Received: 22 August 2022 | Accepted: 14 April 2024

DOI: 10.1111/mafi.12434

ORIGINAL ARTICLE

MATHEMATICAL FINANCE An International Journal of Mathematics, Statistics and Financial Economics WILEY

Stability of the Epstein–Zin problem

Michael Monoyios[1] | **Oleksii Mostovyi**[2]

[1]Mathematical Institute, University of Oxford, Oxford, UK

[2]Department of Mathematics, University of Connecticut, Storrs, Connecticut, USA

Correspondence
Oleksii Mostovyi, Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA.
Email: oleksii.mostovyi@uconn.edu

Funding information
National Science Foundation, Grant/Award Number: DMS1848339

Abstract

We investigate the stability of the Epstein–Zin problem with respect to small distortions in the dynamics of the traded securities. We work in incomplete market model settings, where our parametrization of perturbations allows for joint distortions in returns and volatility of the risky assets and the interest rate. Considering empirically the most relevant specifications of risk aversion and elasticity of intertemporal substitution, we provide a condition that guarantees the convexity of the domain of the underlying problem and results in the existence and uniqueness of a solution to it. Then, we prove the convergence of the optimal consumption streams, the associated wealth processes, the indirect utility processes, and the value functions in the limit when the model perturbations vanish.

KEYWORDS

BSDE, Epstein–Zin problem, Epstein–Zin utility, incomplete market, stability, stochastic differential utility

JEL CLASSIFICATION

C61, G11

1 | INTRODUCTION

Recursive utilities of Epstein–Zin type allow for the incorporation of *future* consumption choice into preferences. In the discrete-time environment, this topic goes back to Kreps and Porteus (1978) and Epstein and Zin (1989), whereas in continuous-time stochastic settings, it was originally investigated in Duffie and Epstein (1992). These utilities allowed for the resolution of several asset pricing puzzles; see the introduction to Xing (2017a) for an overview of this topic. The Epstein–Zin problem remains an active research area. Thus, recently, explicit solutions are characterized in Xing (2017a), Kraft et al. (2017), and Matoussi and Xing (2018); for the results in infinite time horizon settings, we refer to Herdegen et al. (2023), Herdegen et al. (2021), and Aurand and Huang (2023); a finite yet random horizon is considered in Aurand and Huang (2021).

The continuous-time counterpart of a recursive utility is also known as a stochastic differential utility. Two constants govern its parametrization. One is the usual risk aversion, and the other is an elasticity of intertemporal substitution (EIS) that specifies the willingness to interchange consumption over time. As pointed out in Xing (2017a, Remark 2.1, p. 231), the empirically most relevant case corresponds to the relative risk aversion $\gamma > 1$ and the EIS $\psi > 1$.

The notion of the well-posedness of a mathematical problem goes back to Hadamard (1902), and it comprises the following three properties for a solution to a given problem to hold: existence, uniqueness, and continuous dependence on the initial data, where the last property is loosely known as stability. While the existence and uniqueness results (and various characterizations of the solution) for the Epstein–Zin problem are established in the papers mentioned above, the questions of stability in the context of this problem, to the best of our knowledge, have not been answered before.

An additional motivation for studying stability comes from the fact that in many cases, for example, in the factor model considered in Kraft et al. (2017), the explicit solution ceases to exist under general perturbations of the model parameters, where such perturbations can be associated with a procedure of calibration. In this case, it is important to understand whether the outputs of the problem, such as the optimal consumption, the optimal wealth process, the indirect utility process, and the value function, differ only slightly from the solution to the unperturbed problem admitting an explicit solution.

In the case of the more traditional additive utility, which corresponds to a particular case of the Epstein–Zin problem ($\gamma = \frac{1}{\psi}$, in the present notations), the questions of stability are studied more, and historically, and they have also followed establishing the existence and uniqueness results. The results on the stability of the outputs to the optimal investment problem with respect to various perturbations and in varying formulations are contained in Jouini and Napp (2004), Carasus and Rasonyi (2007), Kardaras and Žitković (2011), Xing (2017b), Veraguas and Silva (2018), and Mostovyi (2021), among others. These works do not establish any stability to BSDEs result, in contrast to the present paper, as the analysis of the stability of the optimal investment problem in many formulations relies on different techniques, despite the BSDE-base approach pioneered in Hu et al. (2005). Thus, compared to the papers on the stability of the traditional utility maximization in various formulations mentioned in this paragraph, we rely on the analysis of BSDE and establish related approximation and stability results in the present work.

In view of the previously listed works, one can argue that the literature on the Epstein–Zin problem does not contain its stability analysis. The aim of the present paper is to give insight into this problem, and thus, here, we investigate the stability of the Epstein–Zin problem with respect to perturbations of the dynamics of the traded securities. Our parametrization of perturbations allows us to include joint or separate distortions of the interest rate as well as of the return and volatility of the risky assets. We consider the above-described case when both the relative risk aversion and EIS exceed one. Our analysis is performed under a weak no-arbitrage condition, no unbounded profit with bounded risk (NUPBR) introduced in Karatzas and Kardaras (2007), which still allows for the meaningful structure of the underlying problem.

Our results include a sufficient condition for the convexity of the domain of the primal problem and for the existence and uniqueness of the optimizer to this problem. This condition can be stated as nonemptiness of the dual domain, that is, the existence of a state price density satisfying an integrability condition, which guarantees a unique solution of class D to the dual BSDE, see Lemma 2.1. We also show the convergence of the value functions, the optimal consumption streams, the associated wealth processes, and the indirect utility processes as perturbations vanish.

One of the difficulties in the analysis involves establishing stability-type estimates for the solutions to BSDEs with an unbounded terminal condition and non-Lipschitz generator, with respect to particular perturbations of both the terminal condition and the generator. Here, we establish a ucp convergence result for the family of solutions to such BSDEs, see Lemma 5.6. Further, it is crucial for the proof to show the strict r(and stronger than strict) concavity of the rvalue function, in a sense Lemma 5.5. All these estimates are needed to establish the convergence of the optimal consumption streams, whereas the convergence of the value functions relies on conjugacy results from Matoussi and Xing (2018) and on a particular construction of the nearly optimal consumption streams also combined with localization.

The remainder of this paper is organized as follows: in Section 2, we specify the model and Section 3 contains the main results. In Section 4, we discuss the integrability condition on the perturbations, and the proofs are given in Section 5.

2 | MODEL

2.1 | Market

Let $\left(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}, \mathbb{P}\right)$ be a complete stochastic basis, where $T \in (0,\infty)$ is the time horizon, $\mathcal{F}_0$ is the completion of the trivial σ-field, $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmented filtration generated by a $(k+n)$-dimensional Brownian motion $B = \left(W, W^{\perp}\right)$, where W represents the first k components and $W^{\perp}$ the remaining n components. For an $\mathbb{R}^{n\times n}$-valued $\mathcal{F}^W$-adapted process ρ taking values in $\mathbb{R}^{n\times k}$ and for an $\mathbb{R}^{n\times n}$-valued adapted process[1] $\rho^{\perp}$ satisfying $\rho\rho^{\top} + \rho^{\perp}(\rho^{\perp})^{\top} = I_{n\times n}$, the n-dimensional identity matrix, we set

$$W^{\rho} := \int_0^{\cdot} \rho_s dW_s + \int_0^{\cdot} \rho_s^{\perp} dW_s^{\perp}.$$

We consider a family of markets parametrized by $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$. Thus, for a fixed ε, the traded assets are $(S^{\varepsilon,0}, \dots, S^{\varepsilon,n})$, where $S^{\varepsilon,0}$ is the price process of the riskless asset and $(S^{\varepsilon,1}, \dots, S^{\varepsilon,n})$ are the prices of the risky assets. Their evolution is given by

$$dS_t^{\varepsilon,0} = S_t^{\varepsilon,0} r_t^{\varepsilon} dt, \quad dS_t^{\varepsilon,i} = S_t^{\varepsilon,i}\left(\left(r_t^{\varepsilon} + \mu_t^{\varepsilon,i}\right)dt + \sum_{j=1}^{n} \sigma_t^{\varepsilon,i,j} dW_t^{\rho,j}\right), \quad i \in \{1,\dots,n\}, \tag{1}$$

where the processes $r^{\varepsilon} \geq 0$, $\mu^{\varepsilon,i}$, and $\sigma^{\varepsilon,i}$ are $\mathcal{F}^W$-adapted processes such that the integrals in Equation (1) are well-defined and such that σ_t^{ε} is invertible, $t \in [0,T]$, $\mathbb{P}$-a.s..

[1] Through process ρ, one, in particular, can include stochastic volatility-type models as in Kraft et al. (2017, Section 4). We note that the model in Kraft et al. (2017) allows for an explicit solution to the Epstein–Zin problem via Hamilton–Jacobi–Bellman equations, and under perturbations of the model parameters, the structure allowing for explicit solutions can be lost.

In particular, our parametrization of perturbations allows us to include the following cases:

- Perturbations of the drift μ only. This corresponds to setting $r^\varepsilon \equiv r^0$, and $\sigma^{\varepsilon,i,j} \equiv \sigma^{0,i,j}$, for every $i, j \in \{1, \dots, n\}$.
- Perturbations of the volatility σ only.
- Similarly, we can consider perturbations of the interest rate only. In many works in mathematical finance, the riskless asset is assumed to be constant-valued. While this gives the correct structure to many problems of mathematical finance, having nonzero interest rates can also be significant and leads to extra technicalities.
- Perturbations of the numéraire, where the parametrization of such perturbations can follow the ones in Mostovyi (2020).
- Combinations of perturbations above.

2.2 | The Epstein–Zin problem

For every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, let $\pi^\varepsilon = \left(\pi^{\varepsilon,0}, \dots, \pi^{\varepsilon,n}\right)$ be an S^ε-integrable $\mathbb{R}^{n+1}$-valued process representing the proportions of the total wealth invested in the respective assets, thus, satisfying r $\sum_{i=0}^n \pi_t^{\varepsilon,i} = 1$, $t \in [0, T]$. Let c^ε be a non-negative progressively measurable process representing the consumption rate in the εth market. Let κ be a deterministic consumption clock given by $\kappa_t = t + 1_{\{T\}}(t)$, $t \in [0, T]$. We specify the dynamics of the wealth process $X^{\varepsilon,\pi^\varepsilon,c^\varepsilon}$ associated with consumption–investment pair $(\pi^\varepsilon, c^\varepsilon)$ and starting from an initial wealth x as follows

$$dX_t^{\varepsilon,\pi^\varepsilon,c^\varepsilon} = X_t^{\varepsilon,\pi^\varepsilon,c^\varepsilon} \sum_{i=0}^n \pi_t^{\varepsilon,i} \frac{dS_t^{\varepsilon,i}}{S_t^{\varepsilon,i}} - c_t^\varepsilon d\kappa_t, \quad X_0^\varepsilon = x. \tag{2}$$

We call a consumption process c^ε admissible from $x > 0$ for the εth market, if there exists an S^ε-integrable process π^ε, such that $\sum\limits_{i=0}^n \pi_t^{\varepsilon,i} = 1$, $t \in [0, T]$, and the associated wealth process in Equation (2) is non-negative, $\mathbb{P}$-a.s.. We denote the family of *admissible* consumptions from $x > 0$ in the εth market by $\mathcal{A}(x, \varepsilon)$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

An agent, starting with an initial capital $x > 0$, invests and consumes in the market in a way to maximize his or her expected utility with Epstein–Zin preferences. With $\delta > 0$ representing the discount rate, $0 < \gamma \neq 1$ being the relative risk aversion, and $0 < \psi \neq 1$ specifying the EIS, one can define the Epstein–Zin aggregator f via

$$f(c, u) = \delta \frac{c^{1-\frac{1}{\psi}}}{1 - \frac{1}{\psi}} ((1-\gamma)u)^{1-\frac{1}{\theta}} - \delta\theta u, \quad c > 0 \quad \text{and} \quad (1-\gamma)u > 0, \tag{3}$$

where $\theta := (1-\gamma)/\left(1 - \frac{1}{\psi}\right)$. Given the bequest utility $U_T(c) = c^{1-\gamma}/(1-\gamma)$, $c > 0$, the Epstein–Zin utility for a non-negative consumption stream c is a process $\left(U_t^c\right)_{t \in [0,T]}$, which satisfies the BSDE

$$U_t^c = \mathbb{E}_t \left[U_T(c_T) + \int_t^T f\left(c_s, U_s^c\right) ds \right], \quad t \in [0, T], \tag{4}$$

where $\mathbb{E}_t$ is $\mathbb{E}[\,\cdot\,|\mathcal{F}_t]$. Sufficient conditions for the existence of U^c for a given c are contained in Matoussi and Xing (2018, Proposition 2.1).

The agent aims to maximize his or her Epstein–Zin utility at time zero over all admissible strategies, that is

$$\sup_{c\in\mathcal{A}(x,\varepsilon)} U^c_0, \quad (x,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0). \tag{5}$$

This formulation, however, does not guarantee that for a given $c\in\mathcal{A}(x,\varepsilon)$, U^c in Equation (4) is well-defined. As pointed out in Matoussi and Xing (2018, Remark 2.2), one needs some mild integrability properties on the elements of $\mathcal{A}(x,\varepsilon)$, $(x,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0)$. Below, we provide some insights on this issue. For this, we need to introduce the state price densities.

2.3 | State price density processes

The family of state price density processes is defined as

$$\begin{aligned}\mathcal{D}(y,\varepsilon) :=\{D>0 : D_0 = y, DX^{\varepsilon,\pi,c} + \int_0^\cdot D_s c_s d\kappa_s \\ \mathit{is\ a\ supermartingale\ for\ every}\ c\in\mathcal{A}(1,\varepsilon)\}, \\ (y,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0),\end{aligned} \tag{6}$$

where (π,c) is the investment–consumption pair, such that $X^{\varepsilon,\pi,c}$ in Equation (2) is non-negative. Thus, one can see that the family of minimal state price densities

$$D^{\varepsilon,0} := \mathcal{E}\left(-\int_0^\cdot r^\varepsilon_s ds - \int_0^\cdot \left((\sigma^\varepsilon_s)^{-1}\mu^\varepsilon_s\right) dW^\rho_s\right), \quad \varepsilon\in(-\varepsilon_0,\varepsilon_0), \tag{7}$$

is well-defined, where $\mathcal{E}$ denotes the stochastic exponential. In particular, since, for every ε, the set of state price densities is nonempty, and this also applies to the set of supermartingale deflators, this precludes the arbitrage opportunities in the sense of unbounded profit with bounded risk (UPBR) introduced in Karatzas and Kardaras (2007), we also refer to Karatzas and Kardaras (2021, Chapter 2) for its multiple equivalent characterizations. In other words,

$$\mathit{NUPBR\ \ holds\ \ for\ \ every}\ \ \varepsilon\in(-\varepsilon_0,\varepsilon_0). \tag{8}$$

For the BSDE characterizations, as in Xing (2017a), it is important to restrict the admissible consumptions to the ones that are also integrable in a sense made precise below. Thus, one can define

$$\widetilde{\mathcal{A}}(x,\varepsilon) := \left\{c\in\mathcal{A}(x,\varepsilon) :\ \mathbb{E}\left[\int_0^T c_s^{1-\frac{1}{\psi}} ds\right] < \infty\right\}. \tag{9}$$

Formally, in Xing (2017a), also $\mathbb{E}\left[c_T^{1-\gamma}\right] < \infty$ is imposed. However, for every constant $\delta>0$, a consumption plan satisfying $c_T\geq\delta$ satisfies $\mathbb{E}\left[c_T^{1-\gamma}\right]<\infty$. In particular, the plans such that

$\mathbb{E}\left[c_T^{1-\gamma}\right] = \infty$ correspond to small values of c_T, and thus are suboptimal. By setting the associate $U^c \equiv -\infty$ for every c such that $\mathbb{E}\left[c_T^{1-\gamma}\right] = \infty$, one can rule them out. If all consumption plans allow for $\mathbb{E}\left[c_T^{1-\gamma}\right] = \infty$, then intuitively, the problem is degenerate. This, however, does not happen if the interest rate $r^0 \geq 0$, in which case constant-valued consumptions are admissible and integrable in the sense above.

Having ruled out the possibility of $\mathbb{E}\left[c_T^{1-\gamma}\right] = \infty$ for all consumption plans, as in the paragraph above, one can provide a sufficient condition for $\mathbb{E}\left[\int_0^T c_s^{1-\frac{1}{\psi}} ds\right] < \infty$ to hold for every $c \in \mathcal{A}(x, \varepsilon)$. It is related to a characterization via the reverse Hölder inequality in the spirit of Nutz (2010, Proposition 4.5) and Kazamaki (1994).

The following lemma provides a sufficient condition for $\mathcal{A}(x, \varepsilon) = \widetilde{\mathcal{A}}(x, \varepsilon)$.

Lemma 2.1. *Let $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ be fixed and suppose that there exists $D \in \mathcal{D}(1, \varepsilon)$, such that*

$$\mathbb{E}\left[\int_0^T D_s^{1-\psi} ds\right] < \infty. \tag{10}$$

Then, we have

$$\mathcal{A}(x, \varepsilon) = \widetilde{\mathcal{A}}(x, \varepsilon), \quad x > 0. \tag{11}$$

Proof. Let us fix $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Then, for every $D \in \mathcal{D}(1, \varepsilon)$, along the lines of Mostovyi (2015, Proposition 4.2), one can show that

$$\mathbb{E}\left[\int_0^T D_s c_s ds\right] \leq 1, \quad for\ every\ c \in \mathcal{A}(1, \varepsilon). \tag{12}$$

Next, let us consider $D \in \mathcal{D}(1, \varepsilon)$, satisfying Equation (10). Then, for an arbitrary $c \in \mathcal{A}(1, \varepsilon)$, using Hölder's inequality, we get

$$\begin{aligned}
\mathbb{E}\left[\int_0^T c_s^{1-\frac{1}{\psi}} ds\right] &= \mathbb{E}\left[\int_0^T c_s^{1-\frac{1}{\psi}} D_s^{1-\frac{1}{\psi}} D_s^{\frac{1}{\psi}-1} ds\right] \\
&\leq C\mathbb{E}\left[\int_0^T D_s c_s ds\right]^{1-\frac{1}{\psi}} \mathbb{E}\left[\int_0^T D_s^{1-\psi} ds\right]^{\frac{1}{\psi}} < \infty,
\end{aligned}$$

for some constant $C \in (0, \infty)$, where the last inequality follows from Equations (10) and (12). Therefore, $\mathbb{E}\left[\int_0^T e^{-\delta s} c_s^{1-\frac{1}{\psi}} ds\right] < \infty$, and we conclude that $c \in \widetilde{\mathcal{A}}(1, \varepsilon)$. □

As pointed out in Matoussi and Xing (2018, Remark 2.2), instead of verifying the integrability conditions in Equation (9), it is enough to check for the optimal consumption stream, c^*, that the associated U^{c^*} exists and is of class (D). A similar argument can be provided for the dual problem below.

With the integrability conditions in Equation (9), one can restate Equation (5) as

$$u(x,\varepsilon) = \sup_{c\in\widetilde{\mathcal{A}}(x,\varepsilon)} U^c_0, \quad (x,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0). \tag{13}$$

We call u—the value function and $U^{\hat{c}(x,\varepsilon)}$—the value process if $\hat{c}(x,\varepsilon)$ is an optimizer in Equation (13) for a given pair $(x,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0)$, provided that such an optimizer exists. Next, following Matoussi and Xing (2018), let us define

$$g(d,v) := \delta^{\psi}\frac{d^{1-\psi}}{\psi-1}((1-\gamma)v)^{1-\frac{\gamma\psi}{\theta}} - \delta\theta v, \quad d>0, \quad (1-\gamma)v>0, \tag{14}$$

and a function V_T, the convex conjugate of U_T, which is given by

$$V_T(d) := \frac{\gamma}{1-\gamma}d^{\frac{\gamma-1}{\gamma}}, \quad d>0. \tag{15}$$

Next, for a given pair $(y,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0)$ and $D\in\mathcal{D}(y,\varepsilon)$, one defines the Epstein–Zin *stochastic differential dual* for D to be a process V^D satisfying the BSDE

$$V^D_t = \mathbb{E}_t\left[V_T(D_T) + \int_t^T g\left(D_s, \frac{1}{\gamma}V^D_s\right)ds\right], \quad t\in[0,T]. \tag{16}$$

Sufficient conditions for the existence of V^D are presented in Matoussi and Xing (2018, Proposition 2.5). We state the family of the dual minimization problems as

$$\inf_{D\in\mathcal{D}(y,\varepsilon)} V^D_0, \quad (y,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0). \tag{17}$$

Similarly to Equation (13), to ensure that for a given state price density D, V^D is well-defined, one needs some integrability conditions, and following Matoussi and Xing (2018, Proposition 2.5), one can set

$$\widetilde{\mathcal{D}}(y,\varepsilon) := \left\{D\in\mathcal{D}(y,\varepsilon) : \mathbb{E}\left[\int_0^T D_s^{1-\psi}ds\right]\right\} < \infty. \tag{18}$$

Technically in Matoussi and Xing (2018, Proposition 2.5), it is also required that $\mathbb{E}\left[D_T^{\frac{\gamma-1}{\gamma}}\right] < \infty$, which however holds in our settings for every state price density, by an application of Hölder's inequality, as $\frac{\gamma-1}{\gamma}\in(0,1)$, and since D (under nonnegative interests rates) is a supermartingale. This allows us to restate Equation (17) as

$$v(y,\varepsilon) := \inf_{D\in\widetilde{\mathcal{D}}(y,\varepsilon)} V^D_0, \quad (y,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0). \tag{19}$$

We conclude this section by pointing out that, by Equation (10), if $\widetilde{\mathcal{D}}(1,\varepsilon) \neq \emptyset$, then $\mathcal{A}(1,\varepsilon) = \widetilde{\mathcal{A}}(1,\varepsilon)$, and thus, the convexity of $\widetilde{\mathcal{A}}(1,\varepsilon)$ holds. Thus, for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the nonemptiness of the dual feasible set implies the convexity of the primal domain.

3 | MAIN RESULTS

3.1 | Model assumptions

We will need two assumptions. To ensure that the dual problem (19) is nondegenerate in a neighborhood of $\varepsilon = 0$, we impose the following assumption.

Assumption 3.1. For every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\widetilde{\mathcal{D}}(1,\varepsilon) \neq \emptyset$.

The second assumption allows for the additional structure for the base model corresponding to $\varepsilon = 0$.

Assumption 3.2. Let $x > 0$ be fixed and suppose that, for $\varepsilon = 0$, a conjugacy relation in the following sense holds:

$$u(x,0) = \min_{\widetilde{y}>0} \left(v(\widetilde{y},0) + x\widetilde{y}\right) = v(y,0) + xy, \tag{20}$$

for some $y > 0$. Further, assume that, for $\varepsilon = 0$, there exist optimizers $\widehat{c}(x,0)$ to Equation (13) and $\widehat{D}(y,0)$ to Equation (19), such that $U^{\widehat{c}(x,0)}$ and $V^{\widehat{D}(y,0)}$ are of class D.

3.2 | Sufficient conditions for Assumption 3.2

Sufficient conditions for Equation (20) are contained in Matoussi and Xing (2018, Section 3). Explicit solutions are contained (Xing, 2017a, Theorem 2.14), see also Kraft et al. (2017), where optimal strategies are obtained in Markovian settings. To be more precise, Kraft et al. (2017) contain the explicit solution for the primal problem (13), and the optimal state price density could be identified via the utility gradient approach, following, for example, Duffie and Skiadas (1994).

Proposition 3.3 (Matoussi & Xing, 2018, Theorem 3.6). *Suppose that $\gamma\psi \geq 1$, $\psi > 1$, or $\gamma\psi \leq 1$, $\psi < 1$ and the processes r^0, $(\mu^0)^\top \left(\left(\sigma^0\right)^\top\right)^{-1} \left(\sigma^0\right)^{-1} \mu^0$ are bounded. Then Equation (20) holds.*

For models with unbounded market price of risk, we refer to Matoussi and Xing (2018, Section 3.4) for the exact conditions that guarantee Assumption 3.2. In a Markov setting, we refer to Kraft et al. (2017, Theorem 5.1), where, in the one-dimensional stock prices process and a factor model for the dynamics of both riskless and risky assets, boundedness of μ^0 and r^0 as well as boundedness away from 0 and ∞ of σ^0, guarantee that Matoussi and Xing (2018, Theorem 3.6) applies.

To analyze the behavior of the primal and dual problems under perturbations, we introduce a family of $\mathbb{R}^n$-dimensional processes λ^ε, defined by

$$\lambda^\varepsilon_t := \left(\left(\sigma^0_t\right)^\top\right)^{-1}\left(\left(\sigma^\varepsilon_t\right)^{-1}\mu^\varepsilon_t - \left(\sigma^0_t\right)^{-1}\mu^0_t\right), \quad t \in [0,T], \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0). \tag{21}$$

We also set $R := (R^1, \dots, R^n)$, where

$$dR^i_t = \mu^{0,i} dt + \sum_{j=1}^n \sigma^{0,i,j}_t dW^{\rho,j}_t, \quad R^i_0 = 0, \quad i \in \{1, \dots, n\}, \tag{22}$$

Along the lines of Mostovyi (2020), let us introduce the family of processes N^ε, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, given via

$$dN^\varepsilon{}_t = N^\varepsilon_t\left(\left(r^0_t - r^\varepsilon_t\right)dt - \lambda^\varepsilon_t dR_t\right), \quad t \in [0,T], \quad N^\varepsilon_0 = 1, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0). \tag{23}$$

We recall that κ is given by $\kappa_t = t + 1_{\{T\}}(t)$, $t \in [0,T]$. Let $\mathbb{L}^0(d\kappa \times \mathbb{P})$ be the linear space of (equivalence classes of) real-valued measurable processes on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ equipped with the topology of convergence in measure $(d\kappa \times \mathbb{P})$.

3.3 | Stability theorems

The first theorem establishes convergence of the value functions.

Theorem 3.4. *Let $x > 0$ be fixed, $\gamma > 1$ and $\psi > 1$ in Equation (3). Let us further suppose that Assumptions 3.1 and 3.2 hold and for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, σ^ε is invertible, λ^ε appearing in Equation (21) is R-integrable and $\lim\limits_{\varepsilon\to 0} N^\varepsilon = 1$, in measure $(d\kappa \times \mathbb{P})$.*

Then for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we have

(i) the value functions are finite-valued, that is

$$u(z,\varepsilon) \in \mathbb{R} \quad \text{and} \quad v(z,\varepsilon) \in \mathbb{R}, \quad (z,\varepsilon) \in (0,\infty) \times (-\varepsilon_0, \varepsilon_0); \tag{24}$$

(ii) the value functions converge

$$\lim_{(x',\varepsilon)\to(x,0)} u(x',\varepsilon) = u(x,0), \quad x > 0, \tag{25}$$

$$\lim_{(y',\varepsilon)\to(y,0)} v(y',\varepsilon) = v(y,0), \quad y > 0; \tag{26}$$

(iii) for every $(x,\varepsilon) \in (0,\infty) \times (-\varepsilon_0, \varepsilon_0)$, there exists a unique optimizer to Equation (13).

Remark 3.5. For the problem in Equation (13), a condition of the finiteness of the value functions condition is typically imposed. In the present settings, as we deal with nonpositive value functions

$u(x, \varepsilon)$ finiteness from above (by zero) holds. For the finiteness from below,

$$u(x, \varepsilon) > -\infty, \quad (x, \varepsilon) \in (0, \infty) \times (-\varepsilon_0, \varepsilon_0), \tag{27}$$

we remark that this also holds as $r^0 \geq 0$, and thus $c \equiv \frac{x}{T+1}$ is an admissible consumption for the initial wealth x, for which one can use comparison results for BSDEs to show that the value function is finite-valued. Similar arguments can be employed to show the finiteness of $v(y, \varepsilon)$, as it is also bounded by zero from above, and by $(u(1, \varepsilon) - y)$ from below.

The next theorem addresses the convergence of the optimizers. The assumptions of Theorem 3.4 ensure that for every $(x, \varepsilon) \in (0, \infty) \times (-\varepsilon_0, \varepsilon_0)$, there exists a unique $\hat{c}(x, \varepsilon)$, such that $u(x, \varepsilon) = U_0^{\hat{c}(x,\varepsilon)}$, and that $u(x, \varepsilon)$ is finite-valued for every such (x, ε). To prove convergence of the optimizers, we need to ensure finiteness for the value processes in the sense below.

Theorem 3.6. *Let $x > 0$ be fixed. Let the conditions of Theorem 3.4 hold and suppose that there exists $\varepsilon' > 0$, such that*

$$\operatorname{ess\,sup}_{(x,\varepsilon)\in B_{\varepsilon'}(0,0)} \hat{c}(x, \varepsilon) \in \mathbb{L}^0(d\kappa \times \mathbb{P}), \quad \operatorname{ess\,inf}_{(x,\varepsilon)\in B_{\varepsilon'}(0,0)} U^{\hat{c}(x,\varepsilon)} \in \mathbb{L}^0(d\kappa \times \mathbb{P}),$$

where $B_{\varepsilon'}(0, 0)$ is a Euclidean ball of radius ε' in $\mathbb{R}^2$.

We then have that

$$\lim_{(x',\varepsilon)\to(x,0)} \hat{c}(x', \varepsilon) = \hat{c}(x, 0), \tag{28}$$

where the convergence is in measure $(d\kappa \times \mathbb{P})$.

Let us recall that under the conditions of Theorem 3.4, the existence and uniqueness of the optimizer to Equation (13) follows from Theorem 3.4, item (*iii*). Let us also recall that, for a given nonnegative consumption stream c, U^c was defined in Equation (4). The following theorem establishes the convergence of the indirect utility processes.

Theorem 3.7. *Let $x > 0$ be fixed. Then, under the conditions of Theorem 3.6, we have*

$$\lim_{(x',\varepsilon)\to(x,0)} U^{\hat{c}(x',\varepsilon)} = U^{\hat{c}(x,0)}, \quad ucp.$$

Next, for a fixed $x > 0$ and $\varepsilon = 0$, let $y > 0$ be as in Assumption 3.2 and suppose that the dual minimizer has the form

$$\hat{D}_t(y, 0) = C \exp\left(\int_0^t \partial_u f\left(\hat{c}_t(x, 0), U_t^{\hat{c}(x,0)}\right) ds\right) \partial_c f\left(\hat{c}_t(x, 0), U_t^{\hat{c}(x,0)}\right), \quad t \in [0, T], \tag{29}$$

for some constant $C > 0$ and

$$X^{\hat{c}(x,0)} \hat{D}(y, 0) + \int_0^\cdot \hat{D}_s(y, 0) \hat{c}_s(x, 0) d\kappa_s \quad \text{is a martingale}, \tag{30}$$

where $X^{\hat{c}(x,0)}$ is the wealth process starting from x financing $\hat{c}(x, 0)$ (given by Equation 2 at $\varepsilon = 0$). We note that sufficient conditions for Equations (29) and (30) are similar to the ones for

Assumption 3.2 to hold; see the discussion after Assumption 3.2. In particular, both Equations (29) and (30) hold if the market price of risk $(\mu^0)^\top \left(\left(\sigma^0\right)^\top\right)^{-1} \left(\sigma^0\right)^{-1} \mu^0$ process is bounded as well as $\gamma > 1$ and $\psi > 1$. Then, the conditions of Matoussi and Xing (2018, Theorem 3.6, p. 1002), apply and Matoussi and Xing (2018, Corollary 3.7, p. 1002) implies Equations (29) and (30). Furthermore, representation (29) for the optimal state-price density goes back to Duffie and Skiadas (1994) and is known as the utility gradient approach.

Theorem 3.8. *Let $x > 0$ be fixed. Let the assumptions of Theorem 3.6, Equation (29), and Equation (30) hold. Then, $\lim_{(x',\varepsilon)\to(x,0)} X^{\widehat{c}(x',\varepsilon)} = X^{\widehat{c}(x,0)}$, in measure $(d\kappa \times \mathbb{P})$, where $X^{\widehat{c}(x',\varepsilon)}$ is any wealth process financing $\widehat{c}(x',\varepsilon)$ starting from the initial capital x' in the market where the traded assets are given by Equation (1). Furthermore, if $\lim_{\varepsilon\to 0} N^\varepsilon = 1$, ucp, then $\lim_{(x',\varepsilon)\to(x,0)} X^{\widehat{c}(x',\varepsilon)} = X^{\widehat{c}(x,0)}$, ucp.*

4 | ON THE INTEGRABILITY CONDITION ON PERTURBATIONS

Let us revisit Assumption 3.1. For a fixed $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, in order for $\widetilde{\mathcal{D}}(1,\varepsilon) \neq \emptyset$, where $\widetilde{\mathcal{D}}(1,\varepsilon)$ are defined in Equation (18), there must exist a supermartingale state price density $D^\varepsilon \in \mathcal{D}(1,\varepsilon)$, such that

$$\mathbb{E}\left[\int_0^T \left(D_s^\varepsilon\right)^{1-\psi} ds\right] < \infty. \tag{31}$$

The natural candidate for Equation (31) to hold is to check whether the minimal state price density given by Equation (7) satisfies the integrability condition (31). Another sufficient condition for Assumption 3.1 to hold is given by

$$\mathbb{E}\left[\int_0^T \left(\widehat{D}_s(y,0) N_s^\varepsilon\right)^{1-\psi} ds\right] < \infty, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$

where $\widehat{D}_s(y,0)$ is the dual minimizer at $(y,0)$, which exists by Assumption 3.2 and N^ε are given by Equation (23). Condition (31) is the only integrability condition needed on the perturbations to ensure that the dual domain incorporating the additional integrability for perturbed models is nonempty, that is, $\widetilde{\mathcal{D}}(1,\varepsilon) \neq \emptyset$, for $\varepsilon \neq 0$. Perhaps the most surprising feature in our analysis (at least for the authors) was that other than Equation (31), no further integrability needs to be imposed. This can be explained as follows, where the key is in the utility maximization considerations. It is well-known that for the expected utility maximization from terminal wealth, the key role is played by the finiteness of the value functions, see Kramkov and Schachermayer (1999), where the finiteness of the primal value function (from above) is assumed, and Kramkov and Schachermayer (2003), where the finiteness of the dual value function (from above) is required. To be more precise, both conditions require the value functions to be less than ∞ (in Kramkov and Schachermayer (1999, Theorem 2.2), under the asymptotic elasticity). In Mostovyi (2015), the finiteness of both primal and dual value functions (from below and above) is introduced and proven to be necessary and sufficient for the standard assertions of the utility maximization theory in the case of additive and stochastic utility.

In the present setting, in view of the choice of the parameters $\gamma > 1$ and $\psi > 1$, we obtain that the associated value function is negative-valued. This follows from the analysis of the associated BSDEs as in rLemma 5.2. In particular, the base model exhibits a finiteness conditions for both the primal and dual value functions. For the perturbed models, as γ and ψ do not change, we still obtain that the primal value function is negative-valued, and the dual one too. Here, the primal gives a lower bound for the dual via the conjugacy relations. Thus the blow-up to ∞ is not possible under perturbations of the models. In turn, the blow-up to $-\infty$ is also not possible, as Equation (13) is a *maximization* problem, and thus finiteness of the base model guarantees that we do not have a blow-up as long as the processes N^ε's appearing in Equation (23) are well-defined, and without any further integrability conditions needed on this family. This situation can be compared to the counterexample in Mostovyi (2020), where blow-up does happen for a particular form of perturbations, as the utility function there can take positive values.

The connection between the last two paragraphs can be further illustrated by the case of $\gamma = \frac{1}{\psi}$ (going outside the scope of the analysis in this paper). Then the problem (13) reduces to the one with an *additive utility*, given by

$$U_0^c = \mathbb{E}\left[\int_0^T \delta e^{-\delta s}\frac{c_s^{1-\gamma}}{1-\gamma}ds + e^{-\delta T}\frac{c_T^{1-\gamma}}{1-\gamma}\right].$$

In this case, and with $\gamma > 1$, the value function is negative-valued, and so is the dual one, thus precluding the blow-up to ∞. The blow-up to $-\infty$ is not possible by the feasibility of the constant-valued consumptions, which also gives a lower bound for the dual problem.

5 | PROOFS

5.1 | Preliminary results

We begin with the following structural lemma.

Lemma 5.1. *Let the conditions of Theorem 3.4 hold, let $x > 0$ be fixed, and $y > 0$ be given through (20). Then, for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we have*

$$\begin{aligned} c^\varepsilon &:= \hat{c}(x,0)\frac{1}{N^\varepsilon} \in \mathcal{A}(x,\varepsilon), \quad x > 0, \\ D^\varepsilon &:= \hat{D}(y,0)N^\varepsilon \in \mathcal{D}(y,\varepsilon), \quad y > 0, \end{aligned} \tag{32}$$

where $\hat{c}(x,0)$ and $\hat{D}(y,0)$ are the optimizers, for $\varepsilon = 0$, to (13) and (19), respectively.

Proof. First, we observe that for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the process N^ε is progressively measurable by Karatzas and Shreve (1998, Proposition 1.13, p. 5). Now, the assertion of the lemma follows from Itô's lemma. □

Let us introduce some notations used in this section's remaining part.

- Let $\mathcal{S}^2$ be the space of one-dimensional continuous adapted processes $(Y_t)_{t\in[0,T]}$ such that $\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^2\right] < \infty$.
- Let $\mathcal{S}^\infty = \left\{Y \in \mathcal{S}^2 : ||\sup_{t\in[0,T]}|Y_t|||_\infty < \infty\right\}$.

- Let $\mathcal{M}^2$ denote the space of predictable multidimensional processes $(Z_t)_{t\in[0,T]}$, such that $\mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < \infty$.

With f given in Equation (3), let us consider the BSDE

$$U_t^c = \frac{c_T^{1-\gamma}}{1-\gamma} + \int_t^T f\left(c_s, U_s^c\right) ds - \int_t^T Z_s^c dB_s, \quad t \in [0,T]. \tag{33}$$

Next, with the transformation

$$(Y,Z) := e^{-\delta\theta t}(1-\gamma)(U^c, Z^c), \quad t \in [0,T], \tag{34}$$

we obtain a BSDE for (Y,Z) of the form

$$Y_t = e^{-\delta\theta T} c_T^{1-\gamma} + \int_t^T F(s, c_s, Y_s) ds - \int_t^T Z_s dB_s, \quad t \in [0,T], \tag{35}$$

where, for $\theta < 0$, $F(t,x,y) := \delta\theta e^{-\delta t} x^{1-\frac{1}{\psi}} y^{1-\frac{1}{\theta}} \leq 0$ is monotonically decreasing in y.

Lemma 5.2. *Under the conditions of Theorem 3.4, for every $(z,\varepsilon) \in (0,\infty)\times(-\varepsilon_0,\varepsilon_0)$, $u(z,\varepsilon)$ and $v(z,\varepsilon)$ are finite-valued.*

Proof. Let us fix $(z,\varepsilon) \in (0,\infty)\times(-\varepsilon_0,\varepsilon_0)$. From Lemma 5.1, we deduce that $\frac{z}{x}\hat{c}(x,0)\frac{1}{N^\varepsilon} \in \mathcal{A}(z,\varepsilon)$. Therefore, for a fixed $m \geq 1$,

$$c := \frac{z}{x + \frac{1}{m}} \frac{1}{m} \vee \left(\hat{c}(x,0)\frac{1}{N^\varepsilon}\right) \wedge m \in \mathcal{A}(z,\varepsilon). \tag{36}$$

In particular, we have

$$\mathbb{E}[(c_T)^{1-\gamma}] < \infty.$$

Next, with $F^k(t,c_t,y) := \delta\theta e^{-\delta t} c_t^{1-\frac{1}{\psi}} (|y| \wedge k)^{1-\frac{1}{\theta}}$ (note that the process c is bounded from above by m here), let us consider

$$Y_t^k = e^{-\delta\theta T} c_T^{1-\gamma} + \int_t^T F^k(s, c_s, Y_s^k) ds - \int_t^T Z_s^k dB_s, \quad t \in [0,T], \quad k \in \mathbb{N}.$$

This is a BSDE with a Lipschitz generator and a bounded terminal condition. Therefore, by Cohen and Elliot (2012, Theorem 5.1), this BSDE admits a unique solution $(Y^k, Z^k) \in \mathcal{S}^2 \times \mathcal{M}^2$. Furthermore, as c in Equation (36) is bounded away from 0 and ∞, we have $\frac{1}{\bar{C}} \leq Y_T^k \leq \bar{C}$, for some constant $\bar{C} > 0$. As, additionally, F^k is nonpositive-valued, using the comparison result for BSDEs (Pardoux, 1999, Theorem 2.4, p. 517), one can show that Y^k takes values in $[0,\bar{C}]$. Therefore, with c in Equation (36) and the associated Y given via Equation (35), for every $k \geq \bar{C}$, $F^k(t,c_t,Y^k) = F(t,c_t,Y^k)$, $t \in [0,T]$, $\mathbb{P}$-a.s. As a result, $(Y,Z) := (Y^k, Z^k)$ is a solution to Equation (35) (for c given in Equation 36).

Changing variables back to (U^c, Z^c), that is from Equation (34), and with

$$\left(U_t^c, Z_t^c\right) := \frac{e^{\delta\theta t}}{1-\gamma}(Y_t, Z_t), \quad t \in [0,T],$$

one can show that this pair satisfies Equation (33) and further, following the proof of Xing (2017a, Proposition 2.2), that U^c satisfies Equation (4), U^c is nopositive-valued and is bounded away from $-\infty$. As in Equation (13), we take the supremum over all admissible consumptions, $u(z,\varepsilon) \geq U_0^c$ (for c as above). Next, also similarly to the proof of Xing (2017a, Proposition 2.2) and relying on the localization technique from Briand and Hu (2006), one can see that for every admissible consumption, $U_0^c \leq 0$. We conclude that $u(z,\varepsilon) \leq 0$ and is finite-valued for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Now, by Matoussi and Xing (2018, Theorem 2.7), $v(z,\varepsilon) \geq u(x,\varepsilon) - xz$, $(x,z) \in (0,\infty)^2$, and thus $v(z,\varepsilon)$ is bounded away from $-\infty$. Furthermore, similarly to showing that $u(z,\varepsilon) \leq 0$, one can show that $v(z,\varepsilon) \leq 0$. We conclude that $v(z,\varepsilon)$ is finite-valued. □

Lemma 5.3. *Let $x > 0$ be fixed. Then, under the conditions of Theorem 3.4, we have*

$$\liminf_{(x',\varepsilon)\to(x,0)} u(x',\varepsilon) \geq u(x,0). \tag{37}$$

Proof. Let us fix $\varepsilon' > 0$ and let $\hat{c}(x,0)$ be the optimizer to Equation (13) at $(x,0)$, which belongs to $\widetilde{\mathcal{A}}(x,0)$, as this results from Assumption 3.2. Let (x_k, ε_k), $k \in \mathbb{N}$, be a sequence which converges to $(x,0)$ and such that

$$\lim_{k\to\infty} u(x_k,\varepsilon_k) = \liminf_{(x',\varepsilon)\to(x,0)} u(x',\varepsilon). \tag{38}$$

For $c = \hat{c}(x,0)$, let us consider the BSDE (35) (which is related to Equation 33 via Equation 34). As, by Assumption 3.2, U^c is of class D, one can show (see the discussion in Matoussi and Xing (2018, Remark 2.2)) that $c \in \widetilde{\mathcal{A}}(x,0)$. Furthermore, as established in the proof of Xing (2017a, Proposition 2.2), r(for $\gamma, \psi > 1$) (35) admits a *unique* solution (Y,Z), such that Y is continuous, strictly positive, and of class D, with $\int_0^T |Z_t|^2 dt < \infty$, $\mathbb{P}$-a.s. Moreover, $U^c := e^{\delta\theta t} Y_t \frac{1}{1-\gamma}$, $t \in [0,T]$, satisfies Equations (33) and (4). Next, using the approximation procedure as in step 2 of the proof of Xing (2017a, Proposition 2.2), one can show that there exists $n_0 \in \mathbb{N}$, such that

$$|Y_0^n - Y_0| < \frac{\varepsilon'}{3}, \quad n \geq n_0, \tag{39}$$

where Y^n solves

$$Y_t^n = \left(e^{-\delta\theta T} c_T^{1-\gamma}\right) \wedge n + \int_t^T F(s, c_s, Y_s^n)\, ds - \int_t^T Z_s^n dB_s, \quad t \in [0,T].$$

The latter BSDE admits a solution $(Y^n, Z^n) \in \mathcal{S}^\infty \times \mathcal{M}^2$, where using the comparison argument, one can show that $0 \leq Y^n \leq n$, and $Y^n = \downarrow \lim_{m\to\infty} Y^{n,m}$, $n \in \mathbb{N}$, where $Y^{n,m}$ solves

$$Y_t^{n,m} = \left(e^{-\delta\theta T} c_T^{1-\gamma}\right) \wedge n + \int_t^T F^m(s, c_s, Y_s^{n,m}) ds - \int_t^T Z_s^{n,m} dB_s, \quad t \in [0,T], \tag{40}$$

with $F^m(t,c_t,y) := \delta\theta e^{-\delta t}\left(c_t^{1-\frac{1}{\psi}} \wedge m\right)(|y| \wedge m)^{1-\frac{1}{\theta}}$. Likewise, one can show that $0 \le Y^{n,m} \le n$. Therefore, for $m \ge n$, we obtain

$$\begin{aligned} F^m\left(t,c_t,Y_t^{n,m}\right) &= \delta\theta e^{-\delta t}\left(c_t^{1-\frac{1}{\psi}} \wedge m\right)\left(Y_t^{n,m} \wedge m\right)^{1-\frac{1}{\theta}} \\ &= \delta\theta e^{-\delta t}\left(c_t^{1-\frac{1}{\psi}} \wedge m\right)\left(Y_t^{n,m}\right)^{1-\frac{1}{\theta}}. \end{aligned} \tag{41}$$

For every $n \in \mathbb{N}$, one can show that $\lim_{m\to\infty}\sup_{t\in[0,T]}|Y_t^n - Y_t^{n,m}| = 0$ in probability $\mathbb{P}$, and thus, we deduce that there exists $m'(n) \in \mathbb{N}$, such that

$$\left|Y_0^n - Y_0^{n,m}\right| < \frac{\varepsilon'}{3}, \quad n \in \mathbb{N}, \quad m \ge m'(n). \tag{42}$$

It follows from Lemma 5.1 that the process $c^k = \hat{c}(x,0)\frac{1}{N^{\varepsilon_k}} \in \mathcal{A}(x,\varepsilon_k)$. Next, for every $M_1 > 0, M_2 > 0$ the process $\bar{c}^k$ defined as $\bar{c}_t^k := \frac{x_k}{x+\frac{1}{M_1}}\frac{1}{M_1} \vee c_t^k \wedge M_2$, $t \in [0,T]$, satisfies $\bar{c}^k \in \widetilde{\mathcal{A}}(x_k,\varepsilon_k)$. Now let us consider the sequence of BSDEs

$$\bar{Y}_t^k = e^{-\delta\theta T}\left(\bar{c}_T^k\right)^{1-\gamma} + \int_t^T F\left(s,\bar{c}_s^k,\bar{Y}_s^k\right)ds - \int_t^T \bar{Z}_s^k dB_s, \quad t \in [0,T], \quad k \in \mathbb{N}. \tag{43}$$

Cohen and Elliot (2012, Theorem 5.1) ensure that there exists a unique solution to BSDE (43), $\left(\bar{Y}^k,\bar{Z}^k\right) \in \mathcal{S}^2 \times \mathcal{M}^2$. Further, by replacing F with F^k as in the previous step, and using the comparison for BSDEs (with Lipschitz generator) results, see, for example, Pardoux (1999, Theorem 2.4), we deduce that the first component of the solution is in $\mathcal{S}^\infty$.

Let us consider Equations (40) and (43). These are BSDEs with bounded terminal conditions and Lipschitz generators. Therefore, the stability of BSDEs, as in Cohen and Elliot (2015, Theorem 19.1.6, p. 472), implies that, for some n satisfying Equation (39) and for $m = n(m)$ satisfying Equation (42), one can first pick M_1 and M_2 and then k_0, such that

$$\left|\bar{Y}_0^k - Y_0^{n,m}\right| < \frac{\varepsilon'}{3}, \quad k \ge k_0. \tag{44}$$

Comparing Equations (39), (42), and (44), we deduce that

$$\left|\bar{Y}_0^k - Y_0\right| < \varepsilon', \quad k \ge k_0.$$

Therefore, as $\bar{c}^k \in \widetilde{\mathcal{A}}(x_k,\varepsilon_k)$, via Equation (34), we obtain

$$\liminf_{k\to\infty} u(x_k,\varepsilon_k) \ge \liminf_{k\to\infty}\frac{\bar{Y}_0^k}{1-\gamma} \ge \frac{Y_0}{1-\gamma} - \frac{\varepsilon'}{|1-\gamma|} = u(x,0) - \frac{\varepsilon'}{|1-\gamma|}.$$

Consequently, as ε' is arbitrary, via Equation (38), we deduce that Equation (37) holds. □

The next lemma establishes a result similar to Lemma 5.3 for the dual value function. The proof is similar to the proof of Lemma 5.3, so we only outline the main steps.

Lemma 5.4. *Under the conditions of Theorem 3.4, we have*

$$\limsup_{(y',\varepsilon)\to(y,0)} v(y',\varepsilon) \leq v(y,0). \tag{45}$$

Proof. Let us consider a sequence (y_k, ε_k), $k \in \mathbb{N}$, convergent to $(y,0)$ and such that

$$\lim_{k\to\infty} v(y_k,\varepsilon_k) = \limsup_{(y',\varepsilon)\to(y,0)} v(y,0).$$

By Assumption 3.1, for every $k \in \mathbb{N}$, there $\widetilde{D}^k \in \widetilde{\mathcal{D}}(1,\varepsilon_k)$, that is, $\widetilde{D}^k$, such that

$$\mathbb{E}\left[\int_0^T \left(\widetilde{D}^k_s\right)^{1-\psi} ds\right] < \infty, \quad k \in \mathbb{N}. \tag{46}$$

Let us set

$$D^k = (1-(-1\vee\varepsilon_k\wedge 1))\frac{y_k}{y}\widehat{D}(y,0)N^{\varepsilon_k} + (-1\vee\varepsilon_k\wedge 1)y_k\widetilde{D}^k, \quad k\in\mathbb{N}.$$

Then, as $N^{\varepsilon_k}\to 1$, in measure $(d\kappa\times\mathbb{P})$, by the assumption of Theorem 3.4, we deduce that $D^k \to \widehat{D}(y,0)$, in measure $(d\kappa\times\mathbb{P})$. Moreover, it follows from Equation (46), and since $1-\psi<0$, that $D^k \in \widetilde{\mathcal{D}}(y_k,\varepsilon_k)$, $k\in\mathbb{N}$. Next, applying the approximation procedure entirely similarly to Lemma 5.3, we obtain the assertion of the lemma. □

We now show the concavity of U^c_0 in c in the following sense, which is closely related to the notion of strong concavity.

Lemma 5.5. *Let us suppose that c' and c'' are in $\bigcup_{(x,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0)}\widetilde{\mathcal{A}}(x,\varepsilon)$ and are such that*

$$(d\kappa\times\mathbb{P})\left[|c'-c''|\geq\delta,\ c'+c''\leq\frac{1}{\delta}\right]\geq\delta, \quad \mathit{for\ some}\ \delta>0. \tag{47}$$

Further, let us suppose that for a given constant $\lambda\in(0,1)$, we have

$$c := \lambda c' + (1-\lambda)c'' \in \bigcup_{(x,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0)} \widetilde{\mathcal{A}}(x,\varepsilon).$$

Then, there exists a constant $\bar{\eta}>0$, such that

$$\lambda U^{c'}_0 + (1-\lambda)U^{c''}_0 + \bar{\eta} \leq U^c_0. \tag{48}$$

Proof. Let us show that

$$\lambda Y^{c'}_0 + (1-\lambda)Y^{c''}_0 - \eta_0 \leq Y^c_0, \tag{49}$$

where Ys satisfy Equation (35) with respective cs and η_0 is some positive constant. As the generator of Y is not *jointly* concave in (c,Y,Z), with $p := 1-\frac{1}{\psi}$, following Xing (2017a), one can set

$$(\mathbb{Y},\mathbb{Z}) := \frac{1}{p}\left(Y^{\frac{1}{\theta}}, \frac{1}{\theta}Y^{\frac{1}{\theta}-1}Z\right).$$

Then $\mathbb{Y}$ satisfies

$$\mathbb{Y}_t = e^{-\delta T}\frac{c_T^p}{p} + \int_t^T \left(\delta e^{-\delta s}\frac{c_s^p}{p} + \frac{1}{2}(\theta - 1)\frac{\mathbb{Z}_s^2}{\mathbb{Y}_s}\right)ds - \int_t^T \mathbb{Z}_s dB_s, \quad t \in [0, T], \tag{50}$$

where the generator is *jointly* concave in $(c, \mathbb{Y}, \mathbb{Z})$ when $\theta < 1$.

For $\mathbb{Y}'$ and $\mathbb{Y}''$, let $\Delta\mathbb{Y} := \lambda\mathbb{Y}' + (1-\lambda)\mathbb{Y}''$, $\Delta c := \lambda c' + (1-\lambda)c''$, and $\Delta\mathbb{Z} := \lambda\mathbb{Z}' + (1-\lambda)\mathbb{Z}''$. We observe that

$$\Delta Y := (p\Delta\mathbb{Y})^\theta \quad and \quad \Delta Z := (1-\gamma)(p\Delta\mathbb{Y})^{\theta-1}\Delta\mathbb{Z},$$

satisfy

$$\begin{aligned}\Delta Y_t = (p\Delta\mathbb{Y}_T)^\theta &+ \int_t^T \left(\delta\theta e^{-\delta s}(\Delta c_s)^p - (1-\gamma)A_t\right)\Delta Y_s^{1-\frac{1}{\theta}} ds \\ &- \int_t^T \Delta Z_s dB_s, \quad t \in [0, T],\end{aligned}$$

where

$$\begin{aligned}A_t = \frac{\delta e^{-\delta t}}{p}&\left((\Delta c_t)^p - \lambda(c_t')^p - (1-\lambda)\left(c_t''\right)^p\right) \\ &+ \tfrac{1}{2}(\theta-1)\left(\frac{\Delta\mathbb{Z}_t^2}{\Delta\mathbb{Y}_t} - \lambda\frac{\mathbb{Z}_t'^2}{\mathbb{Y}_t'} - (1-\lambda)\frac{\mathbb{Z}_t''^2}{\mathbb{Y}_t''}\right) \geq 0,\end{aligned} \tag{51}$$

as both terms on the right-hand side are non-negative by the joint concavity of the generator to Equation (50). Additionally, the function $x \to x^p$, $x > 0$ is strictly concave. Therefore, on $\left\{c_t' + c_t'' \leq \frac{1}{\varepsilon}, |c_t' - c_t''| \geq \varepsilon\right\}$, we have $\left((\Delta c_t)^p - \lambda\left(c_t'\right)^p - (1-\lambda)\left(c_t''\right)^p\right) \geq \delta_1$, for some constant $\delta_1 > 0$, which depends only on ε and λ in the statement of the lemma (also on $p = 1 - \frac{1}{\psi}$, but ψ is fixed throughout the paper). Similarly, we obtain

$$pe^{\delta T}\Delta\mathbb{Y}_T \leq (\Delta c_T)^p - \delta_1 1_{\{c_T' + c_T'' \leq \frac{1}{\varepsilon}, |c_T' - c_T''| \geq \varepsilon\}}.$$

Therefore, for some constant $\delta_2 > 0$, we have

$$\Delta Y_T \geq e^{-\delta\theta T}(\Delta c_T)^{1-\gamma} + \delta_2 1_{\{c_T' + c_T'' \leq \frac{1}{\varepsilon}, |c_T' - c_T''| \geq \varepsilon\}}. \tag{52}$$

It follows from Equations (51) and (52), that $(\Delta Y, \Delta Z)$ is a supersolution to

$$Y_t^{\Delta c} = e^{-\delta\theta T}(\Delta c_T)^{1-\gamma} + \int_t^T F(s, \Delta c_s, Y_s^{\Delta c})ds - \int_t^T Z_s^{\Delta c} dB_s, \quad t \in [0, T].$$

Setting $\xi_t := -(1-\gamma)A_t\Delta Y_t^{1-\frac{1}{\theta}}$, $t\in[0,T)$, $\xi_T := \Delta Y_T - Y_T^{\Delta c}$, we observe that $\xi_t \geq 0$, $t\in[0,T]$, and, moreover, for some constant $\widetilde{\delta}_1>0$, we have

$$\begin{aligned} \xi_t &\geq \widetilde{\delta}_1 \Delta Y_t^{1-\frac{1}{\theta}} 1_{\{c_t'+c_t''\leq\frac{1}{\varepsilon},|c_t'-c_t''|\geq\varepsilon\}}, \quad t\in[0,T), \\ \xi_T &\geq \delta_2 1_{\{c_T'+c_T''\leq\frac{1}{\varepsilon},|c_T'-c_T''|\geq\varepsilon\}}. \end{aligned} \tag{53}$$

We stress here that $\widetilde{\delta}_1$ and δ_2 depend only on λ and ε in the statement of the lemma.

Further, let us define $\eta_t := \Delta Y_t - Y_t^{\Delta c}$, and $\zeta_t := \Delta Z_t - Z_t^{\Delta c}$, $t\in[0,T]$, we deduce that

$$\begin{aligned} \eta_t = \xi_T &+ \int_t^T \left\{ \left(\delta\theta e^{-\delta s}(\Delta c_s)^p - (1-\gamma)A_t\right)\Delta Y_s^{1-\frac{1}{\theta}} - F\left(s,\Delta c_s, Y_s^{\Delta c}\right)\right\} ds \\ &- \int_t^T \zeta_s dB_s. \end{aligned} \tag{54}$$

Let us rewrite the latter generator as

$$\begin{aligned} &\left(\delta\theta e^{-\delta s}(\Delta c_s)^p - (1-\gamma)A_t\right)\Delta Y_s^{1-\frac{1}{\theta}} - F\left(s,\Delta c_s, Y_s^{\Delta c}\right) \\ &= -(1-\gamma)A_t\Delta Y_s^{1-\frac{1}{\theta}} + F(s,\Delta c_s,\Delta Y_s) - F\left(s,\Delta c_s, Y_s^{\Delta c}\right). \end{aligned}$$

Setting $\alpha_t := \frac{F(t,\Delta c_s,\Delta Y_t) - F(t,\Delta c_t, Y_t^{\Delta c})}{\eta_s} 1_{\{\eta_t\neq 0\}}$, $t\in[0,T]$, we can rewrite Equation (54) as

$$\eta_t = \xi_T + \int_t^T (\alpha_s\eta_s + \xi_s)ds - \int_t^T \zeta_s dB_s,$$

With $\Gamma_t := \exp\left(\int_0^t \alpha_s ds\right)$, we get

$$\eta_t = \frac{1}{\Gamma_t}\mathbb{E}_t\left[\Gamma_T\xi_T + \int_t^T \xi_s\Gamma_s ds\right].$$

In particular, at $t=0$, we get

$$\eta_0 = \mathbb{E}\left[\Gamma_T\xi_T + \int_0^T \xi_s\Gamma_s ds\right] = \mathbb{E}[(\Gamma\xi)\cdot\kappa_T]. \tag{55}$$

As both ΔY and $Y^{\Delta c}$ are finite-valued, $\Gamma>0$. Next, as from Equations (70) and (55), we deduce the strict positivity of $\bar{U}_0$ (by the strict comparison and Equation 47) that

$$\Delta Y_0 - Y_0^{\Delta c} = \eta_0 > 0. \tag{56}$$

Moreover, as in Xing (2017a, eq. (A.6), p. 247), we get

$$\Delta Y_t \leq \lambda Y_t^{c'} + (1-\lambda)Y_t^{c''}, \quad t\in[0,T], \quad \mathbb{P}\text{-a.s.}. \tag{57}$$

Combining Equations (56) and (57), we deduce that

$$\lambda Y_0^{c'} + (1-\lambda) Y_0^{c''} \geq \Delta Y_0 \geq Y_0^{\Delta c} + \eta_0,$$

and thus Equation (49) holds, which, via Equation (34) and Xing (2017a, Proposition 2.2), implies Equation (48), where $\bar{\eta} = \frac{\eta_0}{\gamma - 1}$. □

We will need the following technical lemma.

Lemma 5.6. *Let $x > 0$ be fixed. Under the conditions of Theorem 3.6, let ε_k, $k \in \mathbb{N}$, be a sequence or real numbers converging to zero. Let us set*

$$c_t^{k,\delta',M} := \frac{x}{x+\delta'}\delta' \vee \hat{c}_t(x,0)\frac{1}{N_t^{\varepsilon_k}} \wedge M, \quad t \in [0,T], \quad k \in \mathbb{N}, \quad \delta' > 0, \quad M > 0. \tag{58}$$

Then, for every $k \in \mathbb{N}$, there exist $\delta'(k)$, $M(k)$, such that for

$$\tilde{c}^k := c^{k,\delta'(k),M(k)}, \quad k \in \mathbb{N}, \tag{59}$$

the associated solutions to Equation (35) satisfy

$$\lim_{k\to\infty} Y^{\tilde{c}^k} = Y^{\hat{c}(x,0)}, \quad ucp.$$

Proof. First, we observe that $\tilde{c}^k \in \mathcal{A}(x, \varepsilon_k)$, $k \in \mathbb{N}$. Fixing an $\varepsilon' > 0$, reutilizing the argument from the proof of Xing (2017a, Proposition 2.2), one can first show that there exists $n' \in \mathbb{N}$, such that

$$\mathbb{P}\left[\sup_{t\in[0,T]} \left|Y_t^n - Y_t^{\hat{c}(x,0)}\right| \geq \frac{\varepsilon'}{3}\right] < \frac{\varepsilon'}{3}, \quad n \geq n', \tag{60}$$

where Y^n is the first component of the solution to

$$Y_t^n = \left(e^{-\delta\theta T}(\hat{c}_T(x,0))^{1-\gamma}\right) \wedge n + \int_t^T F\left(s, \hat{c}_s(x,0), Y_s^n\right) ds - \int_t^T Z_s^n dB_s, \ t \in [0,T].$$

Further following the proof of Xing (2017a, Proposition 2.2), one can show that $(Y^n, Z^n) \in \mathcal{S}^\infty \times \mathcal{M}^2$ to the BSDE above exists and is unique. Furthermore, via the comparison result, see, for example, Pardoux (1999, Theorem 2.4), we deduce that $0 \leq Y^n \leq n$, and $Y^n = \downarrow \lim_{m\to\infty} Y^{n,m}$, $m \in \mathbb{N}$, where $Y^{n,m}$ solves

$$\begin{aligned} Y_t^{n,m} = &\left(e^{-\delta\theta T}(\hat{c}_T(x,0))^{1-\gamma}\right) \wedge n + \int_t^T F^m\left(s, \hat{c}_s(x,0), Y_s^{n,m}\right) ds \\ &- \int_t^T Z_s^{n,m} dB_s, \quad t \in [0,T], \end{aligned} \tag{61}$$

where $F^m(t, c_t, y) := \delta\theta e^{-\delta t}\left(c_t^{1-\frac{1}{\psi}} \wedge m\right)(|y| \wedge m)^{1-\frac{1}{\theta}}$. Here, by comparison, we have $0 \leq Y^{n,m} \leq n$. As a result, for $m \geq n$, we get

$$
\begin{aligned}
F^m\left(t, c_t, Y_t^{n,m}\right) &= \delta\theta e^{-\delta t}\left(c_t^{1-\frac{1}{\psi}} \wedge m\right)\left(Y_t^{n,m} \wedge m\right)^{1-\frac{1}{\theta}} \\
&= \delta\theta e^{-\delta t}\left(c_t^{1-\frac{1}{\psi}} \wedge m\right)\left(Y_t^{n,m}\right)^{1-\frac{1}{\theta}}.
\end{aligned} \tag{62}
$$

Further, as we can show that, for every $n \in \mathbb{N}$, we have $\lim_{m\to\infty} \sup_{t\in[0,T]} |Y_t^n - Y_t^{n,m}| = 0$ in probability $\mathbb{P}$. Therefore, we conclude that there exists $m'(n)$, such that

$$
\mathbb{P}\left[\sup_{t\in[0,T]} |Y_t^n - Y_t^{n,m}| \geq \frac{\varepsilon'}{3}\right] \leq \frac{\varepsilon'}{3}, \quad m \geq m'(n). \tag{63}
$$

Next, for $c^{k,\delta',M}$ given by Equation (58), $k \in \mathbb{N}$, $\delta' > 0$, and $M > 0$, let us consider the following family of BSDEs:

$$
\begin{aligned}
\bar{Y}_t^{c^{k,\delta',M}} &= e^{-\delta\theta T}\left(c_T^{k,\delta',M}\right)^{1-\gamma} + \int_t^T F\left(s, c_s^{k,\delta',M}, \bar{Y}_s^{c^{k,\delta',M}}\right) ds \\
&\quad - \int_t^T \bar{Z}_s^{k,\delta',M} dB_s, \quad t \in [0,T], \quad k \in \mathbb{N}, \quad \delta' > 0, \quad M > 0.
\end{aligned} \tag{64}
$$

By Cohen and Elliot (2012, Theorem 5.1), for every choice of k, δ', M, there exists a unique solution to Equation (64), $\left(\bar{Y}^{c^{k,\delta',M}}, \bar{Z}^{k,\delta',M}\right) \in \mathcal{S}^2 \times \mathcal{M}^2$. Further, by replacing F with F^k as in Equation (62), and using the comparison for BSDEs results as in Pardoux (1999, Theorem 2.4), we deduce that the first component of the solution is in $\mathcal{S}^\infty$.

Let us consider Equations (61) and (64). These are BSDEs with bounded terminal conditions and Lipschitz generators. Therefore, for a given n satisfying Equation (60) and m satisfying Equation (61), Cohen and Elliot (2015, Theorem 19.1.6, p. 472) allow to pick δ' and M and then k_0, such that

$$
\mathbb{P}\left[\sup_{t\in[0,T]} \left|\bar{Y}_t^{c^{k,\delta',M}} - Y_t^{n,m}\right| \geq \frac{\varepsilon'}{3}\right] < \frac{\varepsilon'}{3}, \quad k \geq k_0. \tag{65}
$$

Comparing Equations (60), (63), and (65), we deduce that

$$
\mathbb{P}\left[\sup_{t\in[0,T]} \left|\bar{Y}_t^{c^{k,\delta',M}} - Y_t^{\widehat{c}(x,0)}\right| \geq \varepsilon'\right] < \varepsilon', \quad k \geq k_0.
$$

As ε' is arbitrary, we deduce that there exists $\widetilde{c}^k$, $k \in \mathbb{N}$, as in Equation (59), for which the assertion of this lemma holds. $\square$

5.2 | Proofs of the main theorems

Proof of Theorem 3.4. First, we observe that Equation (24) follows from Lemma 5.2. Next, from Lemma 5.3, we get

$$\liminf_{(x',\varepsilon)\to(x,0)} u(x',\varepsilon) \geq u(x,0). \tag{66}$$

Applying Lemma 5.4, we obtain

$$v(y,0) + xy \geq \limsup_{(y',\varepsilon)\to(y,0)} (v(y',\varepsilon) + xy'). \tag{67}$$

Now, using Matoussi and Xing (2018, Theorem 2.7), we have

$$\limsup_{(y',\varepsilon)\to(y,0)} (v(y',\varepsilon) + x'y') \geq \limsup_{\varepsilon\to 0} u(x',\varepsilon), \quad x' > 0. \tag{68}$$

From the assumption of the theorem (Equation 20), we deduce

$$u(x,0) = v(y,0) + xy. \tag{69}$$

Combining Equations (66)–(69), we conclude

$$\begin{aligned} u(x,0) &\leq \liminf_{(x',\varepsilon)\to(x,0)} u(x',\varepsilon) \leq \limsup_{(x',\varepsilon)\to(x,0)} u(x',\varepsilon) \\ &\leq \limsup_{(x',\varepsilon)\to(x,0)} \left(v(y,\varepsilon) + x'y\right) \leq v(y,0) + xy = u(x,0). \end{aligned}$$

Therefore, all inequalities above are equalities, and we get Equation (25). Next, from Equations (69), (67), (68), and (66), we obtain

$$\begin{aligned} u(x,0) = \ & v(y,0) + xy \geq \limsup_{(y',\varepsilon)\to(y,0)} \left(v(y',\varepsilon) + xy'\right) \\ &\geq \liminf_{(y',\varepsilon)\to(y,0)} \left(v(y',\varepsilon) + xy'\right) \geq \liminf_{(y',\varepsilon)\to(y,0)} u(x,\varepsilon) \geq u(x,0). \end{aligned}$$

Therefore, all inequalities above are actually equalities. This implies Equation (26).

Finally, the existence and uniqueness of the optimizers follow from Lemma 5.5 and convexity and closedness in $\mathbb{L}^0(d\varkappa \times \mathbb{P})$ of the set $\widetilde{\mathcal{A}}(x,\varepsilon)$, $(x,\varepsilon) \in (0,\infty)\times(-\varepsilon_0,\varepsilon_0)$, note that the convexity of $\widetilde{\mathcal{A}}(x,\varepsilon)$, $(x,\varepsilon) \in (0,\infty)\times(-\varepsilon_0,\varepsilon_0)$, follows from Assumption 3.1 and Lemma 2.1. □

Proof of Theorem 3.6. *Step 1.* Assume by contradiction that the assertion of this theorem, that is Equation (28), fails. Then, there exists $\delta > 0$, such that

$$\limsup_{n\to\infty} (d\varkappa \times \mathbb{P})[|\widehat{c}(x^n,\varepsilon^n) - \widehat{c}(x,0)| > \delta] > \delta.$$

As $\frac{1}{N^{\varepsilon_n}}$, $n \in \mathbb{N}$, converges to 1, in measure $(d\varkappa \times \mathbb{P})$, consequently $\frac{1}{N^{\varepsilon_n}}$, $n \in \mathbb{N}$, is bounded in $\mathbb{L}^0(d\varkappa \times \mathbb{P})$. Next, following Equation (8) and the argument in Mostovyi (2015, Proposition 4.2), one can see that the set $\mathcal{A}(1,0)$ is bounded in $\mathbb{L}^0(d\varkappa \times \mathbb{P})$. Therefore, since $\widehat{c}(x^n,\varepsilon^n)\frac{1}{N^{\varepsilon^n}} \in \mathcal{A}(x^n,0)$, by possibly passing to smaller δ, we deduce that

$$\limsup_{n\to\infty}(d\kappa\times\mathbb{P})\left[\left|\widehat{c}(x^n,\varepsilon^n)-\widehat{c}(x,0)\frac{1}{N^{\varepsilon^n}}\right|\geq\delta,\ \ \widehat{c}(x^n,\varepsilon^n)+\widehat{c}(x,0)\frac{1}{N^{\varepsilon^n}}\leq\frac{1}{\delta}\right]\geq\delta.$$

With $\widetilde{c}^k$, $k\in\mathbb{N}$, as in Equation (59) (in Lemma 5.6), by passing to even smaller δ, we get

$$\limsup_{n\to\infty}(d\kappa\times\mathbb{P})\left[\left|\widehat{c}(x^n,\varepsilon^n)-\widetilde{c}^n\right|\geq\delta,\ \ \widehat{c}(x^n,\varepsilon^n)+\widetilde{c}^n\leq\frac{1}{\delta}\right]\geq\delta. \tag{70}$$

Let us set

$$\bar{c}^n := \frac{1}{2}\left|\widehat{c}(x^n,\varepsilon^n)+\widetilde{c}^n\right|\in\mathcal{A}\left(\frac{x^n+x}{2},\varepsilon^n\right),\quad n\in\mathbb{N}. \tag{71}$$

Furthermore, one can show that $\bar{c}^n\in\widetilde{\mathcal{A}}\left(\frac{x^n+x}{2},\varepsilon^n\right)$, as for every $t\in[0,T]$, we have

$$\begin{aligned}\left(\bar{c}^n_t\right)^{1-\frac{1}{\psi}} &= \left(\frac{1}{2}\left|\widehat{c}_t(x^n,\varepsilon^n)+\widetilde{c}^n_t\right|\right)^{1-\frac{1}{\psi}}\\ &\leq \max\left(\widehat{c}_t(x^n,\varepsilon^n),\widetilde{c}^n_t\right)^{1-\frac{1}{\psi}}\leq\left(\widehat{c}_t(x^n,\varepsilon^n)\right)^{1-\frac{1}{\psi}}+\left(\widetilde{c}^n_t\right)^{1-\frac{1}{\psi}},\end{aligned}$$

and at maturity, we have

$$\left(\bar{c}^n_T\right)^{1-\gamma}=\left(\frac{1}{2}\left|\widehat{c}_T(x^n,\varepsilon^n)+\widetilde{c}^n_T\right|\right)^{1-\gamma}\leq\left(\frac{1}{2}\frac{x}{x+\varepsilon^n}\frac{1}{\varepsilon^n}\right)^{1-\gamma},$$

and thus, $\bar{c}^n$s satisfy both integrability conditions in the definition of $\widetilde{\mathcal{A}}'$s in Equation (9).

Step 2. Let us use Lemma 5.5 along a subsequence from *Step 1* that we do not relabel and such that

$$\lim_{n\to\infty}(d\kappa\times\mathbb{P})\left[\left|\widehat{c}(x^n,\varepsilon^n)-\widetilde{c}^n\right|\geq\delta,\ \ \widehat{c}(x^n,\varepsilon^n)+\widetilde{c}^n\leq\frac{1}{\delta}\right]\geq\delta.$$

In the argument below, the notations from the proof of Lemma 5.5 are used. For

$$\eta^n := \mathbb{E}[(\Gamma^n\xi^n)\cdot\kappa_T],\quad n\in\mathbb{N}, \tag{72}$$

one can show that[2]

$$\Gamma^n_t\geq\exp\left(a\theta\int_0^t\left(\bar{c}^n_s\right)^{1-\frac{1}{\psi}}\left(\Delta Y^n_s\right)^{-\frac{1}{\theta}}ds\right),$$

for some constant $a>0$ (and where $\theta<0$ and rΔY^n is as in the proof of Lemma 5.5 corresponding to $c'=\widehat{c}(x^n,\varepsilon^n)$, $c''=\widetilde{c}^n$, and $\lambda=\frac{1}{2}$.)

Next, from Lemma 5.6, along a subsequence, which we do not relabel, we have

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\left|Y^{\widetilde{c}^k}_t-Y^{\widehat{c}(x,0)}_t\right|=0,\quad \mathbb{P}\text{-a.s.}$$

[2] The lower bounds on α^ns are obtained through estimates on the slopes of F.

Further, $Y^{\widehat{c}(x^n,\varepsilon^n)}$ is bounded from above by a real-valued process, by the assumption of this theorem. Therefore, as $\Delta Y^n_t \leq \frac{1}{2}Y^{\widehat{c}(x^n,\varepsilon^n)}_t + \frac{1}{2}Y^{\widetilde{c}^n}_t$, $t \in [0,T]$, $\mathbb{P}$-a.s., by the proof of Lemma 5.5, we deduce that

$$\begin{aligned}\Gamma^n_t &\geq \exp\left(a\theta \int_0^t \left(\widetilde{c}^n_s\right)^{1-\frac{1}{\psi}} \left(\Delta Y^n_s\right)^{-\frac{1}{\theta}} ds\right) \\ &\geq \exp\left(a\theta \int_0^t \left(\widehat{c}_s(x^n,\varepsilon^n) + \widetilde{c}^n_s\right)^{1-\frac{1}{\psi}} \left(Y^{\widehat{c}(x^n,\varepsilon^n)}_s + Y^{\widetilde{c}^n}_s\right)^{-\frac{1}{\theta}} ds\right).\end{aligned} \tag{73}$$

From the assumptions of this theorem and Lemma 5.6, we obtain that

$$\liminf_{n\to\infty} \Gamma^n_t =: \widetilde{\Gamma}^\infty_t > 0, \quad t \in [0,T], \quad \mathbb{P}\text{-a.s.} \tag{74}$$

Let us consider the sequence ξ^n, $n \in \mathbb{N}$. Following the proof of Lemma 5.5 (see Equation 70), we observe that

$$\xi^n_t \geq \widetilde{\delta}_1 \left(\Delta Y^n_t\right)^{1-\frac{1}{\theta}} 1_{\{|\widehat{c}_t(x^n,\varepsilon^n)-\widetilde{c}^n_t|\geq\delta,\ \widehat{c}_t(x^n,\varepsilon^n)+\widetilde{c}^n_t\leq\frac{1}{\delta}\}}, \quad t \in [0,T), \tag{75}$$

and

$$\xi^n_T \geq \delta_2 1_{\{\left|\widehat{c}_T(x^n,\varepsilon^n)-\widetilde{c}^n_T\right|\geq\delta,\ \widehat{c}_T(x^n,\varepsilon^n)+\widetilde{c}^n_T\leq\frac{1}{\delta}\}}, \tag{76}$$

where constant $\widetilde{\delta}_1 > 0$ and $\delta_2 > 0$ depend on δ appearing in Equation (70) only. As $\Delta Y^n_t \geq \frac{1}{2^\theta}Y^{\widetilde{c}^n}_t$, $t \in [0,T]$, $\mathbb{P}$-a.s., by the argument in Lemma 5.6, we have

$$\liminf_{n\to\infty} \left(\Delta Y^n_t\right) \geq \liminf_{n\to\infty} \frac{1}{2^\theta} Y^{\widetilde{c}^n}_t = \frac{1}{2^\theta} Y^{\widehat{c}(x,0)}_t > 0, \ \ t \in [0,T], \ \ \mathbb{P}\text{-a.s..} \tag{77}$$

By the Dunford–Pettis compactness criterion (see, e.g., Karatzas and Shreve (1998, p. 26)), there exists a weakly (in $\mathbb{L}^1(d\kappa \times \mathbb{P})$) convergent subsequence of $\xi^n \wedge 1$, $n \in \mathbb{N}$, whose limit is denoted by ξ^∞. In view of Equations (75)–(77), we have $(d\kappa \times \mathbb{P})[\xi^\infty > 0] > 0$.

Let us pass to this subsequence that we do not relabel again. The non-negativity of $\Gamma^n\xi^n$ (by the construction above) allows invoking Fatou's lemma, which implies that

$$\begin{aligned}&\liminf_{n\to\infty} \mathbb{E}[(\Gamma^n(\xi^n \wedge 1)) \cdot \kappa_T] \\ =&\liminf_{n\to\infty} \left(\mathbb{E}\left[(\widetilde{\Gamma}^\infty(\xi^n \wedge 1)) \cdot \kappa_T\right] + \mathbb{E}\left[\left((\Gamma^n - \widetilde{\Gamma}^\infty)(\xi^n \wedge 1)\right) \cdot \kappa_T\right]\right) \\ \geq&\mathbb{E}\left[(\widetilde{\Gamma}^\infty\xi^\infty) \cdot \kappa_T\right] > 0,\end{aligned}$$

as $(d\kappa \times \mathbb{P})[\xi^\infty > 0] > 0$ and $\widetilde{\Gamma}^\infty > 0$, $(d\kappa \times \mathbb{P}) - a.e.$, as well as

$$\lim_{n\to\infty} \mathbb{E}\left[(\widetilde{\Gamma}^\infty(\xi^n \wedge 1)) \cdot \kappa_T\right] = \mathbb{E}\left[(\widetilde{\Gamma}^\infty\xi^\infty) \cdot \kappa_T\right],$$

by the weak convergence in $\mathbb{L}^1(d\kappa \times \mathbb{P})$ (here we recall that Γs take values in $[0,1]$ as $\theta < 0$), by Fatou's lemma (here, $\Gamma^\infty(\xi^n \wedge 1)$ is bounded from below by -1) and Equation (74), we have

$$\liminf_{n\to\infty} \mathbb{E}\Big[\big((\Gamma^n - \widetilde{\Gamma}^\infty)(\xi^n \wedge 1)\big) \cdot \kappa_T\Big] \geq 0.$$

We conclude that

$$\liminf_{n\to\infty} \eta^n > 0. \tag{78}$$

Step 3. For $\bar{c}^n$, $n \in \mathbb{N}$, defined in *Step 1* (see Equation 71), let us consider the subsequence from *Step 2*. By Lemma 5.5, we have

$$\liminf_{n\to\infty} U_0^{\bar{c}^n} \geq \liminf_{n\to\infty} \left(\frac{1}{2}u(x^n,\varepsilon^n) + \frac{1}{2}U_0^{c^n} + \bar{\eta}^n\right), \tag{79}$$

where $\bar{\eta}^n = \frac{\eta^n}{\gamma - 1}$ and η^n are given in Equation (72). It follows from Lemma 5.6 that

$$\lim_{n\to\infty} U_0^{c^n} = u(x,0). \tag{80}$$

On the other hand, as $\bar{c}^n \in \widetilde{\mathcal{A}}\left(\frac{x^n+x}{2}, \varepsilon^n\right)$, we get

$$U_0^{\bar{c}^n} \leq u\left(\frac{x^n + x}{2}, \varepsilon^n\right). \tag{81}$$

By Theorem 3.4, we have

$$\liminf_{n\to\infty} u(x^n,\varepsilon^n) = u(x,0). \tag{82}$$

Therefore, in Equation (79), via Equations (80)–(82), we conclude that

$$u(x,0) \geq \liminf_{n\to\infty} U_0^{\bar{c}^n} \geq \liminf_{n\to\infty} \left(\frac{1}{2}u(x^n,\varepsilon^n) + \frac{1}{2}U_0^{\bar{c}^n} + \bar{\eta}^n\right) \geq u(x,0) + \liminf_{n\to\infty} \bar{\eta}^n,$$

which is impossible, as $\liminf_{n\to\infty} \bar{\eta}^n = \liminf_{n\to\infty} \frac{\eta^n}{\gamma-1} > 0$ by Equation (78). □

Proof of Theorem 3.7. Let us recall that, for a given nonnegative consumption stream c, U^c was defined in Equation (4) and Y^c in Equation (35). The proof of Theorem 3.7 is entirely similar to the proof of Lemma 5.6. It relies on the truncation and the stability of BSDEs result as in Cohen and Elliot (2015, Theorem 19.1.6, p. 472), so that we can show that

$$\lim_{(x',\varepsilon)\to(x,0)} Y^{\widehat{c}(x',\varepsilon)} = Y^{\widehat{c}(x,0)}, \quad ucp \quad \text{and} \quad \lim_{(x',\varepsilon)\to(x,0)} U^{\widehat{c}(x',\varepsilon)} = U^{\widehat{c}(x,0)}, \quad ucp.$$

We omit further details for brevity. □

Proof of Theorem 3.8. Let us consider a sequence (x_n, ε_n), $n \in \mathbb{N}$, convergent to $(x, 0)$. Without loss of generality, we will suppose that $x_n > 0$ and $\varepsilon_n \in (-\varepsilon_0, \varepsilon_0)$, $n \in \mathbb{N}$. Let us denote

$$X^n = X^{\widehat{c}(x_n,\varepsilon_n)}, \quad N^n = N^{\varepsilon_n}, \quad c^n = \widehat{c}(x_n,\varepsilon_n), \quad n \in \mathbb{N}, \quad D = \widehat{D}(y,0),$$

and set

$$D^n := DN^n, \quad L^n := \frac{1}{x_n}\left(X^n D^n + \int_0^\cdot D^n_s c^n_s d\kappa_s\right). \tag{83}$$

Then, by Lemma 5.1, $D^n \in \mathcal{D}(y, \varepsilon_n)$ and thus L^n, $n \in \mathbb{N}$, is a sequence of non-negative supermartingales. Since $(d\kappa \times \mathbb{P})\text{-}\lim_{n\to\infty} c^n = \widehat{c}(x,0)$ by Theorem 3.6 and $(d\kappa \times \mathbb{P})\text{-}\lim_{n\to\infty} D^n = D$ by Equation (83) and the assumption that $(d\kappa \times \mathbb{P})\text{-}\lim_{\varepsilon\to 0} N^\varepsilon = 1$, we pass to a subsequence, which we do not relabel and suppose that $\lim\limits_{n\to\infty} D^n c^n = D\widehat{c}(x,0)$, $(d\kappa \times \mathbb{P})$-a.e. Therefore, using Fatou's lemma, we get

$$\liminf_{n\to\infty} \int_0^T D^n_s c^n_s d\kappa_s \geq \int_0^T D_s \widehat{c}_s(x,0) d\kappa_s, \quad \mathbb{P}\text{-a.s.} \tag{84}$$

Let us further set

$$L := \frac{1}{x}\left(X^{\widehat{c}(x,0)} D + \int_0^\cdot D_s \widehat{c}_s(x,0) d\kappa_s\right). \tag{85}$$

The optimality of $\widehat{c}(x,0)$ implies that $X^{\widehat{c}(x,0)}_T = 0$, $\mathbb{P}$-a.s., as it is optimal to consume everything that is left at maturity. Likewise, the optimality of c^n implies that $X^n_T = 0$, $\mathbb{P}$-a.s., for every $n \in \mathbb{N}$. Therefore, Equations (83)–(85) result in

$$\liminf_{n\to\infty} L^n_T \geq L_T, \quad \mathbb{P}\text{-a.s.} \tag{86}$$

From the respective definitions of L^n, $n \in \mathbb{N}$, and L, we conclude that

$$L^n_0 = D_0 = L_0, \quad n \in \mathbb{N}. \tag{87}$$

Let us recapitulate that L^n, $n \in \mathbb{N}$, are non-negative càdlàg supermartingales and L is a non-negative càdlàg martingale satisfying Equations (86) and (87). Let us consider a probability measure $\mathbb{R}$, whose density is $\frac{d\mathbb{R}}{d\mathbb{P}} = \frac{1+L_T}{1+L_0}$. Then $\mathbb{R} \sim \mathbb{P}$, and, under $\mathbb{R}$, $\frac{1+L^n}{1+L} = \left(\frac{1+L^n_t}{1+L_t}\right)_{t\in[0,T]}$, $n \in \mathbb{N}$, are non-negative supermartingales. Consequently, from Equations (86) and (87), we conclude that $\frac{1+L^n}{1+L}$, $n \in \mathbb{N}$, satisfies

$$\frac{1+L^n_0}{1+L_0} = 1 \quad and \quad \lim_{n\to\infty} \frac{1+L^n_T}{1+L_T} = 1, \quad \mathbb{P}\text{-a.s.}$$

Therefore, applying Kardaras (2013, Lemma 2.11) (under $\mathbb{R}$), we deduce that

$$\lim_{n\to\infty} \frac{1+L^n}{1+L} = 1, \quad ucp,$$

and thus

$$\lim_{n\to\infty} L^n = L, \quad ucp. \tag{88}$$

Consequently, and in particular, passing to another subsequence, which we do not relabel, we get

$$\lim_{n\to\infty} \int_0^T D_s^n c_s^n d\kappa_s = \int_0^T D_s \widehat{c}_s(x,0) d\kappa_s, \quad \mathbb{P}\text{-a.s.} \tag{89}$$

Next, similarly to Equation (84), for every $t \in [0,T]$, we deduce that

$$\liminf_{n\to\infty} \int_t^T D_s^n c_s^n d\kappa_s \geq \int_t^T D_s \widehat{c}_s(x,0) d\kappa_s, \quad \mathbb{P}\text{-a.s.}, \tag{90}$$

and

$$\liminf_{n\to\infty} \int_0^t D_s^n c_s^n d\kappa_s \geq \int_0^t D_s \widehat{c}_s(x,0) d\kappa_s, \quad \mathbb{P}\text{-a.s.} \tag{91}$$

Therefore, from Equations (89) and (90), we get

$$\begin{aligned} \limsup_{n\to\infty} \int_0^t D_s^n c_s^n d\kappa_s &= \limsup_{n\to\infty} \left(\int_0^T D_s^n c_s^n d\kappa_s - \int_t^T D_s^n c_s^n d\kappa_s \right) \\ &\leq \int_0^t D_s \widehat{c}_s(x,0) d\kappa_s, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{92}$$

In turn, Equations (91) and (92) imply that

$$\lim_{n\to\infty} \int_0^t D_s^n c_s^n d\kappa_s = \int_0^t D_s \widehat{c}_s(x,0) d\kappa_s, \quad \mathbb{P}\text{-a.s.}, \tag{93}$$

where the last equality holds for every $t \in [0,T]$. As the processes $\int_0^{\cdot} D_s^n c_s^n d\kappa_s$, $n \in \mathbb{N}$, and $\int_0^{\cdot} D_s \widehat{c}_s(x,0) d\kappa_s$, are cádlág monotone, from Equation (93), we get

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_0^t D_s^n c_s^n d\kappa_s - \int_0^t D_s \widehat{c}_s(x,0) d\kappa_s \right| = 0, \quad \mathbb{P}\text{-a.s.}. \tag{94}$$

Finally, as D^ns and D are strictly positive and $(d\kappa \times \mathbb{P})$-$\lim_{n\to\infty} D^n = D$, from Equations (88) and (94), using Durret (2005, Thoerem 1.6.2, p. 46), we deduce that $X^n = \frac{L^n - \int_0^{\cdot} D_s^n c_s^n d\kappa_s}{D^n}$, $n \in \mathbb{N}$, converges to $X^{\widehat{c}(x,0)} = \frac{L - \int_0^{\cdot} D_s \widehat{c}_s(x,0) d\kappa_s}{D}$ in measure $(d\kappa \times \mathbb{P})$. If $\lim_{\varepsilon\to 0} N^\varepsilon = 1$, ucp, then, similarly, from Equations (83), (88), and (94), using Durret (2005, Thoerem 1.6.2, p. 46), we conclude that

$$\lim_{n\to\infty} X^n = X^{\widehat{c}(x,0)}, \quad ucp.$$

□

ACKNOWLEDGMENTS

We would like to thank Kostas Kardaras for his invaluable suggestions several years ago in the context of a different problem that helped with the proof of Theorem 3.8. Oleksii Mostovyi has been supported by the National Science Foundation under Grant No. DMS1848339. Part of this work was completed while he was visiting the University of Oxford.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest related to this publication.

DATA AVAILABILITY STATEMENT

N/A.

ORCID

Oleksii Mostovyi https://orcid.org/0000-0003-4780-3751

REFERENCES

Aurand, J., & Huang, Y.-J. (2021). Mortality and healthcare: A stochastic control analysis under Epstein-Zin preferences. *SIAM Journal of Control Optimization*, *59*(5), 4051–4080.

Aurand, J., & Huang, Y.-J. (2023). Epstein-Zin utility maximization on a random horizon. *Mathematical Finance*, *33*(2023), 1370–1411.

Briand, P., & Hu, Y. (2006). BSDE with quadratic growth and unbounded terminal value. *Probability Theory Related Fields*, *136*(4), 604–618.

Carasus, L., & Rasonyi, M. (2007). Optimal strategies and utility-based prices converge when agents' preferences do. *Mathematical Operations Research*, *32*(1), 102–117.

Cohen, S., & Elliot, R. (2012). Existence, uniqueness and comparisons for BSDEs in general spaces. *Annals of Applied Probability*, *40*(5), 2264–2297.

Cohen, S., & Elliot, R. (2015). *Stochastic calculus and applications*. Springer.

Duffie, D., & Epstein, L. (1992). Asset pricing with stochastic differential utility. *Review of Financial Studies*, *5*(3), 411–436.

Duffie, D., & Skiadas, C. (1994). Continuous-time security pricing: A utility gradient approach. *Journal Mathematical Economic*, *23*(2), 107–131.

Durret, R. (2005). *Probability: Theory and examples. Duxbury advanced series* (3rd ed.). Thomson Brooks/Cole.

Epstein, L., & Zin, S. (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, *57*, 937–969.

Hadamard, J. (1902). Sur les problèmes aux dérivées partielles et leur signification physique. *Princeton University Bulletin*, *13*, 49–52.

Herdegen, M., Hobson, D., & Jerome, J. (2021). Proper solutions for Epstein-Zin stochastic differential utility. arXiv:2112.06708 [q-fin.MF].

Herdegen, M., Hobson, D., & Jerome, J. (2023). The infinite horizon investment-consumption problem for Epstein-Zin stochastic differential utility. *Finance Stochastic*, *27*, 127–158.

Hu, Y., Imkeller, P., & Müller, M. (2005). Utility maximization in incomplete markets. *Annals of Applied Probability*, *15*(3), 1691–1712.

Jouini, E., & Napp, C. (2004). Convergence of utility functions and convergence of optimal strategies. *Finance Stochastic*, *8*, 133–144.

Karatzas, I., & Kardaras, K. (2007). The numéraire portfolio in semimartingale financial models. *Finance Stochastic*, *11*(4), 447–493.

Karatzas, I., & Kardaras, K. (2021). *Portfolio theory and arbitrage: A course in mathematical finance*. AMS.

Karatzas, I., & Shreve, S. (1998). *Brownian motion and stochastic calculus* (2nd ed.). Springer.

Kardaras, K. (2013). On the closure in the Emery topology of semimartingale wealth-process sets. *Annals of Applied Probability*, *23*(4), 1355–1376.

Kardaras, K., & Žitković, G. (2011). Stability of the utility maximization problem with random endowment in incomplete markets. *Mathematical Finance*, *21*(2), 313–333.

Kazamaki, N. (1994). *Continuous exponential martingales and BMO. Lecture notes in mathematics* (Vol. 1579). Springer-Verlag.

Kraft, H., Seiferling, T., & Seifried, F. T. (2017). Optimal consumption and investment with Epstein-Zin recursive utility. *Finance Stochastic*, *17*, 187–226.

Kramkov, D., & Schachermayer, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, *9*(3), 904–950.

Kramkov, D., & Schachermayer, W. (2003). Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Annals of Applied Probability*, *13*(4), 1504–1516.

Kreps, D., & Porteus, E. (1978). Temporal resolution of uncertainty and dynamic choice theory. *Econometrica*, *46*(1), 185–200.

Matoussi, A., & Xing, H. (2018). Convex duality for Epstein-Zin stochastic differential utility. *Mathematical Finance*, *28*(4), 991–1019.

Mostovyi, O. (2015). Necessary and sufficient conditions in the problem of optimal investment with intermediate consumption. *Finance Stochastic*, *19*(1), 135–159.

Mostovyi, O. (2020). Asymptotic analysis of the expected utility maximization problem with respect to perturbations of the numéraire. *Stochastic Processes Applied*, *130*(7), 4444–4469.

Mostovyi, O. (2021). Stability of the indirect utility process. *SIAM. Journal of Financial Mathematical*, *12*(2), 641–671.

Nutz, M. (2010). The opportunity process for optimal consumption and investment with power utility. *Mathematical Financial Economics*, *3*(3), 139–159.

Pardoux, E. (1999). BSDEs, weak convergence and homogenization of semilinear PDEs. In F. H. Clark (Ed.), *Nonlinear analysis, differential equations and control. NATO Science Series, Series C: Mathematical and Physical Sciences* (Vol. 528, pp. 503–549). Kluwer Academic Publishers.

Veraguas, J., & Silva, F. (2018). Sensitivity analysis for expected utility maximization in incomplete Brownian market models. *Mathematical Financial Economics*, *12*(3), 387–411.

Xing, H. (2017a). Consumption-investment optimization with Epstein-Zin utility in incomplete markets. *Finance Stochastic*, *21*(1), 227–262.

Xing, H. (2017b). Stability of the exponential utility maximization problem with respect to preferences. *Mathematical Finance*, *27*(1), 38–67.

How to cite this article: Monoyios, M., & Mostovyi, O. (2024). Stability of the Epstein–Zin problem. *Mathematical Finance*, 1–28. https://doi.org/10.1111/mafi.12434