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Absolute and relative Weyl theorems for generalized eigenvalue problems

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ABSTRACT

Weyl-type eigenvalue perturbation theories are derived for Hermitian definite pencils $A - \lambda B$, in which B is positive definite. The results provide a one-to-one correspondence between the original and perturbed eigenvalues, and give a uniform perturbation bound. We give both absolute and relative perturbation results, defined in the standard Euclidean metric instead of the chordal metric that is often used.

1. Introduction

We are concerned with eigenvalue perturbations of a generalized Hermitian eigenvalue problem $A\boldsymbol{x} = \lambda B\boldsymbol{x}$, in which $A, B \in \mathbb{C}^{n\times n}$ are Hermitian and B is positive definite. For a standard Hermitian eigenvalue problem $A\boldsymbol{x} = \lambda\boldsymbol{x}$, Weyl's theorem [16,13,3] is perhaps the best-known perturbation result. We denote the spectral norm of a matrix by $\|\cdot\|_2$ (the largest singular value, or matrix 2-norm).

Theorem 1.1 (Weyl's theorem). *Let the eigenvalues of the Hermitian matrices A and $A+E$ be $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ and $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \cdots \leqslant \tilde{\lambda}_n$ respectively. Then* $\max_i |\lambda_i - \tilde{\lambda}_i| \leqslant \|E\|_2$.

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Despite being merely a special case of the Lidskii-Mirsky-Wielandt theorem [9], Weyl's theorem stands out as a simple and useful result that

- Orders and pairs up the original and perturbed eigenvalues, so that we can discuss in terms of the matching distance [15, pp. 167].
- Gives a bound on the largest distance between a perturbed and exact eigenvalue.

Owing to its simple expression and wide applicability, the theorem has been used in many contexts, e.g., in the basic forward error analysis of standard eigenvalue problems [1, Chapter 4.8]. In order to distinguish this theorem from the variants discussed below, in this paper we refer to it as the *absolute Weyl theorem*.

The *relative Weyl theorem*, which can provide much tighter bounds for small eigenvalues, is also known [3].

Theorem 1.2 (*Relative Weyl theorem*). *Let A be Hermitian and X be nonsingular. Let the eigenvalues of A be $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, let the eigenvalues of X^HAX be $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \cdots \leqslant \tilde{\lambda}_n$, and let $\epsilon = \|X^HX - I\|_2$. Then the eigenvalues differ by ϵ in the relative sense, i.e.,*

$$\frac{|\lambda_i - \tilde{\lambda}_i|}{|\lambda_i|} \leqslant \epsilon, \quad i = 1, 2, \ldots, n.$$

This important observation leads to a number of relative perturbation results, along with algorithms that compute eigenvalues/singular values to high relative accuracy including small ones [4,7,5].

For eigenvalues of a Hermitian definite pencil $A - \lambda B$, in which A, B are Hermitian and B is positive definite, many properties analogous to the eigenvalues of a Hermitian matrix carry over. For example, the pencil has n real and finite eigenvalues, which satisfy a min-max property similar to that for Hermitian matrices [14].

Some perturbation results for generalized eigenvalue problems are known, mostly in the chordal metric [15, Chapter 6.3], [10,12,2]. Using the chordal metric is a natural choice for a general matrix pencil because it deals uniformly with infinite eigenvalues. However, this metric is not invariant under scaling, and bounds in this metric may be less intuitive than those defined in the standard Euclidean metric. Most importantly, for a Hermitian definite pencil we know a priori that no infinite eigenvalues exist, so in this case the Euclidean metric may be a more natural choice.

The goal of this paper is to derive Weyl-type theorems for Hermitian definite matrix pencils, both the absolute (Section 2) and relative (Section 3) versions. Our results employ the Euclidean metric, and have the two Weyl-type properties described above. Compared to known results, our absolute Weyl theorem is simpler than some known bounds (e.g., [10]), and our relative Weyl theorem assumes no condition on A. By contrast, the relative perturbation results for Hermitian definite pairs obtained in [9,11] are derived under the assumption that A and B are both positive definite, which limits their applications; Hermitian definite pencils that arise in practice may not have this property (e.g., [6]).

We deal only with the case in which B is positive definite, and refer to such pencils as *Hermitian definite pencils*. This type of problem appears in practice most often, and is sometimes simply called a *generalized Hermitian eigenvalue problem* [1, Chapter 5]. In the literature a matrix pencil is often called *Hermitian definite* if $\alpha A + \beta B$ is positive definite for some scalars α and β [15, pp. 281]. When $\alpha A + \beta B$ is positive definite, we can reduce the problem to the positive definite case $A - \theta(\alpha A + \beta B)$, noting that this pencil has eigenvalues $\theta_i = \lambda_i/(\beta + \alpha\lambda_i)$.

Notations: $\lambda_i(A)$ denotes the ith smallest eigenvalue of a Hermitian matrix A, and $\lambda_{\min}(A) = \lambda_1(A)$. We use only the spectral norm $\|\cdot\|_2$, and $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$ is the condition number of A.

2. Absolute Weyl theorem for generalized eigenvalue problems

For a Hermitian definite pencil $A - \lambda B$, we have the following generalization of the absolute Weyl theorem [1, Chapter 5.7], when only A is perturbed.

Theorem 2.1. *Suppose that the Hermitian definite pencils $A - \lambda B$ and $(A + \Delta A) - \lambda B$ have eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ and $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \cdots \leqslant \tilde{\lambda}_n$, respectively. Then for all $i = 1, 2, \ldots, n$,*

$$|\lambda_i - \tilde{\lambda}_i| \leqslant \frac{\|\Delta A\|_2}{\lambda_{\min}(B)}. \tag{1}$$

Proof. Define $Z = B^{-1/2}$ (the matrix square root [8, Chapter 6] of B). A congruence transformation that multiplies by Z from both sides shows that the pencil $A - \lambda B$ is equivalent to the pencil $ZAZ - \lambda I$. Hence, these pencils and the Hermitian matrix ZAZ have the same eigenvalues. Similarly, the pencil $(A + \Delta A) - \lambda B$ and the Hermitian matrix $Z(A + \Delta A)Z$ have the same eigenvalues.

Now, to compare the eigenvalues of ZAZ and $Z(A + \Delta A)Z$, we observe that $\|Z\Delta AZ\|_2 \leqslant \|Z^2\|_2\|\Delta A\|_2 = \|B^{-1}\|_2\|\Delta A\|_2 = \|\Delta A\|_2/\lambda_{\min}(B)$, so we obtain (1) by using the absolute Weyl theorem applied to ZAZ and $Z(A + \Delta A)Z$. $\square$

Theorem 1 takes into account only perturbations in the matrix A. In practical problems, the matrix B may be obtained from data that may include errors, or may be subject to floating-point representation errors. Therefore, we are also interested in the impact of perturbations in B. The following result takes such perturbations into account.

Theorem 2.2 (Absolute Weyl theorem for generalized eigenvalue problems). *Suppose that a Hermitian definite pencil $A - \lambda B$ has eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. If $\Delta A, \Delta B$ are Hermitian and $\|\Delta B\|_2 < \lambda_{\min}(B)$, then $(A + \Delta A) - \lambda(B + \Delta B)$ is a Hermitian definite pencil whose eigenvalues $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \cdots \leqslant \tilde{\lambda}_n$ satisfy*

$$|\lambda_i - \tilde{\lambda}_i| \leqslant \frac{\|\Delta A\|_2}{\lambda_{\min}(B)} + \frac{\|A\|_2 + \|\Delta A\|_2}{\lambda_{\min}(B)(\lambda_{\min}(B) - \|\Delta B\|_2)}\|\Delta B\|_2, \quad i = 1, 2, \ldots, n. \tag{2}$$

Proof. The fact that $(A + \Delta A) - \lambda(B + \Delta B)$ is a Hermitian definite pencil follows directly from applying the absolute Weyl theorem to B and $B + \Delta B$, from which we see that $\lambda_{\min}(B + \Delta B) \geqslant \lambda_{\min}(B) - \|\Delta B\|_2 > 0$.

Define $Z = B^{-1/2}$. By a congruence transformation that multiplies by Z from both sides, we see that the pencil $A - \lambda B$ has the same eigenvalues as the Hermitian matrix ZAZ. This transformation also shows that the pencil $A + \Delta A - \lambda(B + \Delta B)$ is equivalent to the pencil $Z(A + \Delta A)Z - \lambda(I + Z\Delta BZ)$.

Note that $\|Z\Delta BZ\|_2 \leqslant \|Z\|_2^2\|\Delta B\|_2 = \|B^{-1}\|_2\|\Delta B\|_2 = \|\Delta B\|_2/\lambda_{min}(B) < 1$. Therefore, by the absolute Weyl theorem, all the eigenvalues of $I + Z\Delta BZ$ are positive and lie in $[1 - \|\Delta B\|_2/\lambda_{min}(B), 1 + \|\Delta B\|_2/\lambda_{min}(B)]$. Defining $Z_1 = (I + Z\Delta BZ)^{-1/2}$, we see that

$$\|Z_1Z_1 - I\|_2 \leqslant \frac{1}{1 - \|\Delta B\|_2/\lambda_{\min}(B)} - 1 = \frac{\|\Delta B\|_2}{\lambda_{\min}(B) - \|\Delta B\|_2}. \tag{3}$$

We also see that $Z(A + \Delta A)Z - \lambda(I + Z\Delta BZ)$ is equivalent to $Z_1Z(A + \Delta A)ZZ_1 - \lambda I$, a standard Hermitian eigenvalue problem. Thus, the original eigenvalue comparison between the pencils $A - \lambda B$ and $A + \Delta A - \lambda(B + \Delta B)$ can be reduced to a comparison between the eigenvalues of two Hermitian matrices, ZAZ (whose eigenvalues are λ_i) and $Z_1Z(A + \Delta A)ZZ_1$ (whose eigenvalues are $\tilde{\lambda}_i$). This comparison can be done by using both the absolute and the relative Weyl theorems, as follows.

First, using the absolute Weyl theorem for ZAZ and $Z(A + \Delta A)Z$, we have

$$|\lambda_i - \lambda_i(Z(A + \Delta A)Z)| \leqslant \|Z\Delta AZ\|_2 \leqslant \frac{\|\Delta A\|_2}{\lambda_{\min}(B)}. \tag{4}$$

Next, using the relative Weyl theorem for $Z(A + \Delta A)Z$ and $Z_1Z(A + \Delta A)ZZ_1$, we have

$$\begin{aligned} |\lambda_i(Z(A + \Delta A)Z) - \tilde{\lambda}_i| &\leqslant |\lambda_i(Z(A + \Delta A)Z)| \cdot \|Z_1Z_1 - I\|_2 \\ &\leqslant \left(|\lambda_i| + \frac{\|\Delta A\|_2}{\lambda_{\min}(B)}\right) \cdot \frac{\|\Delta B\|_2}{\lambda_{\min}(B) - \|\Delta B\|_2}, \end{aligned}$$

in which we used (3) and (4). Combining these two results gives

$$|\lambda_i - \tilde{\lambda}_i| \leqslant \frac{\|\Delta A\|_2}{\lambda_{\min}(B)} + \frac{|\lambda_i|\lambda_{\min}(B) + \|\Delta A\|_2}{\lambda_{\min}(B)(\lambda_{\min}(B) - \|\Delta B\|_2)}\|\Delta B\|_2.$$

Using $|\lambda_i| \leqslant \|ZAZ\|_2 \leqslant \|A\|_2/\lambda_{\min}(B)$ for all i, we get (2). $\square$

Several points are worth noting regarding Theorem 2.2.

- Theorem 2.2 reduces to Theorem 2.1 when $\Delta B = 0$. Moreover, for the standard Hermitian eigenvalue problem ($B = I$ and $\Delta B = 0$), Theorem 2.2 becomes $|\lambda_i(A) - \lambda_i(A + \Delta A)| \leqslant \|\Delta A\|_2$, the absolute Weyl theorem.
- The result is sharp. This can be seen by the simple example
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Delta A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \Delta B = \begin{pmatrix} -0.8 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5}$$
The eigenvalues of $A - \lambda B$ are {2,1} and those of $A + \Delta A - \lambda(B + \Delta B)$ are {20,0}, so $\max_i |\lambda_i - \tilde{\lambda}_i| = 18$. On the other hand, applying $\|A\|_2 = 2$, $\|\Delta A\|_2 = 2$, $\|\Delta B\|_2 = 0.8$, $\lambda_{\min}(B) = 1$ to (2) gives $\max_i |\lambda_i - \tilde{\lambda}_i| \leqslant 2/1 + 0.8(2 + 2)/(1 - 0.8) = 18$, matching the actual perturbation.
- It is worth comparing our result with that of Stewart and Sun [15, Cor VI.3.3]. They give a bound
$$\rho(\lambda_i, \tilde{\lambda}_i) \leq \frac{\sqrt{\|\Delta A\|_2^2 + \|\Delta B\|_2^2}}{\gamma(A, B)}, \tag{6}$$
in which $\gamma(A, B) = \min_{\|x\|_2=1} \sqrt{(x^H Ax)^2 + (x^H Bx)^2}$. Here $\rho(a, b) = |a - b|/\sqrt{(1 + a^2)(1 + b^2)}$ is the chordal metric. Noting that the distance between any two numbers a and b is less than 1 in the chordal metric, we see that (6) does not provide any information when $\|\Delta A\|_2^2 + \|\Delta B\|_2^2 > (\gamma(A, B))^2$. In fact, (6) is useless for the matrices in (5), because $\|\Delta A\|_2^2 + \|\Delta B\|_2^2 = 4.64$ while $(\gamma(A, B))^2 = 2$. On the other hand, Theorem 2.2 gives a nontrivial bound as long as $\|\Delta B\|_2 < \lambda_{\min}(B)$. However, when $\|\Delta B\|_2 \geq \lambda_{\min}(B)$ our result is not applicable, whereas it may still be that $\|\Delta A\|_2^2 + \|\Delta B\|_2^2 > (\gamma(A, B))^2$, in which case (6) is a nontrivial bound. Therefore the two bounds are not comparable in general. An advantage of our result is that it is defined in the Euclidean metric, making its application more direct and intuitive.
- In [10] a result similar to Theorem 2.2 is proved, using the chordal metric but directly applicable to the Euclidean metric:
$$\begin{aligned} |\lambda_i - \tilde{\lambda}_i| \leqslant\; & \frac{1}{\sqrt{\lambda_{\min}(B)\lambda_{\min}(B + \Delta B)}}\|\Delta A\|_2 \\ & + \frac{\|A\|_2/\sqrt{\lambda_{\min}(B)} + \|A + \Delta A\|_2/\sqrt{\lambda_{\min}(B + \Delta B)}}{\lambda_{\min}(B)\lambda_{\min}(B + \Delta B)(\|B\|_2^{-1/2} + \|B + \Delta B\|_2^{-1/2})}\|\Delta B\|_2. \end{aligned}$$
Compared to this bound, our result is simpler and requires less information.

3. Relative Weyl theorem for generalized eigenvalue problems

We now discuss a generalization of the relative Weyl theorem to Hermitian definite pencils. We show two classes of perturbations that preserve relative accuracy of eigenvalues.

First we observe that a simple analogy from the relative Weyl theorem for standard eigenvalue problems does not work, in the sense that the pencils $X^T AX - \lambda B$ and $A - \lambda B$ can have totally different eigenvalues for X such that $\|X^H X - I\|_2$ is small. This is seen by the simple example $A = B = \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; the second pencil has eigenvalues $\{1, 1\}$ while those of the first are $\{100, 0.01\}$.

Therefore, the allowed types of multiplicative perturbations have to be more restricted. The following result claims that perturbations of the form $(I+\Delta A)^T A(I+\Delta A)$ are acceptable.

Theorem 3.1 (Relative Weyl theorem for generalized eigenvalue problems 1). *Let a Hermitian definite pencil $A-\lambda B$ have eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, and let $\sqrt{\kappa_2(B)}\|\Delta A\|_2 = \epsilon$. If $\|\Delta B\|_2 < \lambda_{\min}(B)$, then $(I+\Delta A)^T A(I+\Delta A) - \lambda(B+\Delta B)$ is a Hermitian definite pencil whose eigenvalues $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \cdots \leqslant \tilde{\lambda}_n$ satisfy*

$$|\lambda_i - \tilde{\lambda}_i| \leqslant \left(\epsilon(2+\epsilon) + (1+\epsilon)^2 \frac{\|\Delta B\|_2}{\lambda_{\min}(B) - \|\Delta B\|_2}\right)|\lambda_i|, \quad i = 1, 2, \ldots, n. \tag{7}$$

Proof. First, $(I+\Delta A)^T A(I+\Delta A) - \lambda(B+\Delta B)$ is Hermitian definite for the same reason as in Theorem 2.2.

Define $Z = B^{-1/2}$ and $Z_1 = (I + Z\Delta BZ)^{-1/2}$. By an argument similar to that in the proof of Theorem 2.2, we see that the comparison between the eigenvalues of $A-\lambda B$ and $(I+\Delta A)^T A(I+\Delta A) - \lambda(B+\Delta B)$ is equivalent to a comparison between the eigenvalues of the Hermitian matrices ZAZ and $Z_1 Z(I+\Delta A)^H A(I+\Delta A)ZZ_1$, so our goal is to compare the eigenvalues of these two matrices.

The key idea is to consider the matrix $X = I + Z^{-1}\Delta AZ$, which satisfies $Z(I+\Delta A)^H A(I+\Delta A)Z = X^H ZAZX$. Note that $\|Z^{-1}\Delta AZ\|_2 \leqslant \kappa_2(Z)\|\Delta A\|_2 = \sqrt{\kappa_2(B)}\|\Delta A\|_2 (\equiv \epsilon)$, so

$$\|X^H X - I\|_2 = \|Z^{-1}\Delta AZ + (Z^{-1}\Delta AZ)^H + (Z^{-1}\Delta AZ)^H Z^{-1}\Delta AZ\|_2 \leqslant \epsilon(2+\epsilon).$$

Therefore, by using the relative Weyl theorem for ZAZ and $Z(I+\Delta A)^H A(I+\Delta A)Z$ and recalling that $\lambda_i(ZAZ) = \lambda_i$, we obtain

$$\begin{aligned}|\lambda_i(Z(I+\Delta A)^H A(I+\Delta A)Z) - \lambda_i(ZAZ)| &= |\lambda_i(X^H ZAZX) - \lambda_i(ZAZ)| \\ &\leqslant |\lambda_i(ZAZ)| \cdot \|X^H X - I\|_2 \\ &\leqslant \epsilon(2+\epsilon)|\lambda_i|.\end{aligned} \tag{8}$$

Now to compare the eigenvalues between $Z(I+\Delta A)^H A(I+\Delta A)Z$ and $Z_1 Z(I+\Delta A)^H A(I+\Delta A)ZZ_1$, we use the relative Weyl theorem again to get

$$\begin{aligned}&|\lambda_i(Z_1 Z(I+\Delta A)^H A(I+\Delta A)ZZ_1) - \lambda_i(Z(I+\Delta A)^H A(I+\Delta A)Z)| \\ &\leqslant |\lambda_i(Z(I+\Delta A)^H A(I+\Delta A)Z)| \cdot \|Z_1 Z_1 - I\|_2 \\ &\leqslant 1+\epsilon)^2|\lambda_i| \cdot \frac{\|\Delta B\|_2}{\lambda_{\min}(B) - \|\Delta B\|_2} \quad (\because (8) \text{ and } (3)).\end{aligned}$$

Combining the above yields (9):

$$\begin{aligned}&|\lambda_i(Z_1 Z(I+\Delta A)^H A(I+\Delta A)ZZ_1) - \lambda_i| \\ &\leqslant \epsilon(2+\epsilon)|\lambda_i| + (1+\epsilon)^2|\lambda_i| \cdot \frac{\|\Delta B\|_2}{\lambda_{\min}(B) - \|\Delta B\|_2} \\ &\leqslant \left(\epsilon(2+\epsilon) + (1+\epsilon)^2 \frac{\|\Delta B\|_2}{\lambda_{\min}(B) - \|\Delta B\|_2}\right)|\lambda_i|. \quad \square\end{aligned}$$

The next result shows that a simpler result can be obtained when both perturbations are multiplicative and the pencil can be expressed as $(I+\Delta A)^H A(I+\Delta A) - \lambda(I+\Delta B)^H B(I+\Delta B)$.

Theorem 3.2 (Relative Weyl theorem for generalized eigenvalue problems 2). *Let the Hermitian definite pencils $A-\lambda B$ and $(I+\Delta A)^T A(I+\Delta A) - \lambda(I+\Delta B)^H B(I+\Delta B)$ have eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ and $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \cdots \leqslant \tilde{\lambda}_n$, respectively. Suppose that $\sqrt{\kappa_2(B)}\|\Delta A\|_2 = \epsilon$ and $\sqrt{\kappa_2(B)}\|\Delta B\|_2 = \delta < 1$. Then, $\tilde{\lambda}_i$ $(1 \leqslant i \leqslant n)$ satisfy*

$$|\lambda_i - \tilde{\lambda}_i| \leqslant \left(\epsilon(2+\epsilon) + \frac{(1+\epsilon)^2\delta(2-\delta)}{(1-\delta)^2}\right)|\lambda_i|, \quad i = 1, 2, \ldots, n. \tag{9}$$

Proof. Define $Z = B^{-1/2}$ and consider $Y = I + Z^{-1}\Delta BZ$, which satisfies $(I + \Delta B)^T B(I + \Delta B) = Z^{-1}Y^H YZ^{-1}$. We observe that the pencil $(I + \Delta A)^T A(I + \Delta A) - \lambda(I + \Delta B)^T B(I + \Delta B) = (I + \Delta A)^T A(I + \Delta A) - \lambda(Z^{-1}Y^H YZ^{-1})$ has the same eigenvalues as the matrix $Y^{-H}Z(I + \Delta A)^T A(I + \Delta A)ZY^{-1}$. Hence we shall compare the eigenvalues of the matrices ZAZ and $Y^{-H}Z(I + \Delta A)^T A(I + \Delta A)ZY^{-1}$.

Using the same argument as in the proof of Theorem 3.1, we have (cf. (8))

$$|\lambda_i(Z(I + \Delta A)^H A(I + \Delta A)Z) - \lambda_i| = \epsilon(2+\epsilon)|\lambda_i|. \tag{10}$$

Next we recall that $Y = I + Z^{-1}\Delta BZ$, and see that $\|Z^{-1}\Delta BZ\|_2 \leqslant \kappa_2(Z)\|\Delta B\|_2 = \sqrt{\kappa_2(B)}\|\Delta B\|_2 (\equiv \delta)$. It follows that the singular values of Y^{-1} lie in $[1/(1+\delta), 1/(1-\delta)]$, so we have

$$\|Y^{-H}Y^{-1} - I\|_2 \leqslant 1/(1-\delta)^2 - 1 = \frac{\delta(2-\delta)}{(1-\delta)^2}.$$

Therefore, using the relative Weyl theorem and (10) we have

$$\begin{aligned}
&|\lambda_i(Y^{-H}Z(I + \Delta A)^H A(I + \Delta A)ZY^{-1}) - \lambda_i(Z(I + \Delta A)^H A(I + \Delta A)Z)| \\
&\quad\leqslant |\lambda_i(Z(I + \Delta A)^H A(I + \Delta A)Z)| \cdot \|Y^{-H}Y^{-1} - I\|_2 \\
&\quad\leqslant (1+\epsilon)^2 \frac{\delta(2-\delta)}{(1-\delta)^2}|\lambda_i|.
\end{aligned}$$

Therefore, (9) is obtained by

$$\begin{aligned}
&|\lambda_i(Y^{-H}Z(I + \Delta A)^H A(I + \Delta A)ZY^{-1}) - \lambda_i| \\
&\quad\leqslant \epsilon(2+\epsilon)|\lambda_i| + \frac{(1+\epsilon)^2\delta(2-\delta)}{(1-\delta)^2}|\lambda_i|. \quad \square
\end{aligned}$$

Theorems 3.1 and 3.2 do not directly match the relative Weyl theorem for standard eigenvalue problems by letting $B = I$ and $\Delta B = 0$, because a general unitary transformation on A is not allowed.

Nonetheless, our results are consistent, as the following argument indicates. Consider the pencil $X^H AX - \lambda I$. If $\|X^H X - I\|_2 = \epsilon$ and $\epsilon < 1$ then the singular values of X must lie in $[\sqrt{1-\epsilon}, \sqrt{1+\epsilon}]$. Hence, X can be written as $X = U + \Delta U$, in which U is the unitary polar factor of X (the closest unitary matrix to X [8, pp. 197]) and $\|\Delta U\|_2 \leqslant 1 - \sqrt{1-\epsilon}$. Then, the pencil $X^H AX - \lambda I$ is rewritten as $U^H\left(I + \left(\Delta UU^H\right)^H\right)A\left(I + \Delta UU^H\right)U - \lambda I$, which a unitary transformation shows is equivalent to $(I + (\Delta UU^H)^H)A(I + \Delta UU^H) - \lambda I$. Noting that $\|\Delta UU^H\|_2 = \|\Delta U\|_2 \leqslant 1 - \sqrt{1-\epsilon}$ and using Theorem 3.1 (or 3.2) for the pencils $A - \lambda I$ and $(I + (\Delta UU^H)^H)A(I + \Delta UU^H) - \lambda I$, we see that the pencil $X^H AX - \lambda I$ has eigenvalues that match those of the pencil $A - \lambda I$ to relative accuracy $(1 - \sqrt{1-\epsilon})(2 + 1 - \sqrt{1-\epsilon})$. Notice that $(1 - \sqrt{1-\epsilon})(2 + 1 - \sqrt{1-\epsilon}) \simeq \epsilon$ when $\epsilon \ll 1$, yielding the relative Weyl theorem. Hence, Theorems 3.1 and 3.2 become equivalent to the relative Weyl theorem when $B = I$, $\Delta B = 0$ and $\epsilon \ll 1$.

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