IOP PUBLISHING NONLINEARITY

Nonlinearity **23** (2010) 409–427 doi:10.1088/0951-7715/23/2/010

Self-similar rupture of viscous thin films in the strong-slip regime

D Peschka[1], **A Münch**[2] **and B Niethammer**[2]

[1] Weierstrass Institute for Applied Analysis and Stochastics, 10117 Berlin, Germany
[2] Mathematical Institute, University of Oxford, 24-29 St Giles, Oxford OX1 3LB, UK

Received 23 February 2009, in final form 7 December 2009
Published 11 January 2010
Online at stacks.iop.org/Non/23/409

Recommended by L Mahadevan

Abstract
We consider rupture of thin viscous films in the strong-slip regime with small Reynolds numbers. Numerical simulations indicate that near the rupture point viscosity and van der Waals forces are dominant and that there are self-similar solutions of the second kind. For a corresponding simplified model the self-similar behaviour is analysed rigorously. There exists a continuous one-parameter family of exact self-similar solutions. In a certain parameter regime necessary and sufficient conditions for convergence to a self-similar solution are established. A conjecture on the domains of attraction of all self-similar solutions is presented and supported by numerical simulations.

Mathematics Subject Classification: 76D08, 35A20, 45J05, 35B40

1. Introduction

In this paper we investigate convergence to self-similar form in models describing thin-film rupture or jet pinch-off.

The term *jet pinch-off* or *break-up* refers to the process where a liquid cylindrical thread/jet pinches, i.e. owing to capillary forces the radius of the jet goes to zero and then the jet decays into drops. Similarly the term *film rupture* refers to a flat hydrophobic liquid layer which consequently ruptures, i.e. the thickness of the layer goes to zero, and shows various (spinodal) dewetting patterns [1]. Review papers for both situations are, for example, by Eggers and Villermaux [2] and by Oron *et al* [3].

Numerical simulations of a variety of models for pinching of jets and for rupture of hydrophobic thin films indicate self-similar behaviour near the pinching or rupture point, respectively, e.g. [2, 4–9]. Despite this large number of results based on numerical simulations and, partly, formal asymptotic analysis, there seem to be almost no rigorous results on the convergence to self-similar solutions for general data.

 © 2010 IOP Publishing Ltd and London Mathematical Society Printed in the UK

Our focus in this paper is on the self-similar dynamics towards rupture in a strong-slip model with van der Waals forces [10]. For small Reynolds numbers numerical simulations suggest that the evolution towards rupture passes through a self-similar regime where viscosity and intermolecular forces are dominant [11]. This regime is only transient because inertia cannot be neglected in the final stages of rupture. For freely suspended liquid sheets this was pointed out by Vaynblat *et al* in [12]. However, for a small Reynolds number this transient regime dominates the rupture dynamics over several orders of magnitude (film thickness goes to zero, hydrodynamic velocity goes to infinity). Within that regime the evolution of the film thickness and the velocity are governed by the usual power-law-type self-similar dynamics, that is the film thickness h and the horizontal velocity u follow

$$h(t,y) \sim (t_* - t)^{\alpha} \bar{H}(\eta) \qquad \text{and} \qquad u(t,y) \sim (t_* - t)^{\beta-1} \bar{U}(\eta),$$

where $\eta = (y - y_*)/(t_* - t)^{\beta}$ and y_*, t_* denote the point and time of rupture, respectively. The scaling exponent α is $1/3$, while β is not determined by the balance of the dominant terms and therefore one speaks of self-similarity of the second kind [13]. In the context of freely suspended sheets this indeterminacy was noted in [14].

Second-kind similarity solutions also appear in simplified models for inertialess jet pinch-off. Self-similar solutions for pinching jets are studied in [15], while in [16] the existence of countably many self-similar solutions is established numerically. In [17] convergence to these solutions is discussed. Under strong assumptions on the evolution of the jet the authors show that the selection of the self-similar solution in this model is determined solely by the behaviour of the initial data around their minimum. This behaviour is expected, but a rigorous analysis without *a priori* assumptions on the solutions has still been elusive.

The goal of this paper is to provide such an analysis for viscosity dominated thin-film rupture. Note that this analysis is also directly applicable to the model of jet pinch-off considered in [17].

The paper is organized as follows. In section 2 the strong-slip equation is introduced. In the regime where surface tension and inertia can be neglected one can simplify the strong-slip equation to an integro-differential equation for the Lagrangian variables following the ideas in [18, 19]. For this model finite time rupture is established in [11] by extending the methods introduced by Renardy [17, 19].

We introduce similarity variables and characterize self-similar solutions in section 3. As described above, we encounter the situation that the scaling for the spatial variable β is not determined by dimensional analysis. It turns out that for each $\beta > 1/3$ there exists a unique self-similar solution. Equivalently these self-similar solutions can be uniquely characterized by their behaviour $\bar{H}(\eta) = \bar{H}_0 + \bar{H}_\rho \eta^\rho$ as $\eta \to 0$, where $\rho = \rho(\beta) \in (0, \infty)$. Note that the model studied here neglects surface tension and thereby it is not necessary for the self-similar profile to be smooth at the origin. Doing so one cannot explain why certain smooth self-similar solutions are selected, but therefore one can explain why the dynamics is self-similar in the first place.

In section 4 we investigate whether solutions of the time-dependent problem converge to a self-similar shape and, in the case they do, which self-similar solution, i.e. which ρ (or β), is selected. As expected, the long-time asymptotic behaviour is completely determined by the behaviour of the initial profile $h(0, y)$ at its minimum. We establish a necessary and sufficient condition for convergence to any of the self-similar solutions with $0 < \rho < 3/2$ (for the pinch-off model it would be for $0 < \rho < 2/3$). The precise criterion is that the solution converges if and only if the initial data are regularly varying at their minimum with index ρ. The corresponding scaling is the one associated with the self-similar solution up to some slowly varying function implicitly given by the initial data. These results are very similar in nature to the dynamics in mean-field models for domain coarsening [20–22] and coagulation [23], where the long-time behaviour depends sensitively on the tail of the initial distribution functions.

We can prove the analogous characterization of domains of attraction for every positive ρ under an additional assumption, namely that an associated nonlocal functional converges sufficiently fast. Presently we have no proof whether this assumption is satisfied for regularly varying data. Numerical results in section 5 indicate that it is, but they also show that the situation is much more involved than in the case of $0 < \rho < 3/2$ and the details of the convergence proof must be different.

Since our mathematical results and proofs are given in Lagrangian variables we finally state in section 6 our main results in terms of the original variables h in the Eulerian coordinate system.

2. Model and simplification

The starting point for the following considerations is the 1D strong-slip model

$$\partial_t h + \partial_y(hu) = 0, \tag{1a}$$

$$Re^*(\partial_t u + u\partial_y u) = \frac{4}{h}\partial_y(h\partial_y u) + \partial_y\left(\partial_{yy}^2 h - V(h)\right) - b^{-1}\frac{u}{h}, \tag{1b}$$

where h represents the film thickness and u is the depth-averaged velocity. This model contains the Reynolds number Re^*, the non-dimensional slip-length b and the van der Waals interaction $V(h) = A/h^3$. Boundary conditions at $y = 0$ and $y = L$ are $\partial_y h(t, 0) = \partial_y h(t, L) = 0$ and $u(t, 0) = u(t, L) = 0$. This model emerges as a limit of a 2D Navier–Stokes equation with a free boundary if the ratio $\varepsilon = [h]/[y]$ of the typical height scale $[h]$ and the length scale $[y]$ is small and if the dimensional slip-length scales as $B = \varepsilon^{-2}b[h]$. In [10] various models with different scalings of B are derived. A model for the dynamics of freely suspended films can be found in [24].

As discussed in [11] the slip-length as well as surface tension are subdominant near the point of rupture and have no direct influence on the dynamics of rupture. Hence it is reasonable to neglect the corresponding terms in (1a) and (1b). Furthermore we consider the case of small Reynolds number and consequently also neglect the effect of inertia. With $L = A = 1$ this leads to the following equations for $h(t, y)$ and $u(t, y)$

$$\partial_t h + \partial_y(hu) = 0, \tag{2a}$$

$$\frac{4}{h}\partial_y(h\partial_y u) - \partial_y\left(h^{-3}\right) = 0, \tag{2b}$$

for all $y \in (0, 1)$, supplemented with boundary conditions $u(t, 0) = u(t, 1) = 0$ and initial data $h(0, y) = h_0(y)$ for all $y \in [0, 1]$. Note that no boundary conditions for h are necessary. Equation (2a) describes transport of fluid volumes whereas the momentum equation (2b) describes the quasi-steady balance due to van der Waals forces and dissipation due to Trouton viscosity.

For the analysis to come it is convenient to convert (2a) and (2b) to Lagrangian coordinates [18]. Let x be the Lagrangian coordinate, and $y = y(t, x)$ the map to Eulerian coordinates defined by $\partial_t y \equiv u(t, y)$ with some initial value $y(0, x)$.

Below equations (2a) and (2b) are written in terms of the stretching variable defined as $s(t, x) \equiv \partial_x y(t, x)$. The transformation $y(t, \cdot)$ preserves the orientation and it maps $(0, 1)$ onto itself, which implies $s(t, x) > 0$ and $y(t, 1) = 1$ implies

$$\int_0^1 s(t, x)\,\mathrm{d}x = 1. \tag{3}$$

With $\bar{h}(t,x) \equiv h(t,y)$, $\bar{u}(t,x) \equiv u(t,y)$ and (2*a*) one obtains

$$\partial_t\big(\bar{h}(t,x)\,s(t,x)\big) = 0.$$

This implies that the product $c(x) \equiv \bar{h}(t,x)s(t,x)$ is constant along characteristic curves. Integrating (2*b*) with respect to y and going over to Lagrangian coordinates yields

$$\partial_t s(t,x) = \frac{3\,c(x)}{8}\left(\frac{1}{\bar{h}^4(t,x)} - \frac{\sigma^2(t)}{\bar{h}^2(t,x)}\right), \tag{4}$$

with a constant of integration $\sigma^2(t)$. The explicit dependence on $c(x)$ is removed by choosing $y(0,x)$ such that $c(x)$ is constant, i.e. $c(x) \equiv c$. Thus, after rescaling time by an appropriate constant, (4) becomes

$$\partial_t s(t,x) = s(t,x)^2\big(s(t,x)^2 - \sigma^2(t)\big), \qquad x \in [0,1], \tag{5a}$$

where finally $\sigma^2(t)$ is determined by the constraint (3), which implies

$$\sigma^2(t) = \int_0^1 s(t,x)^4\,\mathrm{d}x \bigg/ \int_0^1 s(t,x)^2\,\mathrm{d}x. \tag{5b}$$

Equations (5*a*) and (5*b*) have to be supplemented with initial conditions

$$s(0,x) = s_0(x) \geqslant 0 \qquad \text{for all } x \in [0,1]\,, \tag{5c}$$

where $\int_0^1 s_0(x)\,\mathrm{d}x = 1$. For large times the behaviour of the solution $s(t,x)$ of (5*a*)–(5*c*) is determined by its behaviour around the maximum $s_{\max}(t) \equiv \operatorname{ess\,sup}_{x\in(0,1)} s(t,x)$.

The main assumption on s_0 used throughout this paper is that only decreasing initial data are considered. This is certainly justified if h_0 is symmetric around the rupture point. Furthermore, for non-monotone initial data one can work with the monotone rearrangement [25] defined as

$$s_0^*(x) \equiv \inf\big\{s : \mu\{x' : s_0(x') > s\} \leqslant x\big\}, \qquad x \in \mathbb{R}_+,$$

which leaves important properties of the dynamics unchanged, e.g. $s_{\max}(t) = s^*_{\max}(t)$.

Otherwise s_0 should be right-continuous such that $s_{\max}(t) = s(t,0)$ for all $t \geqslant 0$ and for convenience it is assumed that s_0 is strictly positive. Since the right-hand side of (5*a*) is locally Lipschitz and contains x only as a parameter one obtains existence and uniqueness of solutions locally in time. It is easily seen that $s_{\max}(t)$ is strictly increasing in time if $s_0(x)$ is not constant. In [11] it is shown that if $s_0(x) < s_{\max}(0)$ for all $x > 0$, then $s_{\max}(t) \to \infty$ as $t \to t_*$ where t_* can be finite or infinite.

From now on it is assumed that $s_0(x) \leqslant s_{\max}(0) - cx^\rho$ for some $c, \rho > 0$. As established in [11] this implies that t_* is finite, i.e. the blow-up occurs after a finite time. The goal of this paper is to study whether this blow-up occurs in a self-similar fashion.

For a later interpretation of the theorems that are proven for the solution $s(t,x)$ of (5*a*) it is useful to remember that $h(t,y)$ and $u(t,y)$ can be recovered back by

$$h(t,y) = \frac{c}{s(t,x)} \qquad \text{and} \qquad u(t,y) = \int_0^x \partial_t s(t,x')\,\mathrm{d}x', \tag{6}$$

where the Eulerian coordinate is by definition $y = \int_0^x s(t,x')\,\mathrm{d}x'$. The constant c appeared in the definition of the transformation.

3. Scaling and exact self-similar solutions

3.1. Scaling of the solution

Roughly speaking, a scaling denotes the time-dependent factor by which one stretches the graph of the function $s(t,x)$ at $x = 0$. If that renormalized graph has a nontrivial limit one speaks of self-similarity. Commonly one would seek self-similar solutions of (5a)–(5c) with power-law-type scaling

$$s(t,x) = \big(t_* - t\big)^{-\alpha}\varphi\big(\eta\big) \tag{7}$$

with $\eta = x\big(t_* - t\big)^{-\gamma/3}$ (the factor 1/3 is for later convenience) and $\theta(t) = \sigma^2(t)(t_* - t)^{2/3}$. The notation t_* stands for the blow-up time. The numbers $\alpha, \gamma > 0$ and the profile φ have to be determined. By plugging (7) into (5a) one observes that $\alpha = 1/3$ and φ must satisfy

$$\frac{\gamma}{3}\varphi'(\eta) = \varphi^4 - \theta\varphi^2 - \frac{1}{3}\varphi. \tag{8}$$

If one tries to fix γ in (8) using the condition imposed by the constraint (3), which is valid for any time-dependent solution $t \to t_*$, one arrives at

$$1 = \lim_{t\to t_*}\big(t_* - t\big)^{(\gamma-1)/3}\int_0^{(t_*-t)^{-\gamma/3}}\varphi(\eta)\,\mathrm{d}\eta. \tag{9}$$

This suggests to set $\gamma = 1$, but then equation (8) implies that $\varphi(\eta) \sim \eta^{-1}$ as $\eta \to \infty$, which is inconsistent with (9). Therefore condition (3) does not fix the similarity scale. Instead one needs to look for second-kind similarity solutions. It will be shown that for any $\gamma > 1$ there exists a second-kind similarity solution.

Instead of using definition (7) it will be more convenient to rescale $s(t,x)$ with $s_{\max}(t)$ as specified in the following two definitions.

Definition 3.1 (height scaling). *For any solution $s(t,x)$ of (5a)–(5c) define the normalized solution $\psi(\tau,x)$ and the new time scale τ via*

$$\psi(\tau,x) \equiv \frac{s(t,x)}{s_{\max}(t)} \qquad \textit{and} \qquad \tau \equiv \log\left(\frac{s_{\max}(t)}{s_{\max}(0)}\right). \tag{10}$$

The corresponding initial data are $\psi_0(x) = s_0(x)/s_{\max}(0)$. The definition of τ makes sense as $s_{\max}(t)$ is strictly increasing and unbounded.

Definition 3.2 (similarity scaling). *Let $\lambda(\tau)$ be a rescaling (or scaling) function with $\lambda(0) = 1$ and $\lambda \to \infty$ as $\tau \to \infty$. For any solution $s(t,x)$ of (5a)–(5c) define the normalized and rescaled solution $\varphi(\tau,\eta)$ via*

$$\varphi(\tau,\eta) \equiv \frac{s(t,x)}{s_{\max}(t)}, \qquad \textit{where } \eta = x\,\lambda(\tau), \tag{11}$$

and the initial data are $\varphi_0(\eta) = s_0(\eta)/s_{\max}(0)$. Two scalings $\lambda(\tau)$ and $\bar{\lambda}(\tau)$ are said to be equivalent, if there exists a positive number C such that

$$\lim_{\tau\to\infty}\left(\frac{\lambda(\tau)}{\bar{\lambda}(\tau)}\right) = C.$$

3.2. *Solution formulae*

The solution of the integro-differential equation (5a)–(5c) in the height scaling (10) solves

$$\partial_\tau\psi = K(\tau)\big(\psi^4 - \psi^2\big) + \psi^4 - \psi \equiv f\big(K(\tau),\psi\big), \tag{12}$$

where

$$K(\tau) \equiv \theta(\tau)\big(1 - \theta(\tau)\big)^{-1} \qquad \text{and} \qquad \theta(\tau) \equiv \frac{\int_0^1 \psi(\tau,x)^4\,\mathrm{d}x}{\int_0^1 \psi(\tau,x)^2\,\mathrm{d}x}. \tag{13}$$

This form of $K(\tau)$ is equivalent to the constraint (5*b*). To integrate (12) the function

$$H(\xi) \equiv \int_{1/2}^{\xi} \frac{\mathrm{d}r}{f(K_*, r)}$$

is introduced, where f is as in (12). This function is strictly decreasing, $H(1/2) = 0$. Next insert ψ from (10) into H and compute the derivative with respect to τ

$$\frac{\mathrm{d}}{\mathrm{d}\tau} H(\psi(\tau, x)) = \frac{\partial_\tau \psi}{f(K_*, \psi)} = \frac{f\big(K(\tau), \psi(\tau, x)\big)}{f\big(K_*, \psi(\tau, x)\big)}.$$

Integrating in time this yields the solution formula for ψ

$$H\big(\psi(\tau, x)\big) - H\big(\psi(\tau_0, x)\big) = \tau - \tau_0 + \int_{\tau_0}^{\tau} \big(K(t) - K_*\big) g\big(\psi(t, x)\big)\, \mathrm{d}t \tag{14}$$

for all $x \in (0, 1)$ and $0 \leqslant \tau_0 \leqslant \tau$ with g being defined as

$$g(\xi) \equiv \frac{\xi(\xi + 1)}{1 + (K_* + 1)\xi(\xi + 1)} \in \left[0, \frac{2}{2K_* + 3}\right]. \tag{15}$$

Rescaled solutions can be conveniently studied in this formulation since the dependence on $K(\tau)$ and the dependence on initial data are encoded separately. It will be repeatedly used that the leading order singular behaviour of H as $\psi \to 1$ is

$$H(\psi) = \frac{1}{2K_* + 3} \log(1 - \psi) + \mathcal{O}(1)\,, \tag{16}$$

which follows from the fact that

$$\frac{1}{2K_* + 3} \log\left(\frac{1 - \psi}{1/2}\right) - H(\psi) = \int_{1/2}^{\psi} \left(\frac{1}{(2K_* + 3)(\xi - 1)} - \frac{1}{f(K_*, \xi)}\right) \mathrm{d}\xi \tag{17}$$

and that the integrand on the right-hand side is bounded as $\psi \to 1$.

Now consider solutions $\varphi(\tau, \eta)$ under the similarity scaling. For further reference note that the functions $\theta(\tau)$ and $K(\tau)$ defined in (13) are independent of the rescaling $\lambda = \lambda(\tau)$ and can also be written as $K = \theta(1 - \theta)^{-1}$, where

$$\theta(\tau) \equiv \frac{\int_0^{\lambda} \varphi(\tau, \eta)^4\, \mathrm{d}\eta}{\int_0^{\lambda} \varphi(\tau, \eta)^2\, \mathrm{d}\eta}. \tag{18}$$

Recalling (11), equation (14) gives the solution formula for φ

$$\begin{aligned} H\big(\varphi(\tau, \eta)\big) = \tau - \tau_0 + H\left(\varphi\left(\tau_0, \eta\frac{\lambda(\tau_0)}{\lambda(\tau)}\right)\right) \\ + \int_{\tau_0}^{\tau} \big(K(t) - K_*\big)\, g\left(\varphi\left(t, \eta\frac{\lambda(t)}{\lambda(\tau)}\right)\right) \mathrm{d}t \end{aligned} \tag{19}$$

for all $\eta \in \big(0, \lambda(\tau)\big)$ and $0 \leqslant \tau_0 \leqslant \tau$. This formulation admits less regular solutions than

$$\partial_\tau \varphi + (\partial_\tau \log \lambda)\, \eta \partial_\eta \varphi = f\big(K(\tau), \varphi\big), \tag{20}$$

which is equivalent to (19) but only meaningful for differentiable φ and λ.

3.3. *Exact self-similar solutions*

Exact self-similar solutions are time-independent rescaled solutions $\varphi_*(\eta)$ of (5*a*)–(5*c*) with constant $K(\tau) = K_*$. That is, in view of (19) they satisfy

$$H\big(\varphi_*(\eta)\big) = \tau + H\left(\varphi_*\left(\frac{\eta}{\lambda(\tau)}\right)\right) \tag{21}$$

for all $\eta > 0$ and for some scaling $\lambda(\tau)$.

All strictly monotone solutions of (21) are smooth and hence, using (20), one finds that $\gamma \equiv \partial_\tau \log \lambda(\tau)$ is constant and $\lambda(\tau) = \exp(\gamma\tau)$. Since φ_* is decreasing and $f < 0$ it follows that $\gamma > 0$ and that φ_* solves

$$\gamma\eta \frac{\mathrm{d}\varphi_*}{\mathrm{d}\eta} = K_*\big(\varphi_*^2 - \varphi_*^2\big) + \varphi_*^4 - \varphi_* = f(K_*, \varphi_*). \tag{22}$$

Note that for any $K_* \in \mathbb{R}$ equation (22) also has the two homogeneous solutions $\varphi_*(\eta) = 1$ and $\varphi_*(\eta) = 0$ for all $\eta > 0$. The latter is discontinuous for $\eta \to 0$. These are obviously not relevant solutions and correspond to rescalings which either grow too fast, such that one sees only $\varphi_* = 1$ in the limit, or too slow in the other case.

Equation (22) is equivalent to

$$H(\varphi_*(\eta)) = \frac{1}{\gamma}\log\eta + C \tag{23}$$

for some constant $C \in \mathbb{R}$. The undetermined constant C in (23) is due to the invariance of equation (22) under a constant rescaling of η which is also valid for equation (20) and is also related to the equivalence of rescalings $\lambda(\tau)$.

As in the definition of the similarity scaling two stationary solutions φ_*^1 and φ_*^2 are said to be equivalent if there exists $c > 0$ such that $\varphi_*^1(c\eta) = \varphi_*^2(\eta)$ for all $\eta > 0$. Sometimes it is convenient to fix one member of each equivalence class, for example by requiring that

$$\varphi_*(1/2) = 1/2. \tag{24}$$

Then equation (23) becomes

$$H(\varphi_*(\eta)) = \frac{1}{\gamma}\log\left(\frac{\eta}{1/2}\right).$$

Before proceeding, some properties of a stationary solution $\varphi_*(\eta)$ are collected. First, (23) implies that φ_* is decreasing and satisfies $\lim_{\eta\to 0}\varphi_*(\eta) = 1$ and $\lim_{\eta\to\infty}\varphi_*(\eta) = 0$. Furthermore, as $\eta \to \infty$ equation (22) implies that

$$\varphi_*(\eta) \sim \frac{1}{\eta^{1/\gamma}} \qquad \text{as } \eta \to \infty. \tag{25}$$

Using Taylor's expansion $f(K, \xi) = -(2K+3)(1-\xi) + O\big((1-\xi)^2\big)$, one finds that every solution of (22) satisfies

$$\varphi_*(\eta) \sim 1 - C\eta^{\rho} \qquad \text{for } \eta \to 0 \quad \text{where } \gamma\rho = 2K_* + 3. \tag{26}$$

Next, one has to ask which combinations of γ and K_* are meaningful. First note that the constraint (3) poses further restrictions on γ. As discussed before it is requested that (3) is valid in the limit as $\tau \to \infty$, that is

$$\frac{1}{s_{\max}(0)} = \lim_{\tau\to\infty} \mathrm{e}^{-(\gamma-1)\tau} \int_0^{\mathrm{e}^{\gamma\tau}} \varphi_*(\eta)\,\mathrm{d}\eta. \tag{27}$$

Thus a nontrivial self-similar solution which satisfies (3) can only exist if $\gamma \geqslant 1$. Furthermore, due to (25), the choice $\gamma = 1$ leads to a contradiction with (27). Hence self-similar solutions satisfy $\gamma > 1$. Instead of working with (27) the following analogue of (18) will be requested.

Definition 3.3 (exact similarity solutions). *A function φ_* which satisfies (23) for some $\gamma > 1$ and for which satisfies*

$$\overline{K}(K_*, \gamma) = K_*, \tag{28a}$$

where

$$\overline{K}(K_*, \gamma) \equiv \frac{\int_0^\infty \varphi_*^4(\eta)\,\mathrm{d}\eta}{\int_0^\infty \left(\varphi_*^2(\eta) - \varphi_*^4(\eta)\right)\mathrm{d}\eta}, \tag{28b}$$

is called an exact self-similar solution of (5a)–(5c). In (28b) $\overline{K} = 0$ is set if the second moment of φ_* is infinite. One can easily show that convergence of φ to a self-similar solution together with $\int \varphi^2\,\mathrm{d}\eta \to \infty$ implies $K(\tau) \to 0$.

The main result for exact self-similar solutions is the following.

Theorem 3.1 (existence and uniqueness). *For any $\gamma > 1$ there exists an exact self-similar solution φ_*. That solution is unique up to equivalence. The requirement $\gamma > 1$ is equivalent to $\rho > 0$ with ρ defined in (26).*

Strategy: given $\gamma > 1$ and $K_* \geqslant 0$ find a φ_* via (23) and impose (24). For such a solution consider $\overline{K}(K_*, \gamma)$ as in (28b) and find a unique fixed point K_* of $\overline{K}(K_*, \gamma)$.

Proof.

(i) Proof for $K_* = 0$ ($\gamma \geqslant 2$ or $0 < \rho \leqslant 3/2$).
Integrate (22) explicitly using (23). For $K_* = 0$ one finds that

$$H\big(\varphi_*(\eta)\big) = \frac{1}{3}\log\left(\frac{s^3 - 1}{s^3}\right)\Bigg|_{1/2}^{\varphi_*(\eta)}$$

such that $\varphi_*(\eta) = (1 + c_0\eta^{3/\gamma})^{-1/3}$, where the choice of c_0 ensures $\varphi_*(1/2) = 1/2$.
For φ_* to be a self-similar solution one needs $\overline{K}(0, \gamma) = 0$, more precisely $\int_0^\infty \varphi_*^2(\eta)\,\mathrm{d}\eta = \infty$. This is the case for $\gamma \geqslant 2$. Solutions of (22) with the same γ are equivalent and the relation between γ and ρ is $\gamma\rho = 3$.

(ii) Proof for $K_* > 0$ ($\gamma \in (1, 2)$ or $\rho > 3/2$): a rigorous proof can be found in [26]. The idea is to show $\overline{K}(K_*, \gamma) < K_*$ for fixed γ and large K_*. Since $\overline{K} > 0$ for $K_* = 0$ and since $\overline{K}$ depends continuously on K_* this proves the existence of a fixed point $\overline{K} = K$. As this proof is somewhat laborious only a numerical reasoning to the problem is presented.
Figure 1 shows for every fixed $\gamma \in (0, 1)$ that $\overline{K}$ has a unique intersection point with the K_* plane.

The relation between ρ and γ is one-to-one since $\gamma\rho = 2\overline{K} + 3$ admits exactly one solution ($\overline{K}$ in figure 1 is decreasing with γ for fixed K_*). □

4. Convergence to self-similar solutions

In this section convergence of solutions $s(t, x)$ of (5a)–(5c) to self-similar solutions is proven. More precisely, we speak of convergence to a self-similar solution if the similarity scaling $\varphi(\tau, \eta)$ of (11) converges to some $\varphi_*(\eta)$ as $\tau \to \infty$. The scaling $\lambda(\tau)$ is yet unknown. In the following lemma it is shown that convergence of φ to a self-similar solution implies uniqueness

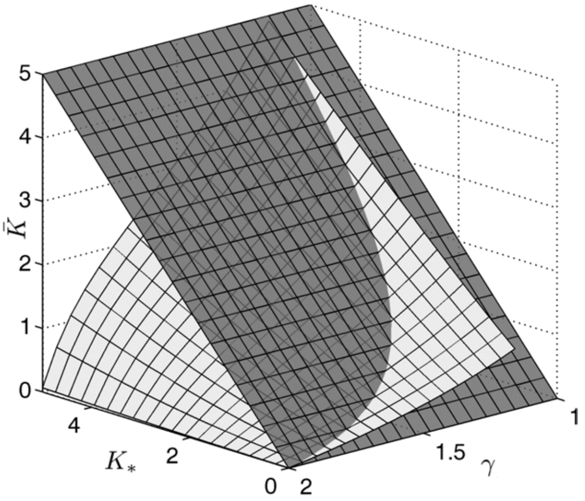

Figure 1. $\bar{K}(K_*, \gamma)$ surface by numerical quadrature (bright) and K plane (dark); self-similar solutions with $\gamma \in (1, 2)$ lie on the intersection curve of both surfaces. Note that $\gamma \in (1, 2)$ increases from right to left.

of the corresponding scalings up to the equivalence if both trivial limits $\varphi_* = 1$ and $\varphi_* = 0$ a.e. are excluded.

Lemma 4.1 (uniqueness of scalings). *Let $s(t, x)$ solve (5a)–(5c) with decreasing initial data. Given $\lambda(\tau)$, assume that the associated similarity scaling $\varphi(\tau, \eta)$ converges (pointwise) to a non-constant continuous function $\varphi_*(\eta)$ as $\tau \to \infty$, i.e. φ converges to a self-similar solution φ_*.*
Then λ is unique up to equivalence. Furthermore, one can always select a scaling $\bar{\lambda}(\tau)$ such that $\bar{\varphi}(\tau, 1/2) = 1/2$ for all $\tau \geqslant 0$.

Proof. First note that φ also converges uniformly. Assume there exist two similarity scalings $\lambda_{1,2}$ so that the corresponding $\varphi_{1,2}$ converge to self-similar solutions. The relation between both solutions is $\varphi_2(\tau, \eta) = \varphi_1(\tau, \eta\lambda_1(\tau)/\lambda_2(\tau))$. Then if $\lambda_1/\lambda_2 \to 0$ for $\tau \to \infty$, this implies $\varphi_2(\tau, \eta) \to 1$ which contradicts the assumption that the solutions are not constant. Similarly one can exclude that $\lambda_1/\lambda_2 \to \infty$ for $\tau \to \infty$ since this gives the second trivial solution $\varphi_2(\tau, \eta) = 0$ for $\eta > 0$ as $\tau \to \infty$. Finally note that it is impossible that both $\varphi_{1,2}$ converge to a nontrivial limit if λ_1/λ_2 does not converge.

The second part of the lemma follows by choosing $\bar{\lambda}(\tau) \equiv \lambda(\tau)/2\chi(\tau, 1/2)$ where $\chi(\tau, \cdot)$ denotes the (generalized) inverse of $\psi(\tau, \cdot)$ for any $\tau \geqslant 0$. □

The questions addressed next are under which condition a solution of (20) does exist that converges to a self-similar solution, how does the $\lambda(\tau)$ need to be chosen, and which stationary state is selected?

The main result on convergence, theorem 4.3, gives a necessary and sufficient criterion for convergence to any of the self-similar solutions under the assumption that $K(\tau)$ converges sufficiently fast. If the data satisfy $\varphi_0(\eta) \leqslant 1 - c\eta^\rho$ for $0 < \rho < 3/2$ then fast convergence of $K(\tau) \to 0$ is proven (cf proposition 4.4). Thereby for $\rho \in (0, 3/2)$ the domain of attraction of the self-similar solutions is completely characterized (cf corollary 4.5).

4.1. Weak characterization of scalings

Before the main result is proven a weak characterization of scale transformations λ is given. If, depending on the initial data, certain lower and upper bounds for λ are not fulfilled there can never be convergence to a nontrivial self-similar solution.

Theorem 4.2. *Consider $\varphi(\tau, \eta)$ a solution of (5a)–(5c) rescaled by $\lambda(\tau)$ using (11). Assume $K \to K_*$ and for some $\gamma > 1$ that $\gamma\tau - \log\lambda(\tau) = o(\tau)$ as $\tau \to \infty$.*

(i) If the initial data satisfy

$$\varphi_0(\eta) \geqslant 1 - \alpha\eta^\rho \qquad \text{for some } \alpha, \rho > 0$$

and $\rho > (2K_* + 3)/\gamma$, then $\varphi(\tau, \eta) \to 1$ for all $\eta \geqslant 0$.

(ii) Conversely, if the initial data satisfy

$$\varphi_0(\eta) \leqslant 1 - \alpha\eta^\rho \qquad \text{for some } \alpha, \rho > 0$$

and $\rho < (2K_* + 3)/\gamma$, then $\varphi(\tau, \eta) \to 0$ for all $\eta > 0$.

Remark. Rescalings which satisfy the bounds of the previous theorem are, for example, $\lambda(\tau) = e^{\gamma\tau}$, but also $\lambda(\tau) = e^{\gamma\tau}(1 + \tau)^\mu$ with $\mu \in \mathbb{R}$. It is immediately apparent from the proof that the first and second statements remain true if the positive and negative parts fulfil $\big(\gamma\tau - \log\lambda(\tau)\big)_\pm = o(\tau)$ as $\tau \to \infty$, respectively.

Proof. (Part 1.) For $0 < \varepsilon < (\rho\gamma/(2K_* + 3)) - 1$ choose τ_1 such that $\sup_{\tau \geqslant \tau_1} |K(\tau) - K_*|2/2K_* + 3 < \varepsilon$. Then $\big|\int_0^{\tau_1}(K(t) - K_*)g\,dt\big| \leqslant C(\tau_1)$ due to (15), where the value of the expression $C(\tau_1)$ does not depend on τ. Now we use (19) for $\tau_0 = 0$ and (16) to find

$$\begin{aligned}
H(\varphi(\tau, \eta)) &\leqslant \tau + H\left(\varphi_0\left(\eta\frac{1}{\lambda(\tau)}\right)\right) + \varepsilon\tau + C(\tau_1) \\
&= \tau(1 + \varepsilon) + \frac{1}{2K_* + 3}\log\left(\eta^\rho\left(\frac{1}{\lambda(\tau)}\right)^\rho\right) + C(\tau_1) \\
&= \tau(1 + \varepsilon) - \frac{\rho}{2K_* + 3}\log\big(\lambda(\tau)\big) + C(\tau_1) \\
&= \tau\left(1 - \frac{\rho\gamma}{2K_* + 3} + \varepsilon\right) + \frac{\rho}{2K_* + 3}(\gamma\tau - \log\lambda) + C(\tau_1).
\end{aligned}$$

The first term converges to $-\infty$ whereas the second term is $o(\tau)$ as $\tau \to \infty$. Hence $\varphi(\tau, \eta) \to 1$ as $\tau \to \infty$ for all $\eta > 0$. The proof of the second part is similar. □

4.2. Criterion for convergence

In the rest of this paper the self-similar solution corresponding to the behaviour $\varphi_*(\eta) = 1 - \eta^\rho + o(\eta^\rho)$ as $\eta \to 0$ will be denoted by φ_*^ρ and the corresponding value of K_* by K_*^ρ.

Then the main result on convergence to self-similar solutions is the following. If for given ρ one has $K(\tau) \to K_*^\rho$ sufficiently fast, then $\varphi(\tau, \eta)$ converges to $\varphi_*^\rho(\eta)$ if and only if the data $\varphi_0(\eta)$ are regularly varying at zero with power ρ, that is more precisely if $1 - \varphi_0(\eta) \sim \big(\eta L(\eta)\big)^\rho$ for a slowly varying function L. Moreover, $K(\tau)$ converges exponentially fast for any initial data $\varphi_0(\eta) \leqslant 1 - c\eta^\rho$ with $c > 0$ and $0 < \rho < 3/2$.

Definition 4.1 (slow variation). *A function L is said to be slowly varying at zero if*

$$\lim_{\eta\to 0}\frac{L(t\eta)}{L(\eta)} = 1 \qquad \text{for all } t > 0.$$

Typical examples of slowly varying functions are $\log x$, $\log(\log x)$, *etc and all powers of logarithms. We refer to the book [27] for a full characterization of slowly varying functions, as well as further generalizations and examples.*

Theorem 4.3. *Let $s(t,x)$ be a solution of (5a), let $\rho > 0$ and assume $K(\tau) \to K_*^\rho$ as $\tau \to \infty$ such that $\int_0^\infty |K(\tau) - K_*^\rho|\,\mathrm{d}\tau < \infty$.*
Then there exists a rescaling $\lambda(\tau)$ (11) such that with $\varphi(\tau,\eta)$ as in (11) one has that $\varphi(\tau,\eta) \to \varphi_*^\rho(\eta)$ for all $\eta > 0$ as $\tau \to \infty$ if and only if the data satisfy

$$1 - \varphi_0(\eta) \sim \big(\eta L(\eta)\big)^\rho \qquad \text{as } \eta \to 0 \tag{29a}$$

with slowly varying function L. The scaling is equivalent to $\lambda(\tau)$ implicitly defined as

$$\varphi_0\left(\frac{1}{\lambda(\tau)}\right) = 1 - \mathrm{e}^{-(2K_*^\rho+3)\tau}. \tag{29b}$$

Remark. Note first that (29b) defines a proper rescaling since φ_0 is decreasing and $\varphi_0(0) = 1$. Equation (29b) implies that with $\gamma = (2K_*^\rho + 3)/\rho$ one has

$$\frac{1}{\lambda(\tau)}L\Big(\frac{1}{\lambda(\tau)}\Big) \sim \mathrm{e}^{-\gamma\tau} \qquad \text{as } \tau \to \infty$$

which can be rewritten, using the de Bruijn conjugate $L^\#$ of L. This is a slowly varying function which satisfies $L(x)L^\#\big(xL(x)\big) \sim 1$ as $x \to 0$ and comes into play, for example, if one wants to invert slowly varying functions. With this definition and the inversion formula (see chapter 1.5.7 of [27]), one can write λ as

$$\frac{1}{\lambda(\tau)} \sim \mathrm{e}^{-\gamma\tau}L^\#\big(\mathrm{e}^{-\gamma\tau}\big) \qquad \text{as } \tau \to \infty\,.$$

Thus it can be seen that $\lambda(\tau)$ is essentially the time scale for the self-similar solution up to a slowly varying correction which is given by the data.

Convergence for rescalings equivalent to $\mathrm{e}^{-\gamma\tau}$ only occurs if $1 - \varphi_0(\eta) \sim c\eta^\rho$ as $\eta \to 0$, that is if the data behave exactly as a power-law.

Proof. Let $\lambda(\tau)$ be a similarity scaling and $\varphi(\tau,\eta)$ as in (11). Within this proof $K_*^\rho \equiv K_*$ for convenience.

(i) The solution formula (19) for $\tau_0 = 0$ implies

$$H\big(\varphi(\tau,\eta)\big) = \tau + H\Big(\varphi_0\Big(\frac{\eta}{\lambda(\tau)}\Big)\Big) + \int_0^\tau \big(K(t) - K_*\big)\,g\Big(\varphi\Big(t, \eta\frac{\lambda(t)}{\lambda(\tau)}\Big)\Big)\,\mathrm{d}t.$$

We first argue that

$$I(\tau) \equiv \int_0^\tau \big(K(t) - K_*\big)\,g\Big(\varphi\Big(t, \eta\frac{\lambda(t)}{\lambda(\tau)}\Big)\Big)\,\mathrm{d}t \to g(1)\int_0^\infty \big(K(t) - K_*\big)\,\mathrm{d}t \tag{30}$$

as $\tau \to \infty$. Indeed, for any $\tau_0 > 0$ write $I(\tau) \equiv \int_0^{\tau_0} \cdots + \int_{\tau_0}^{\tau} \cdots \equiv I_1(\tau) + I_2(\tau)$. Due to the assumptions on $K(\tau)$ and the boundedness of g (cf (15)), one can for any given $\varepsilon > 0$ choose τ_0 so large that $|I_2(\tau)| < \varepsilon$ for all $\tau > \tau_0$. Furthermore, since $\lambda(\tau) \stackrel{\tau\to\infty}{\to} \infty$ one finds

$$\lim_{\tau\to\infty} g(\varphi(t, \eta\lambda(t)/\lambda(\tau))) = g(\varphi(1,0)) = g(1)$$

for all $0 \leqslant t < \tau$. Since $|(K(t) - K_*)g(\varphi(t, \eta\lambda(t)/\lambda(\tau)))| \leqslant C|K(\tau) - K_*|$ and since $|K(t) - K_*|$ is integrable by assumption, Lebesgue's dominated convergence theorem implies $I_1(\tau) \overset{\tau\to\infty}{\to} g(1)\int_0^{\tau_0}(K(t) - K_*)\,\mathrm{d}t$ and thereby proves (30).

(ii) Next recall (16) and (17) to write

$$H\left(\varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)\right) = \frac{1}{2K_* + 3}\log\left(1 - \varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)\right) + \frac{\log 2}{2K_* + 3} + \int_{1/2}^{\varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)}\left\{\frac{1}{(2K_* + 3)(\xi - 1)} - \frac{1}{f(K_*, \xi)}\right\}\mathrm{d}\xi. \tag{31}$$

Since the integrand in the last term is bounded as $\varphi_0\big(\eta/\lambda(\tau)\big) \to 1$ observe that

$$\int_{1/2}^{\varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)}\left\{\frac{1}{(2K_* + 3)(\xi - 1)} - \frac{1}{f(K_*, \xi)}\right\}\mathrm{d}\xi \to \int_{1/2}^{1}\cdots\,\mathrm{d}\xi = \text{const} \tag{32}$$

as $\tau \to \infty$. Thus, (30), (31) and (32) imply

$$H\big(\varphi(\tau, \eta)\big) = \tau + \frac{1}{2K_* + 3}\log\left(1 - \varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)\right) + \omega(\tau), \tag{33}$$

with some function $\omega(\tau)$ that satisfies $\omega(\tau) \to$ const as $\tau \to \infty$ with a constant independent of η.

(iii) Now assume that $\varphi_0(\cdot)$ satisfies (29*a*) and choose $\lambda(\tau)$ as in (29*b*). Then (33) gives

$$\begin{aligned} H\big(\varphi(\tau, \eta)\big) &= \frac{1}{2K_* + 3}\log\left(\frac{1 - \varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)}{1 - \varphi_0\left(\frac{1}{\lambda(\tau)}\right)}\right) + \omega(\tau) \\ &= \frac{\rho}{2K_* + 3}\left(\log\eta + \log\left(\frac{L\left(\frac{\eta}{\lambda(\tau)}\right)}{L\left(\frac{1}{\lambda(\tau)}\right)}\right)\right) + \omega(\tau). \end{aligned} \tag{34}$$

Since L is slowly varying at zero it can be seen

$$\lim_{\tau\to\infty}\log\left(\frac{L\left(\frac{\eta}{\lambda(\tau)}\right)}{L\left(\frac{1}{\lambda(\tau)}\right)}\right) \to 0$$

and thus

$$H\big(\varphi(\tau, \eta)\big) \to \frac{\rho}{2K_* + 3}\log\eta + C \qquad \text{as } \tau \to \infty,$$

which indeed implies that $\varphi(\tau, \eta) \to \varphi_*^{\rho}(\eta)$ for all $\eta > 0$ as $\tau \to \infty$ (recall (23)).

(iv) Conversely, assume that $\varphi(\tau, \eta) \to \varphi_*^{\rho}(\eta)$ as $\tau \to \infty$ for all $\eta > 0$. This implies, using (23) and (33), that

$$\tau + \frac{1}{2K_* + 3}\log\left(1 - \varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)\right) \to \frac{\rho}{2K_* + 3}\log\eta + C \tag{35}$$

as $\tau \to \infty$ for all $\eta > 0$ and some constant $C \in \mathbb{R}$. In particular, for $\eta = 1$ one obtains

$$\tau + \frac{1}{2K_* + 3}\log\left(1 - \varphi_0\left(\frac{1}{\lambda(\tau)}\right)\right) \to C \tag{36}$$

as $\tau \to \infty$ for all $\eta > 0$. This implies also that $\lambda(\tau)$ is indeed equivalent to the choice in (29*b*). Subtracting (36) from (35) yields

$$\frac{1}{2K_* + 3}\log\left(\frac{1 - \varphi_0\left(\frac{\eta}{\lambda(\tau)}\right)}{1 - \varphi_0\left(\frac{1}{\lambda(\tau)}\right)}\right) \to \frac{\rho}{2K_* + 3}\log\eta + C \tag{37}$$

as $\tau \to \infty$ for all $\eta > 0$.

Defining $L(\eta) \equiv \frac{(1-\varphi_0(\eta))^{1/\rho}}{\eta}$ then (37) is equivalent to

$$\log\left(\frac{L\left(\frac{\eta}{\lambda(\tau)}\right)}{L\left(\frac{1}{\lambda(\tau)}\right)}\right) \to C$$

as $\tau \to \infty$ for all $\eta > 0$. Choosing $\eta = 1$ one finds that this constant must be zero and hence L is slowly varying. □

4.3. *Fast convergence of $K(\tau)$ for non-flat data*

An important issue in the proof of convergence to a self-similar solution is establishing convergence of $K(\tau)$. In general, we have yet very little control over $K(\tau)$. However, in the following proposition it is shown that $K(\tau) \to 0$ exponentially fast if the initial data are bounded above by $1 - cx^\rho$ for some $\rho \in (0, 3/2)$. The proof is based on a simple comparison argument.

Proposition 4.4. *Assume that $\psi_0(x) \leqslant 1 - cx^\rho$ for all $x \in (0,1)$ for some $c > 0$ and $\rho \in (0, 3/2)$. Then there exists a $C > 0$ such that*

$$0 < K(\tau) \leqslant C \begin{cases} \exp\left(\tau\left(2 - \dfrac{3}{\rho}\right)\right) & \dfrac{3}{4} < \rho < \dfrac{3}{2}, \\ \exp\big(-2\tau\big) & 0 < \rho \leqslant \dfrac{3}{4}. \end{cases}$$

Proof. Because of $\psi \in [0,1]$ the derivative fulfils $\partial_\tau \psi \leqslant \psi^4 - \psi^2$ and hence

$$\psi(\tau, x) \leqslant \bar{\psi}(\tau, x) \equiv \left(1 + \frac{1 - \psi_0(x)^3}{\psi_0(x)^3}\exp(3\tau)\right)^{-1/3},$$

where $\bar{\psi}$ is the solution of $\partial_\tau \bar{\psi} = \bar{\psi}^4 - \bar{\psi}$ with initial data $\bar{\psi}_0(x) = \psi_0(x)$. Furthermore one can easily check that

$$\begin{aligned} \psi(\tau, x)^4 \leqslant \bar{\psi}(\tau, x)^4 &\leqslant \Big(1 + (1 - \psi_0(x)^3)\, \mathrm{e}^{3\tau}\Big)^{-4/3} \\ &\leqslant \Big(1 + \big(1 - (1 - cx^\rho)^3\big)\mathrm{e}^{3\tau}\Big)^{-4/3} \\ &\leqslant \Big(1 + cx^\rho \mathrm{e}^{3\tau}\Big)^{-4/3}. \end{aligned}$$

By integrating equation (12) one finds $\|\psi(\tau,\cdot)\|_1 = \|\psi_0\|_1\, \mathrm{e}^{-\tau}$ and by applying the Cauchy–Schwarz inequality $\|\psi^2(\tau,\cdot)\|_1 \geqslant \|\psi_0\|_1^2\, \mathrm{e}^{-2\tau}$. Using the abbreviation $a = \mathrm{e}^{-3\tau/\rho}$ and $y \equiv x/a$ the function $\theta(\tau)$ can be estimated from above as follows

$$\begin{aligned} \theta(\tau) &\leqslant \frac{\int_0^1 \bar{\psi}(\tau, x)^4\, \mathrm{d}x}{\|\psi_0\|_1^2\, \mathrm{e}^{-2\tau}} \leqslant \frac{\mathrm{e}^{2\tau}}{\|\psi_0\|_1^2}\left(\int_0^a \mathrm{d}x + \int_a^1 (1 + cx^\rho \mathrm{e}^{3\tau})^{-4/3}\, \mathrm{d}x\right) \\ &= \frac{\exp\left(\tau\left(2 - \frac{3}{\rho}\right)\right)}{\|\psi_0\|_1^2}\left(1 + \int_1^{1/a} (1 + cy^\rho)^{-4/3}\, \mathrm{d}y\right). \end{aligned}$$

For $\rho > 3/4$ the integral converges, whereas for $\rho \leqslant 3/4$ the integrand can be bounded by a multiple of $y^{4\rho/3}$ (depending on c) and hence it grows proportional to $\exp(\tau(3/\rho - 4))$. Thus $\theta(\tau) \leqslant C\exp\big(\tau \max(2 - 3/\rho, -2)\big)$ and in the same manner for $K(\tau) = \theta(1-\theta)^{-1}$. □

Corollary 4.5. *Assume that $\varphi_0(\eta) \leqslant 1 - cx^\mu$ for some $0 < \mu < 3/2$. Then there exists a rescaling $\lambda(\tau)$ such that $\varphi(\tau,\eta) \to \varphi_*^\rho(\eta)$ for all $\eta > 0$ as $\tau \to \infty$ if and only if the data satisfy (29a) for some $0 < \rho \leqslant \mu$.*

Proof. This is a consequence of theorem 4.3 and proposition 4.4. □

5. Numerical examples

Whether a solution $\varphi(\tau,\eta)$ of the time-dependent problem converges to a self-similar solution is fully understood if $\varphi_0(\eta) \leqslant 1 - c\eta^\rho$ for some $c > 0$ and $0 < \rho < 3/2$. In the general case, however, it is not clear under which conditions the associated $K(\tau)$ converges, and if so, whether it does so sufficiently fast. In order to shed some light on this issue we present the results of numerical simulations in this section. They confirm the conjecture that convergence to the self-similar solution φ_*^ρ occurs if and only if the data satisfy that $1 - \varphi_0$ is regularly varying with power ρ.

The numerical solutions presented here are computed as follows. For the initial data $\varphi_0 \equiv \psi_0$ we compute the time-dependent solution of (12). It turns out to be more precise to compute the solution for $1 - \psi$ to avoid roundoff errors. Equation (12) is discretized in x on a non-uniform mesh and then solved explicitly using a standard fourth order Runge–Kutta method, whereby K is computed using a trapezoidal rule.

In lemma 4.1 it was proven that if a self-similar solution and a rescaling exist, then it is always possible to select λ such that $\varphi(\tau, 1/2) = 1/2$. Consequently we select λ such that in every time-step this condition holds and check whether the solution converges.

5.1. Initial data with $\rho < 3/2$

This regime is completely covered by the results of section 4. One has exponentially fast convergence $K(\tau) \to 0$ and convergence to self-similar solutions if and only if the initial data $1-\varphi_0$ are regularly varying. First the convergence rate of $K(\tau)$ with initial data $\varphi_0(\eta) = 1-\eta^\rho$ with $1/2 \leqslant \rho < 3/2$ is computed.
In figure 2 the numerical convergence rates are compared with the computed bound of proposition 4.4 and it is found that $K(\tau)$ decreases with the predicted bound

$$K(\tau) \sim \exp\big(\tau \max(2 - 3/\rho, -2)\big).$$

Now φ converges to a self-similar solution if and only if $1 - \varphi_0$ is regularly varying at zero. Three examples of such behaviour are presented, namely for initial data

$$\varphi_0(\eta) = 1 - \eta, \tag{38a}$$
$$\varphi_0(\eta) = 1 + \eta(\log(\eta) - 1), \tag{38b}$$
$$\varphi_0(\eta) = 1 - \eta\big(\sin(\log\eta) + 2\big). \tag{38c}$$

The initial data in examples (38a) and (38b) are regularly varying, while this is not the case for (38c). Correspondingly, figure 3 shows convergence for (38a) and (38b) and oscillation of the rescaled solution $\varphi(\tau,\eta)$ for example (38c). In addition, figure 3 indicates that the rate of convergence $|\varphi - \varphi_*|$ as $\tau \to \infty$ is faster for example (38a), which is supported by the fact that the rescaled solution (dotted curves) tends to be closer to the exact solution (full line) for comparable times τ in figure 3.

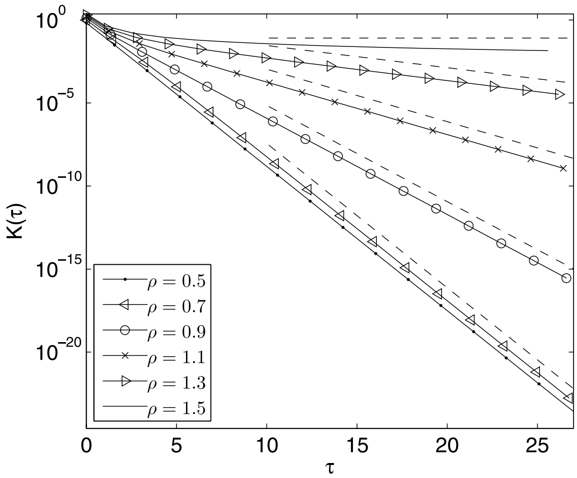

Figure 2. $K(\tau)$ for solutions of $\varphi(\tau, \eta)$ with initial data $\varphi_0(\eta) = 1 - \eta^\rho$.

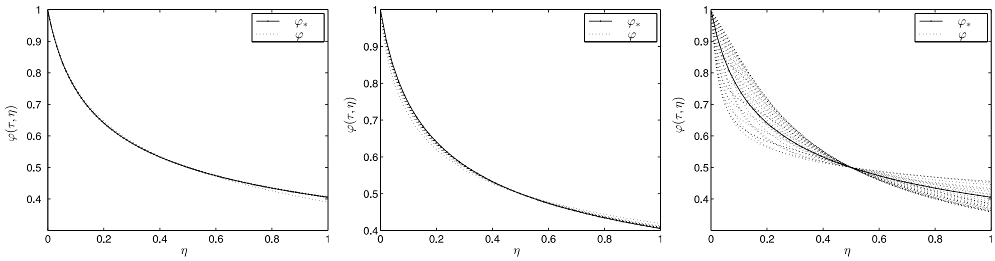

Figure 3. $\varphi(\tau, \eta)$ at various times showing convergence for (38*a*) (left) and (38*b*) (middle) and no convergence (oscillations) in (38*c*) (right); the full line shows the similarity solution φ_* with $\rho = 1$, the scaling $\lambda(\tau)$ is such that $\varphi(\tau, 1/2) = 1/2$.

5.2. *Initial data $\rho > 3/2$*

For $\rho < 3/2$ convergence $K(\tau) \to 0$ is known due to proposition 4.4. Now we present three more examples where $\rho = 2$. Qualitatively similar results are obtained for other $\rho > 3/2$. This part indicates in which cases one can expect convergence of $K(\tau)$ and what the limit and the rate of convergence will be.

We choose

$$\varphi_0(\eta) = 1 - \eta^2, \tag{39a}$$

$$\varphi_0(\eta) = 1 + \eta^2(\log(\eta) - 1), \tag{39b}$$

$$\varphi_0(\eta) = 1 - \eta^2\big(\sin(\log\eta) + 2\big), \tag{39c}$$

and expect convergence for (39*a*) and (39*b*) and no convergence for (39*c*) (figure 4). In addition we show the convergence rates for $K(\tau)$ in these three cases. A closer look reveals that convergence for (39*a*) is exponentially fast, while for (39*b*) it is much slower (figure 5(right)).

In terms of the original variables h and u examples (39*c*) (no convergence) and (39*a*) (convergence) give rise to the numerical solutions shown in figure 6.

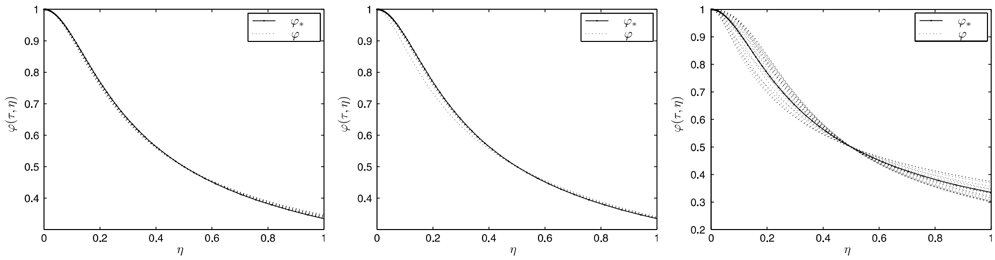

Figure 4. $\varphi(\tau, \eta)$ at various times shows convergence for (39*a*) (left) and (39*b*) (middle) and no convergence (oscillations) in (39*c*) (right); the full line shows the similarity solution with $\rho = 2$, the scaling $\lambda(\tau)$ is such that $\varphi(\tau, 1/2) = 1/2$.

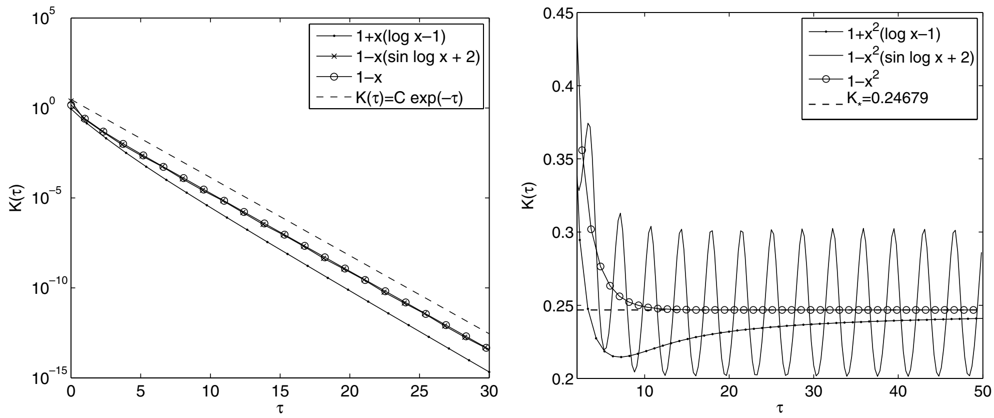

Figure 5. While there is fast convergence of $K(\tau)$ for all three examples (38*a*)–(38*c*) (left), there is fast convergence for (39*a*), slow convergence for (39*b*) and no convergence/oscillation for (39*c*) (right).

6. Main results in terms of h

This section is intended for those who wish to have an immediate interpretation of the main results in terms of the original variable h.

For the following we assume without loss of generality that $h(t, y)$ has increasing initial data such that our new time scale τ is given by $\tau = \log(h(0, 0)/h(t, 0))$.

First note that the existence of a self-similar solution for $s(t, x)$ implies existence of a self-similar solution for h as follows. By definition $s(t, x)$ evolves self-similarly if $\varphi(\tau, \eta) = s(t, x)/s_{\max}(t)$ converges pointwise in η for $\lambda(\tau)$ with $\eta = x\,\lambda(\tau)$ as $\tau \to \infty$. Due to the transformation from the Eulerian to the Lagrangian reference frame this implies

$$\varphi(\tau, \eta) = \frac{s(t, x = \eta/\lambda)}{s_{\max}(t)} = \frac{h(t, 0)}{h\big(t, y(t, x)\big)}, \tag{40}$$

where the argument $y(t, x)$ in h fulfils

$$y = \int_0^{\eta/\lambda} s(t, x)\,\mathrm{d}x = \frac{\eta c}{\lambda(\tau)\mathrm{e}^{-\tau}} + o(\lambda^{-1}) \equiv \frac{\eta}{\tilde{\lambda}(\tau)} + o(\lambda^{-1}). \tag{41}$$

Hence (40) and (41) provide the similarity scaling for h. Due to the right-continuity of the data one can neglect the $o(\lambda^{-1})$ term in the limit.

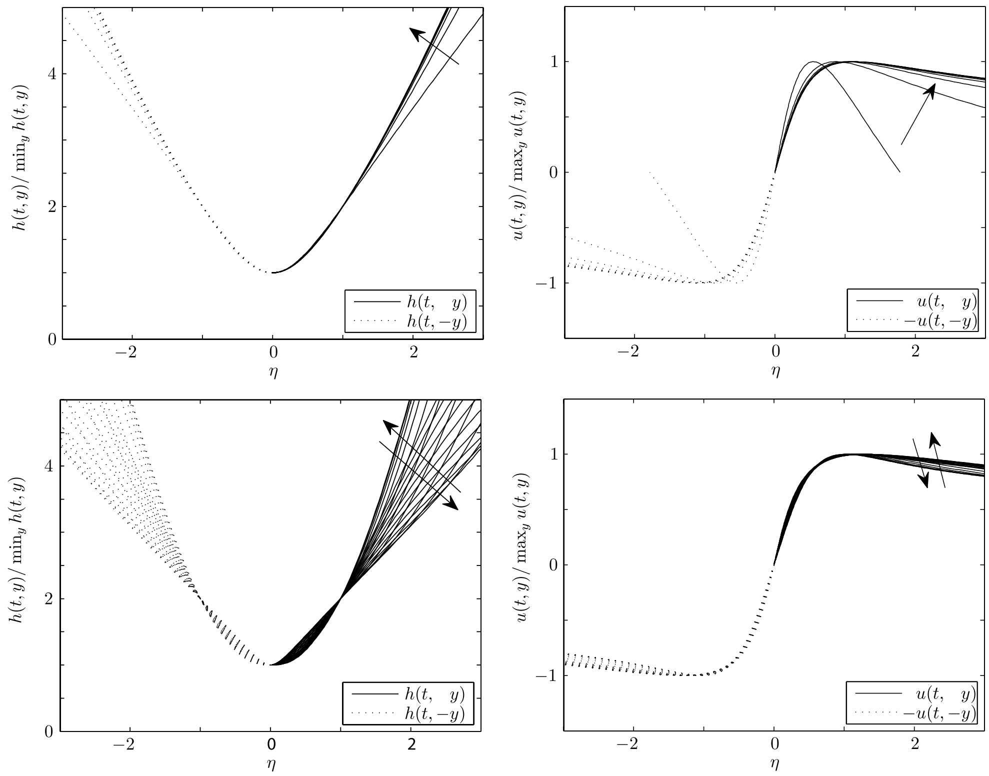

Figure 6. Numerical solution (39*a*) (upper) and (39*c*) (lower) transformed to h (left) and u (right) via (6) show rupture after finite time, i.e. $\min_y h \to 0$ as $t \to t^*$. In order to obtain a full solution on $[-1, 1]$ h and u are reflected at $x = 0$ with and without change of sign, respectively (dotted lines), whereby this only yields a smooth solution for initial data $\varphi_0 = 1 - y^n + o(y^n)$ with even n. To show that only (39*a*) converges to a similarity solution the rescaled quantities $h/\min_y h$ and $u/\max_y u$ are plotted with λ' such that $h(t, 1/\lambda')/h(t, 0) = 2$. Oscillations or convergence is indicated by the arrows.

Theorem 3.1 on the existence of self-similar solutions can be formulated as follows.

Corollary 6.1. *Let $t^* \in (0, \infty)$ be arbitrary. For any $\rho > 0$ there exists a unique $\beta = \beta(\rho) > 0$ and a self-similar solution*

$$h_\rho(t, y) = (t^* - t)^{1/3} \frac{C}{\varphi_\rho(\eta)} \qquad \text{with} \quad \eta = y(t^* - t)^{-\beta}$$

that is an exact solution of (2a) and (2b) on $\mathbb{R}^+$. The function φ_ρ is unique up to rescaling with a constant and satisfies $\varphi_\rho(\eta) = 1 + C'\eta^\rho + o(\eta^\rho)$ as $\eta \to 0$.

Remark. The proof is as in theorem 3.1, where β in theorem 6.1 is related to γ in theorem 3.1 via the formula $\beta = (\gamma - 1)/3$.

The following corollary is equivalent to lemma 4.1 on the uniqueness of rescalings.

Corollary 6.2. *Let $h(t, y)$ be a solution of (2a) and (2b) and assume that there exists a scaling $\tilde{\lambda}(\tau)$ such that $\varphi(\tau, \eta) = h(t, 0)/h(t, y)$ with $\eta = \tilde{\lambda}(\tau)y$ converges pointwise in η, i.e. h converges to a self-similar solution.*
Then $\tilde{\lambda}$ is unique up to equivalence. One can select $\tilde{\lambda}$ such that $\varphi(\tau, 1/2) \equiv 1/2$.

Remark. The proof is as in lemma 4.1.

The first main result is a necessary and sufficient condition for convergence under the assumption that the nonlocal quantity K converges sufficiently fast.

Corollary 6.3. *Suppose for some $\rho > 0$ that $K(\tau)$ as defined in (13) converges to some K_*^ρ as $\tau \to \infty$ such that $\int_0^\infty \left|K(\tau) - K_*^\rho\right| \mathrm{d}\tau < \infty$. Then h converges to the self-similar solution h_ρ if and only if $h(0, y) - h(0, 0) = (yL(y))^\rho$ with L being slowly varying at zero. The scaling $\tilde{\lambda}$ is defined by the initial data through (29b) and the transformation formula (41).*

Remark. The proof is as in theorem 4.3.

Finally, in the regime that the initial data are not flat we have full characterization of the domain of attraction of self-similar solutions.

Corollary 6.4. *Assume that $h(0, y) \geqslant h(0, 0) + cy^\mu$ for a $0 < \mu < 3/2$ and $c > 0$. Then $K(\tau) \to 0$ exponentially fast and by the previous theorem $h(t, y)$ converges to the self-similar solution characterized by $\rho \in (0, \mu)$ if and only if $h(0, y) - h(0, 0) = (yL(y))^\rho$ with L being slowly varying at zero.*

Remark. The proof is as in corollary 4.5.

An interpretation in terms of the original variable u can be easily found by applying similar arguments to the transformation formula (6).

7. Conclusions

We studied the structure of inertialess thin-film rupture, where van der Waals forces and viscosity are the dominant driving forces. For a simplified model the existence of a continuous one-parameter family of self-similar solutions was shown. In terms of the rescaled solution $\varphi(\tau, \eta)$ each of them is characterized by its behaviour near the singularity, i.e. by a number ρ such that

$$\varphi_*(\eta) = 1 - c\eta^\rho + o(\eta^\rho)$$

and by a corresponding value of K_*. The main purpose of this paper is to rigorously study convergence to these self-similar solutions. For $0 < \rho < 3/2$ where $K_* = 0$, a complete characterization of their domains of attraction is established. It turns out that for given initial data $h_0(t, y)$ there exists a spatial rescaling λ' such that the solution converges to a self-similar solution if and only if for some ρ and $\min_y h_0 - h_0 = (yL(y))^\rho$ the function $L(y)$ is slowly varying at zero. The spatial rescaling is already given by the initial data and can always be arranged such that

$$\frac{h(t, 1/\lambda')}{h(t, 0)} \equiv 2.$$

It is worth pointing out that λ' corresponds to a pure power-law only if $L(y) \sim \text{const}$ as $y \to 0$, which is the case if and only if the data behave like an exact power-law at the origin.

We can prove the result described above for all data under the assumption that K converges sufficiently fast. It is easy to prove by a comparison principle that this is true for $0 < \rho < 3/2$, for $\rho \geqslant 3/2$ a proof is still lacking. Numerical simulations indicate that this convergence is satisfied if and only if the initial data are regularly varying. This is strikingly different from the case $0 < \rho < 3/2$ where $K \to 0$ exponentially fast whenever the initial data are bounded above by $1 - c\eta^\rho$ for some $0 < \rho < 3/2$ independent of whether they are regularly varying or not.

Acknowledgments

D Peschka acknowledges support by the *Deutsche Forschungsgemeinschaft* through the Research Training Group *Analysis, Numerics, and Optimization of Multiphase Problems*. A Münch acknowledges support by a Heisenberg Scholarship DFG-grant MU 1626/3-2.

References

[1] Herminghaus S *et al* 1998 Spinodal dewetting in liquid crystal and liquid metal films *Science* **282** 916–9
[2] Eggers J and Villermaux E 2008 Physics of liquid jets *Rep. Prog. Phys.* **73** 036601
[3] Oron A, Davis S H and Bankoff S G 1997 Long-scale evolution of thin liquid films *Rev. Mod. Phys.* **69** 931–80
[4] Constantin P, Dupont T F, Goldstein R E, Kadanoff L P, Shelley M J and Zhou S M 1993 Droplet breakup in a model of the Hele–Shaw cell *Phys. Rev.* E **47** 4169–81
[5] Eggers J 1993 Universal pinching of 3d axisymmetric free-surface flow *Phys. Rev. Lett.* **71** 3458–60
[6] Eggers J 1997 Nonlinear dynamics and breakup of free-surface flows *Rev. Mod. Phys.* **69** 865–930
[7] Witelski T P and Bernoff A J 1999 Stability of self-similar solutions for van der Waals driven thin film rupture *Phys. Fluids* **11** 2443–5
[8] Zhang W W and Lister J R 1999 Similarity solutions for van der Waals rupture of a thin film on a solid substrate *Phys. Fluids* **11** 2454–62
[9] Eggers J and Fontelos M A 2009 The role of self-similarity in singularities of partial differential equations *Nonlinearity* **22** R1–44
[10] Münch A, Wagner B and Witelski T P 2005 Lubrication models with small to large slip lengths *J. Eng. Math.* **53** 359–83
[11] Peschka D, Münch A and Niethammer B 2009 Thin film rupture for large slip *J. Eng. Math.* doi:10.1007/s10665-009-9342-7
[12] Vaynblat D, Lister J R and Witelski T P 2001 Rupture of thin viscous films by van der Waals forces: evolution and self-similarity *Phys. Fluids* **13** 1130–40
[13] Barenblatt G I 1996 *Scaling, Self-Similarity, and Intermediate Asymptotics* (*Cambridge Texts in Applied Mathematics* vol 14) (Cambridge: Cambridge University Press)
[14] Ida M P and Miksis M J 1996 Thin film rupture *Appl. Math. Lett.* **9** 35–40
[15] Papageorgiou D T 1995 On the breakup of viscous liquid threads *Phys. Fluids* **7** 1529–44
[16] Brenner M P, Lister J R and Stone H A 1996 Pinching threads, singularities and the number $0.0304\cdots$ *Phys. Fluids* **8** 2827–36
[17] Renardy M 2005 A comment on self-similar breakup for inertialess Newtonian liquid jets *IMA J. Appl. Math.* **70** 353–8
[18] Renardy M 1994 Some comments on the surface-tension driven break-up (or the lack of it) of viscoelastic jets *J. Non-Newtonian Fluid Mech.* **51** 97–107
[19] Renardy M 2001 Finite time breakup of viscous filaments *Z. Angew. Math. Phys.* **52** 881–7
[20] Niethammer B and Pego R L 1999 Non-self-similar behavior in the LSW theory of Ostwald ripening *J. Stat. Phys.* **95** 867–902
[21] Carr J 2006 Stability of self-similar solutions in a simplified LSW model *Physica* D **222** 73–9
[22] Menon G, Niethammer B and Pego R L 2009 Dynamics and self-similarity in min-driven clustering *Trans Am. Math. Soc.* at press
[23] Menon G and Pego R L 2004 Approach to self-similarity in Smoluchowski's coagulation equations *Commun. Pure Appl. Math.* **57** 1197–232
[24] Erneux T and Davis S H 1993 Nonlinear rupture of free films *Phys. Fluids* A **5** 1117–22
[25] Hardy G H, Littlewood J E and Pólya G 1934 *Inequalities* (Cambridge: Cambridge University Press) p 314
[26] Dirk P 2008 Self-similar rupture of thin liquid films with slippage *PhD Thesis* Humboldt Universität zu Berlin, Berlin
[27] Bingham N H, Goldie C M and Teugels J L 1989 Regular variation *Encyclopedia of Mathematics and its Applications* vol 27