Multilinear Hyperquiver Representations

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(Joint work with Vidit Nanda and Anna Seigal)

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Quiver representations

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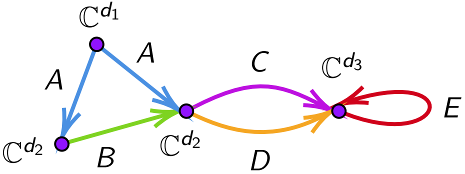

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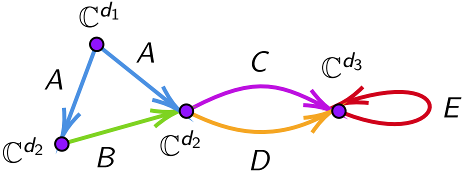

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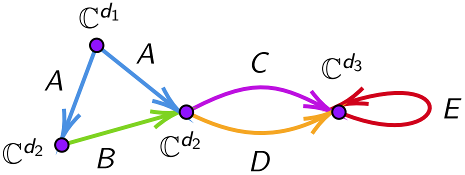

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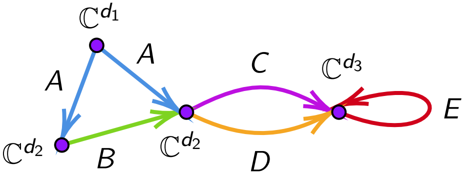

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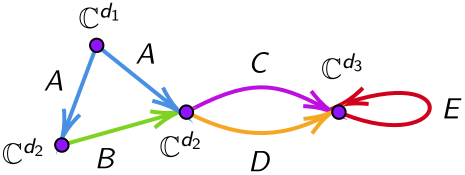

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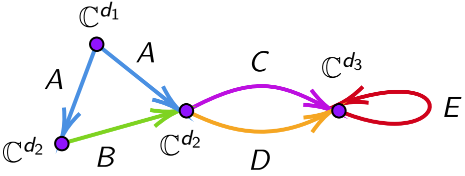

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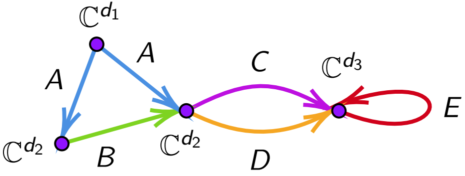

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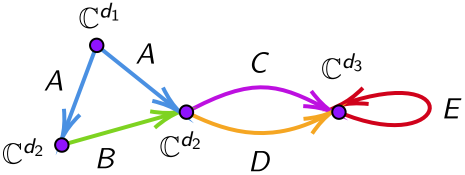

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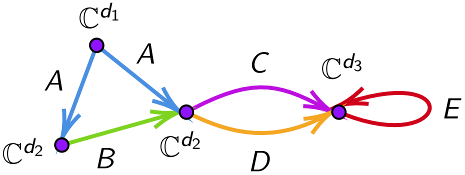

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Singular vectors of a quiver representation

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A tuple of vectors $([\mathbf{x}_1], \ldots, [\mathbf{x}_n]) \in \prod_{i=1}^{n} \mathbb{P}(\mathbb{C}^{d_i})$ on each vertex is a **singular vector tuple** of the representation if for all $e \in E$,

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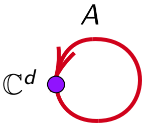

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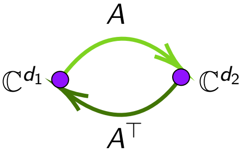

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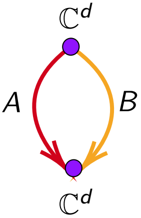

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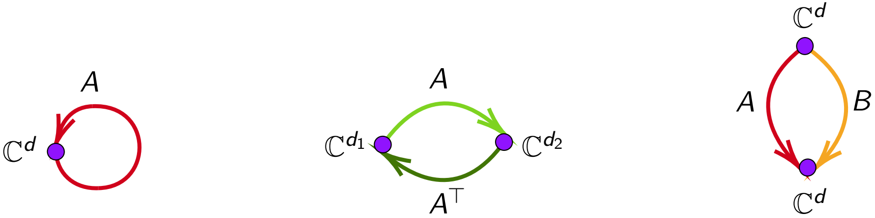
From quivers to hyperquivers

We can generalize quiver representations from graphs to hypergraphs, and from matrices to tensors, giving us **hyperquiver representations**



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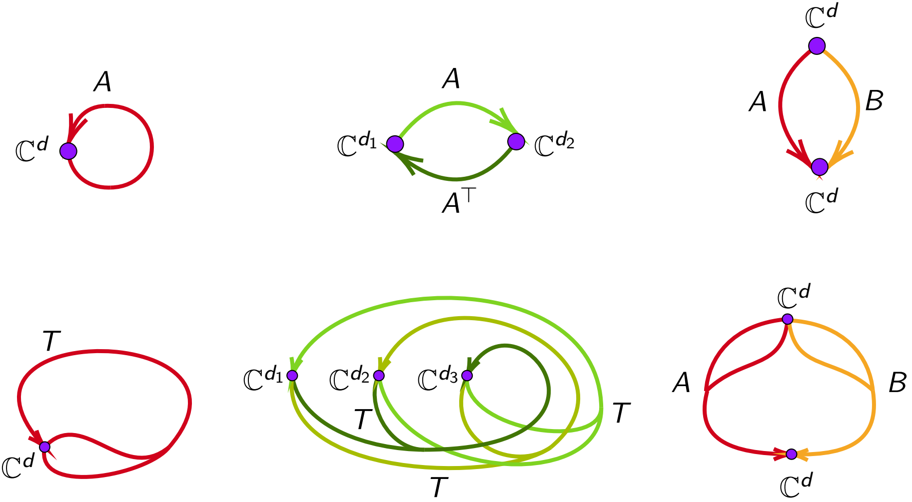
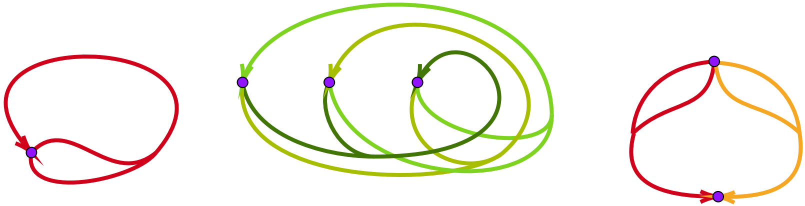


Hyperquiver representations

Definition

A **hyperquiver** $H = (V, E)$ is a set of *vertices* V of size $|V| = n$ and a set of *hyperedges* E such that for each hyperedge $e \in E$, there is an integer $m = m(e) > 1$ and a tuple of vertices $v(e) \in V^m$



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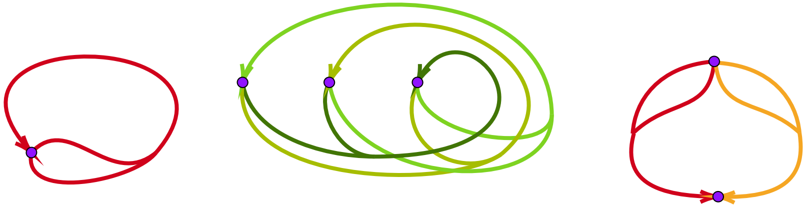

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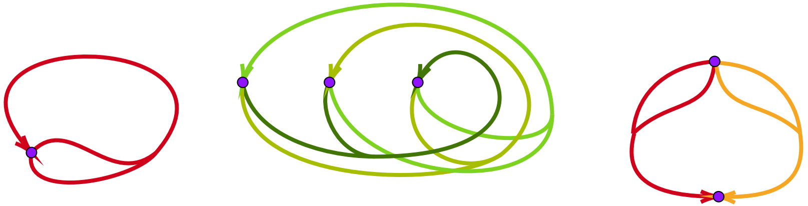
- The tuple $s(e) := (s_1(e), \ldots, s_{m-1}(e))$ are the **sources** of e, and the vertex $t(e) := s_m(e)$ is the **target** of e



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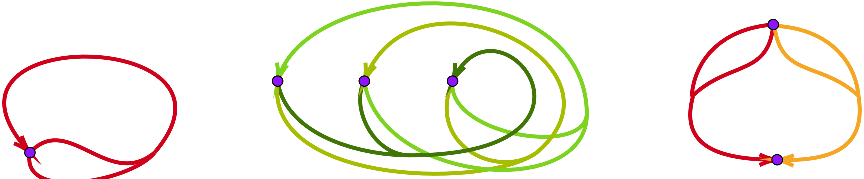

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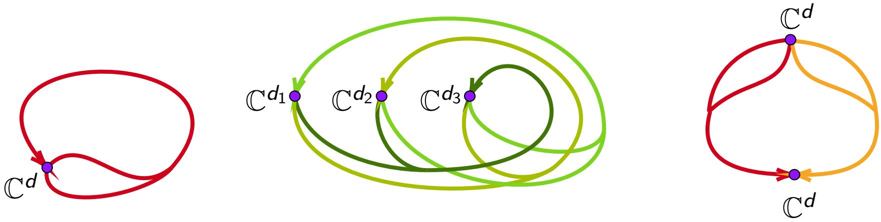

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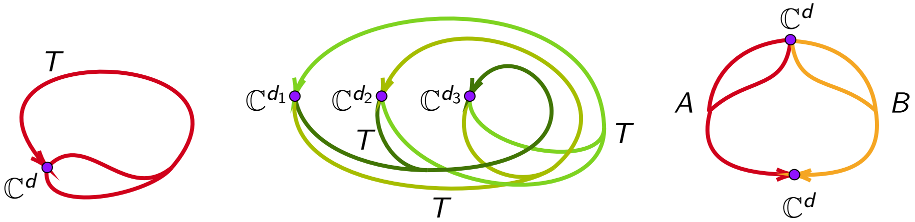

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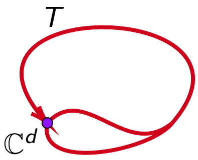

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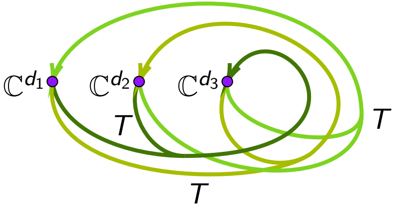

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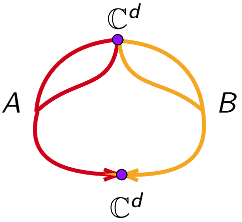

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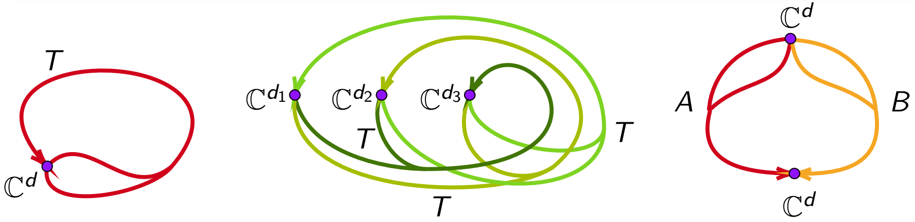

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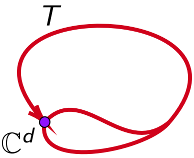

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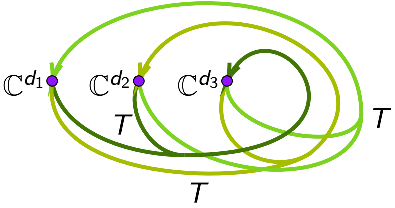

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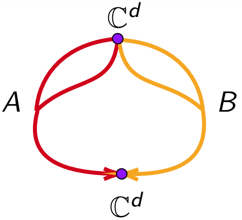

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The singular vector variety

$\mathcal{S}(\mathbf{R})$ is a subvariety of $X = \prod_{i=1}^{n} \mathbb{P}(\mathbb{C}^{d_i})$ whose equations are given by the vanishing of the 2×2 minors of the $d_{t(e)} \times 2$ matrix

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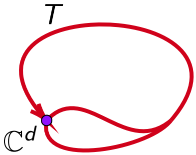
By *degree*, here we mean the degree of its image under the Segre embedding $s : X \hookrightarrow \mathbb{P}^D$, for $D = \prod_{i=1}^{n} d_i - 1$.

The number of eigenvectors of a tensor

Theorem (Cartwright-Sturmfels)

A generic tensor $T \in \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d$ of order m has $\frac{(m-1)^d - 1}{m-2}$ eigenvectors, each occurring with multiplicity 1.



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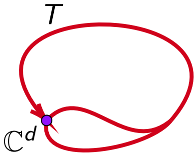
If the tensor T is generic in the hyperquiver representation below, then the singular vector variety $\mathcal{S}(\mathbf{R})$ has dimension 0 and degree $\frac{(m-1)^d-1}{m-2}$.



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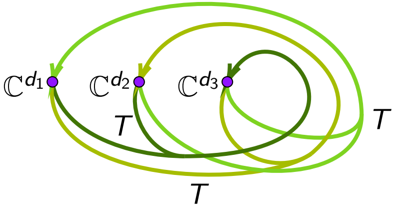
The number of singular vector tuples of a tensor

Theorem (Friedland-Ottaviani)

The number of singular vectors of a generic tensor $T \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$ is the coefficient of the monomial $h_1^{d_1-1} \dots h_n^{d_n-1}$ in the polynomial

$$\prod_{i\in[n]} \frac{\widehat{h_i}^{d_i} - h_i^{d_i}}{\widehat{h_i} - h_i}, \qquad \text{where} \quad \widehat{h_i} := \sum_{j\in[n]\setminus\{i\}} h_j,\ i \in [n]$$

Each singular vector tuple occurs with multiplicity 1.



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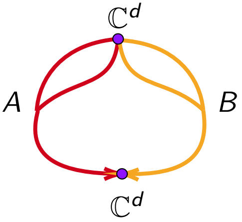
Solutions to the generalized tensor eigenvalue problem

Theorem (Friedland-Ottaviani, Ding-Wei)

If $A, B \in \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d$ are a generic pair of tensors of order m, then the number of solutions to the generalized tensor eigenvalue problem

$$A(\,\cdot\,, \mathbf{x}, \ldots, \mathbf{x}) = \lambda B(\,\cdot\,, \mathbf{x}, \ldots, \mathbf{x})$$

is $d(m-1)^{d-1}$. Each solution occurs with multiplicity 1.



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Problem

What should be the hyperquiver representation analogue of a *generic* tensor? How should one define a *generic* representation?

Generic hyperquiver representations

We say that two tensors T_e and $T_{e'}$ **agree up to permutation** if the tuples $v(e)$ and $v(e')$ agree up to a permutation σ and

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$$(T_e)_{i_m, i_1, \dots, i_{m-1}} = (T_{e'})_{i_{\sigma(m)}, i_{\sigma(1)}, \dots, i_{\sigma(m-1)}}$$

Definition

Let $H = (V, E)$ be a hyperquiver and $\mathbf{R} = (\mathbf{d}, T)$ a representation.

1. The **partition** of $\mathbf{R}$ is the unique partition of hyperedges $E = \coprod_{r=1}^{M} E_r$ such that for any $e, e' \in E_r$, T_e and $T_{e'}$ agree up to a permutation, and no two edges in E_r contract T along the same component

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2. The representation $\mathbf{R}$ is **generic** if given hyperedges $e_r \in E_r$ for $r \in [M]$, the tuple $(T_{e_1}, T_{e_2}, \ldots, T_{e_M})$ is a generic point in $\prod_{r=1}^{M} \mathbb{C}^{e_r}$

Main Theorem

Main Theorem (M-Nanda-Seigal)

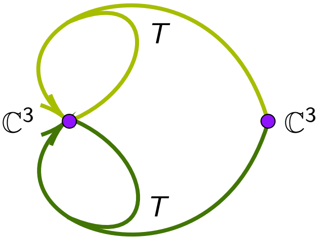
Let $\mathbf{R} = (\mathbf{d}, T)$ be a generic hyperquiver representation and $\mathcal{S}(\mathbf{R})$ be the singular vector variety of $\mathbf{R}$. Let $N = \sum_{i \in V}(d_i - 1) - \sum_{e \in E}(d_{t(e)} - 1)$ and D be the coefficient of the monomial $h_1^{d_1 - 1} \cdots h_n^{d_n - 1}$ in the polynomial

$$\left(\sum_{i \in V} h_i\right)^N \cdot \prod_{e \in E}\left(\sum_{k=1}^{d_{t(e)}} h_{t(e)}^{k-1} h_{s(e)}^{d_{t(e)}-k}\right), \quad \text{where} \quad h_{s(e)} := \sum_{j=1}^{m(e)-1} h_{s_j(e)}$$

Then $\mathcal{S}(\mathbf{R}) = \varnothing$ if and only if $D = 0$. Otherwise, $\mathcal{S}(\mathbf{R})$ is of pure dimension N and has degree D. If $\mathbf{R}$ has finitely many singular vector tuples, then each singular vector tuple occurs with multiplicity 1.

Example

Let **R** be the hyperquiver representation below, with $T \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ a generic tensor.



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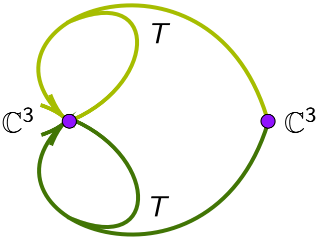

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The dimension of $\mathcal{S}(\mathbf{R})$ is $N = (2+2) - (2+2) = 0$ and

$$\left((h_1 + h_2)^2 + h_1(h_1+h_2) + h_1^2\right)^2 = 9h_1^4 + 18h_1^3h_2 + \mathbf{15}h_1^2h_2^2 + 6h_1h_2^3 + h_2^4.$$

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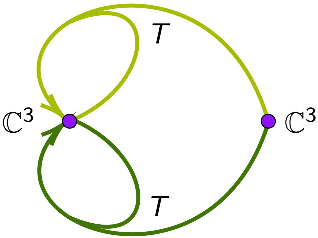

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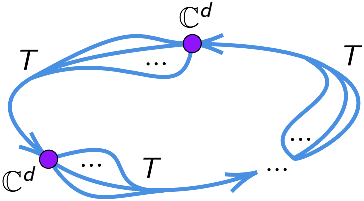
The coefficient of $h_1^2h_2^2$ is 15. Hence the singular vector variety $\mathcal{S}(\mathbf{R})$ consists of 15 distinct singular vector tuples.

Open problems

Let $T \in \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d$ be a generic tensor of order m. The following hyperquiver representation with n vertices is not generic because T is assigned to edges in different partitions, so our main theorem does not apply. Its singular vector tuples are n-periodic points.



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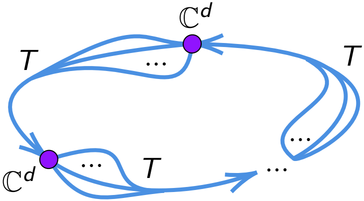

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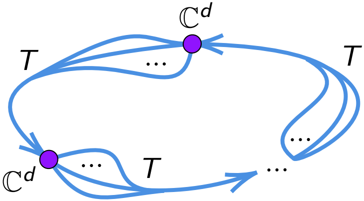

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The main theorem would predict that $\mathcal{S}(\mathbf{R})$ has dimension 0 and degree $\frac{(m-1)^{nd}-1}{(m-1)^n-1}$. This is fact the correct answer! (Fornaess-Sibony, 1994).

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Problem

Is there a meaningful representation theory or moduli theory for hyperquiver representations like there is for quiver representations?