Perfect numbers.

$\sigma(n)$ sum of factors

$\sigma(6) = 1+2+3+6 = 12$

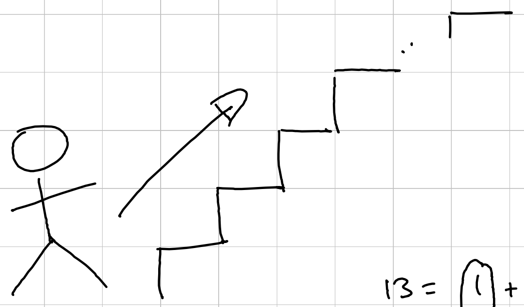
$\sigma(10) = 1+2+5+10 = 18$

$\dfrac{\sigma(n)}{n} = 2$ perfect number 6

$\dfrac{\sigma(n)}{n} = 3$ multi-perfect number 120

Some Vicky Neale problems.

- Stairs problem
- Bracket quadratic
- Double rubber circle thing.



13 steps

Either go up 1 step or 2 steps with each foot.

How many ways are there?

$13 = 1 + 2 + 2 + 2 + 1 + 2 + 2 + 1$

$13 = 2 + 1 + 2 + 2 + 1 + 2 + 2 + 1$

$13 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

In general n steps, call ~~&~~ $f(n)$ for number of ways.

$$f(n) = \underbrace{f(n-1)}_{\text{1-step}} + \underbrace{f(n-2)}_{\text{2-steps}}$$

$f(1) = 1$ $f(2) = 2$ $f(3) = 3$ $f(4) = 5$ $f(5) = 8$ ...

1+1 | 1+1+1 | 1+1+1+1
2 | 1+2 | 2+1+1
 | 2+1 | 1+2+1
 | | 1+1+2
 | | 2+2

$f(13) =$ ~~...~~ 377.

To make 13 ...

All 1s $\binom{13}{0}$ 1

$1+1+1+\dots+2+1+1\dots+1$ || One 2, 11 1s $\binom{12}{1}$ 12

$2+2+1+1+\dots+1$ Two 2s, 9 1s $\binom{11}{2}$ 55

$2+2+2+1+1+\dots+1$ Three 2s $\binom{10}{3}$ $\frac{10\times9\times8}{6}$ $5\times3\times8$ 120

$\frac{9\times8\times7\times6}{4\times3\times2\times1}$ $\binom{9}{4}$ 126

$\frac{8\times7\times6}{3\times2\times1}$ $\binom{8}{5}$ 56

⋮

Six 2s, 1 1 $\binom{7}{6}$ 7

$14 \times 9 = 126$

$$F_{14} = 377 = \binom{13}{0} + \binom{12}{1} + \binom{11}{2} + \dots + \binom{7}{6}$$

1 1

1 2 1

1 = 1 3 3 1

2 = 1 4 6 4 1

3 = 1 5 10 10 5 1

5 = 1 6 15 20 15 6 1

8 = 1 7 21 35 35 21 7 1

13 =

21 =

Extension: What if steps of 1, 2, or 3 at a time are allowed?

- How many ways?
- What's that sequence? ~~Three~~ tri-bonacci numbers OEIS
- Is this sequence hidden in Pascal's Δ?

"Add together previous k terms" with k starting numbers almost always grows exponentially like $(A_k)^n$ for n^{th} term, where A_k is a number between ϕ and 2.

Classic result from number theory: some numbers can be written as the sum of 2 squares (and we know which).

$5^2+0^2 = 3^2+4^2$

Vicky Neale: what about a different sequence?

Sequence : $n \times \lfloor \sqrt{2}\, n \rfloor$

floor rounds down to an integer.

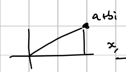
$\lfloor x \rfloor$ is largest integer less than or equal to x.

$\lfloor -\pi \rfloor = -4$.

Theorem (Neale): For any r, there is a number which be written in the form $n_1 \lfloor \sqrt{2} n_1 \rfloor + n_2 \lfloor \sqrt{2} n_2 \rfloor$ in r different ways.

Which numbers are the sum of two squares? — could be same — could be zero

$0=0^2+0^2$	$1 = 1^2+0^2$	$2=1^2+1^2$	$3=$	$4=$	$5=$	$6=$	$7=$
$8=2^2+2^2$	$9=$	$10=$	$11=$	$12=$	$13=$	$14=$	$15=$



$|a+bi|^2 = a^2+b^2$

x, Remainder when $\div 8$	x^2, remainder when $\div 8$
0	0
1	1
2	4
3	1
4	0
5	1
6	4
7	1

$(8k+1)^2$

$64k^2 + 2\times 8k + 1$

If p, q can each be written as S.O.T.S. Then pq can.

Proof: $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$

Expand and check.

Real proof: let $z_1 = a+bi$, $z_2 = c+di$, Then $z_1 z_2 = ac-bd + (ad+bc)i$

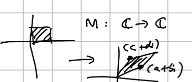
and $|z_1 z_2|^2 = |z_1|^2 |z_2|^2$

$\square$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$M: \mathbb{C} \to \mathbb{C}$



Det M gives area of parallelogram.

Matrices + Fib ?

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \text{ means } \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \times \dots \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{n}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(there are methods for this)

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$$

$(PDP^{-1})^n$

$= PD^n P^{-1}$

Fib: keep track of previous two numbers.

Operation: add the numbers, throw away smaller one.

In storage $\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{M} \begin{pmatrix} a+b \\ a \end{pmatrix}$

Linear transformation

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Sequence: double the previous term and add 10 times the term before last

$$\begin{pmatrix} 2 & 10 \\ 1 & 0 \end{pmatrix}$$

$a_{n+1} + a_n = 2$

or $a_{n+1} = 2 - a_n$

$M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

eigenvalues of matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ ??? } \frac{1 \pm \sqrt{5}}{2}$$

$$= M \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} M^{-1}$$

for some M

$$A \left(\frac{1+\sqrt{5}}{2}\right)^n + B \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$n=0$ $A+B=0$

$n=1$ $(A-B)\frac{\sqrt{5}}{2} = 1$

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$$\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}}$$

$n=3$
$$\frac{\left(1 + 3\sqrt{5} + 3(\sqrt{5})^2 + (\sqrt{5})^3\right) - \left(1 - 3\sqrt{5} + 3(\sqrt{5})^2 - (\sqrt{5})^3\right)}{2^3 \sqrt{5}}$$

$$= \frac{6\sqrt{5} + 2(\sqrt{5})^3}{2^3 \sqrt{5}}$$

$$= \frac{6 + 2 \times 5}{2^3}$$

$$= 2$$

Xavi's T-shirt.