Perfect Numbers

$$6 = 1 + 2 + 3$$

$$28 = 1 + 2 + 4 + 7 + 14$$

Zoë/Nesinator Theorem: 6 is the only perfect number of the form pq (p, q distinct primes)

Proof: If $N = pq$, the factors are $1, \underbrace{p, q}_{\text{distinct}}, pq$
(say $p<q$)

Sum of those is $1+p+q+pq$.
Is that $2pq$?
Only if $(1+p)(1+q) = 2pq$
so only if $\left(\frac{1}{p}+1\right)\left(\frac{1}{q}+1\right) = 2$

But! Oh no. $1+\frac{1}{p}$ and $1+\frac{1}{q}$

either $\frac{3}{2}$ (if $p=2$)

or $\frac{4}{3}$ (if $p=3$)

or, if $p>3$ (so $p \geq 5$)

if $p=2$ $q \geq 3$

$\frac{3}{2} \times \left(\leq \frac{6}{5}\right) \leq 1.8$

if $p \geq 2$ $q \geq 3$

$\left(\leq \frac{4}{3}\right)\left(\leq \frac{4}{3}\right) \leq \frac{16}{9} < 2$

! wherefore
! how

In general, the sum of the factors of N factorises.

e.g. $28 = \underbrace{1 + 2 + 4} + \underbrace{7 + 14 + 28}$

$= (1+7)(1+2+4)$

$= (1+7)(1+2+2^2)$

$28 = 2^2 \times 7$

factor $2^a 7^b$

$0 \le a \le 2$

$0 \le b \le 1$

Factors of $60 = 2^2 \times 3 \times 5$ are $(1+2+2^2)(1+3)(1+5)$

Ratio of that sum to 60 is $\left(\frac{1+2+2^2}{2^2}\right)\left(\frac{1+3}{3}\right)\left(\frac{1+5}{5}\right)$

$(1+\frac{1}{2}+\frac{1}{4})\ (1+\frac{1}{3})\ (1+\frac{1}{5})$

$\frac{7}{4} \times \frac{4}{3} \times \frac{6}{5}$

$= \frac{14}{5}$ (almost three!)

Claim:

? $\frac{\sigma(n!)}{n!}$ larger than all previous values of $\frac{\sigma(k)}{k}$?

~~maybe...~~ No. :(

$\sigma(120)$

factors
↓

$\frac{\text{Sum of factors of } 120}{120} = (1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8})(1+\frac{1}{3})(1+\frac{1}{5})$

$\frac{15}{8} \times \frac{4}{3} \times \frac{6}{5}$

$= 3$

:O wow!

"No two people see the same Hero"

| Haven't seen exactly 4 |

$1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \dots = \frac{1}{1-1/p} = \frac{p}{p-1}$

Is there an upper bound if I have lots of different primes?

? Is $\prod_{p \text{ prime}} \frac{p}{p-1}$ bounded ?

No, looks like it grows like log log n or something due to PNT.

If instead of primes, one looks at all numbers >1 $\prod_{n \ge 2} \frac{n}{n-1} = \underbrace{\underbrace{\underbrace{\frac{2}{1}}_{2} \times \frac{3}{2}}_{2} \times \frac{4}{3}}_{3} \times \dots$

4

not bounded

If, instead of primes, we look at square numbers >1

$$\prod_{\substack{n=k^2 \\ k>1}} \frac{n}{n-1} = \prod_{k>1} \frac{k^2}{k^2-1} = \frac{2^2}{\underbrace{2^2-1}_{1\times\cancel{3}}} \times \frac{\cancel{3^2}}{\underbrace{3^2-1}_{2\times\cancel{4}}} \times \frac{\cancel{4^2}}{\underbrace{4^2-1}_{\cancel{3}\times 5}} \times \frac{5^2}{\underbrace{5^2-1}_{\cancel{4}\times 6}} \times \dots$$

$= 2$

Wilson's Theorem!

$(p-1)!$ is one less than a multiple of p.

(for example $4!$ is 24 is one less than a multiple of 5)

$6!$ is 720 is one less than a multiple of 7.

$10!$ is one less than a multiple of 11.

"Homework": - try to find n such that sum of factors is $4n$.

- Understand CA numbers

- Wilson's Theorem.

Try $2^k \times q$ with q prime

Sum of factors is $(1+2+2^2+\dots+2^k)(1+q)$

$$(2^{k+1}-1)(1+q)$$

Cool observation: if $2^{k+1}-1=q$ then this is

$$(q)\quad(2^{k+1})$$
$$= 2\times(2^k\times q)$$

so then $2^k q$ is a perfect number.

All we need is $2^{k+1}-1$ to be prime

then $N = 2^k(2^{k+1}-1) = \dfrac{2^{k+1}(2^{k+1}-1)}{2}$

These are (currently) all the perfect numbers triangle number (s) we know.

all humans on or near Earth.

$$2^k m = (2^k - 1)\,\sigma(m)$$

$$\frac{m}{2^k - 1} = \frac{\sigma(m)}{2^k} \quad (= M)$$

Then $\sigma(m) \geq m+1$?

If N is an odd perfect number, then it's <u>not</u> a multiple of 105.

5×21

$3 \times 5 \times 7$

$$\left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots\right)\left(1 + \frac{1}{5} + \frac{1}{5^2} + \dots\right) = 2$$

$$\left(1 + \frac{1}{3} + \dots\right)\left(1 + \frac{1}{5} + \dots\right)\left(1 + \frac{1}{7} + \dots\right)(\dots)$$

~~Ad breaks?~~ TODAY

- 6 is only perfect number of form pq (so 35 is out)
- $\sigma(n)$ can be exactly 3
- Proof that even perfect numbers are all of a particular form.
- Maybe don't broadcast your screen on Twitch?
- Computers are good (but not magic).

$pqrst$ (all prime) not perfect?

6 is only squarefree prime.