Parity for the ~~egg~~ maze problem.

Take positive integers n, r.

Consider vectors $(x_1, x_2, \dots, x_r)$ with $1 \le x_1 < x_2 < \dots < x_r \le n$
all positive integers

For example $n=6$ $r=3$ vectors $(1,2,3)$ $(1,2,4)$ $(1,2,5)$ $(1,2,6)$
$(1,3,4)$ $(1,3,5)$ $(1,3,6)$
...
$(2,3,6)$
$(3,4,5)$
$(4,5,6)$

Claim: There are $\binom{n}{r}$ such vectors.

Proof: There are r integers in the vector, all different, chosen from $1, 2, \dots, n$.
And they're sorted $x_1 < x_2 < \dots < x_r$ so order of choice doesn't matter.

Think of these $\binom{n}{r}$ vectors as vertices of a graph, with an edge between two vertices if and only if the (Euclidean) distance between the vectors is 1.

Example $n=4$ $r=2$

$(1,2)$ — $(1,3)$ — $(1,4)$
$(1,3)$ | $(2,3)$; $(1,4)$ | $(2,4)$
$(2,3)$ — $(2,4)$
$(2,4)$ | $(3,4)$

Question: Is this graph traversable?

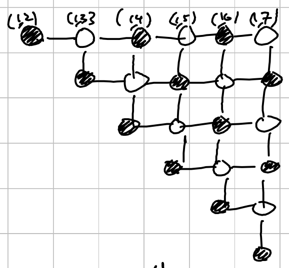
For $n=4$ $r=2$ No.

For $n=3$ $r=2$ Yes. $(1,2)$ — $(1,3)$ | $(2,3)$

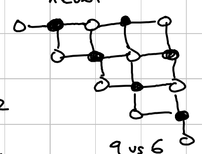
Claim (Thursday MAT Livestream) $n=6$ $r=3$ is traversable.

123
124
134
234
235
236
246
256
146 on flip ..

Claim (Thu) $n \ge 4$ $r=2$ is not traversable.



n even



$n=7$ $r=2$

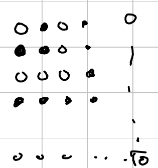
(21) vertices

12 ● 9 ○

9 vs 6
● ○

In general ($n \ge 4$ $r=2$), n odd, there are $\frac{(n-1)n}{2}$ vertices of which

$$1 + 3 + 5 + \dots + (n-2) = \left(\frac{n-1}{2}\right)^2 \quad (n \text{ odd})$$



$\left(\frac{n-3}{2}\right) + \left(\frac{n-3}{2}\right) + 1$ get

Let's investigate $n=7$ $r=3$

$x_1 = 1$

123
124 134
125 135 145
126 136 146 156
127 137 147 157 167

+3

$x_1 = 2$

sgn$(3x_1+3)$

19 even-sum.
16 odd-sum

234
235 245
236 246 256
237 247 257 267

−2

$x_1 = 3$

345
346 356
347 357 367

+2

$x_1 = 4$

456
457 467

−1

$x_1 = 5$

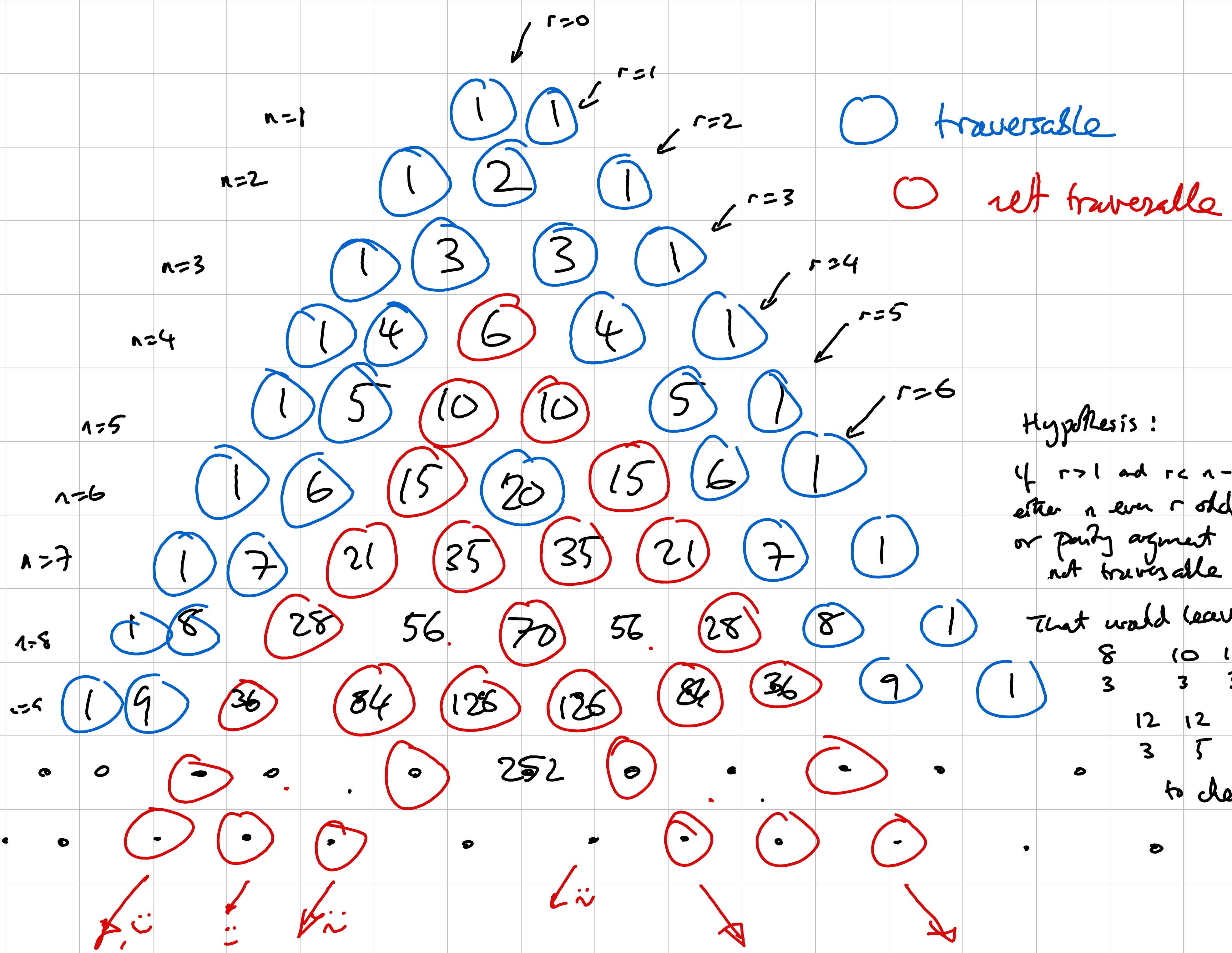
567

+1

Not traversable

Idea for $n=9$ $r=3$ $+1$ -1 $+2$ -2 $+3$ -3 $+4$ I think not traversable.

But this doesn't work for $n=8$ $r=3$. I think you get = #s of ●/○
even-sum / odd-sum



Hypothesis:

If $r > 1$ and $r < n-1$
either n even r odd
or parity argument proves
not traversable.

That would leave
8 10 10
3 3 5

12 12
3 5

to check.

Special cases $n = 2r$. ($r=2$ or $r=4 \Rightarrow$ not traversable
$r=1$ or $r=3 \Rightarrow$ is traversable)

I suspect that parity arguments proves that even r, $n=2r$ not traversable.
e.g. prove $n=12$, $r=6$ is not traversable.

$\underline{1, 2, 3}, \boxed{4, 5}, \underline{6, 7}, \boxed{8, 9, 10}$ Sum $=55$

If r odd select five things. Switch, with out other 5.
Sum of those has the opposite parity!
Same number of each parity. ☺

Consider r even write $r = 2k$ (then $n = 4k$).

$1 \quad 3 \quad 5 \quad 7 \quad 4k-3 \quad 4k-1$
$2 \quad 4 \quad 6 \quad 8 \quad \ldots \quad 4k-2 \quad 4k$

select $2k$ of them. Idea: sum is ~~frequently~~ even.

How many ways to choose with an even sum. $\binom{n}{r} = \binom{4k}{2k}$ total

might choose all odd numbers $\binom{2k}{2k}\binom{2k}{0}$ ways to do that
might choose $(2k-2)$ of them $\binom{2k}{2k-2}\binom{2k}{2}$ ways to do that
might choose $(2k-4)$ of them $\binom{2k}{2k-4}\binom{2k}{4}$
...

\# even sum total $\binom{2k}{2k}\binom{2k}{0} + \binom{2k}{2k-2}\binom{2k}{2} + \binom{2k}{2k-4}\binom{2k}{4} + \ldots + \binom{2k}{0}\binom{2k}{2k}$

$= \binom{2k}{2k}^2 + \binom{2k}{2k-2}^2 + \binom{2k}{2k-4}^2 + \ldots + \binom{2k}{0}^2$

$k=2$
$1\,4\,6\,4\,1$ $\quad 1^2+6^2+1^2 = 38 \quad \binom{8}{4} = 70 \quad 38 > 70/2$. $\quad +3$

$k=3$
$1\,6\,15\,20\,15\,6\,1$ $\quad 1^2+15^2+15^2+1^2 = 452 \quad \binom{12}{6} = 924 \quad 462 > 452 \quad -10$

$k=4$ $\quad 1^2 + \ldots + 1^2 = 6470 \quad \binom{16}{8} = 12870 \quad 6435 < 6470 \quad +35$

6
20
70

\# odd sum
$\binom{2k}{2k-1}\binom{2k}{1} + \binom{2k}{2k-3}\binom{2k}{3} + \ldots + \binom{2k}{1}\binom{2k}{2k-1}$

$1^2 - 6^2 + 15^2 - 20^2 + 15^2 - 6^2 + 1^2$