CAT 2019

Problem Sheet 3

Cohomology and Persistence

Problems to Hand In

(1) Prove the cup product identity. Namely, given a simplicial complex K and cochains $\phi \in C^i(K)$ and $\psi \in C^j(K)$, show that

$$d^{i+j}(\phi \smile \psi) = (d^i\phi \smile \psi) + (-1)^j(\phi \smile d^j\psi).$$

(2) Let $f : K \to L$ be (what else?) a simplicial map. There is a certain commuting diagram of vector spaces, a part of which is shown below:

$$\begin{array}{ccccccc} H^i(K) & \times & H_j(K) & \xrightarrow{\frown} & H_{j-i}(K) \\ \\ H^i(L) & \times & H_j(L) & \xrightarrow{\frown} & H_{j-i}(L) \end{array}$$

- draw three vertical arrows representing maps induced by f which connect the top row to the bottom row. What are natural candidates for these maps?
- formulate an identity based on the fact that your diagram commutes. You do not have to prove that this identity holds (but it is a good exercise to meditate on).

(3) Let $[a, b)$ and $[c, d)$ be two intervals (i.e., bounded connected subsets of the real line) so that $a < c < b < d$. What is the interleaving distance between the two corresponding interval modules $\mathbb{I}_\bullet^{[a,b)}$ and $\mathbb{I}_\bullet^{[c,d)}$?

(4) Show that the set of all global sections of a sheaf $\mathcal{F}$ on a simplicial complex X has a natural vector space structure. Then show that the dimension of this vector space is exactly the dimension of the zeroth sheaf cohomology $H^0(K; \mathcal{F})$.

(5) Show that if the Hausdorff distance between two finite sets P and Q of $\mathbb{R}^n$ is less than ϵ, then the persistence modules of the Čech filtrations around P and Q are ϵ-interleaved.

This problem may require you to dig deep through the material of Lectures 1 and 2, so I'd like to help you a bit. Take a deep breath — here is a guideline to one possible solution.

- Use the Haursdorff condition to show that every $p \in P$ has a Q-friend, i.e., some $q \in Q$ so that the Euclidean distance $\|p - q\|$ is at most ϵ. Similarly, for each q there is a P-friend. [Note that this does *not* imply a one-to-one mapping between points of P and points of Q!]

- now use Q-friends to build a chain map ϕ_t from the chain complex of $\check{\mathbf{C}}_t(P)$ to that of $\check{\mathbf{C}}_{t+\epsilon}(Q)$ for every $t \geq 0$. Similarly build a chain map ψ_t going from the chains of $\check{\mathbf{C}}_t(Q)$ to those of $\check{\mathbf{C}}_{t+\epsilon}(P)$.
- Now we have to show that certain triangles and parallelograms commute. This is **hard** because we don't have a one-one matching of friends. But up to homotopy, this is not so bad! Show that the composite chain map $\psi_{t+\epsilon} \circ \phi_t$ is homotopic to the chain map induced by inclusion $\check{\mathbf{C}}_t(P) \to \check{\mathbf{C}}_{t+2\epsilon}(P)$ — we saw one good way of doing this in Corollary 9 of Lecture 2.

Problems to Think About (and not Hand In)

Some of these will only make sense after Week 6. Don't panic!

(1) Draw the **Petersen graph**, and construct an acyclic partial matching on it so that all but one of its vertices are paired. What can you conclude about the first Betti number of the Petersen graph?

(2) Let $K_0 \subset K_1$ be a pair of simplicial complexes, treated as a two-step filtration of K_1. How would find the dimension of the i-th relative homology group $H_i(K_1, K_0)$ from the i-th persistent homology barcode of this filtration?

(3) Consider the filtered simplicial complex shown below (the numbers next to simplices indicate the index at which they are born). Draw an acyclic partial matching compatible with this filtration, describe its critical cells and gradient paths, and use it to simplify the construction of the barcodes in dimensions 0 and 1 of this filtered complex.