CAT 2019

Problem Sheet 2

Homology

(1) Find the smallest open cover of the circle with contractible intersections. What is the nerve of this good cover? Find a cover of the circle containing at least two open sets which violates the hypotheses of the nerve lemma. What is the nerve of this bad cover?

(2) Let K be a simplicial complex. Show that for any oriented n-simplex $\sigma = (v_0, \ldots, v_n)$ in K, the squared algebraic boundary $\partial^K_{n-1} \circ \partial^K_n(\sigma)$ is the zero chain in C^K_{n-2}.

(3) Show that the Euler characteristic of a finite simplicial complex, usually defined as the alternating count of simplices, also equals the alternating sum of dimensions of homology groups:

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i \dim H_i(K)$$

(You can use $\mathbb{R}$ coefficients for the homology).

(4) A simplicial complex is *connected* if any two vertices u and v can be connected by a path of consecutive edges, i.e.,

$$u \leftrightarrow w_0 \leftrightarrow w_1 \leftrightarrow \cdots \leftrightarrow w_k \leftrightarrow v$$

Show that $H_0(K)$ has dimension one whenever K is connected.

(5) Show that for any simplicial map $f : K \to L$, the induced linear maps $C_i f : C^K_i \to C^L_i$ constitute a chain map.

(6) Compute the homology of a 2-sphere modelled as the boundary ∂K of a 3-simplex K.

(7) Let K and L be simplicial complexes. Identify a vertex v of K with a vertex w of L to create a new simplicial complex $K \wedge L$. Prove that $H_i(K \wedge L) \simeq H_i(K) \oplus H_i(L)$ for all $i > 0$. Hint: Mayer-Vietoris.