Maths problem (not MAT) (not A-level)

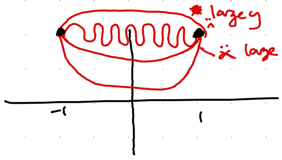
✓ • Kind of do-able
✓ • Links to a cool method
✓ • Proof to do with tanh 1

Given a function $y(x)$, we could work out

$$I(y) = \int_{-1}^{1} (y(x))^2 + (y'(x))^2 \, dx \;>\; 0$$

I for integral [value of y] [derivative of y]

Among all smooth functions $y(x)$ with $y(1)=1$, $y(-1)=1$, which function has the smallest $I(y)$?

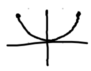


not too high
not too steep.

Try $y = x^n$ n even $n^2 x^{2n-2}$

$$I(y) = \int_{-1}^{1} x^{2n} + (nx^{n-1})^2 \, dx$$

$$= \left[\frac{x^{2n+1}}{2n+1} + \frac{n^2 x^{2n-1}}{2n-1} \right]_{-1}^{1} = \frac{2}{2n+1} + \frac{2n^2}{2n-1}$$



$n=2$?

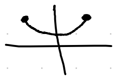
$$\frac{2}{5} + \frac{8}{3}$$

$$= \frac{46}{15} \;?$$

Try $y = 1$

$$I(y) = \int_{-1}^{1} 1^2 + 0^2 \, dx = 2$$

Try



parabola $y = Ax^2 + B$ $A+B=1$

$= 1 + A(x^2-1)$

? dunno what A is.

$= (1-A) + Ax^2$

$A=0$ we've seen (2)

$A=1$ we've seen ($46/15$)

$y^2 = A^2x^4 + 2A(1-A)x^2 + (1-A)^2$

$y' = 2Ax$ $(y')^2 = 4A^2x^2$

$$\int_{-1}^{1} A^2x^4 + (2A + 2A^2)x^2 + (1-A)^2 \, dx$$

$$\frac{2A^2}{5} + \frac{4A(1+A)}{3} + 2(1-A)^2$$

New task! Choose A to minimise this. (or $15\times$ this)

$6A^2 + 20A + 20A^2 + 30 - 60A + 30A^2$

$56A^2 - 40A + 30$

Min. @ $112A - 40 = 0$ || $A = \frac{5}{14}$

Value at $\frac{5}{14}$ is $\dfrac{56\left(\frac{5}{14}\right)^2 - 40\left(\frac{5}{14}\right) + 30}{15}$

$$\frac{\frac{4\times5^2}{14} - \frac{200}{14}}{15} + 2$$

$2 \;\; \frac{\cancel{100}}{\cancel{14}\times15}$ $\frac{10}{7 \;\; 3}$ $\frac{20}{}$

$2 - \frac{10}{21} = \frac{32}{21}$

≈ 1.5

Try $\boxed{y = \dfrac{\cosh(x)}{\cosh(1)}}$ [through $(1,1)$]

Let $A = \frac{1}{\cosh 1}$

$$\frac{\sinh 2}{2}$$

$$\int_{-1}^{1} y^2 + (y')^2$$

$$\left[\frac{\sinh 2x}{2}\right]_{-1}^{1} \; ?$$

$$\int_{-1}^{1} A^2\cosh^2 x + A^2 \sinh^2 x \, dx = \left[A^2 \int_{-1}^{1} \cosh 2x \, dx\right]$$

$\cosh^2 x = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$

$\sinh^2 x = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$

[Add] $= \frac{1}{2}(e^{2x} + e^{-2x})$

$$= \frac{A^2}{2} \int_{-1}^{1} e^{2x} + e^{-2x} \, dx$$

$$= \frac{A^2}{2} \left[\frac{e^{2x}}{2} - \frac{e^{-2x}}{2} \right]_{-1}^{1}$$

$$= \frac{A^2}{\cancel{2}} \frac{(e^2 - e^{-2})}{2}$$

$$= \frac{\sinh 2}{\cancel{2}\cosh^2 1} \approx 1.5$$

We now know that $\frac{\cosh(x)}{\cosh(1)}$ is the best.

So $\dfrac{\sinh(2)}{\cosh^2(1)} < \dfrac{32}{21}$

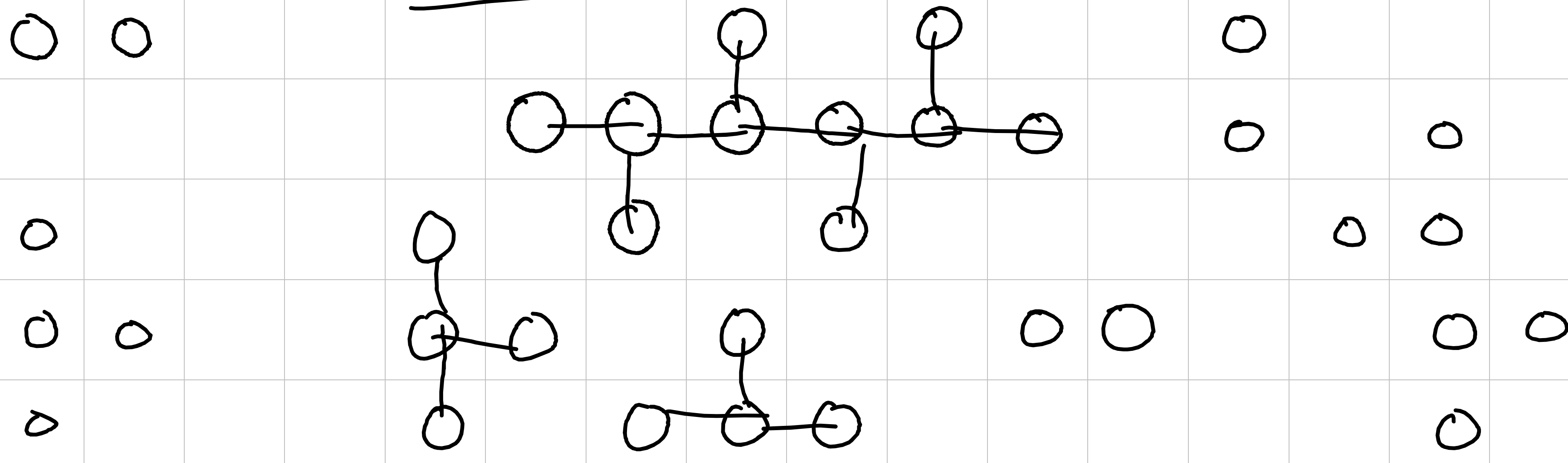
Now $\sinh(2) = 2\sinh(1)\cosh(1)$

so $\tanh(1) < 16/21$

OOMC

Stop here!

1 or 3 <u>neighbours</u> (diagonals don't count)



Can you make a loop?

Among all smooth functions $y(x)$ with $y(1)=1$, $y(-1)=1$, the one which minimises $I(y) = \int_{-1}^{1} (y(x))^2 + (y'(x))^2 \, dx$

is $y = \dfrac{\cosh(x)}{\cosh(1)}$.

Proof. Write $y(x) = \dfrac{\cosh(x)}{\cosh(1)} + f(x)$ with $f(x)$ smooth fn with $f(1)=0$, $f(-1)=0$.

Let $A = \dfrac{1}{\cosh(1)}$

Now.

$$I(y) = \int_{-1}^{1} (A\cosh(x) + f)^2 + (A\sinh x + f')^2 \, dx$$

$$= \int_{-1}^{1} \overbrace{A^2\cosh^2 x + A^2 \sinh^2 x}\,+ \overbrace{f^2 + f'^2} \geq 0$$

$$+\; 2A\cosh(x)\, f + 2A\sinh x\, f' \, dx$$

$$\geq I\left(\frac{\cosh(x)}{\cosh(1)}\right) + 2A\int_{-1}^{1} \cosh(x) f + \sinh(x) f' \, dx$$

$A x^2 + Bx^4 = \dfrac{\cosh(x)}{\cosh(1)} + \left(Ax^2 + Bx^4 - \dfrac{\cosh(x)}{\cosh(1)}\right)$

$f(x)$

$$\int_{-1}^{1} \underset{u}{\sinh(x)}\, \underset{dv}{f'(x)}\, dx = \underbrace{\left[\sinh(x) f(x)\right]_{-1}^{1}}_{\text{but that's zero}} - \int_{-1}^{1} \underset{du}{\cosh(x)}\, \underset{v}{f(x)}\, dx$$

$$= I\left(\frac{\cosh(x)}{\cosh(1)}\right)$$

some maple
some no.
40 mins!

General Theory $\qquad \int_{-1}^{1} y^2 + y'^2 \, dx \qquad y(1)=1$, $y(-1)=1$

To minimize $\displaystyle\int L(y, y')\, dx$,

solve $\qquad \dfrac{\partial L}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial L}{\partial y'}\right) = 0 \qquad$ Euler-Lagrange equation

|| just look at y-dependence $\qquad$ just look at y'-dependence ||

$$2y - \frac{d}{dx}(2y') = 0$$

$$y - y'' = 0$$

$\cosh$, $\sinh$ $\quad$ ($y(1)=1$, $y(-1)=1$)

Minimise $\displaystyle\int_{-1}^{1} y^4 + (y')^4 \, dx$ $\quad$:($\quad$ subject to $y(1)=1$, $y(-1)=1$

Euler-Lagrange says to solve

$$4y^3 - \frac{d}{dx}\left(4y'^3\right) = 0$$

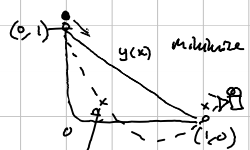
$$y^3 - \frac{d}{dx}\left(y'^3\right) = 0$$

$$y^3 - 3y'^2 y'' = 0 \leftarrow \text{oops that's hard.}$$

$$y^3 = 3y'^2\, y''$$

$$\frac{3y'}{y^3} = \frac{1}{y'' y'} \qquad \text{bleh}$$

Roller coaster



minimize time taken $\qquad$ Brachistochrone problem

$g=1$
$m=1$

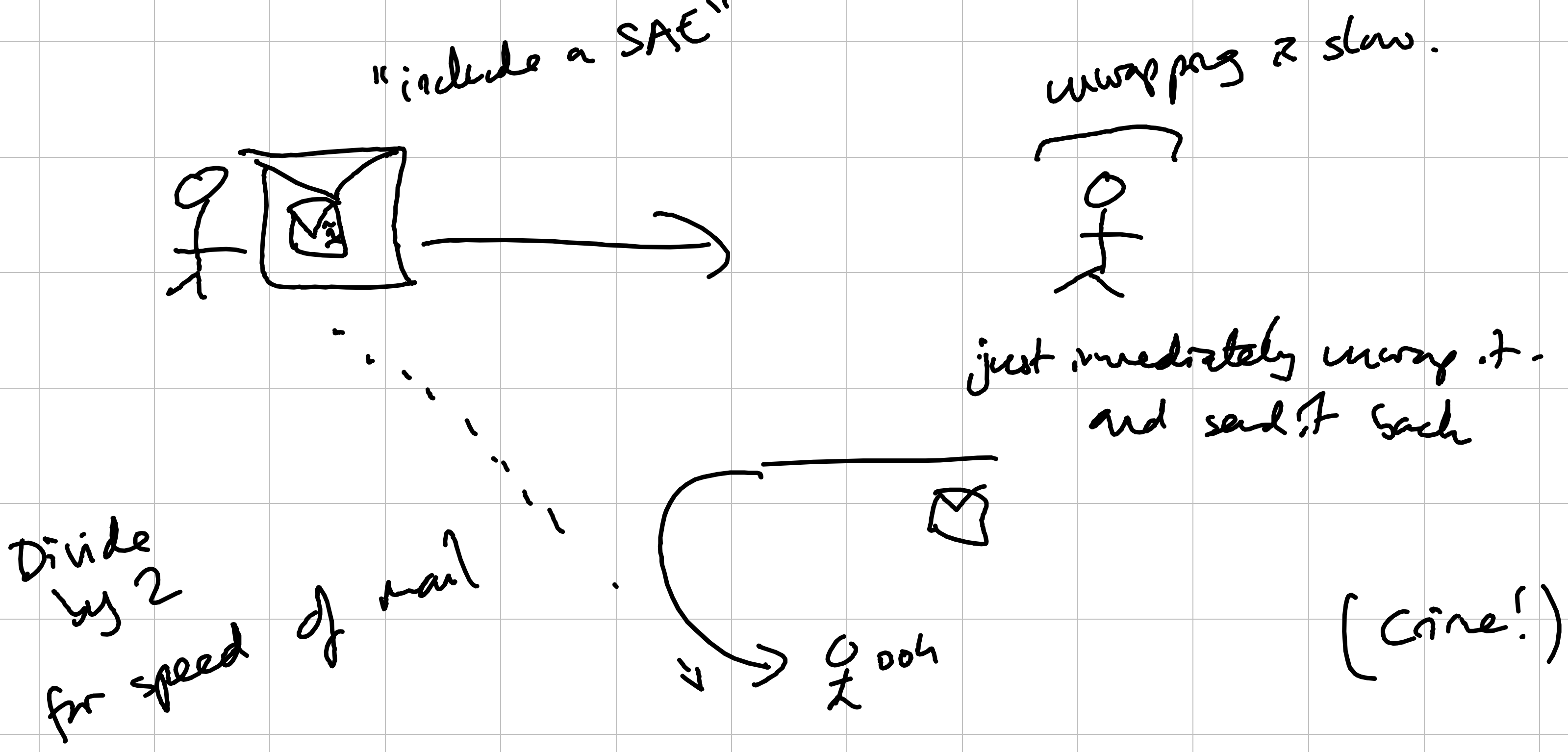
Energy? velocity? $\dfrac{dx}{dt}$? $\qquad$ Total time is $\displaystyle\int_0^1 \frac{dt}{dx}\, dx$

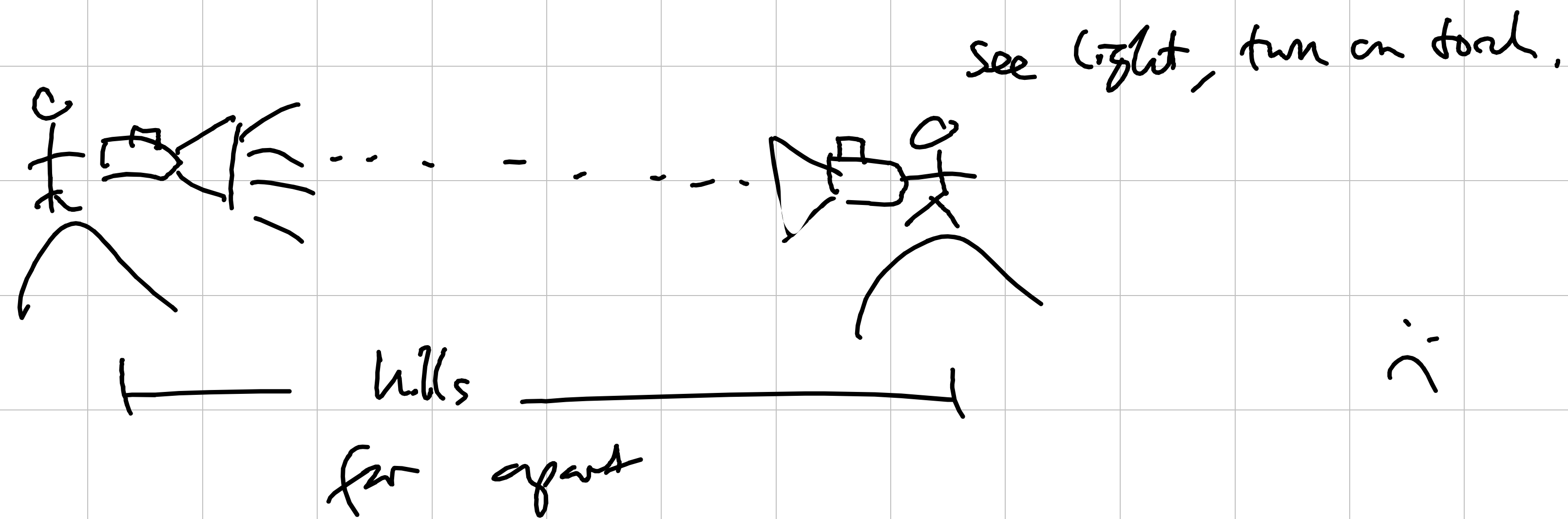
$\frac{1}{2}mv^2 + mgy = mg\,1$

$v = \sqrt{1-y}\,\sqrt{2}$ $\qquad$ $\sqrt{2}$ Time $= \displaystyle\int_0^1 \frac{\sqrt{1+y'^2}}{\sqrt{1-y}}\, dx$

horizontal velocity

$\dfrac{1}{\sqrt{1+y'^2}}$ $\qquad$ ... bleh


"include a SAE"
unwrapping is slow.
just immediately unwrap it.
and send it back
Divide by 2
for speed of mail
ooh
(Crime!)


see light, turn on torch.
hills
far apart

$= \frac{22}{7} - \pi$ but it's positive! So $\frac{22}{7} > \pi$

(1) $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$.

(2) Deduce a fact about a famous irrational number.

Strategy for $\int_0^1 \frac{p(x)}{1+x^2}\,dx$: write $p(x) = q(x)(1+x^2) + Ax + B$

$\int q(x)\,dx$:)

$A\int \frac{x}{1+x^2}\,dx$

$\frac{1}{2}\ln(1+x^2)$

$B\int_0^1 \frac{1}{1+x^2}\,dx$

$\left[\tan^{-1}(x)\right]_0^1$

$\frac{\pi}{4}$

$\frac{\pi}{4}$ vs. ~~$\ln 2$~~ and fractions?

$\frac{-4}{x^2+1}$ gives $-\pi$

$p(x) = x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$

$= (x^2+1)(x^6) - 4x^7 + 5x^6 - 4x^5 + x^4$

$= (x^2+1)(x^6 - 4x^5) + 5x^6 + x^4$

$\cdots$ $-4x^4$

$= (x^2+1)(x^6 - 4x^5 + 5x^4 - 4x^2 + 4) - 4$

no x term

$\int_0^1 x^6 - 4x^5 + 5x^4 - 4x^2 + 4\,dx$

$\frac{1}{7} - \frac{4}{6} + 1 - \frac{4}{3} + 4$ (too fast!)

$-\frac{2}{3}$

$\int$ is equal to

$3\frac{1}{7}$

TODAY

- ✓ $\frac{16}{21} > \tanh(1)$
- ✓ $\frac{22}{7} > \pi$
- ✓ Theory for $\int_0^1 y^2 + y'^2 \, dx$
- ✓ loops?