Improved exploration of temporal graphs

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Abstract

A temporal graph G is a sequence $(G_t)_{t\in I}$ of graphs on the same vertex set of size n. The *temporal exploration problem* asks for the length of the shortest sequence of vertices that starts at a given vertex, visits every vertex, and at each time step t either stays at the current vertex or moves to an adjacent vertex in G_t. Bounds on the length of a shortest temporal exploration have been investigated extensively. Perhaps the most fundamental case is when each graph G_t is connected and has bounded maximum degree. In this setting, Erlebach, Kammer, Luo, Sajenko, and Spooner [ICALP 2019] showed that there exists an exploration of G in $\mathcal{O}(n^{7/4})$ time steps. We significantly improve this bound by showing that $\mathcal{O}(n^{3/2}\sqrt{\log n})$ time steps suffice.

In fact, we deduce this result from a much more general statement. Let the *average temporal maximum degree* D of G be the average of $\max_{t\in I} d_{G_t}(v)$ over all vertices $v \in V(G)$, where $d_{G_t}(v)$ denotes the degree of v in G_t. If each graph G_t is connected, we show that there exists an exploration of G in $\mathcal{O}(n^{3/2}\sqrt{D\log n})$ time steps. In particular, this gives the first subquadratic upper bound when the underlying graph has bounded average degree. As a special case, this also improves the previous best bounds when the underlying graph is planar or has bounded treewidth and provides a unified approach for all of these settings. Our bound is subquadratic already when $D = o(n/\log n)$.

1 Introduction

A great variety of topics, ranging from communication networks to the design of power grids or the study of metabolism, can be represented via graphs [New18]. For plenty of these applications, the underlying graph evolves with time, with some connections appearing or disappearing. The dynamics of these changes is fundamental to the study of such problems and is not captured by static networks. To address this issue, temporal graphs were introduced, as they provide a framework for modelling networks that evolve over time. There has been recent interest in studying complexity-theoretic problems on temporal graphs [BM23, FMN+20, HKMS22, MS21, MS22], specifically related to reachability questions [AKM14, AMSR21, AFGW23, BZ19, EHK21, ES20, ES23, MS16], as well as in proving combinatorial bounds for such questions [AGMZ22, BGK+25, BST25, DEK+24, EHK21, ES18, ES22, IKW14, IW13, TA20]. We refer the readers to [CFQS12, Mic16] for surveys on temporal graphs.

Formally, a temporal graph G over an interval $I \subseteq \mathbb{N}$ is a sequence of graphs $(G_t)_{t\in I}$ that all have the same vertex set V, but possibly different edge sets. We call G_t the *snapshot* at time step t, we say that $(V, \bigcup_{t\in I} E(G_t))$ is the *underlying graph* of G, and we say that G is *always-connected* if G_t is connected for all t.

In the *temporal exploration problem* (TEXP), introduced by Michail and Spirakis [MS16], an agent starts at time step one at a given vertex of a temporal graph G and wants to visit all vertices of V as quickly as possible. At each time step t, the agent can either stay at the current vertex or use an edge of the snapshot G_t to move to an adjacent vertex. We assume that the agent knows the sequence of graphs in advance.

Deciding if a temporal exploration exists is NP-hard in general [MS16]. Even when the graph is always-connected, approximating the shortest length of a temporal exploration up to a factor of $\mathcal{O}(n^{1-\varepsilon})$ is NP-hard for any $\varepsilon > 0$ [EHK21]. Moreover, the problem of deciding if an an exploration of a given length exists remains NP-hard when the underlying graph has pathwidth 2 [BZ19].

Due to the difficulty of approximating a shortest exploration in an always-connected temporal graph, there has been significant interest in determining upper and lower bounds on the length of such an exploration in terms of the number of vertices n. The fact that the temporal graph is always-connected ensures that any vertex

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can reach any other vertex within $n-1$ time steps, and therefore the length of a shortest exploration is always between $n-1$ and n^2. There are temporal graphs that require $\Omega(n^2)$ time steps [EHK15], but the snapshots G_t in these constructions are stars that contain vertices of degree $\Omega(n)$, and the underlying graph is very dense. It is natural to ask whether restricting either the maximum degree of each snapshot G_t or the density of the underlying graph can guarantee short explorations. The star construction can be extended to show that for every d, it is possible that each snapshot G_t has maximum degree d, yet any shortest exploration takes at least $\Omega(dn)$ time steps [EHK15]. There are also examples where the underlying graph is planar of maximum degree 3 and all G_t are paths, but a shortest exploration can still have length $\Omega(n \log n)$ [AGMZ22, EHK15]. Inapproximability results persist even when the underlying graph has bounded maximum degree. For example, for all $\varepsilon, \delta > 0$, if the underlying graph has maximum degree at most d with $d = \Omega(n^\delta)$, it is still NP-hard to approximate the length of a shortest exploration within a factor of $\mathcal{O}(d^{1-\varepsilon})$ [EHK21].

Nonetheless, in the case where each snapshot has bounded maximum degree, Erlebach and Spooner [ES18] gave the first subquadratic upper bound for the shortest exploration. They showed that $\mathcal{O}((n^2 \log d)/\log n)$ time steps suffice if each snapshot has maximum degree at most d. This was substantially improved by Erlebach, Kammer, Luo, Sajenko, and Spooner [EKL+19] who showed that $\mathcal{O}(n^{7/4})$ time steps suffice, provided that either the maximum degree of each snapshot is bounded or the temporal exploration is allowed to make two moves per time step.

In terms of restrictions on the underlying graph, Erlebach, Hoffmann, and Kammer [EHK15, EHK21] showed that temporal graphs can be explored in $\mathcal{O}(k^{1.5} n^{1.5} \log n)$ time steps if the underlying graph has treewidth k and in $\mathcal{O}(n^{1.8} \log n)$ time steps if the underlying graph is planar. Adamson, Gusev, Malyshev, and Zamaraev [AGMZ22] improved these results, showing that $\mathcal{O}(k\, n^{1.5} \log n)$ time steps suffice for treewidth k and $\mathcal{O}(n^{1.75} \log n)$ time steps suffice for planar underlying graphs. In addition, linear or near-linear bounds have been obtained when the underlying graph is a cactus [IKW14], the $2 \times n$-grid [EHK21], a cycle with few chords [AGMZ22, EHK21, TA20], or a graph with small orbit number [DEK+24]. Variations on the temporal exploration problem have also been studied, such as bounding the number of edges used during the exploration [BST25] or finding a short temporal exploration when few edges are removed from the underlying graph in each snapshot [ES22] or when some edges are present for a longer time interval [IW13].

As bounding the length of a shortest exploration in temporal graphs turned out to be quite difficult, this problem has also been studied for a random model by Baguley, Göbel, Klodt, Skretas, Sylvester, and Zamaraev [BGK+25]. Given a probability distribution μ on a collection of trees on n vertices, they sample a random temporal graph by sampling a random tree $G_t \sim \mu$ independently at each time step. In this setting, they showed that there is a temporal exploration of length $\mathcal{O}(n^{3/2})$ with high probability and that this is optimal up to the multiplicative constant factor for a suitable distribution μ on stars. The authors of [BGK+25] also highlighted the study of short explorations of temporal graphs with snapshots of bounded maximum degree as being *"arguably the most fundamental case"* in the deterministic setting.

Our results. We study the temporal exploration problem using a unified approach that applies both to temporal graphs whose snapshots have bounded maximum degree and to temporal graphs whose underlying graph has small edge density. This allows us to simultaneously improve several of the upper bounds mentioned above [AGMZ22, EHK15, EHK21, EKL+19, ES18]. To state our general result, for any vertex $v \in V(G)$, define $d_{\max}(v) \coloneqq \max_{t \in I} d_{G_t}(v)$ as the maximum degree of v over all possible time steps. We define the *average temporal maximum degree* of G as $\sum_{v \in V(G)} d_{\max}(v)/n$. Our main result is the following.

Theorem 1.1. *For any always-connected temporal graph $G = (G_t)_{t \in \mathbb{N}}$ with n vertices and average temporal maximum degree D, there exists a temporal exploration of G spanning at most $\mathcal{O}(n^{3/2}\sqrt{D \log n})$ time steps.*

The average temporal maximum degree encompasses many classes of temporal graphs investigated previously. For example, if the maximum degree of every snapshot G_t is at most d, then the average temporal maximum degree of G is also at most d, and so we obtain the following corollary.

Corollary 1.2. *For any $d \in \mathbb{N}$ and any always-connected temporal graph $G = (G_t)_{t \in \mathbb{N}}$ where G_t has maximum degree at most d for all $t \in \mathbb{N}$, there exists a temporal exploration of G spanning $\mathcal{O}(n^{3/2}\sqrt{d \log n})$ time steps.*

This significantly improves the previous best bound of $\mathcal{O}_d(n^{7/4})$ by Erlebach, Kammer, Luo, Sajenko, and Spooner [EKL+19]. Moreover, in any always-connected temporal graph, the square of each snapshot G_t has a spanning tree of maximum degree at most three [EKL+19, Lemma 4.1], and so the corollary above also implies that $\mathcal{O}(n^{3/2}\sqrt{\log n})$ time steps suffice if the exploration is allowed to make two moves per time step. For this problem, $\mathcal{O}(n^{7/4})$ was the previous best bound as well [EKL+19].

Moreover, observe that if the underlying graph of G has average degree at most k, then the average temporal maximum degree of G is at most k, and so we also obtain the following corollary.

Corollary 1.3. *For any $k \in \mathbb{N}$ and any always-connected temporal graph $G = (G_t)_{t\in\mathbb{N}}$ whose underlying graph has average degree at most k, there exists a temporal exploration of G spanning at most $\mathcal{O}(n^{3/2}\sqrt{k \log n})$ time steps.*

No subquadratic upper bound on the length of a shortest exploration was previously known when the underlying graph has bounded average degree. Using this result as a black box, we also immediately obtain the following consequences for short explorations in temporal graphs whose underlying graph H satisfies certain structural properties.

- If H is planar, then there exists an exploration spanning $\mathcal{O}(n^{3/2}\sqrt{\log n})$ time steps. The previous best bound was $\mathcal{O}(n^{7/4} \log n)$ [AGMZ22].
- If H has treewidth at most k, then there exists an exploration spanning $\mathcal{O}(n^{3/2}\sqrt{k \log n})$ time steps. This removes a factor of order $\sqrt{k \log n}$ from the previous best bound $\mathcal{O}(k\, n^{3/2} \log n)$ [AGMZ22].
- If H is K_t-minor-free, then H has average degree $\mathcal{O}(t\sqrt{\log t})$ [Kos82, Kos84, Tho84], and so there exists an exploration spanning $\mathcal{O}(t^{1/2} \log^{1/4} t \cdot n^{3/2}\sqrt{\log n})$ time steps.
- If H has at most $o(n^2/\log n)$ edges, then there exists an exploration spanning $o(n^2)$ time steps.

Finally, we remark that the proof of Theorem 1.1 can easily be made algorithmic: for all theorems stated above, there is a polynomial-time algorithm which, given a temporal graph, constructs an exploration satisfying the stated bounds.

2 Proof of Theorem 1.1

We write $[\ell, r] \coloneqq \{\ell, \dots, r\}$ and $[r] \coloneqq [1, r]$. All logarithms in this paper have base 2.

Let $G = (G_t)_{t\in\mathbb{N}}$ be a temporal graph. We will always use n to denote the number of vertices of G. For an interval $I \subseteq \mathbb{N}$, let G_I denote the temporal subgraph $(G_t)_{t\in I}$. A *temporal walk* in G is a sequence of vertices $W = (w_\ell, \dots, w_{r+1})$ where $[\ell, r] \subseteq \mathbb{N}$ and for every $t \in [\ell, r]$ either $w_t = w_{t+1}$ or $w_t w_{t+1} \in E(G_t)$. We say that W *spans* $r - \ell + 1$ time steps and *covers* the vertices $w_\ell, \dots, w_{r+1}$. A *temporal exploration* of G is a temporal walk that starts at time step 1 and covers all vertices of G.

Observe that for any k vertices in a connected graph on n vertices, there is always a path between two of these vertices of length at most $2n/k$. The key step in our argument is the following lemma which proves an analogous result for temporal graphs, quantifying the number of time steps required to guarantee that there is a temporal walk between two distinct vertices in a given subset of vertices.

Lemma 2.1. *Let $G = (G_t)_{t\in I}$ be an always-connected temporal graph with average temporal maximum degree D, and let $X \subseteq V(G)$ be a set of at least two vertices. If $|I| \geqslant 2Dn/|X| + 1$, then there exist two distinct vertices $u, v \in X$ such that there is a temporal walk from u to v in G.*

We remark that this lemma is tight up to the constant factor for $|X| \geqslant D$ (see Lemma 2.4 for $|X| < D$). To illustrate that, consider the temporal graph $G = (G_t)_{t\in I}$ where $I \coloneqq [\ell]$ with $\ell \geqslant D$ and for $t \in [\ell]$, the graph G_t consists of an $\ell \times m$ grid with $D - 4$ leaves attached to each vertex in row t, where each snapshot G_t uses the same set X of leaves (see Fig. 1). Then, it can be shown inductively that for any $t \in [\ell]$, a temporal walk in $G_{[t]}$ that starts at some vertex $x \in X$ can only remain at x or must end at a grid vertex in one of the first t rows. This implies that there exists no temporal walk in G between two distinct vertices of X. However, $n = \Theta((\ell + D)m)$ and $|X| = \Theta(Dm)$, and therefore $Dn/|X| = \Theta(\ell + D) = \Theta(|I|)$.

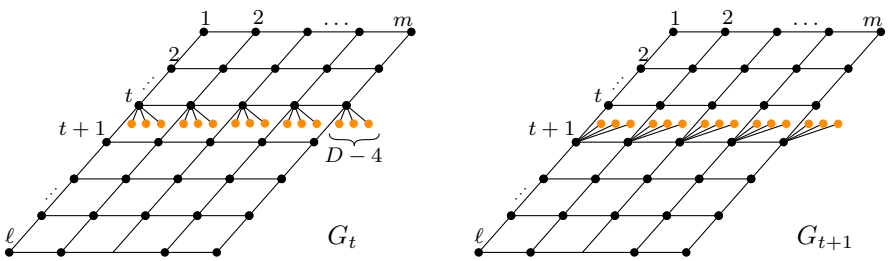

Figure 1: The figure depicts two consecutive snapshots from the construction that shows that Lemma 2.1 is tight. The orange vertices represent the set X.

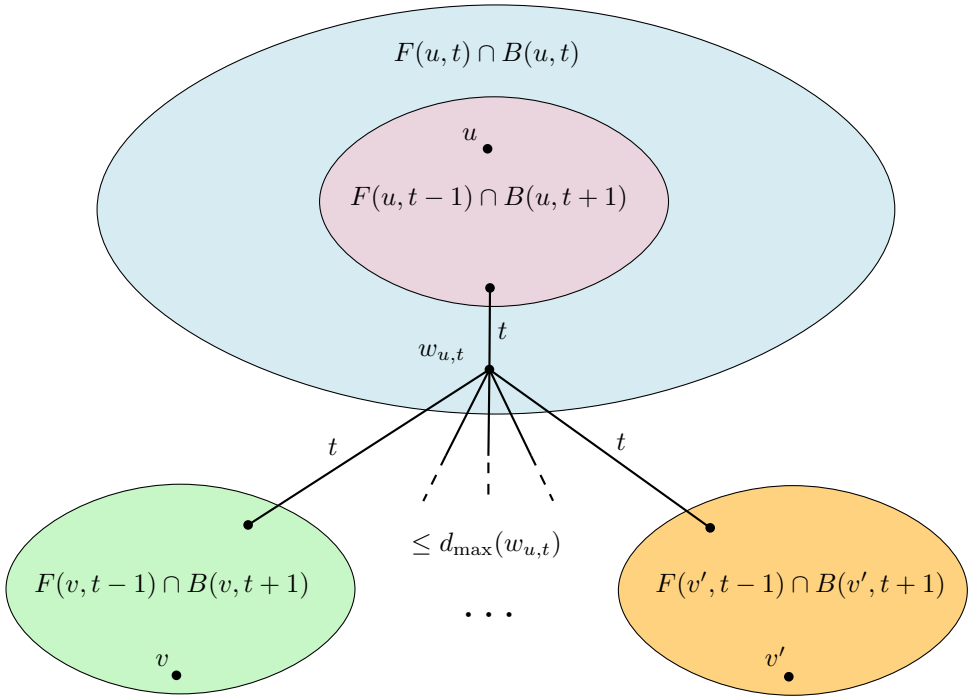

Figure 2: The figure depicts the situation described in the proof of Lemma 2.2. The fact that G_t is connected implies the existence of a vertex $w_{u,t}$ outside of but adjacent to $F(u,t-1) \cap B(u,t+1)$. By construction, $w_{u,t}$ will be in $F(u,t) \cap B(u,t)$. Crucially, $d_{G_t}(w_{u,t}) \leqslant d_{\max}(w_{u,t})$. Therefore, $w_{u,t}$ is *recorded* by at most $d_{\max}(w_{u,t})$ vertices at time step t. We stress that even though some sets are represented to be disjoint for clarity, this is not always the case.

Proof of Lemma 2.1. Let $I = [\ell, r]$. For all $u \in X$ and $t \in I$, define

$$F(u,t) \coloneqq \{w \in V : \text{there is a temporal walk from } u \text{ to } w \text{ in } G_{[\ell,t]}\} \text{ and}$$
$$B(u,t) \coloneqq \{w \in V : \text{there is a temporal walk from } w \text{ to } u \text{ in } G_{[t,r]}\}.$$

Also define $F(u,\ell-1) = B(u,r+1) \coloneqq \{u\}$. For all $u \in X$ and $t \in I$ we have $u \in F(u,t-1) \cap B(u,t+1)$. Since we are done if $F(u,t-1) \cap B(u,t+1) = V(G)$ for some $t \in I$, suppose that $F(u,t-1) \cap B(u,t+1) \neq V(G)$ for all $t \in I$.

Note that for any set S which is not the empty set or $V(G)$, there is an edge between S and $V(G) \setminus S$ in G_t because G_t is connected. So, there exists a vertex $w_{u,t} \in N_{G_t}(F(u,t-1) \cap B(u,t+1))$. Fix such a vertex and say that u *records* $w_{u,t}$ at time step t. By definition, we have

$$w_{u,t} \notin F(u,t-1) \cap B(u,t+1)$$

and

$$w_{u,t} \in N_{G_t}(F(u,t-1) \cap B(u,t+1)) \subseteq F(u,t) \cap B(u,t),$$

where the inclusion holds because $F(u,t-1) \cup N_{G_t}(F(u,t-1)) = F(u,t)$ and, similarly, $B(u,t+1) \cup N_{G_t}(B(u,t+1)) = B(u,t)$ (see Fig. 2).

We claim that for any $w \in V(G)$ and $u \in X$, there are at most two time steps $t \in I$ such that $w_{u,t} = w$. Indeed, if there were three such time steps $t_1 < t_2 < t_3$, then as w is recorded at time steps t_1 and t_3, we get that

$$w \in F(u,t_1) \cap B(u,t_3) \subseteq F(u,t_2-1) \cap B(u,t_2+1).$$

However, since $w = w_{u,t_2}$, we should also have $w \notin F(u,t_2-1) \cap B(u,t_2+1)$, a contradiction.

Since every vertex in X records one vertex at each of $|I|$ different time steps, the total number of records is $|X| \cdot |I| > 2Dn = 2\sum_{w \in V(G)} d_{\max}(w)$. In particular, there exists a vertex $w \in V(G)$ that is recorded more than $2 \cdot d_{\max}(w)$ times. This implies that w must have been recorded by more than $d_{\max}(w)$ distinct vertices from X since each vertex from X records it at most twice.

If all of these vertices record w at the same time step $t \in I$, then as there are more than $d_{\max}(w) \geqslant d_{G_t}(w)$ many of these vertices, there must exist two distinct vertices $u, v \in X$ and a neighbour $x \in N_{G_t}(w)$ such that

$x \in F(u, t-1) \cap B(u, t+1)$ and $x \in F(v, t-1) \cap B(v, t+1)$. But this implies that there is a temporal walk from u to x in $G_{[\ell,t-1]}$ and a temporal walk from x to v in $G_{[t+1,r]}$, and so by waiting at x for one time step, the concatenation of these walks yields a temporal walk from u to v in G, as required.

Otherwise, it follows that there exist two distinct vertices $u, v \in V(G)$ and two time steps $t_1 < t_2$ such that $w = w_{u,t_1} = w_{v,t_2}$. But then there is a temporal walk from u to w in $G_{[\ell,t_1]}$ and a temporal walk from w to v in $G_{[t_2,r]}$, and so by waiting at w for $t_2 - t_1 - 1$ time steps, the concatenation of these walks yields a temporal walk from u to v in G, as required. □

When constructing a temporal exploration, we can apply the lemma above to the set X of unexplored vertices, but also to any subset of X of a fixed size. This provides us with several walks between unexplored vertices. Then, we could try to construct a temporal exploration by repeating this on multiple disjoint time intervals and linking the obtained walks across different time intervals. Unfortunately, it quickly becomes difficult to link these walks in such a way that they contain many *distinct* unexplored vertices.

Instead, we use the preceding lemma to find a small subset of X such that any other vertex in X can be reached by a temporal walk starting from some vertex in this small subset. In later time intervals, we can then simply avoid this small subset entirely to ensure that we only visit distinct unexplored vertices.

Lemma 2.2. *Let $G = (G_t)_{t \in I}$ be an always-connected temporal graph with average temporal maximum degree D, let $X \subseteq V(G)$ be a set of at least two vertices, and let $k \geqslant 2$ be an integer. If $|I| \geqslant 2Dn/k + 1$, then there exists a set of vertices $S \subseteq X$ with*

$$|S| \leqslant \frac{\log|X|}{-\log(1 - 1/(2k))} \leqslant 2k \log|X|$$

such that for all $u \in X$ there is a vertex $v \in S$ with a temporal walk from v to u in G.

Proof. We prove the statement by induction on $|X|$. If $|X| \leqslant 2k$, then the statement holds for $S := X$. Otherwise, for any $v \in X$ define

$$F(v) := \{u \in X \setminus \{v\} : \text{there is a temporal walk from } v \text{ to } u \text{ in } G\} \text{ and}$$
$$B(v) := \{u \in X \setminus \{v\} : \text{there is a temporal walk from } u \text{ to } v \text{ in } G\}.$$

Let $X_0 := X$. Then, for $i \geqslant 1$, if $X_{i-1} \neq \varnothing$, we can select a vertex $v_i \in X_{i-1}$ such that $|F(v_i) \cap X_{i-1}| \geqslant |B(v_i) \cap X_{i-1}|$ and define $X_i := X_{i-1} \setminus (F(v_i) \cup B(v_i) \cup \{v_i\})$. Let ℓ be minimal such that $X_\ell = \varnothing$. Note that for any $i \neq j$, there is no temporal walk from v_i to v_j in G, and so by Lemma 2.1 we get $\ell < k$. This implies that there exists some i such that $|(F(v_i) \cup B(v_i) \cup \{v_i\}) \cap X_{i-1}| \geqslant |X|/(k-1)$ and so

$$|F(v_i) \cap X| \geqslant |F(v_i) \cap X_{i-1}| \geqslant \frac{|(F(v_i) \cup B(v_i)) \cap X_{i-1}|}{2} \geqslant \frac{|X|}{2(k-1)} - \frac{1}{2} \geqslant \frac{|X|}{2k} - 1.$$

In particular, the set $X' := X \setminus (F(v_i) \cup \{v_i\})$ has size at most $(1 - 1/(2k)) \cdot |X|$. By applying the induction hypothesis to X', we obtain a set of vertices $S' \subseteq X'$ with

$$|S'| \leqslant \frac{\log|X'|}{-\log(1 - 1/(2k))} \leqslant \frac{\log((1 - 1/(2k)) \cdot |X|)}{-\log(1 - 1/(2k))} = \frac{\log|X|}{-\log(1 - 1/(2k))} - 1$$

such that for all $u \in X'$ there is a vertex $w \in S'$ with a temporal walk from w to u in G. Therefore, $S := S' \cup \{v_i\}$ satisfies the statement of the lemma. □

We now construct a temporal walk that covers many vertices of X in n time steps. We do this by dividing the n time steps into multiple time intervals. In each time interval, we use the preceding lemma to find a small subset S of X that can reach all other vertices of X within this time interval, and we remove all vertices of S from X. We then show that these sets allow us to construct a temporal walk that visits exactly one vertex in each time interval from the small subset of vertices that we removed during that time interval. This way, all explored vertices will be distinct.

Lemma 2.3. *Let $G = (G_t)_{t \in I}$ be an always-connected temporal graph with average temporal maximum degree D, and let $X \subseteq V(G)$ be a set of at least two vertices. If $|I| \geqslant n$, then there exists a temporal walk in G that covers at least $\frac{1}{8}\sqrt{|X|/(D \cdot \log|X|)}$ vertices of X.*

Proof. Let $m := \left\lfloor \frac{1}{8}\sqrt{|X|/(D \cdot \log|X|)} \right\rfloor$ and $k := \left\lceil \sqrt{D \cdot |X| / \log|X|} \right\rceil$. If $m = 0$, then any temporal walk that stays at a vertex in X satisfies the statement, so we may assume that $m \geqslant 1$. We partition I into m intervals $I_1, \ldots, I_m$ with at least $\lfloor n/m \rfloor$ steps each. Iteratively for $i = 1, \ldots, m$, we apply Lemma 2.2 to obtain a set

$S_i \subseteq X \setminus (S_1 \cup \cdots \cup S_{i-1})$ with $|S_i| \leqslant 2k \log|X|$ such that for all $u \in X \setminus (S_1 \cup \cdots \cup S_{i-1})$ there is a vertex $v \in S_i$ with a temporal walk from v to u in G_{I_i}. This is possible since for every $i \in [m]$,

$$|I_i| \geqslant \left\lfloor \frac{n}{m} \right\rfloor \geqslant \frac{n}{2m} \geqslant 4n\sqrt{\frac{D \cdot \log|X|}{|X|}} = 4Dn\sqrt{\frac{\log|X|}{D \cdot |X|}} \geqslant 4Dn/k \geqslant 2Dn/k + 1.$$

The union of the sets $S_1, \ldots, S_m$ has size at most

$$m \cdot 2k \log|X| \leqslant \frac{1}{8}\sqrt{\frac{|X|}{D \cdot \log|X|}} \cdot 2 \cdot 2\sqrt{\frac{D \cdot |X|}{\log|X|}} \cdot \log|X| = \frac{|X|}{2} < |X|$$

Therefore, there exists a vertex $v_{m+1} \in X \setminus (S_1 \cup \cdots \cup S_m)$. Iteratively for $i = m, \ldots, 1$, since S_i satisfies Lemma 2.2 there is a vertex $v_i \in S_i$ with a temporal walk from v_i to v_{i+1} in G_{I_i}. By concatenating these walks, this yields a temporal walk in G that covers at least $m + 1$ vertices of X. □

Finally, we repeatedly apply the preceding lemma to find a temporal walk which covers many vertices that have not been covered so far. To ensure that we can reach the start of the next temporal walk from the end of the previous temporal walk, we always leave $n - 1$ time steps between these walks as this allows any vertex to reach any other vertex during those time steps. This follows from the following lemma, first shown in [EHK21].

Lemma 2.4 ([EHK21]). *Let $G = (G_t)_{t \in I}$ be an always-connected temporal graph, and let $u, v \in V(G)$ be two vertices. If $|I| \geqslant n - 1$, then there is a temporal walk from u to v in G.*

Using this strategy, we now prove our main result.

Proof of Theorem 1.1. Let $G = (G_t)_{t \in \mathbb{N}}$ be a temporal graph. We iteratively construct a temporal walk W of G and a sequence of sets of vertices $V(G) =: X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ such that every vertex in $V(G) \setminus X_i$ has been covered by W before time step $2in$. Suppose we have already constructed X_i and the temporal walk W up to step $2in$. If $|X_i| \leqslant 2$, then by two consecutive applications of Lemma 2.4, we can extend W to visit all vertices in X_i before time step $2(i+1)n$, and we set $X_{i+1} = \varnothing$. Otherwise, by Lemma 2.3, there exists a temporal walk W' in $G_{[(2i+1)n, 2(i+1)n-1]}$ that covers at least $\sqrt{|X_i|/(D \cdot \log|X_i|)}/8$ vertices of X_i. Observe that there exists a temporal walk between every pair of vertices in $G_{[2in+1,(2i+1)n-1]}$ by Lemma 2.4. Therefore, we can extend W by a temporal walk in $G_{[2in+1,(2i+1)n-1]}$ to the first vertex of W', and then extend it by W' itself. So, if X_{i+1} denotes the set of vertices not covered before time step $2(i+1)n$ by this extended temporal walk, then

$$|X_{i+1}| \leqslant |X_i| - \frac{1}{8}\sqrt{\frac{|X_i|}{D \cdot \log|X_i|}}. \tag{1}$$

We claim that there exists some $i_0 = \mathcal{O}(\sqrt{D \cdot n \log n})$ with $X_{i_0} = \varnothing$. Indeed, note that if ℓ is an integer with $\ell \geqslant 4\sqrt{2D \cdot |X_i| \log|X_i|}$, then we can deduce from (1) that

$$|X_{i+\ell}| \leqslant \frac{|X_i|}{2}.$$

Therefore, if $|X_i| \leqslant n/2^k$ for some k, then for $\ell_k := \left\lceil 4\sqrt{2D \cdot (n/2^k) \log(n/2^k)} \right\rceil$ we get $|X_{i+\ell_k}| \leqslant n/2^{k+1}$. This shows that $X_{i_0} = \varnothing$ for

$$i_0 := \sum_{k=0}^{\lceil \log n \rceil} \ell_k \leqslant \sqrt{D \cdot n \log n} \sum_{k=0}^{\infty} \frac{8\sqrt{2}}{2^{k/2}} = \mathcal{O}\left(\sqrt{D \cdot n \log n}\right).$$

So, at time step $2ni_0 = \mathcal{O}(n^{3/2}\sqrt{D \log n})$, all vertices have been covered. □

3 Conclusion

In this paper, we have shown that any always-connected temporal graph $G = (G_t)_{t \in \mathbb{N}}$ on n vertices admits a temporal exploration spanning at most $\mathcal{O}(n^{3/2}\sqrt{D \log n})$ time steps, where $D := \sum_{u \in V(G)} \max_t d_{G_t}(u)/n$ denotes the average temporal maximum degree of G. Although this significantly reduces the best upper bounds known for various natural classes of temporal graphs, such as when each G_t has bounded maximum degree, when each G_t is a path, or when the underlying graph is planar, the gap to the best lower bounds known remains large,

where the best lower bound is $\Omega(n \log n)$ for each of these three examples. It is unclear to us whether the lower or upper bounds should be correct. There is also a lower bound of $\Omega(dn)$ if d is a bound on the maximum degree of each snapshot G_t, but this construction seems difficult to combine with the $\Omega(n \log n)$ lower bound. We therefore raise the following problem.

Problem 3.1. *Is there a function $f = \omega(1)$ such that for all integers $n, D \geqslant 1$, there is a temporal graph $G = (G_t)_{t \in \mathbb{N}}$ on n vertices with average temporal maximum degree at most D, yet the length of a shortest temporal exploration of G is at least $f(D) \cdot n \log n$?*

One way to obtain such a lower bound construction would be via snapshots of maximum degree at most d, but perhaps this more general setting allows for different approaches.

Another direction for future research is to study whether our techniques can provide further improvements when additional structural constraints are placed on the temporal graph besides bounding its average temporal maximum degree.

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