REDUCTION OF MATRIX POLYNOMIALS TO SIMPLER FORMS

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Abstract. A square matrix can be reduced to simpler form via similarity transformations. Here "simpler form" may refer to diagonal (when possible), triangular (Schur), or Hessenberg form. Similar reductions exist for matrix pencils if we consider general equivalence transformations instead of similarity transformations. For both matrices and matrix pencils, well-established algorithms are available for each reduction, which are useful in various applications. For matrix polynomials, unimodular transformations can be used to achieve the reduced forms but we do not have a practical way to compute them. In this work we introduce a practical means to reduce a matrix polynomial with nonsingular leading coefficient to a simpler (diagonal, triangular, Hessenberg) form while preserving the degree and the eigenstructure. The key to our approach is to work with structure preserving similarity transformations applied to a linearization of the matrix polynomial instead of unimodular transformations applied directly to the matrix polynomial. As an applications, we illustrate how to use these reduced forms to solve parameterized linear systems.

Key words. triangularization, matrix polynomial, quasi-triangular, diagonalization, Hessenberg form, companion linearization, controller form linearization, equivalence, quadratic eigenvalue problem, Schur form, parameterized linear systems.

AMS subject classifications. 15A18, 15A22, 65F15

1. Introduction.

Almost all matrices in $\mathbb{C}^{n\times n}$ can be reduced to diagonal form via a similarity transformation. (The exceptions constitute the measure-zero set of defective matrices.) Furthermore, all matrices in $\mathbb{C}^{n\times n}$ can be reduced to triangular and upper Hessenberg form via unitary similarity transformations. For matrices in $\mathbb{R}^{n\times n}$, we have similar results with the difference that we now have quasi-diagonal and quasi-triangular forms instead of diagonal and triangular forms. Here the prefix "quasi" means that all diagonal blocks are either of size 1×1 or 2×2. Now, consider matrix polynomials with a nonsingular leading coefficient

$$P(\lambda) = \lambda^\ell P_\ell + \cdots + \lambda P_1 + P_0 \quad \text{with} \quad \det(P_\ell) \neq 0, \tag{1}$$

over $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Is it possible to reduce such matrix polynomials to the simpler forms mentioned above while preserving the degree and the eigenstructure, that is, the eigenvalues and their partial multiplicities? If we use only similarity transformations, the answer is, in general, no. Even if we use the broader class of strict equivalence transformations, that is, multiplication by nonsingular matrices from left and right, it is in general not possible. Indeed, if there were to exist nonsingular matrices E and F such that $EP(\lambda)F = T(\lambda)$ is triangular, say, of degree $\ell > 1$ and with $\det P_\ell \neq 0$ then the family of matrices $(P_\ell^{-1}P_{\ell-1}, \ldots, P_\ell^{-1}P_1, P_\ell^{-1}P_0)$ would be simultaneously triangularizable by similarity. This would imply (see for example [6, Thm. 2.4.8.6 and

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Thm. 2.4.8.7]) that for all $i \neq j$, $i, j = 0, 1, \ldots, \ell-1$, the eigenvalues of $P_\ell^{-1} P_i P_\ell^{-1} P_j - P_\ell^{-1} P_j P_\ell^{-1} P_i$ are all equal to zero. This is a very restrictive condition.

A type of transformations that gives us a sufficient amount of freedom while preserving the eigenstructure is multiplication by unimodular matrix polynomials. A matrix polynomial $U(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ is said to be unimodular if $\det U(\lambda) \in \mathbb{F} \setminus \{0\}$, and two matrix polynomials that differ only by multiplication by unimodular matrix polynomials (from the left and the right) are said to be *equivalent*. It was shown in [14] and [15] that unimodular transformations are enough to reduce any square matrix polynomial to triangular form over $\mathbb{C}$ and quasi-triangular form over $\mathbb{R}$, while preserving the degree. Of course, this includes the case of Hessenberg form since (quasi)-triangular matrices are also Hessenberg. Further, it is a straightforward exercise to show that any complex/real matrix polynomial with semisimple eigenstructure is equivalent to a diagonal/quasi-diagonal matrix polynomial of the same degree.

The reduction to diagonal form has applications in structural engineering, where it has been used to decouple systems of second-order differential equations (see for example [2] and [10]). In applications where parametrized linear systems of the form $P(\omega)x = b(\omega)$ with P as in (1) need to be solved for many values of ω over a large range, it may be useful to first reduce P to simpler form before solving the linear systems (see Section 5).

How can we compute these simpler forms in practice? The discussions in [3, Thm. 1.7], [14] are based on applying unimodular transformations to the Smith form, and its numerical implementation is nontrivial. To avoid working with unimodular transformations, which in general affect the degree, we use linearizations. Recall that a pencil $\lambda I - A$ is a monic linearization of the matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ in (1) if $A \in \mathbb{F}^{\ell n\times \ell n}$ and $\lambda I - A$ has the same elementary divisors as $P(\lambda)$. Suppose $P(\lambda)$ has the same eigenstructure as the monic matrix polynomial $R(\lambda) = \lambda^\ell I + \sum_{j=0}^{\ell-1} \lambda^j R_j$ and take any monic linearization $\lambda I - A$ of $P(\lambda)$. Note that $\lambda I - A$ is also a linearization of $R(\lambda)$. The Gohberg, Lancaster, Rodman theory [3, Sec. 1.10] tells us that there is an $\ell n \times n$ matrix X such that (A, X) is a left standard pair for $R(\lambda)$, that is, the $\ell n \times \ell n$ matrix

$$S = [X \ AX \ \ldots \ A^{\ell-1}X] \tag{2}$$

is nonsingular and

$$A^\ell X + A^{\ell-1} X R_{\ell-1} + \cdots + AXR_1 + XR_0 = 0. \tag{3}$$

Taken together, (2) and (3) can be rewritten as

$$S^{-1}AS = \begin{bmatrix} & & & -R_0 \\ I & & & -R_1 \\ & \ddots & & \vdots \\ & & I & -R_{\ell-1} \end{bmatrix} =: C_L(R) \tag{4}$$

showing that A is similar to the left companion matrix associated with $R(\lambda)$. Actually, for any given monic linearization $\lambda I - A$ of $P(\lambda)$ and any nonsingular matrix S of the form (2), $S^{-1}AS$ will always be the left companion matrix of some matrix polynomial, as in (4). This matrix polynomial, $R(\lambda) = \lambda^\ell I + \lambda^{\ell-1}R_{\ell-1} + \cdots + \lambda R_1 + R_0$ will have the same degree and eigenstructure as $P(\lambda)$. The above discussion suggests that in order to reduce $P(\lambda)$ in (1) to a simpler form, it is enough to find an $n\ell \times n$ matrix X such that S in (2) is nonsingular and $S^{-1}AS$ has the desired zero pattern in the

```
n = 5; deg = 3;  % size and degree
P0 = randn(n); P1 = randn(n); P2 = randn(n); % coefficient matrices
C_P = [ zeros(n) zeros(n) -P0 % left companion form
        eye(n)   zeros(n) -P1
        zeros(n) eye(n)   -P2 ]
[U,~] = schur(C_P,'complex');
X = U*kron(eye(n), ones(deg,1));
S = [X C_P*X C_P^(deg-1)*X];
C_R = (S\C_P)*S;
spy(abs(C_R)>1e-12)
```

FIG. 1. *Basic MATLAB M-file that generates a random monic cubic matrix polynomial, then computes the left companion form of an equivalent triangular matrix polynomial, and plots its* (*numerical*)*nonzero pattern.*

coefficient matrices (in the last block column), where A can be any matrix such that $\lambda I - A$ is a linearization of $P(\lambda)$. One of the main contributions in this paper is to give a characterization of such a matrix X in terms of block Krylov subspaces (see Section 2).

In the generic case, when all the eigenvalues are distinct, it turns out to be surprisingly easy to find X such that $S^{-1}AS$ is the left companion matrix of a matrix polynomial in triangular, diagonal or Hessenberg form. We illustrate this with a snippet of MATLAB code in Figure 1. If we replace `schur(C_P,'complex')` by `eig(C_P)`, then `C_R` becomes the companion matrix of an equivalent diagonal matrix polynomial, and if we replace `schur(C_P,'complex')` by `hess(C_P)` and `ones(deg,1)` by `eye(deg,1)`, then `C_R` becomes the companion matrix of an equivalent matrix polynomial in Hessenberg form. The code can be generalized to any degree and works as long as the block Krylov matrix `S` on line 8 is nonsingular, which it is for almost all coefficient matrices, as we will see in Section 3.2. A colored spy plot from one execution of the MATLAB code in Figure 1 is shown on the left of Figure 2. The other plots correspond to the diagonal reduction (middle plot) and the Hessenberg reduction (right plot). We remark that the reduction to Hessenberg form requires no iterative process (such as computing the eigenvalues) and uses a fixed number of arithmetic operations. Our reduction gives a Hessenberg matrix polynomial with all but the second leading coefficient being triangular.

In this paper we discuss why and when the code in Figure 1 works. In the rare cases when it fails, we describe whenever possible what has to be achieved for the reductions to go through. To be precise, one of the main goals of this work is to give a practical procedure to reduce $P(\lambda)$ in (1) to triangular or quasi-triangular form according as $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, while preserving its degree and eigenstructure. This procedure consists of the following steps:

1. Choose a monic linearization $\lambda I - A$ of $P(\lambda)$.
2. Compute a real or complex Schur form, T_0, of A according as $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
3. Reorder the diagonal entries of T_0 and, in the real case, the 2×2 blocks along its diagonal to produce a new Schur form T of A that can be split into blocks that are suited to construct the matrix X of the next step.
4. Use the diagonal blocks of T to produce a matrix $X \in \mathbb{F}^{\ell n \times \ell}$ of full column rank such that S in (2) is nonsingular and $S^{-1}AS$ is the left companion matrix of a monic upper triangular matrix polynomial.

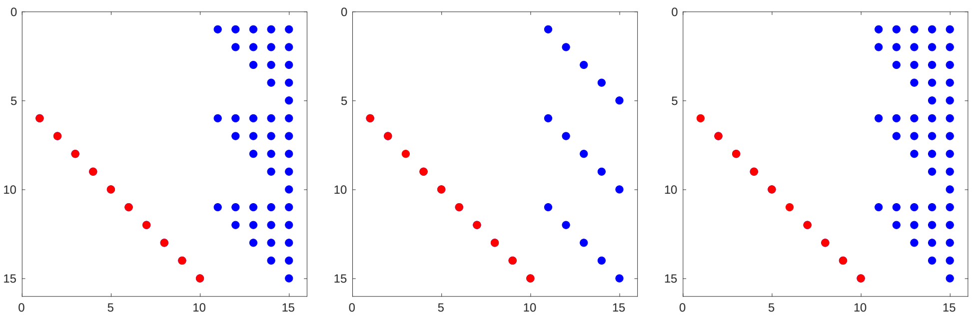

FIG. 2. *Colored spy plots of the left companion linearization of the reduced matrix polynomials (size $n = 5$, degree $\ell = 3$) obtained by the MATLAB code in Figure 1 (or its modification as explained in the text): triangular (left), diagonal (middle), and Hessenberg (right). The red dots are 1s, the blue dots are nonzero entries of the coefficient matrices.*

5. Compute $S^{-1}A^{\ell}X$, i.e., the last block column of $C_L(R)$ in (4), and extract the blocks R_j, $j = 0,\dots,\ell-1$ defining $R(\lambda) = \lambda^{\ell}I + \lambda^{\ell-1}R_{\ell-1} + \cdots + \lambda R_1 + R_0$.

The matrix polynomial $R(\lambda)$ will be upper (quasi-) triangular and have the same eigenstructure as $P(\lambda)$. We remark that the structure of A and S can be exploited to compute $S^{-1}A^{\ell}X$ at a reduced cost in step 5, but this is outside the scope of this work.

We will show in Section 3 how to implement step 3 in a numerically stable manner when all the eigenvalues of $P(\lambda)$ have algebraic multiplicity not greater than n (the size of $P(\lambda)$). Matrices X that are used to implement step 4 are characterized in Section 2; a method to construct them explicitly is provided in Section 3. The quadratic case ($\ell = 2$) is fully examined in Section 4, where a stable way of implementing step 3 is given that works independently of the algebraic multiplicity of the eigenvalues of $P(\lambda)$.

To be slightly more general, we will also study how to construct matrices X to reduce $P(\lambda)$ to one of the following forms:

- block-diagonal form:

$$D(\lambda) = D_1(\lambda) \oplus D_2(\lambda) \oplus \cdots \oplus D_k(\lambda) \in \mathbb{F}[\lambda]^{n\times n} \tag{5}$$

  monic of degree ℓ with $D_i(\lambda) \in \mathbb{F}[\lambda]^{s_i\times s_i}$, $1 \le i \le k$ and $s_1 + \cdots + s_k = n$,
- block-triangular form:

$$T(\lambda) = \begin{bmatrix} T_{11}(\lambda) & T_{12}(\lambda) & \dots & T_{1k}(\lambda) \\ & T_{22}(\lambda) & & \vdots \\ & & \ddots & \vdots \\ & & & T_{kk}(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{n\times n}, \tag{6}$$

  monic of degree ℓ with $T_{jj}(\lambda) \in \mathbb{F}[\lambda]^{s_j\times s_j}$, $1 \le j \le k$ and $s_1 + \cdots + s_k = n$, and
- Hessenberg form:

$$H(\lambda) = \lambda^{\ell}I + \lambda^{\ell-1}H_{\ell-1} + \cdots + \lambda H_1 + H_0 \in \mathbb{F}[\lambda]^{n\times n}, \tag{7}$$

  with coefficient matrices H_i, $i = 0,\dots,\ell-1$ in Hessenberg form.

We will discuss in Section 5 how to use the simpler forms to solve parameterized linear systems $P(\omega)x = b(\omega)$, where x is to be computed for many values of the parameter ω.

2. Conditions for reduction to simpler forms. For matrices $A \in \mathbb{F}^{m\times m}$ and $V \in \mathbb{F}^{m\times j}$ we define the block Krylov matrix

$$K_\ell(A, V) = [V \ AV \ \cdots \ A^{\ell-1}V] \in \mathbb{F}^{m\times \ell j}$$

and the block Krylov subspace

$$\mathcal{K}_\ell(A, V) = \mathrm{range} K_\ell(A, V).$$

For a subspace $\mathcal{X}$ of $\mathbb{F}^m$ and a matrix A operating on that subspace we define $A\mathcal{X} = \{Ax : \ x \in \mathcal{X}\}$.

Assume that $P(\lambda)$ is given by (1) and let $\lambda I - A$ be any monic linearization of $P(\lambda)$, for example, the left companion linearization of $P_\ell^{-1}P(\lambda)$. Recall that we are looking for a matrix $X \in \mathbb{F}^{\ell n\times n}$ such that

(i) $S := K_\ell(A, X) = [X \ AX \ \cdots \ A^{\ell-1}X]$ is nonsingular, and

(ii) $\lambda I - S^{-1}AS$ is the left companion linearization of one of the reduced forms in (5)–(7).

If (i) holds, then $S^{-1}AS$ is the left companion matrix of a monic matrix polynomial, say $R(\lambda) = \lambda^\ell I + \cdots + \lambda R_1 + R_0$, and

$$S^{-1}A^\ell X = S^{-1}AS(e_n \otimes I_\ell) = -\begin{bmatrix} R_0 \\ R_1 \\ \vdots \\ R_{\ell-1} \end{bmatrix}, \tag{8}$$

where e_j denotes the jth column of the identity matrix and $\otimes$ denotes the Kronecker product. Then we see that the (i, j) entry of $R(\lambda)$ with $i \neq j$ is zero if and only if the vector $S^{-1}A^\ell Xe_j$ has zeros in the entries $i, i + n, \ldots, i + (\ell - 1)n$. This means that $S^{-1}A^\ell Xe_j$ is in the span of the columns of the submatrix of $I_{n\ell}$ obtained by deleting the columns $i, i + n, \ldots, i + (\ell - 1)n$. Thus, taking into account that $S^{-1}[X \ AX \ \cdots \ A^{\ell-1}X] = I$ and (8), it follows that

$$[R(\lambda)]_{ij} \equiv 0, \ i \neq j \quad \Longleftrightarrow \quad A^\ell x_j \in \mathcal{K}_\ell(A, [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]), \tag{9}$$

where x_j denotes the jth column of X.

We are now ready to state our main theorem, but before we do so we introduce some new notation. For the block reductions (5) and (6), it is convenient to partition X as

$$X = [X_1 \ X_2 \ \cdots \ X_k],$$

where $X_j \in \mathbb{F}^{\ell n\times s_j}$ and $s_1 + \cdots + s_k = n$. Also, we let $x_{1:i}$ and $X_{1:i}$ denote the matrices $[x_1 \ x_2 \ \cdots \ x_i] \in \mathbb{F}^{\ell n\times i}$ and $[X_1 \ X_2 \ \cdots \ X_i] \in \mathbb{F}^{\ell n\times \sigma_i}$, respectively, where $\sigma_i := s_1 + \cdots + s_i$. Finally, we define $\sigma_0 := 0$.

THEOREM 1. *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be of degree ℓ with nonsingular leading matrix coefficient and let $\lambda I - A$ be any monic linearization of $P(\lambda)$. Then $P(\lambda)$ is equivalent to a monic matrix polynomial $R(\lambda)$ of degree ℓ having one of the reduced forms (5)–(7) if and only if there exists a full rank matrix $X \in \mathbb{F}^{\ell n\times n}$ such that*

(i) *the matrix $[X \ AX \ \ldots \ A^{\ell-1}X] \in \mathbb{F}^{\ell n\times \ell n}$ is nonsingular, and*

(ii) (a) *$\mathcal{K}_\ell(A, X_i)$ for $1 \leq i \leq k$ is A-invariant for block-diagonal form as in (5),*

(b) *$\mathcal{K}_\ell(A, X_{1:i})$ for $1 \leq i \leq k$ is A-invariant for block-triangular form as in (6),*

(c) $A^\ell x_i \in \mathcal{K}_\ell(A, x_{1:i+1})$, $1 \le i \le n-1$, *for Hessenberg form as in* (7).

Proof. ($\Rightarrow$) Suppose that $P(\lambda)$ is equivalent to $R(\lambda)$. Then $\lambda I - A$ is also a monic linearization of $R(\lambda)$ and as explained in the introduction (recall equation (2)-(4)), there is a matrix X such that (A, X) is a left standard pair for $R(\lambda)$, which implies (i) and $AS = SC_L(R)$, where $S = [X\ AX\ \dots\ A^{\ell-1}X] = K_\ell(A, X)$.

(ii)(a): Suppose that $R(\lambda)$ has the block-diagonal form of $D(\lambda)$ in (5). Define a permutation matrix

$$\Pi_i = [e_{\sigma_{i-1}+1}\ \cdots\ e_{\sigma_i}\ e_{n+\sigma_{i-1}+1}\ \cdots\ e_{n+\sigma_i}\ \cdots\ e_{(\ell-1)n+\sigma_{i-1}+1}\ \cdots\ e_{(\ell-1)n+\sigma_i}],$$

where e_i is the ith column of $I_{\ell n}$. Then $S\Pi_i = K_\ell(A, X)\Pi_i = K_\ell(A, X_i)$ and $C_L(D)\Pi_i = \Pi_i C_L(D_i)$. It follows from $AS = SC_L(D)$ that

$$AK_\ell(A, X_i) = AS\Pi_i = SC_L(D)\Pi_i = S\Pi_i C_L(D_i) = K_\ell(A, X_i)C_L(D_i),$$

which proves (ii)(a).

(ii)(b): Suppose that $R(\lambda)$ has the block triangular form of $T(\lambda)$ in (6). Let

$$\Pi_{1:i} = [\, e_1 \quad \cdots \quad e_{\sigma_i} \quad e_{n+1} \quad \cdots \quad e_{n+\sigma_i} \quad \cdots \quad e_{(\ell-1)n+1} \quad \cdots \quad e_{(\ell-1)n+\sigma_i}\,].$$

Then $K_\ell(A, X_{1:i}) = K_\ell(A, X)\Pi_{1:i} = S\Pi_{1:i}$ and $C_L(T)\Pi_{1:i} = \Pi_{1:i}C_L(T_i)$, where $T_i(\lambda)$ is the leading $\sigma_i \times \sigma_i$ principal submatrix of $T(\lambda)$. Then from $AS = SC_L(T)$ we obtain

$$AK_\ell(A, X_{1:i}) = AS\Pi_{1:i} = SC_L(T)\Pi_{1:i} = S\Pi_{1:i}C_L T_i = K_\ell(A, X_{1:i})C_L(T_i).$$

(ii)(c): Suppose that $R(\lambda)$ has the Hessenberg form of $H(\lambda)$ in (6). From $AS = SC_L(H)$ and (9), we see that $A^\ell x_i$ lies in the span of $K_\ell(A, x_{1:i+1})$.

($\Leftarrow$) Suppose that there exists X such that $S = [X\ AX\ \cdots\ A^{\ell-1}X]$ is nonsingular. Then the matrix $S^{-1}AS$ is the left companion form of a monic matrix polynomial of degree ℓ, say $R(\lambda)$, equivalent to $P(\lambda)$.

Now, $AS = SC_L(R)$, (ii)(a) and (9) imply that the $n \times n$ blocks $R_0, \dots, R_{\ell-1}$ in the last block column of $C_L(R)$ (see (8)) are block-diagonal with k diagonal blocks, the ith diagonal block being $s_i \times s_i$, where s_i is the number of columns of X_i, $i = 1\colon k$. The proofs for (ii)(b) and (ii)(c) are similar. $\square$

3. Construction of the matrix X.

We discuss in this section a process to construct the matrix X in Theorem 1 such that properties (i) and (ii) hold.

3.1. Auxiliary results.

We start by proving some technical results that will be needed for the triangularization. Let $\lambda I - C_L$ be the left companion matrix of a monic matrix polynomial $P(\lambda)$ of size $n \times n$ and degree ℓ, and let Π denote the permutation matrix

$$\Pi = [\, \pi_1 \quad \pi_2 \quad \cdots \quad \pi_n \,], \qquad \pi_i = [\, e_i \quad e_{n+i} \quad \cdots \quad e_{(\ell-1)n+i} \,] \quad \text{for} \quad i = 1, \dots, n.$$

Then the permuted linearization $\lambda I - \Pi^T C_L \Pi$ is called the *left companion linearization of* $P(\lambda)$ *in controller form*. If we view this linearization as an $\ell \times \ell$ block pencil, then the zero-block structure of the pencil is the same as the zero structure of $P(\lambda)$. Furthermore, the diagonal $\ell \times \ell$ blocks are the companion matrices of the corresponding scalar polynomials on the diagonal of $P(\lambda)$. To illustrate the controller form, Figure 3

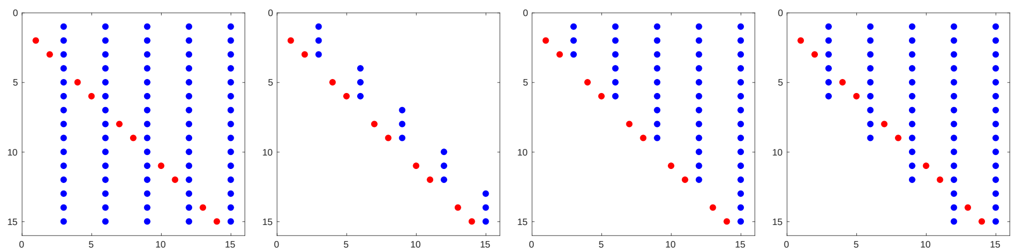

FIG. 3. *Spy plots for the controller form of the left companion matrices for cubic 5×5 matrix polynomials with (from left to right) dense, diagonal, triangular and Hessenberg matrix coefficients, respectively. The red dots are 1's, the blue dots are nonzero entries.*

shows the spy plots of the left companion matrix for $P(\lambda)$ in dense (no structure), diagonal, triangular and Hessenberg forms.

The controller form is useful in the proofs of the following theorems. In these theorems we will work with matrices having eigenvalues of geometric multiplicity at most n. The rationale behind this is that if $\lambda I - A$ is a linearization of an $n \times n$ matrix polynomial $P(\lambda)$ as in (1), then, by [3, Thm. 1.7], the geometric multiplicity of the eigenvalues of A cannot be greater than n.

3.1.1. Existence of Schur form for triangular reduction. Recall that a matrix is called *nonderogatory* if every eigenvalue has geometric multiplicity one.

THEOREM 2 (Schur form with nonderogatory blocks, complex version). *Let $A \in \mathbb{C}^{\ell n \times \ell n}$ be a matrix whose eigenvalues have geometric multiplicity at most n. Then A has a Schur decomposition*

$$A = Q \begin{bmatrix} T_{11} & * & * & * \\ & T_{22} & * & * \\ & & \ddots & * \\ & & & T_{nn} \end{bmatrix} Q^H,$$

where the diagonal blocks $T_{ii} \in \mathbb{C}^{\ell \times \ell}$, $i = 1, \ldots, n$, are nonderogatory.

Proof. Since A has no eigenvalue with geometric multiplicity greater than n, it follows from [3, Proof of Thm. 1.7] that $\lambda I - A$ is a linearization of an $n \times n$ upper triangular monic matrix polynomial $R(\lambda)$ of degree ℓ. This matrix polynomial has a left companion linearization in controller form, which itself must be monic. Denote this linearization by $\lambda I - B$. Then $A = SBS^{-1}$ for some nonsingular S. Furthermore, B is block upper triangular, with blocks of size $\ell \times \ell$, and all diagonal blocks must be nonderogatory (since they are companion matrices). Let $U_i T_i U_i^H$ be a Schur decomposition of the ith diagonal block and set $U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$. Then

$$B = UTU^H, \quad \text{with} \quad T = \begin{bmatrix} T_1 & * & * & * \\ & T_2 & * & * \\ & & \ddots & * \\ & & & T_n \end{bmatrix},$$

is a Schur decomposition. Finally, let $SU = QR$ be a QR factorization of SU and note that since R is upper triangular and nonsingular, $A = Q(RTR^{-1})Q^H$ is a Schur decomposition of A. The theorem follows from the fact that the ith diagonal $\ell \times \ell$ block of RTR^{-1} is similar to T_i. $\square$

We now prove the real analog of Theorem 2.

THEOREM 3 (Schur form with nonderogatory blocks, real version). *Let $A \in \mathbb{R}^{\ell n \times \ell n}$ be a matrix whose eigenvalues have geometric multiplicity at most n . Then A has a real Schur decomposition*

$$A = Q \begin{bmatrix} T_{11} & * & * & * \\ & T_{22} & * & * \\ & & \ddots & * \\ & & & T_{rr} \end{bmatrix} Q^T, \tag{10}$$

where each T_{ii} is either of size $\ell \times \ell$ and nonderogatory or of size $2\ell \times 2\ell$ and such that all eigenvalues have geometric multiplicity one or two.

Proof. Since all eigenvalues of A have geometric multiplicity at most n, it follows that $\lambda I - A$ has a real Smith form $D(\lambda) \oplus I_{(\ell-1)n}$ with $\deg\det D(\lambda) = n\ell$. By [14, Theorem 4.1] $D(\lambda)$ is equivalent to some real quasi-triangular matrix polynomial $T(\lambda)$ of degree ℓ, which may be assumed to be monic. It follows that

$$\lambda I - A \sim \begin{bmatrix} D(\lambda) & \\ & I_{(\ell-1)n} \end{bmatrix} \sim \begin{bmatrix} T(\lambda) & \\ & I_{(\ell-1)n} \end{bmatrix},$$

where $\sim$ denotes the equivalence relation for matrix polynomials. In other words, A is a linearization of some monic quasi-triangular matrix polynomial of degree ℓ. If B denotes the constant matrix of the left companion linearization of $T(\lambda)$ in controller form, then the rest of the proof is essentially the same as the last part of the proof of Theorem 2, with the only difference that we consider the real Schur decomposition instead of the complex one. $\square$

3.1.2. Numerically stable construction of a Schur form for triangular reduction. The above theorems are key stones in the process of constructing the matrix X of Theorem 1 for the (block-) triangular reduction. To be numerically useful we need to overcome the drawback that B is obtained from A via unimodular transformations. In what follows we propose a numerically stable procedure to construct the desired Schur form of A in Theorem 2 or Theorem 3 out of any of its Schur forms. This procedure works as long as all eigenvalues of A have algebraic multiplicity at most n. This will be our assumption.

First compute any (real or complex) Schur decomposition of A. Then reorder the diagonal entries/blocks using the procedure in Bai-Demmel [1] according to the rules described below. We discuss the real and complex case separately.

(I) Complex case. Suppose there are k distinct eigenvalues of algebraic multiplicity n and s distinct eigenvalues of algebraic multiplicity less than n. Note that $k \le \ell$ and $s = 0$ or $s > \ell - k$ according as $k = \ell$ or $k < \ell$, respectively. Reorder the Schur form such that the leading $k \times k$ principal submatrix has one instance of each eigenvalue of algebraic multiplicity n. If there are $k < \ell$ such eigenvalues, pick any $\ell - k (< s)$ distinct eigenvalues of algebraic multiplicity less than n and reorder the diagonal such that these appear after the first k eigenvalues that were deflated. The leading $\ell \times \ell$ submatrix obtained in this way has simple eigenvalues and is thus nonderogatory. By continuing inductively on the lower left $(n-1)\ell \times (n-1)\ell$ part of the matrix, we arrive at the desired Schur form.

(II) Real case. The procedure over $\mathbb{R}$ is more involved because we need to move nonreal eigenvalues in complex conjugate pairs in order to keep the decomposition real. In addition, $2\ell \times 2\ell$ diagonal blocks may appear with eigenvalues of geometric multiplicity two. An example that illustrates the main features of the procedure that follows is given in Example 5.

At this point it is important for us to recall that when applying the Bai-Demmel algorithm [1] to block triangular matrices of size 3×3 (one element and one 2×2 block in the diagonal) and 4×4 (two blocks of size 2×2 in the diagonal), the blocks of size 2×2 before and after applying the algorithm are similar but not necessarily identical. In what follows we will very often use sentences like "reordering" or "moving the diagonal blocks" to mean that consecutive diagonal elements and blocks are swapped to place them in desired diagonal positions.

Let us assume that matrix A has:

- k_r distinct real eigenvalues of algebraic multiplicity n.
- k_c distinct pairs of nonreal complex conjugate eigenvalues of algebraic multiplicity n.
- s distinct real eigenvalues of algebraic multiplicity less than n.
- q_i distinct pairs of nonreal complex conjugate eigenvalues of algebraic multiplicity $i < n$.

Thus, $k_r + 2k_c =: k \leq \ell$ and

$$n(\ell - k) \leq (n-1)s + 2q_1 + 4q_2 + \cdots + 2(n-1)q_{n-1}. \tag{11}$$

In particular, if $n = 2$, then $2(\ell - k) = s + 2q_1$. We claim that if $n \geq 3$, $\ell - k > 0$ and $q := q_1 + q_2 + \cdots + q_{n-1}$ then

$$s + 2q - q_1 > \ell - k. \tag{12}$$

In fact, $\ell - k > 0$ implies $s > 0$ or $q_i > 0$ for some i such that $1 \leq i \leq n-1$. So $(s+2q-q_1)n = (s+q_1+2q_2+\cdots+2q_{n-1})n > (n-1)s+2q_1+4q_2+\cdots 2(n-1)q_{n-1}$ because for $n \geq 3$,

$$s + (n-2)q_1 + (2n-4)q_2 + (2n-6)q_3 + \cdots + (2n-2n+2)q_{n-1} > 0.$$

Hence (12) follows from (11).

Start by reordering the Schur form such that one instance of each of the k_r real eigenvalues and one instance of the 2×2 blocks corresponding to the k_c pairs of nonreal complex conjugate eigenvalues of algebraic multiplicity n appear in the leading $k \times k$ principal submatrix. If $\ell - k = 0$, then the $\ell \times \ell$ leading submatrix is nonderogatory and we can continue the deflating process inductively on the $(n-1)\ell \times (n-1)\ell$ lower right part of the matrix. If $\ell - k > 0$, then we proceed as follows:

(i) If $\ell - k$ is even then let $k_1 = \min\{q, \frac{\ell-k}{2}\}$. Choose k_1 2×2 blocks corresponding to distinct nonreal complex conjugate eigenvalues of algebraic multiplicity less than n and move them so that they appear directly after the deflated $k \times k$ submatrix on the diagonal.

- (i_1) If $k_1 = \frac{\ell-k}{2}$ (that is, $\ell = k + 2k_1$) then the $\ell \times \ell$ leading submatrix is nonderogatory.
- (i_2) If $k_1 = q < \frac{\ell-k}{2}$ then $s > \ell - k - 2q = \ell - k - 2k_1$. This follows from (12) if $n \geq 3$ and from $2(\ell - k) = s + 2q$ if $n = 2$. Move (if necessary) $\ell - k - 2k_1$ distinct real eigenvalues of algebraic multiplicity less than n so that they appear after the k_1 2×2 blocks we just deflated. The $\ell \times \ell$ leading submatrix is nonderogatory.

Continue the deflating process with the $(n-1)\ell \times (n-1)\ell$ lower right part of the matrix as above.

(ii) If $\ell - k$ is odd and $s > 0$ then let $k_1 = \min\{q, \frac{\ell-k-1}{2}\}$. As in the case when $\ell - k$ is even, choose k_1 2×2 blocks corresponding to distinct nonreal complex conjugate eigenvalues of algebraic multiplicity less than n and move them so that they appear directly after the deflated $k \times k$ submatrix on the diagonal.

(ii₁) If $k_1 = q$ then as in case (i), $s > \ell - k - 2k_1$. Move (if necessary) $\ell - k - 2k_1$ distinct real eigenvalues of algebraic multiplicity less than n so that they appear after the k_1 2×2 blocks we just deflated and the resulting $\ell \times \ell$ principal submatrix is nonderogatory.

(ii₂) If $k_1 = \frac{\ell-k-1}{2} < q$ then $\ell = k + 2k_1 + 1$. Since $s > 0$, one real eigenvalue of algebraic multiplicity less than n can be placed after the k_1 2×2 blocks we just deflated so that the $\ell \times \ell$ principal submatrix is nonderogatory.

Continue the deflating process as above.

(iii) If $\ell - k$ is odd and $s = 0$ we aim to form a $2\ell \times 2\ell$ block with eigenvalues of geometric multiplicity at most two. Recall that we already have one instance of each of the k_r real eigenvalues and one instance of the 2×2 blocks corresponding to the k_c pairs of nonreal complex conjugate eigenvalues of algebraic multiplicity $n \geq 2$ in the leading $k \times k$ principal submatrix. Next, move another instance of each of the k_r real eigenvalues and another instance of the 2×2 blocks corresponding to the k_c pairs of nonreal complex conjugate eigenvalues of algebraic multiplicity n, so that they appear just after the $k \times k$ principal submatrix that we just deflated. These eigenvalues may have geometric multiplicity two in the resulting $2k \times 2k$ submatrix.

(iii₁) If $n = 2$, recall that $2(\ell - k) = 2q_1 + s = 2q_1$. This means that there are q_1 2×2 diagonal blocks corresponding to pairs of nonreal complex conjugate eigenvalues of algebraic multiplicity one. Reorder (if necessary) the diagonal blocks so that they appear just after the $2k \times 2k$ principal submatrix that we just deflated. The $2\ell \times 2\ell$ resulting submatrix has all its eigenvalues of geometric multiplicity 2 at the most.

(iii₂) If $n \geq 3$ then by (12), $(\ell - k) < 2q - q_1 + s = 2q - q_1$. This inequality allows us to proceed as follows: Let $k_1 = \min\{q, \ell - k\}$ and select k_1 2×2 blocks corresponding to distinct nonreal complex conjugate eigenvalues of algebraic multiplicity less than n and move them so that they appear directly after the deflated $2k \times 2k$ submatrix. If $k_1 = \ell - k$, then all the eigenvalues of the $2\ell \times 2\ell$ obtained submatrix have geometric multiplicity one or two. Now, if $k_1 = q < \ell - k$ then all nonreal complex conjugate eigenvalues of algebraic multiplicity one have been used. We have to use another instance of nonreal complex conjugate eigenvalues of algebraic multiplicity bigger than one and smaller than n. Since $(\ell - k) < 2q - q_1$ we have that $(\ell - k - q) < q - q_1$. So we have enough 2×2 diagonal blocks associated to distinct nonreal complex conjugate eigenvalues of algebraic multiplicities between 2 and $n-1$ to move to the leading $2\ell \times 2\ell$ principal submatrix in such a way that its eigenvalues have geometric multiplicity at most two.

Continue inductively the deflating process on the $(n-2)\ell \times (n-2)\ell$ lower right part of the matrix.

We note that when $\ell - k$ is odd and $s = 0$ (case *(iii)*), the eigenvalues of the constructed $2\ell \times 2\ell$ block may have all geometric multiplicity one. If this is the case, it may or may not be possible to further move the eigenvalues along the diagonal of the $2\ell \times 2\ell$ block to produce two $\ell \times \ell$ nonderogatory blocks. The following example illustrates the two possibilities.

```
A = [ 1 0 0 1; 0 0 -1 0; 0 1 0 1; 0 0 0 1];
E = ordeig(A);
[Q, T] = ordschur(eye(4),A,imag(E)==0);
```

FIG. 4. *MATLAB M-file that implements Bai-Demmel algorithm to swap diagonal block* $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ *and diagonal element* 1 *in position* $(4,4)$ *of the real Schur matrix* A *of Example 4.*

EXAMPLE 4. (a) Let $n=2$, $\ell=2$ and

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix is in real Schur form. The MATLAB code in Figure 4 implements Bai-Demmel algorithm [1] to swap the diagonal block $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and the last diagonal element of A. The returned matrices Q and T are

$$Q = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & -0.4082 & 0.7071 & 0.5774 \\ 0 & 0.4082 & 0.7071 & -0.5774 \\ 0 & 0.8165 & -0.000 & 0.5774 \end{bmatrix}, \ T = \left[\begin{array}{rr|rr} 1.0000 & 0.8165 & -0.0000 & 0.5774 \\ 0 & 1.0000 & 0.5774 & 0.7071 \\ \hline 0 & 0 & -0.0000 & 1.2247 \\ 0 & 0 & -0.8165 & 0.0000 \end{array}\right].$$

The 2×2 submatrix in the lower right corner of T is (approximately) similar to $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and T is another (approximate) real Schur form of A with two nonderogatory diagonal blocks. Notice that if we replace $a_{14}=1$ by $a_{14}=0$ in A then the eigenvalue 1 of the new matrix would have geometric multiplicity two and there would not be a Schur form of A with two nonderogatory blocks of size 2×2 in the diagonal.

(b) Let $n=2$, $\ell=3$ and

$$A = \operatorname{diag}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \right).$$

Despite the eigenvalues of A being simple, there is no real Schur form of A with two nonderogatory diagonal blocks of size 3×3. If we apply to A the procedure of item **(II)** then $k_r = k_c = k = 0$, $q_1 = q = 3$ and $s=0$. Since $\ell - k = 3$ is odd and $s=0$, we use item (iii_1). In fact, $2(\ell-k) = 2\ell = 6 = 2q_1$ and we must put together three diagonal blocks of size 2×2. This means that A is itself the desired matrix.

The following example clarifies the main features of the procedure for the real case (item **(II)**) to bring a matrix in real Schur form to another one satisfying the requirements in Theorem 3.

EXAMPLE 5. Let $n=4$, $\ell=2$ and let $A\in\mathbb{R}^{8\times 8}$ be a matrix in real Schur form with the following diagonal blocks:

$$B = \operatorname{diag}\left(1, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, 1, 2, 1\right).$$

We can write $A = B + T$ where T is a strict block-upper triangular matrix (block-upper triangular with zero blocks in the diagonal). We are going to apply to A the procedure of item **(II)** to find an orthogonal matrix Q such that $Q^T A Q$ is a real Schur form satisfying the requirements in Theorem 3.

<u>**Step 1**</u>: For A we have: $k_r = k_c = 0$, $s = 2$ and $q_1 = 1$. Then $k = k_r + k_c = 0$ and $\ell - k = 2$. Thus $\ell - k$ is even and $k_1 = \min\left\{q, \frac{\ell-k}{2}\right\} = \frac{\ell-k}{2} = 1$. We use (i_1): use Bai-Demmel algorithm to move the first $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ block to place it in the upper-left corner: there is an orthogonal matrix Q_1 such that $A_1 = Q_1^T A Q = B_1 + T_1$ with T_1 strict block-upper triangular and $B_1 = \operatorname{diag}\left(B_{11}, 1, \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], 1, 2, 1\right)$, where B_{11} is similar to $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

We deflate the 2×2 first rows and columns of A_1 and work with the lower right 6×6 matrix $\widehat{A}_1 = \widehat{B}_1 + \widehat{T}_1$ where $\widehat{B}_1 = \operatorname{diag}\left(1, \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], 1, 2, 1\right)$. Now $n = 3$ and $\ell = 2$.

<u>**Step 2**</u>: For $\widehat{A}_1$ we have: $k_r = 1$, $k_c = 0$, $s = 1$ and $q_1 = 1$. Since $k_r = 1$ for eigenvalue 1, first of all, we must move it to position $(1, 1)$. In this case no action is needed because it is already there. Now, $k = k_r + k_c = 1$, $\ell - k = 1$ is odd, $s > 0$ and $k_1 = \min\left\{q, \frac{\ell-k-1}{2}\right\} = \min\{1, 0\} = 0$. Hence we use (ii_2): move a real eigenvalue of multiplicity less than $n = 3$ to position $(2, 2)$. There is only one choice: use Bai-Demmel algorithm to exchange the block $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and the entry in position $(5, 5)$ (actually, we must swap first diagonal entries 1 and 2 and then swap 2 and block $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$). So, there is an orthogonal matrix $\widehat{Q}_2$ such that $\widehat{A}_2 = \widehat{Q}_2^T \widehat{A}_1 \widehat{Q}_2 = \widehat{B}_2 + \widehat{T}_2$ with $\widehat{T}_2$ strict block- upper triangular and $\widehat{B}_2 = \operatorname{diag}(1, 2, B_{21}, 1, 1)$, where B_{21} is similar to $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

We deflate again the 2×2 first rows and columns of $\widehat{A}_2$ and pay attention to $\widetilde{A}_2 = \widetilde{B}_2 + \widetilde{T}_2$ with $\widetilde{B}_2 = \operatorname{diag}(B_{21}, 1, 1)$. Now $n = 2$ and $\ell = 2$.

<u>**Step 3**</u>: For $\widetilde{A}_2$ we have: $k_r = 1$, $k_c = 0$, $s = 0$ and $q_1 = 1$. Again $k_r = 1$ and we must place the eigenvalue of algebraic multiplicity 2 in position $(1, 1)$. We use Bai-Demmel algorithm to swap the diagonal block B_{21} and the diagonal entry $(3, 3)$. Let $\widehat{B}_{21}$ the resulting 2×2 block.

Now $k = k_r + k_c = 1$, $\ell - k = 1$ is odd and $s = 0$ and so, case (iii) applies: move another copy of the eigenvalues of algebraic multiplicity $n = 2$ to position $(2, 2)$. We use Bai-Demmel algorithm to exchange the diagonal block $\widehat{B}_{21}$ and the entry in position $(4, 4)$. Let B_{21} be the obtained block. We observe that $n = 2$ and $2(\ell - k) = 2 = 2q_1$. So we proceed as indicated in item (iii_1): move the block B_{21} to place it right after the two repeated eigenvalues to get a diagonal block of size 4. In this case, no action is needed. Thus there is an orthogonal $\widetilde{Q}_3$ such that $\widetilde{A}_3 = \widetilde{Q}_3^T \widetilde{A}_2 \widetilde{Q}_3 = \widetilde{B}_3 + \widetilde{T}_3$ with $\widetilde{T}_3$ strict block-upper triangular, $\widetilde{B}_3 = \operatorname{diag}(1, 1, B_{21})$ and B_{21} similar to $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

If we define $Q = Q_1 \operatorname{diag}(I_2, \widehat{Q}_2) \operatorname{diag}(I_4, \widetilde{Q}_3)$ then Q is an orthogonal matrix and

$$Q^T A Q = \left[\begin{array}{c|cc|ccc}
B_{11} & * & * & * & * & * \\ \hline
 & 1 & * & * & * & * \\
 & & 2 & * & * & * \\ \hline
 & & & 1 & * & * \\
 & & & & 1 & * \\ \hline
 & & & & & B_{21}
\end{array}\right]$$

is a matrix in real Schur form. Blocks B_{11} and B_{21} are both similar to $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. The 4×4 block in the lower-right corner will be nonderogatory if its $(1, 2)$ entry is not zero; otherwise, the geometric multiplicity of 1 in that block would be 2.

Matrix A in Example 5 has repeated eigenvalues but even in the generic case of real matrices with simple eigenvalues, the diagonal blocks of a computed real Schur form might need to be rearranged in order to satisfy the requirements of Theorem 3. In addition, as part (b) of Example 4 shows, the diagonal blocks in the Schur form of Theorem 3 for matrices with simple eigenvalues may need to be of size $2\ell \times 2\ell$.

If one eigenvalue has algebraic multiplicity greater than n, the problem of computing the desired Schur forms in a stable manner, using unitary/orthogonal transformations, becomes significantly more complicated. We devote Section 4 to this problem for quadratic matrix polynomials ($\ell = 2$). The higher-degree case $\ell > 2$ is left as an open problem. There is a process to obtain a desired form by manipulating Jordan forms, but we omit the details as it is an unstable process.

3.1.3. Sufficient conditions for nonsingular $K_\ell(A, X)$. Theorem 2 and Theorem 3 will be used in combination with the following lemmas. They show a nice connection with the following known result in the theory of linear control systems: the minimum number of inputs needed to control a linear time-invariant system is the geometric multiplicity of the eigenvalues with highest geometric multiplicity (see for example [17]). Explicit expressions for the needed input controls are provided in the proofs that follow.

LEMMA 6. *If $B \in \mathbb{F}^{\ell\times\ell}$ is nonderogatory, then there exists $x \in \mathbb{F}^\ell$ such that the Krylov matrix $K_\ell(B, x)$ is nonsingular.*

Proof. Since B is nonderogatory it is similar to the left companion matrix C_L of its characteristic polynomial [6, Thm. 3.3.15], that is, $B = SC_LS^{-1}$ for some nonsingular matrix S. It is now easy to see that $K_\ell(C_L, e_1) = I$. Hence letting $x = Se_1$ yields the desired result. ∎

The next lemma is the real counterpart of Lemma 6.

LEMMA 7. *Let $B \in \mathbb{R}^{2\ell\times 2\ell}$ have eigenvalues with geometric multiplicity at most two. Then there exist two real vectors x and y such that $K_\ell(B, [x\ y])$ is nonsingular.*

Proof. We can rearrange the real Jordan decomposition [16, Sec. 2.4] of B so that

$$S^{-1}BS = \begin{array}{c} \\ {\scriptstyle m_1} \\ {\scriptstyle m_2} \end{array}\!\!\begin{array}{c} \begin{array}{cc} {\scriptstyle m_1} & {\scriptstyle m_2} \end{array} \\ \begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix} \end{array} \in \mathbb{R}^{2\ell\times 2\ell}, \quad m_1 \geq m_2 \geq 0$$

with J_1 and J_2 nonderogatory. Note that matrix B is allowed to be nonderogatory: in this case $S^{-1}BS = J_1$, $m_2 = 0$ and J_2 is empty. Since J_1 and J_2 are nonderogatory matrices, they are similar (via real arithmetic) to the left companion matrices $C_1 \in \mathbb{R}^{m_1\times m_1}$ and $C_2 \in \mathbb{R}^{m_2\times m_2}$ of their characteristic polynomials, respectively. Hence there exists a nonsingular $W \in \mathbb{R}^{2\ell\times 2\ell}$ such that

$$W^{-1}BW = C_1 \oplus C_2 =: C.$$

If $m_2 = 0$ then $C := C_1$. It suffices to prove that there exist $u, v \in \mathbb{R}^{2\ell}$ such that $M = [K_\ell(C, u)\ K_\ell(C, v)]$ is nonsingular because we then get the desired result by taking $x = Wu$ and $y = Wv$.

If $m_1 = m_2$ or $m_2 = 0$ then $u = e_1$ and $v = e_{\ell+1}$ yield $M = I_{2\ell}$ and we are done. If $m_1 > m_2 > 0$, we let $u = e_1$ and $v = e_{\ell-m_2+1} + e_{m_1+1}$. Then direct calculations show that

$$M = \begin{array}{c} \\ {\scriptstyle \ell-m_2} \\ {\scriptstyle m_2} \\ {\scriptstyle \ell-m_2} \\ {\scriptstyle m_2} \end{array}\!\!\begin{array}{c} \begin{array}{cccc} {\scriptstyle \ell-m_2} & {\scriptstyle m_2} & {\scriptstyle m_2} & {\scriptstyle \ell-m_2} \end{array} \\ \begin{bmatrix} I & & & \\ & I & I & \\ & & & I \\ & & I & * \end{bmatrix} \end{array},$$

where $*$ is some irrelevant $m_2 \times (\ell - m_2)$ matrix. It is now easy to see that M has full column rank, and thus is nonsingular. ∎

Finally we provide a lemma that can be seen as a block generalization of Lemma 6 and Lemma 7.

Lemma 8. *If all eigenvalues of $A \in \mathbb{F}^{k\ell\times k\ell}$ have geometric multiplicity at most k, then there exists $X \in \mathbb{F}^{k\ell\times k}$ such that $K_\ell(A,X)$ is nonsingular.*

Proof. We will handle the real and complex case simultaneously. Let $A = ZTZ^{-1}$ be the decomposition from Theorem 2 or Theorem 3 and denote the diagonal blocks by T_{ii}, $i = 1\colon r$. For each T_{ii} we define W_i in the following way. If T_{ii} is of size $\ell\times\ell$ take W_i to be the $\ell\times 1$ vector in Lemma 6 such that $K_\ell(T_{ii}, W_i)$ is nonsingular, and if T_{ii} is of size $2\ell\times 2\ell$ take W_i to be the $2\ell\times 2$ matrix whose columns are the two real vectors in Lemma 7. Letting $W = W_1 \oplus W_2 \oplus \cdots \oplus W_r$ and $X = ZW$ yields $K_\ell(A,X) = ZK_\ell(T,W)$, which is of full rank. ∎

3.2. Reduced forms. For a given matrix polynomial with nonsingular leading matrix coefficient and monic linearization $\lambda I - A$, we now discuss how to construct a matrix X such that properties (i) and (ii) in Theorem 1 hold.

3.2.1. Block-triangular form. For the reduction to (block) triangular form we have the following result.

Proposition 9. *Let $s_1,\dots,\ s_k$ be positive integers such that $s_1+\cdots+s_k = n$ and let*

$$T = \begin{bmatrix} T_{11} & * & * & * \\ & T_{22} & * & * \\ & & \ddots & * \\ & & & T_{kk} \end{bmatrix} \in \mathbb{F}^{\ell n\times\ell n},$$

where $T_{ii} \in \mathbb{F}^{\ell s_i\times\ell s_i}$ has no eigenvalues of geometric multiplicity more than s_i for $i = 1,\dots,k$. If $A \in \mathbb{F}^{\ell n\times\ell n}$ is similar to T then there exists $X = [X_1\ X_2\ \cdots\ X_k]$ with $X_i \in \mathbb{F}^{n\ell\times s_i}$ such that $S = K_\ell(A,X)$ is nonsingular and $\mathcal{K}_\ell(A, X_{1:i})$ is A-invariant for $i = 1,\dots,k$.

Proof. By Lemma 8, we can for each T_{ii} pick a $V_i \in \mathbb{F}^{\ell s_i\times s_i}$ such that $K_\ell(T_{ii}, V_i)$ is nonsingular. Thus, if $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$, then $K_\ell(T,V)$ is nonsingular. Let Z be a nonsingular matrix such that $Z^{-1}AZ = T$ and put $X = ZV$. Then $S = K_\ell(A,X) = ZK_\ell(T,V)$ is nonsingular. In addition, if $\sigma_i = s_1+\cdots+s_i$ and

$$W_i = \begin{bmatrix} V_1\oplus\cdots\oplus V_i \\ 0 \end{bmatrix} \in \mathbb{F}^{\ell n\times\sigma_i}, \quad i = 1,\dots,k,$$

then $\mathcal{K}_\ell(A, X_{1:i})$ is A-invariant if and only if $\mathcal{K}_\ell(T, W_i)$ is T-invariant. Since the columns of T^jW_i are also columns of $K_\ell(T,W_i)$ for $j < \ell$, we only have to show that there is a matrix R such that $T^\ell W_i = K_\ell(T,W_i)R$. If T_i is the submatrix of T formed by its $\ell\sigma_i$ first rows and columns and $\widehat{V}_i = V_1\oplus\cdots\oplus V_i$ then

$$K_\ell(T, W_i) = \begin{bmatrix} K_\ell(T_i, \widehat{V}_i) \\ 0 \end{bmatrix}.$$

Since $K_\ell(T_i, \widehat{V}_i)$ is nonsingular, there is a matrix $R \in \mathbb{F}^{\ell n\times\sigma_i}$ such that

$$T^\ell W_i = \begin{bmatrix} T_i^\ell \widehat{V}_i \\ 0 \end{bmatrix} = \begin{bmatrix} K_\ell(T_i, \widehat{V}_i)R \\ 0 \end{bmatrix} = K_\ell(T, W_i)R,$$

as desired. ∎

REMARK 10. The proof of Proposition 9 provides a practical means to construct X. From the proof we see that the columns of $K_\ell(A, X_{1:i})$ must be a basis for the invariant subspace of A corresponding to the eigenvalues of $T_{11}, T_{22}, \ldots, T_{ii}$.

We now explain why the MATLAB M-file in Figure 1 successfully reduced $P(\lambda)$ into triangular form (see the left plot of Figure 2). Since the coefficients are generated randomly, the eigenvalues are all distinct with probability one. Therefore MATLAB's `schur` function computes a Schur decomposition $C_P = ZTZ^H$, where $Z^H = Z^{-1}$ and the $\ell \times \ell$ diagonal blocks are all nonderogatory. Thus each $K_\ell(T_{ii}, V_i) \in \mathbb{R}^{\ell\times\ell}$ becomes nonsingular by taking each $V_i \in \mathbb{F}^{\ell\times 1}$ to be a vector of ones (almost any random vector would do). Hence, $X := Z(V_1 \oplus V_2 \oplus \cdots \oplus V_k)$ is as in Proposition 9, and so the conditions (i) and (ii)(b) in Theorem 1 are fulfilled.

3.2.2. Block-diagonal form. For the reduction to block-diagonal form we have the following result.

PROPOSITION 11. *Let $A \in \mathbb{F}^{\ell n\times \ell n}$ and assume that for some nonsingular Z*

$$Z^{-1}AZ = \begin{bmatrix} D_{11} & & & \\ & D_{22} & & \\ & & \ddots & \\ & & & D_{kk} \end{bmatrix} \in \mathbb{F}^{\ell n\times \ell n}, \tag{13}$$

where $D_{ii} \in \mathbb{F}^{s_i\ell\times s_i\ell}$ has eigenvalues of geometric multiplicity at most $s_i \in \mathbb{N}$, $i = 1, \ldots, k$ with $s_1 + \cdots + s_k = n$. Then there exists $X = [X_1\ X_2\ \ldots\ X_k]$ with $X_i \in \mathbb{F}^{n\ell\times s_i}$ such that $S = K_\ell(A, X)$ is nonsingular and $\mathcal{K}_\ell(A, X_i)$ is A-invariant for $i = 1, \ldots, k$.

The proof is similar to that of Proposition 9 and is omitted. We have the following analog to Remark 10.

REMARK 12. With the notation of Proposition 11, the columns of $K_\ell(A, X_i)$ are a basis for the invariant subspace of A corresponding to the eigenvalues of D_{ii}.

We explain now how the diagonalization corresponding to the middle plot of Figure 2 was achieved. The eigenvalues are again all distinct (with probability one), and the `eig` function computes Λ, Z such that $C_P = Z\Lambda Z^{-1}$ is an eigenvalue decomposition with Λ diagonal. Thus by taking $V_i \in \mathbb{F}^{\ell\times 1}$ to be vectors of ones and letting $X := Z(V_1 \oplus V_2 \oplus \cdots \oplus V_k)$, the conditions (i), (ii)(a) in Theorem 1 are satisfied.

Clearly the number of blocks in the decomposition (13) of Proposition 11 is not arbitrary. Indeed, the linear matrix polynomial $\lambda I - J_\alpha$, where

$$J_\alpha = \begin{bmatrix} \alpha & 1 & & & \\ & \alpha & 1 & & \\ & & \alpha & \ddots & \\ & & & \ddots & 1 \\ & & & & \alpha \end{bmatrix}$$

is of size $2\ell \times 2\ell$, cannot be reduced to a block diagonal structure with smaller block sizes. Further, since $\lambda I - J_\alpha$ is a linearization of

$$P(\lambda) = \begin{bmatrix} (\lambda-\alpha)^\ell & 1 \\ & (\lambda-\alpha)^\ell \end{bmatrix},$$

it may also be the case that for matrix polynomials of degree $\ell > 1$, the block sizes of a block-diagonal form cannot be reduced.

Let $\lambda I - A$ be a linearization of $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ in (1). From Theorem 1 and Proposition 11, we see that a reduction to diagonal form is possible if we can partition the Jordan blocks associated with A into n sets such that

(a) each set has at most one Jordan block of each eigenvalue, and

(b) the sizes of all Jordan blocks in each set sum up to ℓ.

The result also holds in the opposite direction, that is, it is possible to reduce $P(\lambda)$ to diagonal form, *only if* we can partition the Jordan blocks of A such that (a) and (b) hold. To see this, we simply note that any diagonal monic matrix polynomial $D(\lambda) = d_1(\lambda) \oplus d_2(\lambda) \oplus \cdots \oplus d_n(\lambda)$ has left companion linearization in controller form:

$$\lambda I - (C_L(d_1) \oplus C_L(d_2) \oplus \cdots \oplus C_L(d_n)).$$

The following question arises: When is it possible to partition the Jordan blocks such that (a) and (b) are satisfied? This problem was solved by Lancaster and Zaballa [8] for the special case of quadratic matrix polynomials with nonsingular leading matrix coefficient, and by Zúñiga Anaya [18] for general regular quadratics. For matrix polynomials of higher degree the problem is still open.

3.2.3. Hessenberg form. For the reduction to Hessenberg form we have the following result.

PROPOSITION 13. *Let $A \in \mathbb{F}^{\ell n\times \ell n}$ and let Z be a nonsingular matrix such that*

$$Z^{-1}AZ = H, \tag{14}$$

where H is upper Hessenberg and partitioned in $\ell\times\ell$ blocks. Assume that the $\ell\times\ell$ diagonal blocks are unreduced, that is, $H_{i+1,i} \neq 0$ for all i. If we let $V = [e_1\ e_{\ell+1}\ \cdots\ e_{(n-1)\ell+1}]$ and $X = ZV \in \mathbb{F}^{\ell n\times n}$ then $K_\ell(A, X)$ is nonsingular and $A^\ell x_i \in \mathcal{K}_\ell(A, x_{1:i+1})$ for $i = 1, \ldots, n-1$.

Proof. We have $K_\ell(A, X) = ZK_\ell(H, V)$, which is obviously nonsingular. Furthermore, if v_i and x_i are the ith columns of V and X, respectively, then

$$A^\ell x_i = ZH^\ell v_i = ZH^\ell e_{(i-1)\ell+1} \in Z\mathcal{K}_\ell(H, v_{1:\,i+1}) = \mathcal{K}_\ell(A, x_{1:\,i+1}),$$

completing the proof. ∎

In practice we are interested in Hessenberg decompositions $A = UHU^H$, where U is unitary or real orthogonal, depending on whether we work over $\mathbb{C}$ or $\mathbb{R}$. By the implicit Q-theorem [4, Thm. 7.4.2], the Hessenberg matrix H is uniquely defined, up to products by real or complex numbers of absolute value 1, by the first column of U. Hence a random Hessenberg matrix similar to A via unitary/real orthogonal transformations, can be constructed using, e.g., the Arnoldi algorithm with a random starting vector (or equivalently, the standard Hessenberg reduction step [4, Sec. 7.4.2] applied to QAQ^H, where Q is a random orthogonal matrix). If a matrix has distinct eigenvalues, the resulting Hessenberg matrix will be unreduced with probability one. Since this is the generic case for matrix polynomials, Proposition 11 may be used to reduce almost all matrix polynomials to Hessenberg form, without further care. This is how the right plot of Figure 2 was obtained.

If a matrix, on the other hand, has an eigenvalue of geometric multiplicity greater than one, then any similar Hessenberg matrix is necessarily reduced. Now, according to Proposition 13 the reduction of $P(\lambda)$ to Hessenberg form is still valid if H is reduced, as long as the diagonal $\ell\times\ell$ blocks are unreduced. This means that all zeros on the subdiagonal are in some of the positions $(\ell+1,\ell),(2\ell+1,2\ell),\ldots,((n-1)\ell+1,(n-1)\ell)$. If H has a zero in any other position on the subdiagonal (that is, if some Hessenberg diagonal block is reduced), $K_\ell(A,X)$ becomes singular and the reduction will fail with the matrix X selected in the statement of Proposition 13. This raises the following question: Is it possible to move zeros on the subdiagonal, from unwanted to wanted positions, using a finite number of Givens rotations or Householder reflectors? Intuitively, this should not be possible since if moving a zero is possible then we can change the number of "deflated" eigenvalues; a rigorous argument can be found in [13, pp. 104–105].

Proposition 14. *Matrices X of Propositions 9 and 13 can be taken to have orthonormal columns.*

Proof. Let X be the matrix constructed in the proof of either Proposition 9 or that of Proposition 13. Let $X = QR$ be a QR factorization of X. Since X has full column rank, R is nonsingular and $Q = XR^{-1}$. Thus

$$K_\ell(A,X) = K_\ell(A,Q)(R\oplus R\oplus\cdots\oplus R),$$

and so $K_\ell(A,Q)$ is nonsingular,

Now, write $Q = [\,Q_1 \quad Q_2 \quad \cdots \quad Q_k\,]$ with $Q_i \in \mathbb{F}^{\ell n\times s_i}$ and $s_1+\cdots+s_k = n$. Define $Q_{1:i} = [\,Q_1 \quad \cdots \quad Q_i\,] \in \mathbb{F}^{\ell n\times \sigma_i}$ with $\sigma_i = s_1+\cdots+s_i$ and let R_i denote the $i\times i$ upper left principal submatrix of R. Then R_i is nonsingular, $X_{1:i} = Q_{1:i}R_{\sigma_i}$ and

$$K_\ell(A,X_{1:i}) = K_\ell(A,Q_{1:i})(R_{\sigma_i}\oplus R_{\sigma_i}\oplus\cdots\oplus R_{\sigma_i}).$$

Therefore, $\mathcal{K}_\ell(A,X_{1:i}) = \mathcal{K}_\ell(A,Q_{1:i})$, $i = 1,\ldots,k$, and so if X is the matrix of Proposition 9 then $\mathcal{K}_\ell(A,Q_{1:i})$ is A-invariant.

Similarly, the ith column of X and Q are respectively $x_i = q_{1:i}r_i$ and $q_i = x_{1:i}u_i$, where r_i and u_i are the last columns of R_i and R_i^{-1}, respectively. A simple induction argument shows that $A^\ell x_i \in \mathcal{K}_\ell(A,x_{1:i+1})$ if and only if $A^\ell q_i \in \mathcal{K}_\ell(A,q_{1:i+1})$. ∎

In practice, using a matrix X with orthonormal columns to construct $S = K_\ell(A,X)$ may result in a more reliable way of computing $S^{-1}AS$ to obtain the left companion matrix of a block-triangular or Hessenberg matrix polynomial equivalent to $P(\lambda)$. We note that while we have discussed (when possible) how to compute X in a stable manner, finding the reduced matrix polynomial $R(\lambda)$ appears to require futher computing $S^{-1}AS$. We leave the stable computation of $R(\lambda)$ as an open problem.

4. Stable computation of a special Schur form for quadratic matrix polynomials. In this section we show that the Schur decompositions in Theorem 2 and Theorem 3 can always be computed in a numerically stable manner when $\ell = 2$. We collect key tools in lemmas, which all have algorithmic proofs. Examples are provided at the end of the section. We discuss the complex and real cases in different subsections. Recall that the eigenvalues of any linearization of $P(\lambda)$ in (1) have geometric multiplicity at most n [3, Thm. 1.7].

4.1. The complex case. We start by proving two lemmas. Here and below MATLAB notation is used, in which $X(i_1\colon i_2, j_1\colon j_2)$ denotes the submatrix of X formed with the i_1 through i_2 rows and the j_1 through j_2 columns.

Lemma 15. *Assume that $A \in \mathbb{C}^{2n\times 2n}$ has at least two distinct eigenvalues α and β with α of geometric multiplicity at most n. Then there exists a Schur form of A, $A = UTU^H$, such that*

(i) $T(1\colon 2, 1\colon 2) = \begin{bmatrix} \beta & * \\ 0 & \alpha \end{bmatrix}$.

(ii) *α is an eigenvalue of $T(3\colon 2n, 3\colon 2n)$ with geometric multiplicity at most $n-1$.*

Proof. Let $\widehat{T}$ be a Schur form of A. By using, if necessary, the Bai-Demmel algorithm [1] we can assume that the blocks in the diagonal of $\widehat{T}$ are so that $\widehat{T}(1\colon 2, 1\colon 2)$ is as in (i). Then the condition (ii) necessarily holds if the geometric multiplicity of α as eigenvalue of A is less than n. Hence below we suppose that it is equal to n.

Let $m_1 \geq m_2 \geq \cdots \geq m_n$ be the partial multiplicities of α as eigenvalue of A (that is, the sizes of the Jordan blocks) and $s = m_1 + \cdots + m_n$. Since A has at least two distinct eigenvalue, $n \leq s < 2n$ and so $m_n = 1$. We aim to detect one eigenvalue α in the diagonal of $\widehat{T}$ associated with a Jordan block of size 1.

We use again the Bai-Demmel algorithm [1] to reorder the diagonal of $\widehat{T}(2\colon 2n, 2\colon 2n)$ so that in the new matrix T_0 the s eigenvalues α appear in the submatrix $T_1 = T_0(2\colon s+1, 2\colon s+1)$. Observe that T_0 is still a Schur form of A.

Thus α is the only eigenvalue of T_1 and its geometric multiplicity is n. Let $Q_1 \in \mathbb{C}^{s\times n}$ be a matrix whose columns are an orthonormal basis of $\operatorname{Ker}(T_1 - \alpha I_s)$ and complete Q_1 up to a unitary matrix $\widehat{Q} = [Q_1\ \widehat{Q}_1] \in \mathbb{C}^{s\times s}$. Then we have

$$\widehat{Q}^H T_1 \widehat{Q} = \begin{bmatrix} \alpha I_n & C_1 \\ 0 & B \end{bmatrix} \tag{15}$$

for some B, C_1. If $Q_2^H B Q_2 = T_B$ is a Schur decomposition of B and

$$Q = \begin{bmatrix} 1 & & \\ & \widehat{Q} & \\ & & I_{2n-s-1} \end{bmatrix} \begin{bmatrix} I_{n+1} & & \\ & Q_2 & \\ & & I_{2n-s-1} \end{bmatrix}$$

then

$$T_2 = Q^H T_0 Q = \begin{bmatrix} \beta & * & * & * \\ & \alpha I_n & C & * \\ & & T_B & * \\ & & & T_D \end{bmatrix}$$

is a Schur decomposition of A and α is not an eigenvalue of T_D. Now, the size of C is $n \times (s-n)$ and $s < 2n$. It follows that C is row rank deficient and so there is a row of C, say row k, that linearly depends on the other rows of C. In practice, such a row can be detected by using a QR factorization of C^T: the position of a 0 in the diagonal of R is one of such rows.

If $k = 1$ then $\operatorname{null}(T_2(3\colon s+1, 3\colon s+1) - \alpha I_{s-1}) = n-1$ and T_2 is the required Schur form of A satisfying conditions (i) and (ii). If this not the case, there exists a similarity transformation via a Givens rotation[1] in the planes $(2, k+1)$ such that if T_3 is the resulting matrix, it is still triangular and $\operatorname{rank}(T_3(3\colon s+1, 3\colon s+1) - \alpha I_{s-1})) = s-n$. So, $\operatorname{null}(T_3(3\colon s+1, 3\colon s+1) - \alpha I_{s-1}) = n-1$ and T_3 satisfies conditions (i) and (ii). $\square$

We note that the structure in (15) is the first step of a proof of the Jordan canonical form (e.g. [16, Sec. 2.4]), and a further reduction of B establishes the Weyr characteristics [11], leading to the Weyr canonical form.

The next lemma is needed for dealing with a matrix with only one real eigenvalue.

[1] In fact a permutation suffices here.

LEMMA 16. *Let $A \in \mathbb{C}^{2n\times 2n}$ be upper triangular with zero diagonal entries and assume that the geometric multiplicity of the zero eigenvalue is at most n. Then there exists a unitary U such that $U^H AU$ is upper triangular with nonderogatory 2×2 diagonal blocks.*

Proof. Notice that the hypothesis about the geometric multiplicity of zero as an eigenvalue of A is equivalent to rank $(A) \geq n$.

We use induction on n. For $n = 1$, rank $(A) \geq n$ implies that $A = \left[\begin{smallmatrix} 0 & a_{12} \\ 0 & 0 \end{smallmatrix}\right]$ with $a_{12} \neq 0$, that is, A is nonderogatory. Suppose the result holds for $n-1$. Let $A \in \mathbb{C}^{2n\times 2n}$ be upper triangular with zero diagonal and rank $(A) \geq n$. If $a_{12} = 0$ then we can unitarily transform A so that its $(1, 2)$ entry becomes nonzero as follows. Use a sequence of Givens rotations G to transform the first nonzero column of A, say Ae_m, $m \geq 2$, to a multiple of e_1. Then $G^H AG$ is still upper triangular with zero diagonal, first $m-1$ columns equal to zero and the mth column equal to a multiple of e_1. Then we move the $(1, m)$ nonzero entry to the $(1, 2)$ position with a permutation $P_{2,m}$, where $P_{2,m}$ swaps the second and mth row/column of $G^H AG$. The resulting matrix $P_{2,m}^H G^H AGP_{2,m}$ is still upper triangular. Hence below we assume that $a_{12} \neq 0$ in A.

Write $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{smallmatrix}\right]$, where $A_{11} = A(1\colon 2, 1\colon 2) = \left[\begin{smallmatrix} 0 & a_{12} \\ 0 & 0 \end{smallmatrix}\right]$. To use the induction hypothesis, we need to make sure that A_{22} is upper triangular with $\text{rank}(A_{22}) \geq n-1$, that is, the geometric multiplicity of the eigenvalue zero is at most $n-1$. Since it cannot be greater than n, rank $(A_{22}) \geq n-2$ and so care is needed only when rank $(A_{22}) = n-2$. Notice first that, in this case, $n \geq \text{null}(A) \geq \text{null}(A_{22}) = n$ and so rank $(A) = n$. This implies that the second row of A cannot be zero. In order to unitarily transform A so that $\text{rank}(A_{22}) = n-1$, we can use the same technique as that in the proof of Lemma 15: first unitarily transform $\widehat{A} = Q^H AQ$ with $Q = \text{diag}(1, \widehat{Q}_1)$ so that $\widehat{A}$ is upper triangular and the first n columns of $\widehat{A}(2\colon 2n, 2\colon 2n)$ become zero. Notice that the first column of $A(2 : 2n, 2 : 2n)$ is zero and so e_1 can be taken as the first column of $\widehat{Q}_1$. In other words, Q can be chosen to have the form $Q = \text{diag}(I_2, Q_1)$ with Q_1 a unitary matrix of order $2n-2$, implying that the second row of $\widehat{A}$ is not zero.

Now, the size of $\widehat{A}(2\colon n+1, n+2\colon 2n)$ is $n \times (n-1)$. Thus rank $(\widehat{A}(2\colon n+1, n+2\colon 2n)) \leq n-1$ and there exists a row, k say, that linearly depends on the remaining rows of $\widehat{A}(2\colon n+1, n+2\colon 2n)$. Since the first row of $\widehat{A}(2\colon n+1, n+2\colon 2n)$ is not zero, k can be chosen such that $1 < k \leq n+1$. Apply a Givens rotation to $\widehat{A}$ in the planes $(2, k+1)$ so that the obtained matrix $\widetilde{A}$ satisfies the following conditions:

- its $(1, 2)$ entry is again non-zero, and
- rank $(\widetilde{A}(3\colon 2n, 3\colon 2n)) = n-1$.

Such a rotation always exists (almost all rotations suffices) and we still have $\widetilde{A}(2\colon 2n, 2\colon n+1) = 0$. Now the induction hypothesis can be applied to $\widetilde{A}$. □

We are now ready to describe an algorithm that stably computes the Schur form in Theorem 2 when $\ell = 2$. Let $A = U^H TU$ be any computed Schur decomposition of the matrix A in Theorem 2, and suppose that some of the 2×2 diagonal blocks of T are derogatory.

If all eigenvalues have algebraic multiplicity at most n then we can reorder the diagonal entries of T using the Bai-Demmel algorithm [1], as was discussed in Section 3. Thus assume that one eigenvalue α has algebraic multiplicity $n+t$ with $1 \leq t \leq n$. If $t = n$, we use the procedure described in the proof of Lemma 16 to further unitarily reduce $T - \alpha I$ to an upper triangular matrix T_1 with 2×2 nonderogatory diagonal blocks. $T_1 + \alpha I$ is the desired Schur form of A.

If $t < n$, note that all other eigenvalues must have algebraic multiplicity less than n. By using the Bai-Demmel algorithm [1] we pair as many α as possible with eigenvalues other than α thereby forming nonderogatory blocks in the top-left corner of T. In doing so, we use Lemma 15 to ensure that the resulting $2t \times 2t$ bottom-right corner of T has eigenvalue α with geometric multiplicity no larger than t. Thus we are left with

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \tag{16}$$

where $T_{22} \in \mathbb{C}^{2t\times 2t}$ contains 2×2 diagonal blocks with eigenvalue α and rank $(T_{22} - \alpha I) \geq t$. Lemma 16 is then applied to $T_{22} - \alpha I$ as above to obtain a unitarily similar upper triangular matrix with nonderogatory 2×2 diagonal blocks.

4.2. The real case. To describe an algorithm that works in real arithmetic and computes the Schur decomposition in Theorem 3, we need a real version of Lemma 16.

LEMMA 17. *Let $A \in \mathbb{R}^{2n\times 2n}$ and suppose that the spectrum of A contains a pair of nonreal complex eigenvalues $a \pm ib$ and a real eigenvalue α of geometric multiplicity at most n. Then there exists a Schur form T of A, such that*

(i) $T(1\colon 4, 1\colon 4) = \begin{bmatrix} a & b & * & * \\ -b & a & * & * \\ & & \alpha & * \\ & & & \alpha \end{bmatrix}$, *and*

(ii) *α is an eigenvalue of $T(5\colon 2n, 5\colon 2n)$ with geometric multiplicity at most $n-2$.*

Proof. Let $s \leq 2n-2$ and $k \leq n$ be the algebraic and geometric multiplicities of α as an eigenvalue of A. Consider an arbitrary real Schur form of A and reorder the diagonal blocks, using the Bai-Demmel algorithm [1] so as to obtain a Schur form T of A such that the leading 2×2 block is as in (i) and α appears on the diagonal of $\widehat{X} = T(3\colon s+2, 3\colon s+2)$. Thus the nullity of $\widehat{X} - \alpha I$ is $k \leq n$ and, as in Lemma 15, we can find an orthogonal $\widehat{Q}_0$ such that

$$X_0 = \widehat{Q}_0^T \widehat{X} \widehat{Q}_0 = \begin{bmatrix} \alpha I_k & C \\ 0 & T_B \end{bmatrix}$$

is upper triangular. By applying a unitary similarity transformation defined by $Q_0 = I_2 \oplus \widehat{Q}_0 \oplus I_{2n-2-s}$ to T, we get a Schur form T_0 of A whose leading 4×4 block satisfies (i). It also satisfies condition (ii) if $k \leq n-2$. Let us assume that $k = n$ or $k = n-1$ and let $C = \widehat{Q}_1 R$ be a "bottom-up" QR factorization of C (notice that this matrix may be singular). For a matrix $M \in \mathbb{F}^{m\times n}$, we say that $M = QR$ is a bottom-up QR factorization of M if $Q \in \mathbb{F}^{m\times m}$ is a unitary matrix (orthogonal in the real case) and $R(m:-1:1,:)$ is upper triangular. If $\texttt{fud}(M) = M(m:-1:1,:)$ and $\texttt{fud}(M) = Q_f R_f$ is a (full) QR factorization of $\texttt{fud}(M)$, then a bottom-up factorization of M is $M = QR$ where $R = \texttt{fud}(R_f)$ and $Q = \texttt{fud}([\texttt{fud}(Q_f^*)]^*)$. Define $Q_1 = I_2 \oplus \widehat{Q}_1 \oplus I_{2n-2-s}$. Then $T_1 = Q_1^T T_0 Q_1$ is a real Schur form of A and

$$X_1 = T_1(3:s+2, 3:s+2) = \begin{bmatrix} \alpha I_k & R \\ 0 & T_B \end{bmatrix}.$$

We examine the following three cases separately:

$\underline{k = n}$. In this case $\mathrm{null}(X_1 - \alpha I_s) = n$, the size of R is $n \times (s-n)$ and $s - n \leq 2n - 2 - n = n - 2$. Hence the two first rows of R are zero and so $\mathrm{null}(X_1(3:s, 3:s) - \alpha I_{s-2}) = n - 2$. Therefore, T_1 satisfies conditions (i) and (ii).

$\underline{k = n-1 \text{ and } s < 2n-2}$. Now $\mathrm{null}(X_1 - \alpha I_s) = n - 1$, the size of R is $(n-1) \times (s-n+1)$ and $s-n+1 < 2n-2-n+1 = n-1$. The first row of R is zero and since $\mathrm{rank}\,(X_1 - \alpha I_s) = s-n+1$ we can conclude that $\mathrm{rank}\,(X_1(3:s, 3:s) - \alpha I_{s-2}) \geq s - n$ (with equality if the second row of R is not a linear combination of the remaining rows in $\left[\begin{smallmatrix} R \\ T_B \end{smallmatrix}\right]$). Thus $\mathrm{null}(X_1(3:s, 3:s) - \alpha I_{s-2}) \leq (s-2) - (s-n) = n-2$. Again T_1 satisfies the desired conditions.

$\underline{k = n-1 \text{ and } s = 2n-2}$. We have $X_1 = T_1(3:2n, 3:2n)$ and so the only eigenvalue of $T_1(3:2n, 3:2n)$ is α. In addition $\mathrm{null}(X_1 - \alpha I_{2n-2}) = n-1$ and so $X_1 - \alpha I$ fulfils the hypothesis of Lemma 16. It was shown in the proof of that lemma that there is an orthogonal matrix U such that $(\widetilde{X}_1 - \alpha I) = U^T(X_1 - \alpha I)U$ is a matrix in real Schur form and satisfies two conditions: its $(1,2)$ entry is not zero and $\mathrm{rank}\,(\widetilde{X}_1 - \alpha I)(3\colon 2(n-1), 3\colon 2(n-1)) = n-2$. If $V = I_2 \oplus U$ then $V^T T_1 V$ is a real Schur form of A which satisfies conditions (i) and (ii). ∎

It is worth remarking that it is only when $s = 2n-2$ and $\mathrm{null}(X_1 - \alpha I_s) = n-1$ that the 4×4 leading block of the Schur form constructed in Lemma 17 is nonderogatory. This corresponds to the case $k = n-1 = s/2$ in the proof of Lemma 17. In any other case the geometric multiplicity of α in that block is two.

We now have the artillery to describe a stable algorithm that computes the Schur form of A in Theorem 3 for $\ell = 2$. Suppose a real Schur form of A is given. Any 2×2 block on the diagonal associated to a pair of nonreal complex conjugate eigenvalues is obviously nonderogatory, so we only need to take care of the real eigenvalues. The case when all eigenvalues have algebraic multiplicities at most n was discussed in Section 3. In the real case with $\ell = 2$ there cannot be nonreal complex eigenvalues of algebraic multiplicity greater than n. Hence, we only have to deal with the case when exactly one real eigenvalue α has algebraic multiplicity greater than n. We first use Lemma 17 as many times as possible, that is, we pair two copies of α with as many pairs of nonreal complex conjugate eigenvalues as possible. After doing this we are left with real eigenvalues only. Henceforth, Lemma 15 and Lemma 16 can be used as in the complex case to get a Schur form of A with all its diagonal blocks either nonderogatory of size 2×2 or of size 4×4 with eigenvalues whose geometric multiplicity is at most two.

We illustrate this process in the following example.

Example 18. Let $n = 4$, $\ell = 2$ and let A be the following matrix in real Schur form:

$$A = \begin{bmatrix} 1 & * & * & * & * & * & * & * \\ & 0 & -1 & * & * & * & * & * \\ & 1 & 0 & * & * & * & * & * \\ & & & 1 & * & * & * & * \\ & & & & 1 & * & * & * \\ & & & & & 1 & * & * \\ & & & & & & a & * \\ & & & & & & & 1 \end{bmatrix}.$$

where a is either 1 or 2. Thus the distinct eigenvalues of A are 1, i and $-i$ when $a = 1$ and 1, 2, i and $-i$ when $a = 2$. The algebraic multiplicity of 1 is $5 > n$ or $6 > n$ according as $a = 2$ or $a = 1$ but its geometric multiplicity must be at most 4. Let us assume that it is 4. Under these conditions (see [3, Thm. 1.7]) A is a linearization of a 4×4 quadratic matrix polynomial $P(\lambda)$ with nonsingular leading coefficient. Now, by

[15, Th. 3.6], $P(\lambda)$ is not triangularizable over $\mathbb{R}[\lambda]$. Hence there is no real Schur form of A with four nonderogatory blocks of size 2×2 in the diagonal. In other words, any real Schur form of A with the properties of Theorem 3 must have, at least, one block of size 4×4 with 1 as an eigenvalue of geometric multiplicity 2. This is consistent with Theorem 4.1 of [15] and must be revealed by the algorithmic process.

Our goal is to find an orthogonal matrix $Q \in \mathbb{R}^{8\times 8}$ such that $Q^T A Q$ is a real Schur form with two nonderogatory blocks of size 2×2 and one block of size 4×4 with only one real eigenvalue whose geometric multiplicity is 2. We use the procedure developed in Lemmas 15–17 as follows.

Step 1: We are under the hypothesis of Lemma 17. Use Bai-Demmel algorithm to exchange entry $(1,1)$ and block $\left[\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right]$ to obtain a new real Schur form $T_1 = Q_1^T A Q_1$ with diagonal diag $(B_1, 1, 1, 1, 1, a, 1)$ where B_1 is similar to $\left[\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right]$.

Let us assume now that $a = 2$. The case $a = 1$ will be dealt with later on.

Step 2: Use Bai-Demmel algorithm to exchange entries $(7,7)$ and $(8,8)$ so that all equal eigenvalues appear together in the diagonal. We get a new real Schur form

$$T_2 = Q_2^T T_1 Q_2 = \left[\begin{array}{ccc|cccc} B_1 & * & * & * & * & * & * \\ & 1 & * & * & * & * & * \\ & & 1 & * & * & * & * \\ \hline & & & 1 & * & * & * \\ & & & & 1 & * & * \\ & & & & & 1 & * \\ & & & & & & 2 \end{array}\right].$$

T_2 satisfies condition (i) of Lemma 17. We proceed as indicated in steps 1-3 of the proof of that lemma to produce a real Schur form which also satisfies condition (ii).

Step 3: Extract the following submatrix $T_2(3:7, 3:7)$

$$X = \begin{bmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & 1 \end{bmatrix},$$

and obtain an orthonormal basis $\widehat{Q}_1$ of $\mathrm{Ker}(X - I_5)$ (using the singular value decomposition, for instance). By hypothesis, the geometric multiplicity of eigenvalue 1 is 4. Then $\dim \mathrm{Ker}(X - I_5) = 4$ and so the size of $\widehat{Q}_1$ is 5×4. Complete $\widehat{Q}_1$ up to an orthogonal matrix $\widetilde{Q}_1$ (using, for example, the QR factorization of $\widehat{Q}_1$). Then

$$\widetilde{Q}_1^T X \widetilde{Q}_1 = \begin{bmatrix} I_4 & c \\ 0 & 1 \end{bmatrix},$$

where c is not the zero vector. Compute a bottom-up QR factorization of c. In this case we can use a Householder reflection, $\widehat{Q}_2^T (= \widehat{Q}_2)$ so that $\widehat{Q}_2^T c = [0 \quad 0 \quad 0 \quad d]^T$ with $d \neq 0$. Define $\widetilde{Q}_2 = \mathrm{diag}(\widehat{Q}_2, 1)$ and $Q_3 = \mathrm{diag}(I_4, \widetilde{Q}_1 \widetilde{Q}_2)$. Then

$$T_3 = Q_3^T T_2 Q_3 = \left[\begin{array}{ccc|cccc} B & * & * & * & * & * & * \\ & 1 & 0 & 0 & 0 & 0 & * \\ & & 1 & 0 & 0 & 0 & * \\ \hline & & & 1 & 0 & 0 & * \\ & & & & 1 & d & * \\ & & & & & 1 & * \\ & & & & & & 2 \end{array}\right].$$

Since $d \neq 0$, the geometric multiplicity of 1 as eigenvalue of $T_3(5:8, 5:8)$ is 2.

Step 4: We could successively permute the first and second and then the second and third rows and columns of $T_3(5:8,5:8)$ to get the desired matrix. However, $T_3(5:8,5:8)$ satisfies the hypothesis of Lemma 15 and we will proceed as indicated in its proof: use Bai-Demmel algorithm to swap block $\left[\begin{smallmatrix}1 & d\\ & 1\end{smallmatrix}\right]$ and the entry 2 in position $(8,8)$. The resulting matrix is

$$T_4 = \left[\begin{array}{ccc|cc|c} B & * & * & * & * & * \\ & 1 & 0 & 0 & * & * \\ & & 1 & 0 & * & * \\ \hline & & & 1 & * & * \\ & & & & 2 & * \\ \hline & & & & & C \end{array}\right].$$

where C is similar to $\left[\begin{smallmatrix}1 & d\\ & 1\end{smallmatrix}\right]$ and so, it is nonderogatory. If C is itself upper triangular then T_4 is the desired matrix. Otherwise,

Step 5: Reduce C to upper triangular form by orthogonal similarity and apply it to the last two rows and columns of T_4 to obtain:

$$T_5 = \left[\begin{array}{ccc|cc|cc} B & * & * & * & * & * & * \\ & 1 & 0 & 0 & * & * & * \\ & & 1 & 0 & * & * & * \\ \hline & & & 1 & * & * & * \\ & & & & 2 & * & * \\ \hline & & & & & 1 & g \\ & & & & & & 1 \end{array}\right].$$

T_5 is a real Schur form of A with two nonderogatory blocks of size 2×2 and one block of size 4×4 in the diagonal. The geometric multiplicity of 1 as eigenvalue of $T_5(1:4,1:4)$ is two; thus we have a desired Schur form for $a=2$.

Assume now that $a=1$. We skip step 2 and go straight ahead to

Step 3: Now $X = T_1(3:8,3:8)$ is a 6×6 matrix, $\widehat{Q}_1$ is a 6×4 matrix with orthonormal columns. We can complete it to a 6×6 orthogonal matrix $\widetilde{Q}_1$ such that

$$\widetilde{Q}_1^T X \widetilde{Q}_1 = \begin{bmatrix} I_4 & C \\ 0 & T_C \end{bmatrix},$$

where $T_C = \left[\begin{smallmatrix}1 & r_{32}\\ & 1\end{smallmatrix}\right]$. Compute a bottom-up QR factorization of $C = \widehat{Q}_2 R$ and define $Q_3 = \operatorname{diag}(I_4, \widetilde{Q}_1\widetilde{Q}_2)$ with $\widetilde{Q}_2 = \operatorname{diag}(\widehat{Q}_2, I_2)$. Then

$$T_3 = Q_3^T T_2 Q_3 = \left[\begin{array}{c|cccccc} B & * & * & * & * & * & * \\ \hline & 1 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & r_{12} \\ & & & & 1 & r_{21} & r_{22} \\ & & & & & 1 & r_{32} \\ & & & & & & 1 \end{array}\right],$$

where $r_{21}r_{12} \neq 0$ and $r_{21}r_{32} \neq 0$ (that is, $r_{21} \neq 0$ and at least one of r_{12} or r_{32} is not zero) because otherwise $\operatorname{null}(T_3 - I) > 4$. Thus T_3 is a real Schur form of A which satisfies conditions (i) and (ii) of Lemma 17.

Step 4: Deflate $T_3(1:4,1:4)$ and pay attention to $Y = T_3(5:8,5:8)$. This is a 4×4 real matrix with all eigenvalues 1. Its algebraic multiplicity is 4 and its geometric multiplicity is 2 because $r_{21}r_{12} \neq 0$ and $r_{21}r_{32} \neq 0$. We use the proof of Lemma 16 to get a real Schur form with two nonderogatory blocks of size 2×2 in the diagonal.

First we define $Z = Y - I_4$, which is a nilpotent matrix, and notice that if S is a real Schur form of Z then $S + I_4$ is a real Schur form of Y. Now we apply the method proposed in the proof of Lemma 16: observe that the first nonzero column of Z is the second one and so we use a Givens rotation in order to replace that column by a multiple of e_1. In the present case a permutation of the second and third rows and columns suffices:

$$P_1^T Z P_1 = \begin{bmatrix} 0 & 0 & r_{21} & r_{22} \\ 0 & 0 & 0 & r_{12} \\ 0 & 0 & 0 & r_{32} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Next, we permute the second and third rows and columns to get

$$Z_1 = P_2^T P_1^T Z P_1 P_2 = \begin{bmatrix} 0 & r_{21} & 0 & r_{22} \\ 0 & 0 & 0 & r_{32} \\ 0 & 0 & 0 & r_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

With the notation of Lemma 15, $A_{22} = \left[\begin{smallmatrix} 0 & r_{12} \\ 0 & 0 \end{smallmatrix}\right]$ and $n = 2$. Thus, if $r_{12} \neq 0$ then $\operatorname{rank}(A_{22}) = 1 = n - 1$ and no further transformation is needed on Z_1 because it is upper triangular with 2×2 nonderogatory diagonal blocks. But if $r_{12} = 0$ then $\operatorname{rank}(A_{22}) = 0 = n - 2$ and one additional transformation is needed. In fact, as shown above and in the proof of Lemma 16, $r_{32} \neq 0$ and we can perform a Givens rotation on rows and columns two and three to place nonzero elements in $(1, 2)$ and $(3, 4)$ of Z_1.

Summarizing, there is an orthogonal matrix $Q = Q_1 Q_3 Q_4$ with $Q_4 = I_4 \oplus P_1 P_2 G$ where G is an appropriate Givens rotation (the identity if $r_{12} \neq 0$) such that

$$T = Q^T A Q = \left[\begin{array}{ccc|cc|cc} B & * & * & * & * & * & * \\ & 1 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 \\ \hline & & & 1 & s_{12} & s_{13} & s_{14} \\ & & & & 1 & s_{23} & s_{24} \\ \hline & & & & & 1 & s_{34} \\ & & & & & & 1 \end{array}\right],$$

and $s_{12} \neq 0$ and $s_{34} \neq 0$. $Q^T A Q$ is a real Schur form of A with two nonderogatory blocks of size 2×2 and one block of size 4×4 in the diagonal. Again, the geometric multiplicity of 1 as an eigenvalue of $T(1:4, 1:4)$ is two.

5. Parameterized linear systems.

We consider parameterized linear systems of the form

$$P(\omega)x = b(\omega), \quad x = x(\omega). \tag{17}$$

This type of systems appear when computing numerical solutions of differential equations which arise in areas including electromagnetic scattering, wave propagation in porous media or structural dynamics (see, for example [5, 7, 12], and the references therein). The coefficient matrix in (17) is the matrix polynomial in (1) and b may be constant [9, 12] or a (in general, nonlinear) function of the parameter ω [5, 7]. For quadratic matrix polynomials ω is either real or pure imaginary with $|\omega| \in I = [\omega_\ell, \omega_h]$, $\omega_\ell \ll \omega_h$ [5, 7, 9, 12] and the solution of (17) is to be computed for many values of the parameter ω. In particular, in [7] $b(\omega)$ is supposed to be analytic in I except at points ω where $\det P(\omega) = 0$; the solution $x(\omega)$ then inherits the same property. Whether we are interested in analytic solutions of (17) or in solutions for finitely many values of ω, reduced forms $R(\lambda)$ of $P(\lambda)$ can be used to convert system (17) into a simpler equivalent one

$$R(\omega)y = c(\omega), \quad y = y(\omega). \tag{18}$$

In this section we show how to obtain $c(\omega)$ from $b(\omega)$ so that the solution of (17) can be given explicitly in terms of $b(\omega)$ and the solution of (18).

Let $C_L(P)$ and $C_L(R) = S^{-1}C_L(P)S$ be the left companion matrices of $P(\lambda)$ and $R(\lambda)$, respectively, with $S = [X\ C_L(P)X\ \dots\ C_L(P)^{\ell-1}X]$. On using [3, Prop. 1.2], we have that for every $\omega \in \mathbb{C}$ which is not an eigenvalue of P,

$$\begin{aligned} P(\omega)^{-1} &= (e_\ell^T \otimes I_n)(\omega I - C_L(P))^{-1}(e_1 \otimes I_n) \\ &= (e_\ell^T \otimes I_n)S\big(\omega I - C_L(R)\big)^{-1}S^{-1}(e_1 \otimes I_n). \end{aligned} \tag{19}$$

Since $C_L(P) = SC_L(R)S^{-1}$ is a left companion matrix, S^{-1} must be of the form

$$S^{-1} = [Y\ C_L(R)Y\ \dots\ C_L(R)^{\ell-1}Y]$$

for some $\ell n \times n$ matrix Y, and $S^{-1}(e_1 \otimes I_n) = Y = [\,Y_1 \quad \cdots \quad Y_n\,]^T$ Also,

$$\big(\omega I - C_L(R)\big)^{-1} = E(\omega)\begin{bmatrix} R(\omega)^{-1} & 0 \\ 0 & I \end{bmatrix} F(\omega),$$

where

$$E(\omega) = \begin{bmatrix} B_{\ell-1}(\omega) & -I & 0 & \cdots & 0 \\ B_{\ell-2}(\omega) & 0 & -I & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & -I \\ B_0(\omega) & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad F(\omega) = \begin{bmatrix} I & \omega I & \cdots & \omega^{\ell-2}I & \omega^{\ell-1}I \\ 0 & \ddots & \ddots & & \omega^{\ell-2}I \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \omega I \\ 0 & \cdots & \cdots & 0 & I \end{bmatrix}$$

with $B_0(\omega) = I$ and $B_j(\omega) = \omega B_{j-1}(\omega) + R_{\ell-j}$ for $j = 1, \dots, \ell-1$. Let $[Z_1\ \dots\ Z_\ell] = (e_\ell^T \otimes I_n)S$ denote the last n rows of S. Then

$$P(\omega)^{-1} = \Big(\sum_{i=1}^{\ell} Z_i B_{\ell-i}(\omega)\Big)R(\omega)^{-1}\Big(\sum_{i=1}^{\ell}\omega^{i-1}Y_i\Big) - \sum_{j=1}^{\ell-1} Z_j\Big(\sum_{i=j+1}^{\ell}\omega^{i-(j+1)}Y_i\Big).$$

If we let $c(\omega) = \big(\sum_{i=1}^{\ell}\omega^{i-1}Y_i\big)b(\omega)$ and solve (18) for $y(\omega)$ then for the solution $x(\omega)$ to the parameterized linear system (17) we have

$$x(\omega) = \sum_{i=1}^{\ell} Z_i B_{\ell-i}(\omega)y(\omega) - \sum_{j=1}^{\ell-1} Z_j\Big(\sum_{i=j+1}^{\ell}\omega^{i-(j+1)}Y_i b(\omega)\Big).$$

The structure of S and that of the left companion matrix can be exploited to construct the last n rows of S and the first n columns of S^{-1}.

6. Conclusions. All matrix polynomials with nonsingular leading coefficients can be reduced to triangular form while keeping the size, degree, and eigenstructure of the original matrix polynomial by means of unimodular transformations. We do not have a practical way to compute the unimodular transformations, so instead, we have proposed a practical procedure that, starting from a Schur form of any linearization $\lambda I - A$ of a given $n \times n$ matrix polynomial of degree ℓ, consists of three steps:

1. moving the diagonal elements (and the 2×2 diagonal blocks, in the real case) of the Schur form so as to obtain a new Schur form with some specific properties,
2. using the obtained Schur form to construct a full column rank matrix X satisfying the conditions of Theorem 1 (X may be taken to have orthonormal columns), and
3. performing a similarity structure preserving transformation $S = K_\ell(A, X)$ as in (2) so that $S^{-1}AS$ is the left companion matrix of a monic triangular matrix polynomial of degree ℓ (only the last n columns of $S^{-1}AS$ are needed).

We showed how to implement step 1 so that the procedure reduces any quadratic matrix polynomials to triangular form. For $\ell > 2$, however, we only discussed how to succeed with step 1 in the case when no eigenvalue has algebraic multiplicity larger than n.

Reduction to other simple forms like block-diagonal, block-triangular or Hessenberg forms was also considered. In particular, it was shown that if a Hessenberg form of a linearization, when partitioned in $\ell \times \ell$ blocks, has unreduced diagonal blocks then the matrix polynomial can be brought to Hessenberg form using steps 2 and 3 above (with the obvious substitutions " Schur form" by "Hessenberg form" and "triangular matrix" by "Hessenberg matrix").

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