\in\mathcal{I}}|S(Y_i)|! = 2^{|\mathcal{I}|} \geqslant 2^h$. This shows that X is 2^h-shattered.

Moreover, if X is not $2k$-shattered, these arguments imply that $|S(Y_i)| = 2$ for all $i \in \mathcal{I}$ and $|\mathcal{I}| = h$. Suppose that for some $j \in \mathcal{I}$ there exists a set $Z \subseteq X$ with $i(Z) = j$ and $S(Y_j) \neq S(Z)$. Then, by the maximality of Y_j, $|S(Z)| = 2$ and so $|S(Y_j) \cap S(Z)| \leqslant 1$. In particular, for any pair of permutations σ of $S(Y_j)$ and τ of $S(Z)$, there is a permutation σ_j of $S(Y_j) \cup S(Z)$ that induces σ on $S(Y_j)$ and τ on $S(Z)$. By the same argument as in the proof of Lemma 3.3, it follows that $\mathcal{P}$ induces at least $|S(Z)|! \cdot \prod_{i\in\mathcal{I}}|S(Y_i)|! \geqslant 2k$ permutations on X. This contradicts the fact that X is not $2k$-shattered. □

With $b = 2$ and $d = \lceil \log_2 n \rceil$, Theorem 1.6 is an immediate consequence of Lemmas 3.1 and 3.4. We remark that while the description of our construction differs significantly from that of Spencer [12], these constructions are essentially equivalent.

To prove Theorem 1.7, start with a k-lex-shattering family $\mathcal{P}$. By Lemma 3.4, most subsets of size k are $2k$-shattered by $\mathcal{P}$, and even if a subset X is not $2k$-shattered, it is nevertheless $2^{\lceil \log_2 k \rceil}$-shattered and it has a very specific structure. We exploit this structure by adding new permutations to $\mathcal{P}$ which induce at least four additional permutation on X and thereby ensure that X is $(2^{\lceil \log_2 k \rceil} + 4)$-shattered.

To do so, we add a constant number of new permutations for each position $i \in [d]$. Let $<$ and $>$ denote the standard and reverse permutations of $[b]$.[3] Then, for all $\sigma, \tau \in \{<, >\}$, let $\pi_{i,\sigma,\tau}$ be the permutation of $[b]^d$ with $x <_{\pi_{i,\sigma,\tau}} y$ if and only if either $x_i <_\sigma y_i$, or

[3]That is, for $x, y \in [b]$ set $x <_< y$ if and only if $x < y$, and $x <_> y$ if and only if $x > y$.

$x_i = y_i$ and $x_{i(x,y)} <_\tau y_{i(x,y)}$. That is, $\pi_{i,\sigma,\tau}$ first sorts according to position i and only afterwards behaves like a lex-permutation. Define

$$\mathcal{Q}_i = \{\pi_{i,\sigma,\tau} : \sigma, \tau \in \{<, >\}\}.$$

If X is not $2k$-shattered, we show that for an appropriate $i \in [d]$ the permutations of $\mathcal{Q}_i$ induce four additional permutations on X, as required.

Proof of Theorem 1.7. Let $b = 2^d$, $d = \lceil \sqrt{\log_2 n} \rceil$, and $h = \lceil \log_2 k \rceil$. By Lemma 3.1, there exists a k-lex-shattering family $\mathcal{P}$ of $[b]^d$ with size $\mathcal{O}(\sqrt{\log n})$. Define $\mathcal{R} = \mathcal{P} \cup \bigcup_{i \in [d]} \mathcal{Q}_i$ and note that $\mathcal{R}$ has size $\mathcal{O}(\sqrt{\log n})$.

We claim that every subset $X \subseteq [b]^d$ of size k is $\min\{2k, 2^h + 4\}$-shattered by $\mathcal{R}$. Indeed, suppose that X is not $2k$-shattered by $\mathcal{P}$. Then, by Lemma 3.4, we know that the set $\mathcal{I}(X)$ has size h, all $Y \subseteq X$ with $i(Y) < \infty$ satisfy $|S(Y)| = 2$, and all $Y, Z \subseteq X$ with $i(Y) = i(Z) < \infty$ satisfy $S(Y) = S(Z)$.

Write $X = \{x_1, \dots, x_k\}$ with $x_1 < \cdots < x_k$. Note that $i(x_\ell, x_{\ell+1}) \in \mathcal{I}(X)$ for all $\ell \in [k-1]$. Since $k \geqslant 4$, one can check that $k - 1 > h = |\mathcal{I}(X)|$. So there must exist $1 \leqslant \ell < m < k$ with $i(x_\ell, x_{\ell+1}) = i(x_m, x_{m+1})$. Let $i = i(x_\ell, x_{\ell+1})$ and $j = i(x_{\ell+1}, x_m)$.

If $j \geqslant i$, then $i(x_{\ell+1}, x_{m+1}) = i$ and $i(x_\ell, x_{m+1}) = i$. So, $Y = \{x_\ell, x_{\ell+1}, x_{m+1}\} \subseteq X$ satisfies $|S(Y)| = 3$. This contradicts the fact that all $Y \subseteq X$ with $i(Y) < \infty$ satisfy $|S(Y)| = 2$. Therefore, $j < i$. In particular, $(x_\ell)_j = (x_{\ell+1})_j < (x_m)_j = (x_{m+1})_j$, and so $(\{x_\ell, x_{\ell+1}\}, \{x_m, x_{m+1}\})$ is $\mathcal{P}$-ordered.

Note that $i(\{x_\ell, x_{\ell+1}\}) = i = i(\{x_m, x_{m+1}\})$. Since all $Y, Z \subseteq X$ with $i(Y) = i(Z) < \infty$ satisfy $S(Y) = S(Z)$, we must therefore have $(x_\ell)_i = (x_m)_i < (x_{\ell+1})_i = (x_{m+1})_i$, and so $(\{x_\ell, x_m\}, \{x_{\ell+1}, x_{m+1}\})$ is $\mathcal{Q}_i$-ordered.

In particular, $\mathcal{P}$ and $\mathcal{Q}_i$ induce different permutations on X. Moreover, note that $\{x_\ell, x_{\ell+1}, x_m, x_{m+1}\}$ is 4-shattered by $\mathcal{Q}_i$. Since Lemma 3.4 implies that X is 2^h-shattered by $\mathcal{P}$, it follows that X is $(2^h + 4)$-shattered by $\mathcal{R}$. □

4 Open Problems

In this paper, we have shown that at least four regimes exist for the asymptotic behaviour of $f_k(n,t)$ when $k \geqslant 4$, and we narrowed the range of values of t for which the asymptotic behaviour is unknown. The main open problem is to determine the asymptotic behaviour of $f_k(n,t)$ for $k \geqslant 5$ and $\min\{2k, 2^{\lceil \log_2 k \rceil} + 4\} < t \leqslant 2^{k-1}$. The following remains possible.

Question 4.1. *For all integers k and t, is the asymptotic behaviour of $f_k(n,t)$ either $\Theta(1)$, $\Theta(\log\log n)$, $\Theta(\sqrt{\log n})$, or $\Theta(\log n)$?*

We conjecture that Theorem 1.4 determines the entire range of values of t that satisfy $f_k(n,t) = \Theta(\log n)$.

Conjecture 4.2. *Let $k \geqslant 3$ and $t \leqslant 2^{k-1}$. Then, $f_k(n,t) = o(\log n)$.*

Since we did not try to optimise the upper bound on t in Theorem 1.7, we believe that constructions similar to ours could prove that $f_k(n,t) = o(\log n)$ for a larger range of values of t. However, they seem to be far away from reaching $t = 2^{k-1}$.

In this context it seems important to mention the following inspiration for our approach that comes from the Erdős-Gyárfás problem [4]. If $\mathcal{P}$ is a family of permutations of $[n]$, assign a colour $c_{x,y}$ to every pair $x, y \in [n]$ as follows. For $\rho \in \mathcal{P}$, let

$$c_{x,y}(\rho) = \begin{cases} 1 & \text{if } x <_\rho y, \\ 0 & \text{otherwise.} \end{cases}$$

This colouring uses $2^{|\mathcal{P}|}$ colours, and if a subset $X \subseteq [n]$ of size k spans at most ℓ colours, then $\mathcal{P}$ induces at most 2^ℓ permutations on X. We implicitely used this approach to prove Theorems 1.4 and 1.5. Unfortunately, the converse does not hold: X can simultaneously span at least ℓ colours and fail to be 2^ℓ-shattered by $\mathcal{P}$.

Nevertheless, Eichhorn and Mubayi [3] gave a colouring of K_n with $2^{\Theta(\sqrt{\log n})}$ colours such that all subsets of size k span at least $\ell = 2\lceil \log_2 k \rceil - 2$ colours. This inspired our construction for Theorem 1.7, even if our construction does not 2^ℓ-shatter all subsets of size k. Note that Conlon, Fox, Lee, and Sudakov [2] gave a colouring of K_n with $2^{o(\log n)}$ colours such that all subsets of size k span at least $k-1$ colours. We hope that these colourings could inspire further progress towards better upper bounds on $f_k(n,t)$.

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