

Algebraic Geometry Part III

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1 Motivation

Let $I \leq k[x_1, \dots, x_n]$ be an ideal. Then

$$V = V(I) = \{(a_1, \dots, a_n) \in \mathbb{A}_k^n \mid f(a_1, \dots, a_n) = 0 \ \forall f \in I\}$$

We define $K[V] = k[x_1, \dots, x_n]/I(V)$ where

$$I(V) = \{f \in k[x_1, \dots, x_n] \mid f|_V = 0\}$$

This is the classical setting.

Why do we consider schemes, and not just varieties?

1. We may want to work with non-algebraically closed field (eg. in number theory). If we do so with varieties, the coordinate ring is not necessarily recoverable from the geometry: if we work over $\mathbb{R}$ and consider $\langle x^2 + y^2 + 1 \rangle \leq \mathbb{R}[x, y]$. Then $V(I)$ is empty, so $I(V(I)) = \mathbb{R}[x, y]$ and the coordinate ring is the 0 ring.
2. If we care about Diophantine equations, we care about integer solutions, so with to consider $I \leq \mathbb{Z}[x_1, \dots, x_n]$, and we are interested in $V(I) \leq \mathbb{Z}^n$, ie we do not wish to work over a field.
3. Hilbert's Nullstellensatz says that if k is algebraically closed then $I(V(I)) = \sqrt{I}$, so we typically only work with radical ideals, ie those such that $I = \sqrt{I}$. If $I \neq \sqrt{I}$, then $k[x_1, \dots, x_n]/I$ has nilpotent elements (any element of $\sqrt{I} \setminus I$). This doesn't make sense if we think of these as rings of functions. However consider $X_1 = V(x - y^2)$ and $X_2 = V(x)$. We want to consider their intersection as a 'double point' (as the intersection of their defining ideals is $\langle x, y^2 \rangle$), but the classical picture loses this information by reducing this ideal to $\langle x, y \rangle$. if we considered $X_2 = V(x - c)$, we have two roots, and would like to reflect the limiting behaviour.
4. There is a categorical philosophical reason: What is a point? If we let Set be the category of sets. Let $*$ be a singleton. If we look at $\text{Hom}(*, X)$ for any set X , this is in bijection with X . More generally, knowing about X is 'the same thing' as knowing about $\text{Hom}(Y, X)$ for all Y . What happens if we consider this for varieties? For a point, the coordinate ring is k . Giving a morphism $* \rightarrow V$ for V an affine variety is the same as a morphism $\phi: k[V] \rightarrow k$. This is determined by the image of the coordinates on V , with the restriction that they must vanish on $I(V)$. Note since $\phi(1) = 1$, it is surjective, and hence $\ker \phi$ is a maximal ideal. If $k = \bar{k}$, then we know all maximal ideals arise this way. Thus points of V are in bijection with k -algebra maps $k[V] \rightarrow k$ which are in bijection with maximal ideals of $k[V]$ if $k = \bar{k}$. If k is not algebraically closed, what if we consider k -algebra maps $k[V] \rightarrow k'$ where k' is a field extension of k . This allows us to consider solution sets to I inside $(k')^n$ for $k' \supseteq k$. Now these maps need not be surjective, so the kernels are not necessarily maximal, but the image is an integral domain, so the kernels are prime.

5. Why use solutions only in fields? Let R be an arbitrary (commutative, unital) ring, $A = \mathbb{Z}[x_1, \dots, x_n]/I$. An ‘ R -valued point’ of $V(I)$ is a ring homomorphism $A \rightarrow R$, and hence an R -valued point is a solution of the equations in R .

Example 1.1. Considering $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$. This case map to $\mathbb{C}$ (eg by $x \mapsto 0, y \mapsto i$) with image $\mathbb{C}$ and kernel $(x, y^2 + 1)$, which is a maximal ideal. However we would get the same kernel for $y \mapsto -i$ instead, so solutions to the equation $x^2 + y^2 + 1 = 0$ with components in $\mathbb{C}$ correspond to maximal ideal, but not bijectively. In particular Galois orbits of points are sent to the same maximal ideal.

Example 1.2. If $k[V]$ is an integral domain, let $k(V)$ be the field of fractions of $k[V]$. By definition we have an inclusion $k[V] \rightarrow k(V)$, which is non-surjective if $k[V]$ is not a field, and the kernel is 0, so we get a point corresponding to (0) .

Taking the thesis of (5) to its logical extreme, we use any ring in place of $\mathbb{Z}[x_1, \dots, x_n]/I$ for $\mathbb{A}$. The set of ‘ R -valued points’ of $X = \text{Spec } \mathbb{A}$ is $X(R) = \text{Hom}_{\text{Ring}}(A, R)$. A “morphism” from $\text{Spec } A \rightarrow \text{Spec } B$ is just a ring homomorphism $B \rightarrow A$.

Definition 1.1. The category of *affine schemes* is the opposite category to the category of rings (If you’re not taking category theory, give up now, this is literally how he defined it in the lecture)

Remark. All rings are unital and commutative

Definition 1.2. A scheme is something covered by affine schemes (we will define this better later)

Remark. “Something” is a locally ringed space, but we won’t get there for another couple weeks

2 Spectra of Rings

Definition 2.1. Let A be a ring. The *spectrum of A* is $\text{Spec } A = \{\mathfrak{p} \trianglelefteq A \mid \mathfrak{p} \text{ is prime}\}$. For $I \trianglelefteq A$, we define $V(I) = \{\mathfrak{p} \in \text{Spec } A \mid \mathfrak{p} \supseteq I\}$

Remark. Note we can assume I is an ideal, as if it were a set, replacing it by the ideal it generates would not change $V(I)$.

Proposition 2.1. *The subsets $V(I)$ form the closed sets of a topology on $\text{Spec } A$*

Proof. 1. $V(A) = \emptyset$, $V(0) = \text{Spec } A$ so the whole space and empty set are of the form $V(I)$

2. If $\{I_j\}_{j \in J}$ is a collection of ideals then $V(\sum_{j \in J} I_j) = \bigcap_{j \in J} V(I_j)$ so the set of sets $V(I)$ is closed under arbitrary intersection

3. $V(I \cap J) = V(I) \cup V(J)$. The direction $\supseteq$ is obvious. If $\mathfrak{p} \supseteq I \cap J$ and $\mathfrak{p}$ is prime then $\mathfrak{p} \supseteq I$ or $\mathfrak{p} \supseteq J$ (See A-MD 1.11, but try to prove yourself!) $\square$

Definition 2.2. The above topology is called the *Zariski topology*

Example 2.1. If $A = k[x_1, \dots, x_n]$ with k algebraically closed, there is a bijection between points of $\mathbb{A}^n$ and maximal ideals of A . Given $I \subseteq A$, the zero set of I is in bijection with maximal ideals containing I

3 Sheaf Theory

We fix a topological space X .

Definition 3.1. A *presheaf* $\mathcal{F}$ of abelian groups on X consists of the data

1. An abelian group $\mathcal{F}(U)$ for every open set U
2. Whenever $V \subseteq U \subseteq X$ are open, there is a *restriction map* $\rho_{UV}: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ which is a group homomorphism such that $\rho_{UU} = \text{id}$ and when $W \subseteq V \subseteq U$, $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$

Definition 3.2. A morphism of presheaves $f: \mathcal{F} \rightarrow \mathcal{G}$ consists of homomorphisms $f_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for all open $U \subseteq X$ such that

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \rho_{UV} \downarrow & & \rho_{UV} \downarrow \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

commutes

Example 3.1. $\mathcal{F}(U) = \{\phi: U \rightarrow \mathbb{R} \text{ continuous}\}$ with restriction maps being literal restriction is the key example of a presheaf.

Definition 3.3. A presheaf $\mathcal{F}$ is a sheaf if it additionally satisfies

1. If $U \subseteq X$ is open and $\{U_i\}_{i \in I}$ is an open cover of U , and $s \in \mathcal{F}(U)$ has $\rho_{UU_i}(s) = 0$ for all $i \in I$, then $s = 0$
2. If U , $\{U_i\}_{i \in I}$ as in (1), $s_i \in \mathcal{F}(U_i)$ such that $\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j)$ for all $i, j \in I$ (we will abuse notation and use $|_V$ for the restriction to V maps) then there is $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$

Remark.

1. If $\mathcal{F}$ is a sheaf then $\emptyset$ has an empty covering so $\mathcal{F}(\emptyset) = 0$

2. Together, the sheaf axioms can be stated as saying

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_{i \in I} \mathcal{F}(U_i) \xrightleftharpoons[\beta_2]{\beta_1} \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

where $\alpha(s) = (s|_{U_i})_{i \in I}$, $\beta_1((s_i)_{i \in I}) = (s_i|_{U_i \cap U_j})$ and $\beta_2((s_i)_{i \in I}) = (s_j|_{U_i \cap U_j})$, is exact (ie α is injective, and every element of the middle term with agreeing β_1, β_2 image comes from an element of $\mathcal{F}(U)$, and conversely, every element of $\mathcal{F}(U)$ agrees on $\beta_1 \circ \alpha$ and $\beta_2 \circ \alpha$)

Example 3.2.

1. If X is a topological space, then $\mathcal{F}$ as in Example 3.1 is a sheaf
2. If $X = \mathbb{C}$ with the Euclidean topology, and $\mathcal{F}(U) = \{\phi: U \rightarrow \mathbb{C} \text{ bounded holomorphic}\}$. If we cover $\mathbb{C}$ with unit balls, $f(z) = z$ is a function bounded on all of them that agrees on overlaps, but it does not extend to a bounded holomorphic function on all $\mathbb{C}$
3. Let G be a group and set $\mathcal{F}(U) = G, \mathcal{F}(\emptyset) = 0$ is the constant presheaf. It is not a sheaf in full generality as if an open has two disconnected open components, then gluing would give us $\mathcal{F}(U) = \mathcal{F}(U_1) \oplus \mathcal{F}(U_2)$, contradiction. This is called the *constant presheaf*
4. To get the *constant sheaf* on G , we put the discrete topology on G and define $\mathcal{G}(U) = \{f: U \rightarrow G \text{ continuous}\}$, the locally constant functions into G .
5. Let X be a variety. If $U \subseteq X$ is a Zariski open we define $\mathcal{O}_X(U) = \{f: U \rightarrow k \text{ regular}\}$. This is a sheaf, called the *structure sheaf* of X .

Definition 3.4. Let $p \in F$ and let $\mathcal{F}$ be a presheaf on X . Then the *stalk* of $\mathcal{F}$ at p , denoted $\mathcal{F}_p$, is $\{(U, s) \mid p \in U \text{ open}, s \in \mathcal{F}(U)\} / \sim$ where $(U, s) \sim (V, t)$ if there is $W \subseteq U \cap V$ which is an open neighbourhood of p such that $s|_W = t|_W$. We call an equivalence class a *germ*.

Remark. Note that a morphism $F: \mathcal{F} \rightarrow \mathcal{G}$ of presheaves induces a homomorphism $F_p: \mathcal{F}_p \rightarrow \mathcal{G}_p$ by $F_p((U, s)) = (U, F_U(s))$.

Proposition 3.1. Let $f: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. f is an isomorphism if and only if f_p is for every $p \in X$.

Proof. The forward direction is clear, as composition of morphisms induces the composition of their morphism on stalks, on stalks.

For the other direction, assume f_p is an isomorphism for all $p \in X$. We need f_U to be an isomorphism for all $U \subseteq X$ open, as given this we can define $(f^{-1})_U = (f_U)^{-1}$.

Proof that f_U is injective (note later that we need this to prove surjectivity): Suppose $s \in \mathcal{F}(U)$ and $f_U(s) = 0$. We know $f_p(U, s) = (U, f_U(s)) = 0$, so $(U, s) = 0$ in $\mathcal{F}_p$ by injectivity of f_p . Therefore there is some neighbourhood

$U_p \subseteq U$ such that $s|_{U_p} = 0$. The collection of all such U_p is an open cover of U , so we can glue $0 = s|_{U_p}$ to get $s = 0$.

Proof that f_U is surjective: Let $t \in \mathcal{G}(U)$ and write $t_p = (U, t) \in \mathcal{G}_p$. There is some $s_p \in \mathcal{F}_p$ with $f_p(s_p) = t_p$, so there is a neighbourhood $V_p \subseteq U$ of p and germ $(V_p, s_p) \in \mathcal{F}_p$ with $(V_p, f_{V_p}(s_p)) = (U, t)$. Shrinking V_p , we may assume $t|_{V_p} = f_{V_p}(s_p)$. The set of all V_p cover U , and we wish to glue the s_p . On $V_p \cap V_q$, $f_{V_p \cap V_q}(s_p|_{V_p \cap V_q} - s_q|_{V_p \cap V_q}) = t|_{V_p \cap V_q} - t|_{V_p \cap V_q} = 0$, so as $f_{V_p \cap V_q}$ is injective, $s_p|_{V_p \cap V_q} = s_q|_{V_p \cap V_q}$, and we can glue the s_p to get $s \in \mathcal{F}(U)$. Now $f_U(s)|_{V_p} = f_{V_p}(s_p) = t|_{V_p}$, so $f_U(s) = t$ on U and we are done. $\square$

Remark. An element $s \in \mathcal{F}(U)$ is called a *section* of $\mathcal{F}$ over U

Given a presheaf $\mathcal{F}$ there is a sheaf $\mathcal{F}^+$ and a morphism $\theta: \mathcal{F} \rightarrow \mathcal{F}^+$ of presheaves such that if $\mathcal{G}$ is a sheaf and we have a morphism $f: \mathcal{F} \rightarrow \mathcal{G}$, there is a unique morphism $f^+: \mathcal{F}^+ \rightarrow \mathcal{G}$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F} & & \\ \theta \downarrow & \searrow f & \\ \mathcal{F}^+ & \xrightarrow{f^+} & \mathcal{G} \end{array}$$

The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism (by the usual process).

Definition 3.5. We call $\mathcal{F}^+$ the *sheafification* of $\mathcal{F}$

Lemma 3.1. $\mathcal{F}^+$ exists for any sheaf $\mathcal{F}$

Proof. Define

$$\mathcal{F}^+(U) = \left\{ s: U \rightarrow \coprod_{p \in U} \mathcal{F}_p \left| \begin{array}{l} (1) \ s(p) \in \mathcal{F}_p \\ (2) \ \forall p \in U, \exists \text{ a neighbourhood } p \in V, t \in \mathcal{F}(V) \\ \text{and } \forall q \in V, s(q) = (V, t) \in \mathcal{F}_q \end{array} \right. \right\}$$

For $t \in \mathcal{F}(U)$ we define $\theta(t) = (p \mapsto (U, t) \in \mathcal{F}_p)$. It is on the first example sheet to check this works $\square$

Definition 3.6. For $f: \mathcal{F} \rightarrow \mathcal{G}$ a morphism of presheaves, we define

1. The *presheaf kernel* $\ker f$ is $(\ker f)(U) = \ker f_U$
2. The *presheaf cokernel* $\operatorname{coker} f$ is $(\operatorname{coker} f)(U) = \operatorname{coker} f_U$
3. The *presheaf image* $\operatorname{im} f$ is $(\operatorname{im} f)(U) = \operatorname{im} f_U$

Remark. If $f: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves, the $\ker f$ is a sheaf. Any subpresheaf of a sheaf satisfies the first sheaf axiom, and if $s_i \in (\ker f)(U_i)$ with $\{U_i\}$ covering U and s_i agree on overlaps, we get a glued $s \in \mathcal{F}(U)$ which is sent to 0 on each U_i , so it is sent to 0 on U , hence $s \in (\ker f)(U)$

Example 3.3. Take $X = \mathbb{P}^1$, let $P \neq Q \in X$, let $\mathcal{G}$ be the sheaf of regular functions on X , and $\mathcal{F}$ the sheaf of regular functions on X vanishing at P, Q . Now $f: \mathcal{F} \subseteq \mathcal{G}$. Note $\mathcal{F}(U) = \mathcal{G}(U)$ for $P, Q \notin U$. Let $U = \mathbb{P}^1 \setminus \{P\}$ and $V = \mathbb{P}^1 \setminus \{Q\}$. These cover $\mathbb{P}^1$. Note $\mathcal{F}(\mathbb{P}^1) = 0$ and $\mathcal{G}(\mathbb{P}^1) = k$. Thus $\text{coker}(f)(\mathbb{P}^1) = k$, and $(\text{coker}(f))(U) = k[X]/(X) = k$ and similarly $(\text{coker } f)(V) = k$. $(\text{coker } f)(U \cap V) = 0$, so then we would have $(\text{coker } f)(\mathbb{P}^1) = k \oplus k$ so $\text{coker } f$ is not a sheaf.

Definition 3.7. Let $f: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves

1. The *sheaf kernel* of f is the presheaf $\ker f$
2. The *sheaf image* of f is the sheafification of the presheaf $\text{im } f$
3. The *sheaf cokernel* of f is the sheafification of the presheaf $\text{coker } f$

Exercise 3.1. The sheaf image $\text{im } f$ is a subsheaf of $\mathcal{G}$

Definition 3.8. We say $f: \mathcal{F} \rightarrow \mathcal{G}$ is *injective* if $\ker f = 0$ and f is *surjective* if $\text{im } f = \mathcal{G}$.

We say a sequence of sheaves

$$\dots \longrightarrow \mathcal{F}^{i+1} \xrightarrow{f^i} \mathcal{F}^i \xrightarrow{f^{i-1}} \mathcal{F}^{i-1} \longrightarrow \dots$$

is *exact at* $\mathcal{F}$ if $\ker f^{i-1} = \text{im } f^i$.

If $\mathcal{F}' \subseteq \mathcal{F}$ is a subsheaf, we define $\mathcal{F}/\mathcal{F}'$ to be the sheafification of $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$. This is the sheaf cokernel of the inclusion, and

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}' \longrightarrow 0$$

is exact

Lemma 3.2. Let $f: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then $(\ker f)_p = \ker f_p$ for all $p \in X$ and $(\text{im } f)_p = \text{im } f_p$.

Proof. We have a map $(\ker f)_p \rightarrow \ker f_p$ by $(U, s) \mapsto (U, s)$ as $f_p((U, s)) = (U, f_U(s)) = 0$, so $(U, s) \in \ker f_p$.

Injectivity: Suppose $(U, s) \in (\ker f)_p$ with $(U, s) = 0 \in \mathcal{F}_p$. Then there is $p \in V \subseteq U$ with $s|_V = 0$. Then $(U, s) \simeq (V, s|_V) = 0$ in $(\ker f)_p$.

Surjectivity: If $(U, s) \in \ker f_p$, then $(U, f_U(s)) = 0$, so there is $p \in V \subseteq U$ such that $f_V(s|_V) = 0$. Then $(V, s|_V) \in (\ker f)_p$.

For the image, from the first example sheet, if $\mathcal{F}$ is a presheaf then $\mathcal{F}_p = \mathcal{F}_p^+$. Let $\text{im}' f$ be the presheaf image. Then $(\text{im}' f)_p = (\text{im } f)_p$. So we will show $(\text{im}' f)_p \cong \text{im } f_p$ by $(U, s) \mapsto (U, s)$.

Injectivity: If $(U, s) = 0 \in \mathcal{G}_p$ then there is $p \in V \subseteq U$ such that $s|_V = 0$, so $(U, s) = 0$ in $(\text{im}' f)_p$.

Surjectivity: If $(U, s) \in \text{im } f_p$ then there is $(V, t) \in \mathcal{F}_p$ such that $f_p((V, t)) = (U, s)$. Shrinking the open sets, we assume $U = V$ and $f_U(t) = s$. This shows $(U, s) \in (\text{im}' f)_p$ $\square$

Proposition 3.2. $f: \mathcal{F} \rightarrow \mathcal{G}$ is injective if and only if f_p is injective for all points p .

$f: \mathcal{F} \rightarrow \mathcal{G}$ is surjective if and only if $f_p: \mathcal{F}_p \rightarrow \mathcal{G}_p$ is surjective.

Proof. f_p is injective for all p is equivalent to $\ker f_p = 0$ for all p , if and only if $(\ker f)_p = 0$ for all p , if and only if $\ker f = 0$, as the morphism $0 \rightarrow \ker f$ is an isomorphism on stalks.

f_p is surjective for all p if and only if $\text{im} f_p = \mathcal{G}_p$ if and only if $(\text{im } f)_p = \mathcal{G}_p$ if and only if $\text{im } f = \mathcal{G}$. $\square$

3.1 Passing Between Spaces

Suppose $f: X \rightarrow Y$ is a continuous map, $\mathcal{F}$ is a sheaf on X and $\mathcal{G}$ is a sheaf on Y .

Definition 3.9. We define the sheaf $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ on Y , the *push-forward* sheaf.

We define the *pullback* sheaf $f^{-1}\mathcal{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \{(V, s) \mid f(U) \subseteq V \text{ open, } s \in \mathcal{G}(V)\} / \sim$ where $(V, s) \sim (V', s')$ if there is open $f(U) \subseteq W \subseteq V \cap V'$ with $s|_W = s'|_W$.

Example 3.4. Let $i: \{p\} \rightarrow X$. Then $i^{-1}\mathcal{G} = \mathcal{G}_p$. More generally, given $i: Z \rightarrow X$ an inclusion, we write $\mathcal{F}|_Z$ for $i^{-1}\mathcal{F}$.

If i is an inclusion of an open set and $U \subseteq Z$ is open, $(\mathcal{F}|_Z)(U) = \mathcal{F}(U)$.

We often write $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$. We think of $\Gamma(U, \cdot)$ as a functor from the category of presheaves to the category of groups.

4 Schemes

Our initial goal is to define affine schemes. To do this we will define a sheaf of functions over the topological space $\text{Spec } A$.

We recall localisation.

Definition 4.1. Let A be a ring and $S \subseteq A$. We say S is *multiplicatively closed* if $1 \in S$ and if $ab \in S$ whenever $a, b \in S$.

For a multiplicatively closed S , the *localisation* of A at S is $S^{-1}A = \{(a, s) \mid a \in A, s \in S\} / \sim$, where $(a, s) \sim (a', s')$ if there is $s'' \in S$ with $s''(as' - sa') = 0$. We write the equivalence class of (a, s) as a/s , and define $(a, s) + (a', s') = (as' + sa', ss')$ and $(a, s)(a', s') = (aa', ss')$.

Example 4.1. If $f \in A$ and $S = \{1, f, f^2, \dots\}$ we write A_f for $S^{-1}A$ and say that A_f is A localised at f .

If $\mathfrak{p} \subseteq A$ is prime, then $S = A \setminus \mathfrak{p}$ is multiplicatively closed, and we write $A_{\mathfrak{p}}$ for $S^{-1}A$.

We now fix a ring A , and aim to construct a sheaf of rings $\mathcal{O}$ on $\text{Spec } A$, ie $\mathcal{O}(U)$ is a ring for all open sets U and all the restriction maps are ring homomorphisms. We define

$$\mathcal{O}(U) = \left\{ s: U \rightarrow \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \left| \begin{array}{l} (1) \ s(\mathfrak{p}) \in A_{\mathfrak{p}} \\ (2) \ \forall p \in U, \exists \text{ a neighbourhood } p \in V, a, f \in A \\ \text{such that } \forall q \in V, f \notin \mathfrak{q}, s(\mathfrak{q}) = a/f \end{array} \right. \right\}$$

This is clearly a sheaf

Proposition 4.1.

1. For any $\mathfrak{p} \in \text{Spec } A$, $\mathcal{O}_{\mathfrak{p}} = A_{\mathfrak{p}}$
2. For $f \in A$, $\mathcal{O}(D(f)) = A_f$. In particular, $\mathcal{O}(\text{Spec } A) = A$.

Proof. (1) We have a map $\mathcal{O}_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}$ via $(U, s) \mapsto s(\mathfrak{p})$. This is surjective: Any element of $A_{\mathfrak{p}}$ can be written as a/f for $a \in A, f \notin \mathfrak{p}$, then $D(f) = (\text{Spec } A) \setminus V(f)$ so $D(f)$ is open (a “basic” open). Then a/f defines a section $s \in \mathcal{O}(D(f))$ given by $s(q) = a/f \in A_q$, so $s(p) = a/f$, hence $(D(f), s) \mapsto a/f$.

This is injective: Let $\mathfrak{p} \in U \subseteq \text{Spec } A$, $s \in \mathcal{O}(U)$ with $s(p) = 0$ in $A_{\mathfrak{p}}$. Then $(U, s) \mapsto 0$. By shrinking U , we may assume there are $a, f \in A$ such that $s(q) = a/f \in A_q$ for all $q \in U$. So as $a/f = 0/1$ in $A_{\mathfrak{p}}$, there is $h \in A \setminus \mathfrak{p}$ with $ha = 0$. Let $V = D(f) \cap D(h)$. This is a neighbourhood of p , and $(V, s|_V) = 0$ as $a/f = 0$ in A_q for all $q \in V$. Therefore $(U, s) = 0$. (2) Let $\psi: A_f \rightarrow \mathcal{O}(D(f))$ be given by $a/f^n \mapsto (p \mapsto a/f^n \in A_p)$.

Injectivity: If $\psi(a/f^n) = 0$ then for all $\mathfrak{p} \in D(f), a/f^n = 0$ in $A_{\mathfrak{p}}$, so there is $h \notin \mathfrak{p}$ such that $ha = 0$ in A . Let $I = \{g \in A \mid ga = 0\}$. So $h \in I$, so $I \not\subseteq \mathfrak{p}$. This is true for all $p \in D(f)$, so $V(I) \cap D(f) = \emptyset$, so $f \in \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p} = \sqrt{I}$, so $f^m \in I$ for some $m > 0$, so $f^m a = 0$, so $a/f^n = 0$ in A_f .

Surjectivity: let $s \in \mathcal{O}(D(f))$. Cover $D(f)$ with open sets V_i such that $f|_{V_i} = a_i/f_i$. By example sheet 1 q1, $D(h)$ form a basis for the topology on $\text{Spec } A$, so we can assume $V_i = D(h_i)$ for some $h_i \in A$. Note $V_i \subseteq D(f_i)$ so $D(h_i) \subseteq D(f_i)$, or $V(f_i) \subseteq V(h_i)$, so $h_i^n = c_i f_i$ for some $c_i \in A$. Thus $a_i/f_i = c_i a_i/h_i^n$. Replacing each h_i with h_i^n , our fraction now has denominator h_i . We claim $D(f)$ is covered by a finite number of the $D(h_i)$. We know $D(f) \subseteq \bigcup D(h_i)$, which is equivalent to $V(f) \supseteq \bigcap V(h_i) = V((h_i \mid i \in I))$, which is equivalent to $f \in \sqrt{(h_i \mid i \in I)}$ which is equivalent to some $f^n \in \sqrt{(h_i \mid i \in I)}$. So $f^n = \sum_{i=1}^m b_i h_i$, so $D(h_1), \dots, D(h_m)$ cover $D(f)$. On $D(h_i) \cap D(h_j)$ note $a_i/h_i, a_j/h_j$ both represent s and that $D(h_i) \cap D(h_j) = D(h_i h_j)$. By injectivity of ψ , $a_i/h_i = a_j/h_j$ in $A_{h_i h_j}$. Thus for some n , $(h_i h_j)^n (h_j a_i - h_i a_j) = 0$. Pick n sufficiently large as to work for all i, j . We may rewrite our equation as $h_j^{n+1} (h_i^n a_i) - h_i^{n+1} (h_j^n a_j) = 0$. So $(h_j^n a_j)/h_j^{n+1} = (h_i^n a_i)/h_i^{n+1}$. Then $s|_{D(h_i)}$ is still represented by a_i/h_i , but $a_i h_j = a_j h_i$. Let $a = \sum b_i a_i$. Then for any j , $h_j a = \sum h_j b_i a_i = a_j \sum b_i h_i = a_j f^n$, so $a_j/h_j = a/f^n$, so $\psi(a/f^n) = s$, so ψ is surjective. $\square$

Definition 4.2. A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X . A *morphism* of ringed spaces is the data of $f: X \rightarrow Y$ which is continuous and $f^\#: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a morphism of sheaves of rings, ie a ring homomorphism $f_U^\#: \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$ for every open set U of Y .

Example 4.2.

1. Let $\mathcal{O}_X, \mathcal{O}_Y$ be the sheaves of continuous real valued functions on X and Y . Given a continuous $f: X \rightarrow Y$, we get $f_U^\#: \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$ by $\phi \mapsto \phi \circ f$
2. If X, Y are varieties and $\mathcal{O}_X, \mathcal{O}_Y$ the sheaves of regular functions on X, Y . A morphism $f: X \rightarrow Y$ gives in particular $f_U^\#: \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$ by $\phi \mapsto \phi \circ f$

Both of these examples have a further feature, which is that $\mathcal{O}_{X,p}$ are local rings (ie have unique maximal ideals, $\mathfrak{m}_p = \{f \in \mathcal{O}_{X,p} \mid f(p) = 0\}$). Note that the invertible elements of $\mathcal{O}_{X,p}$ are $\mathcal{O}_{X,p} \setminus \mathfrak{m}_p$

Definition 4.3. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that every $\mathcal{O}_{X,p}$ is local. A *morphism of locally ringed spaces* is a morphism of ringed spaces $(f, f^\#)$ such that $f_p^\#: \mathcal{O}_{Y,f(p)} \rightarrow \mathcal{O}_{X,p}$ is a local homomorphism, where a *local homomorphism* is a morphism $g: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{m}')$ of local rings such that $\phi^{-1}(\mathfrak{m}') = \mathfrak{m}$

Remark. Note that for a local ring A , $\phi(A \setminus \mathfrak{m}) = \phi(A^*) \subseteq B^* = B \setminus \mathfrak{m}'$, where A^* is the units of A . In the guiding examples, the condition to be a local homomorphism just says that a function which is 0 at $f(p)$ pulls back to something which is 0 at p .

Example 4.3. $(\text{Spec } A, \mathcal{O})$ is a locally ringed space. This is called an *affine scheme*

Theorem 4.1. *The category whose objects are affine schemes and whose morphisms are morphisms of locally ringed spaces is equivalent to Ring^{op}*

Proof. We need to show

1. If $\phi: A \rightarrow B$ is a morphism of rings, then we obtain a morphism of locally ringed spaces $(f, f^\#): (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \rightarrow (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$
2. Given a morphism $(f, f^\#)$ of affine schemes, it comes from a ring homomorphism

We show these as follows

1. Suppose we are given $\phi: A \rightarrow B$. If $\mathfrak{p} \subseteq B$ is prime, then so is $\phi^{-1}(\mathfrak{p}) \subseteq A$. We define $f: \text{Spec } A \rightarrow \text{Spec } B$ by $p \mapsto \phi^{-1}(\mathfrak{p})$. We want to show f is

continuous. We have

$$\begin{aligned} f^{-1}(V(I)) &= \{\mathfrak{p} \in \operatorname{Spec} B \mid \phi^{-1}(\mathfrak{p}) \subseteq I\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} B \mid \mathfrak{p} \supseteq \phi(I)\} \\ &= V(\phi(I)) \end{aligned}$$

so it is continuous. We now construct $f^\#$. Now for $\mathfrak{p} \in \operatorname{Spec} B$ we obtain a map $\phi_{\mathfrak{p}}: A_{\phi^{-1}(\mathfrak{p})} \rightarrow B_{\mathfrak{p}}$ by $a/s \mapsto \phi(a)/\phi(s)$. We show this is a local homomorphism. The maximal ideal of $B_{\mathfrak{p}}$ is the ideal generated by $\mathfrak{p}$, and similarly for $\phi^{-1}(\mathfrak{p})$ in $A_{\phi^{-1}(\mathfrak{p})}$. We know $\phi_{\mathfrak{p}}^{-1}(\mathfrak{p}B_{\mathfrak{p}}) = \phi^{-1}(\mathfrak{p})A_{\phi^{-1}(\mathfrak{p})}$, so we are done. Now given $V \subseteq \operatorname{Spec} A$, we get $f_V^\#: \mathcal{O}_{\operatorname{Spec} A}(V) \rightarrow \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V))$ by $f_V^\#(\mathfrak{p} \mapsto s(\mathfrak{p})) = (q \mapsto \phi_q(s(f(q))))$. We need to check that the image we have defined is locally coherent (ie if s is locally given by a/h , then $f_V^\#(s)$ is locally given by $\phi(a)/\phi(h)$ - exercise). Since $f^\#$ induces $\phi_{\mathfrak{p}}$ on stalks, we are done.

2. Suppose we are given $(f, f^\#): \operatorname{Spec} B \rightarrow \operatorname{Spec} A$ is a morphism of locally ringed spaces. We have

$$\phi = f_{\operatorname{Spec} A}^\#: \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A \rightarrow B = \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B)$$

We wish to show this gives rise of f . We have $f_p^\#: A_{f(\mathfrak{p})} \rightarrow B_{\mathfrak{p}}$ which is a local homomorphism. This is compatible with the map on global sections, ie

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{\phi^{-1}(\mathfrak{p})} & \xrightarrow{f_{\mathfrak{p}}^\#} & B_{\mathfrak{p}} \end{array}$$

commutes. Then $(f_{\mathfrak{p}}^\#)^{-1}(\mathfrak{p}B_{\mathfrak{p}}) = f(\mathfrak{p})A_{f(\mathfrak{p})}$, and thus $\phi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$ as pulling back around the diagram commutes. Thus f and $f_p^\#$ are induced by ϕ , so we are done. □

Definition 4.4. An *affine scheme* is a locally ringed space isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A . A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ with an open cover $\{(U_i, \mathcal{O}_X|_{U_i})\}$ where each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme. A *morphism of schemes* is a morphism of locally ringed spaces.

Example 4.4. If k is a field, $\operatorname{Spec} k = (*, k)$. If X is a scheme, a morphism $f: \operatorname{Spec} k \rightarrow X$ selects a point $x \in X$ and we have a map $f^\#: \mathcal{O}_{X,x} \rightarrow k$. This induces $f_U^\#$ which is 0 if $x \notin U$ and $s \mapsto (U, s) \mapsto f_*^\#(U, s) \in k$ so as $f^\#$ is local, we get an inclusion $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \hookrightarrow k$, where $k(x)$ is called the *residue field* of $x \in X$. This all reverses, so this data gives a map of schemes

Remark. If $X = \operatorname{Spec} A$, giving $\operatorname{Spec} k \rightarrow X$ is equivalent to giving some $A \rightarrow k$, ie a k -valued point

Example 4.5. Consider $f: X \rightarrow \operatorname{Spec} k$ for k a field. Note that $f^\# : \mathcal{O}_{\operatorname{Spec} k} = k \rightarrow f_* \mathcal{O}_X$, ie a map $k \rightarrow \mathcal{O}_X(X)$, so $\Gamma(X, \mathcal{O}_X)$ is a k algebra, hence so are the $\mathcal{O}_X(U)$ by restriction. Similarly the $\mathcal{O}_{X,x}$ are k -algebras. We say X is a scheme *defined over* $\operatorname{Spec} k$. For example if $A = k[x_1, \dots, x_n]/I$, then $\operatorname{Spec} A$ is our replacement for $V(I)$ and comes with a canonical map $\operatorname{Spec} A \rightarrow \operatorname{Spec} k$. If $k \subseteq k'$ is a field extension, a k' point is a commutative diagram (left) which is the same as a commutative diagram (right)

$$\begin{array}{ccc} k' & \xleftarrow{\quad} & A \\ & \nwarrow \quad \nearrow & \\ & k & \end{array} \qquad \begin{array}{ccc} \operatorname{Spec} k' & \xrightarrow{\quad} & \operatorname{Spec} A \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} k & \end{array}$$

We write $X(k')$ for this set of k' -valued points.

Remark. In general we rarely work with schemes themselves, but schemes over a base scheme S . We define the category Sch/S to have objects morphisms of schemes $X \rightarrow S$ and morphisms commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

We will usually work with $\operatorname{Sch}/k = \operatorname{Sch}/\operatorname{Spec} k$

If $T, X \rightarrow S$ are two objects of Sch/S , a T valued point is a morphism $T \rightarrow X$ over S . We write $X(T)$ for the set of these. By Yoneda, $X(T)$ for all T determines X

Example 4.6. Let $D = \operatorname{Spec} k[t]/(t^2)$ over k and let X be a scheme over $\operatorname{Spec} k$. What is a morphism $f: D \rightarrow X$ over $\operatorname{Spec} k$? $D = (\{(t)\}, k[t]/(t^2))$, so this selects a point $x \in X$ and has a k algebra homomorphism $f^\# : \mathcal{O}_{X,x} \rightarrow k[t]/(t^2)$ where $\mathfrak{m}_x$ maps into (t) and $\mathfrak{m}_x^2 \mapsto 0$, so we have a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \rightarrow (t) = k$ as k vector spaces, ie an element of $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$. We also get an isomorphism of residue fields $k(x) \cong k$, and given this data we get such a map. $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ is known as the *Zariski tangent space* to x in X .

Given schemes X_1, X_2 , $U_i \subseteq X_i$ open and an isomorphism $\phi: U_1 \rightarrow U_2$ we can glue X_1, X_2 to get a scheme with open over X_1, X_2 (see example sheet 1 for details)

Example 4.7.

1. Define $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \dots, x_n]$. Take $X_1, X_2 = \mathbb{A}_k^1$ and $U_i = \mathbb{A}_k^1 \setminus \{(x)\}$ and glue U_i via the identity. We get the two-origin line. We can also glue via $x \mapsto 1/x$, as $U_i = \operatorname{Spec} k[x]_x$. This gives us the projective line over k , a new scheme which we write $\mathbb{P}_k^1$.

4.1 Projective Schemes

Definition 4.5. A *graded ring* is a ring with a specified direct sum decomposition, $S = \bigoplus_{d \geq 0} S_d$ where S_d is an abelian group under addition, and $S_d \cdot S_e = S_{d+e}$.

Example 4.8. $S = k[x_0, \dots, x_n]$ with $\deg x_i = 1$ and S_d the homogeneous polynomials of degree d is a graded ring.

Definition 4.6. The *irrelevant ideal* is $S_+ = \bigoplus_{d > 0} S_d$. A *homogeneous ideal* $I \subseteq S$ is an ideal generated by *homogeneous elements* (elements in one of the S_d).

$\text{Proj } S = \{\mathfrak{p} \in \text{Spec } S \mid \mathfrak{p} \text{ is homogeneous, } \mathfrak{p} \not\supseteq S_+\}$.

If $I \subseteq S$ homogeneous, we define $V(I) = \{\mathfrak{p} \in \text{Proj } S \mid \mathfrak{p} \supseteq I\}$.

Exercise 4.1. (On example sheet 2) Check $V(I)$ for $I \subseteq S$ homogeneous are the closed sets for a topology on $\text{Proj } S$.

For $\mathfrak{p} \in \text{Proj } S$, let $T = \{f \in S \setminus \mathfrak{p} \mid f \text{ homogeneous}\}$. Define $S_{(\mathfrak{p})}$ to be the subring of $T^{-1}S$ consisting of elements of degree 0 (where $\deg f/g = \deg f - \deg g$). Similarly for $f \in S$ homogeneous, we write $S_{(f)}$ to be the subring of S_f of elements of degree 0. We define the structure sheaf $\mathcal{O}$ of $\text{Proj } S$ as

$$\mathcal{O}(U) = \left\{ s: U \rightarrow \prod_{\mathfrak{p} \in U} S_{(\mathfrak{p})} \left| \begin{array}{l} (1) \ s(\mathfrak{p}) \in S_{(\mathfrak{p})} \\ (2) \ \forall p \in U, \exists \text{ a neighbourhood } p \in V, a, f \in S \\ \text{homogeneous such that } \forall q \in V, f \notin \mathfrak{q}, s(\mathfrak{q}) = a/f \end{array} \right. \right\}$$

As before, $\mathcal{O}_{\mathfrak{p}} = S_{(\mathfrak{p})}$

Definition 4.7. For $f \in S$ homogeneous, we define the *distinguished open sets*, $D_+(f) = \{\mathfrak{p} \in \text{Proj } S \mid f \notin \mathfrak{p}\}$. These form a cover of $\text{Proj } S$

Proposition 4.2. $(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \text{Spec } S_{(f)}$ as locally ringed spaces

Proof. Example Sheet 2. □

Definition 4.8. If A is a ring, we write $\mathbb{P}_A^n = \text{Proj } A[x_0, \dots, x_n]$

Remark. If $A = k = \bar{k}$ is a field, then the closed points (the points x with $\overline{\{x\}} = \{x\}$) of $\mathbb{P}_A^n$ are in bijection with points of $(k^{n+1} \setminus \{0\})/\sim$ where $(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n)$. If $a_0 \neq 0$, then $(a_0, \dots, a_n)$ corresponds to $\langle a_0 x_i - a_i x_0 \mid 1 \leq i \leq n \rangle$

Example 4.9. Let $S = k[x_0, \dots, x_n]$ and let $\deg x_i = w_i \in \mathbb{Z}_{>0}$. $\text{Proj } S$ is in general called *weighted projective space* and is written $W\mathbb{P}^n(w_0, \dots, w_n)$. If $n = 2$, $w_0 = w_1 = 1$, $w_2 = 2$, then $D_+(x_2) \cong \text{Spec } S_{(x_2)} = k[x_0^2/x_2, x_0 x_1/x_2, x_1^2/x_2] \subseteq S_{x_2}$. Note we have an isomorphism $k[u, v, w]/(uw - v^2) \rightarrow k[x_0^2/x_2, x_0 x_1/x_2, x_1^2/x_2]$ by $u \mapsto x_0^2/x_2, v \mapsto x_0 x_1/x_2, w \mapsto x_1^2/x_2$, and the former is very much not a subset of $\mathbb{P}^2$

Example 4.10. Let $M = \mathbb{Z}^n$ and let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^n$. Let $\Delta \subseteq M_{\mathbb{R}}$ be a compact convex lattice polytope (ie there is a finite set $V \subseteq M$ such that Δ is the convex hull of V). Let $C(\Delta) = \{(m, r) \mid r \in \mathbb{R}_{\geq 0}, m \in r\Delta\}$. Let $S = k[C(\Delta) \cap (M \oplus \mathbb{Z})] = \bigoplus_{p \in C(\Delta) \cap (M \oplus \mathbb{Z})} k \cdot z^p$. We define a product by $z^p \cdot z^q = z^{p+q}$. We define a grading by $\deg z^{(m, r)} = r$ and define $\text{Proj } S = \mathbb{P}_{\Delta}$.

1. If we take Δ the triangle with vertices $(0, 1), (1, 0), (0, 0)$, then $S = k[x, y, z]$ and $\mathbb{P}_{\Delta} = \mathbb{P}^2$
2. If we take Δ to be the unit square, the degree d monomials are $\{z^{(a, b, d)} \mid 0 \leq a, b \leq d\}$ and in fact $S \cong k[x, y, w, t]/(xy - wt)$ via $x = (0, 0, 1), y = (1, 1, 1), w = (1, 0, 1), t = (0, 1, 1)$, so $\mathbb{P}_{\Delta}$ is a quadric surface in $\mathbb{P}^3$. $\mathbb{P}_{\Delta}$ is called a *projective toric variety*

4.2 Open and Closed Subschemes

Definition 4.9. An *open subscheme* of a scheme $(X, \mathcal{O}_X)$ is $(U, \mathcal{O}_X|_U)$ for $U \subseteq X$ open. An *open immersion* is a morphism $f: X \rightarrow Y$ which induces an isomorphism of X with an open subscheme of Y

Definition 4.10. A *closed immersion* $f: X \rightarrow Y$ is a morphism which is a homeomorphism of X onto a closed subset of Y and the induced morphism $f^{\#}: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a surjection. A *closed subscheme* of Y is an equivalence class of closed immersions where $f \sim g$ if there is an isomorphism of their domains, ϕ , with $f = g \circ \phi$

Example 4.11.

1. If $X = \text{Spec } A$ and $I \subseteq A$ is an ideal, and we take $Y = \text{Spec } A/I$. Then $A \rightarrow A/I$ induces a map $Y \rightarrow X$. For $p \in Y$, we have $f_p^{\#}: \mathcal{O}_{X, f(p)} \rightarrow \mathcal{O}_{Y, p}$ which is $A_{f(p)} \rightarrow (A/I)_p$, which is surjective. This identifies Y with $V(I)$ (primes of A/I are in bijection with primes of A containing I). Note the scheme structure depends on I and not just $V(I)$.
2. If $A = k[x], I = (x^2)$, $\text{Spec } A/I = (\{(t)\}, k[x]/(x^2))$. Note this is a different scheme than if $I = (x)$
3. $\text{Spec } k[x, y]/(x^2, xy)$ is a closed subscheme of $\mathbb{A}_k^2 = \text{Spec } k[x, y]$. Classically, $V(x^2, xy) = V(x)$. Note that on $D(x)$, the ideal I becomes (x) , hence on $D(x)$, $\text{Spec } k[x, y]/(x^2, yx)$ and $\text{Spec } k[x, y]/(x)$ are isomorphic. The primary decomposition of (x^2, xy) is $(x) \cap (x^2, xy, y^2)$

4.3 Fibre Products

Let $\mathbb{C}$ be a category and

$$\begin{array}{ccc} & & Y \\ & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

a diagram in $\mathcal{C}$. Then the *fibre product*, if it exists, is an object W with morphisms $p: W \rightarrow X, q: W \rightarrow Y$ with $g \circ q = f \circ p$ such that for any W' and morphisms $p': W' \rightarrow X, q': W' \rightarrow Y$ with $g \circ q' = f \circ p'$ then there is a unique $h: W' \rightarrow W$ such that $p' = p \circ h, q' = q \circ h$, ie

$$\begin{array}{ccccc}
 & W' & & & \\
 & \swarrow q' & & \searrow q & \\
 & & W & \xrightarrow{\quad} & Y \\
 & \swarrow p' & \downarrow p & & \downarrow g \\
 & & X & \xrightarrow{\quad f \quad} & Z
 \end{array}$$

If this exists it is unique up to unique isomorphism.

In the category of sets, fibre products exist and are $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$, and in general we write fibre products as $X \times_Z Y$

Definition 4.11. Let C be a category. We write h_X for the contravariant functor $h_X: C \rightarrow \text{Set}$ where $h_X(Y) = \text{Hom}(Y, X)$ and $h_X(f: Y \rightarrow Z): \text{Hom}(Z, X) \rightarrow \text{Hom}(Y, X)$ via $\psi \mapsto \psi \circ f$

Definition 4.12. A *natural transformation* of contravariant functors $T: F \rightarrow G$ where $F, G: C \rightarrow D$ consists of the data $T(X): F(X) \rightarrow G(X)$ morphisms in D such that for all $f: X \rightarrow Y$ in C ,

$$\begin{array}{ccc}
 F(X) & \xleftarrow{Ff} & F(Y) \\
 T(X) \downarrow & & \downarrow T(Y) \\
 G(X) & \xleftarrow{Gf} & G(Y)
 \end{array}$$

is commutative

Lemma 4.2. The set of natural transformations from $h_X: C \rightarrow \text{Set}$ and a contravariant functor $G: C \rightarrow \text{Set}$ is $G(X)$

Proof. See Category Theory □

Definition 4.13. A contravariant functor $F: C \rightarrow \text{Set}$ is *representable* if there is an object X such that $F \cong h_X$. We say X *represents* the functor F

Definition 4.14. (Redefinition of fibre product) The fibre product of

$$\begin{array}{ccc}
 & Y & \\
 & \downarrow g & \\
 X & \xrightarrow{\quad f \quad} & Z
 \end{array}$$

in a category C is an object which represents the functor $T \mapsto \text{Hom}(T, X) \times_{\text{Hom}(T, Z)} \text{Hom}(T, Y)$

This redefinition makes it easy to check identities involving fibre products via the Yoneda lemma, eg $(A \times_B C) \times_C D \cong A \times_B D$. Note that a corollary of Yoneda is that $\text{Hom}(h_X, h_Y) = \text{Hom}(Y, X)$.

Theorem 4.3. *Fibre products exist in the category of schemes*

Proof. Step (1): If $X = \text{Spec } A, Y = \text{Spec } B, S = \text{Spec } R$ with ring homomorphisms $R \rightarrow A, B$ giving a diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & S \end{array}$$

We want $X \times_S Y$. Reversing the arrows we get

$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & & \\ & & B \end{array}$$

and the colimit of this is $A \otimes_R B$ of this, so the limit of the former diagram is $X \times_S Y = \text{Spec } A \otimes_R B$ in the category of affine schemes. If T is an arbitrary scheme, giving a morphism $T \rightarrow \text{Spec } A$ is the same as a morphism $A \rightarrow \Gamma(T, \mathcal{O}_T)$. Thus we get a limit in the category of schemes.

Step (2): We use the gluing construction of ES1 Q14. We will also need to glue morphisms. Suppose given X, Y schemes, $\{U_i\}$ an open cover of X and morphisms $f_i: U_i \rightarrow Y$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Then we can glue the f_i to obtain $f: X \rightarrow Y$ such that $f|_{U_i} = f_i$ (exercise). If $X, Y \rightarrow S$ and suppose $X \times_S Y$ exists, with projections p_1, p_2 . Then we claim $p_1^{-1}(U) \cong U \times_S Y$. We draw the diagram

$$\begin{array}{ccccc} & & T & & \\ & \searrow & & \nearrow & \\ & & p_1^{-1}(U) \hookrightarrow X \times_S Y \longrightarrow Y & & \\ & \downarrow & \downarrow & \downarrow & \\ & U \hookrightarrow X \longrightarrow S & & & \end{array}$$

Then the composed map $T \rightarrow U \rightarrow Y$ and the map $T \rightarrow Y$ induce a map $T \rightarrow X \times_S Y$. The image of the induced map lands in U under p_1 by commutativity, and hence it factors through $p_1^{-1}(U)$, giving a unique map $h: T \rightarrow p_1^{-1}(U)$, so we have proven the claim.

Step (3): Suppose $\{X_i\}$ is an open cover of X and $X, Y \rightarrow S$ are given, and suppose the $X_i \times_S Y$ all exist. We claim $X \times_S Y$ exists. Let $X_{ij} = X_i \cap X_j$. We know $U_{ij} = p_1^{-1}(X_{ij}) \subseteq X_i \times_S Y$ is isomorphic to $X_{ij} \times_S Y$ by the previous step. Thus by uniqueness of the fibre product, there is a unique isomorphism $\psi_{ij}: U_{ij} \rightarrow U_{ji}$. It is an exercise to check that the hypotheses of ES1 Q14 hold and so we can glue the U_i to get a scheme we call $X \times_S Y$. We need to check

this is the fibre product. So suppose we have

$$\begin{array}{ccc} T & \xrightarrow{p'_2} & Y \\ p'_1 \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

Set $T_i = p_1'^{-1}(X_i)$, so we get a unique morphism $\theta_i: T_i \rightarrow X_i \times_S Y \subseteq X \times_S Y$. Now $\theta_i|_{T_i \cap T_j} = \theta_j|_{T_i \cap T_j}$ as both morphisms factor through $X_{ij} \times_S Y, X_{ji} \times_S Y$ which are identified by ψ_{ij} , so equality follows from the universal property. Thus we can glue the θ_i to get a $\theta: T \rightarrow X \times_S Y$. Uniqueness follows from the uniqueness for the θ_i

Step (4): Using steps (1), (3), we obtain $X \times_S Y$ when S, Y are affine by covering X in affines. Flipping arguments and doing the same for Y , we obtain existence when only S is affine

Step (5): For X, Y, S arbitrary, take an open affine cover $\{S_i\}$ of S and let $f: X \rightarrow S, g: Y \rightarrow S$ be the morphisms. Set $X_i = f^{-1}(S_i), Y_i = g^{-1}(S_i)$. Then $X_i \times_{S_i} Y_i$ exists and is isomorphic to $X_i \times_S Y_i$ by the universal property, as everything factors through X_i, Y_i so lands in S_i . Now we use step (3) twice to get $X \times_S Y_i$ then $X \times_S Y$, so we are done. $\square$

Example 4.12. $\mathbb{A}_k^{n_1} \times_{\text{Spec } k} \mathbb{A}_k^{n_2} = \text{Spec } k[x_1, \dots, x_{n_1}] \otimes_k k[x_1, \dots, x_{n_2}] = \mathbb{A}_k^{n_1+n_2}$

Definition 4.15. (Fibres of morphisms) For sets, to get a fibre over $\{y\}$ we take $\{y\} \times_Y X$ (expand the definitions). To define the fibre of $f: X \rightarrow Y$ over $y \in Y$. Recall $k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_Y$ and there is a canonical morphism $\text{Spec } k(y) \rightarrow Y$ with image y . We define $X_y = \text{Spec } k(y) \times_Y X$ as the (*Scheme-theoretic*) fibre of f at y

Example 4.13. If $X = \text{Spec } k[x] \rightarrow \text{Spec } k[t] = Y$ by $t \mapsto x^2$ with $y = t - a$ for $a \in k$, $k(x) = k[t]/(t - a)$ so $X_y = \text{Spec } k[t]/(t - a) \times_{\text{Spec } k[t]} \text{Spec } k[x] = \text{Spec } k[x]/(x^2 - a)$. This consists of 2 points if a has two square roots in k , 1 point ‘with fuzz’ if a has one square root and 1 point with residue field k' a degree 2 extension of k if a has no square roots in k . The fibre over $\{(0)\}$ (the generic point, as its closure is the space). $\mathcal{O}_{\mathbb{A}^1, (0)} = k(t)$. To find the scheme-theoretic fibre over (0) , we look at

$$k[x] \otimes_{k[t]} k(t) = k(t)[x]/(t - x^2)$$

The spectrum of this is just a closed subscheme of $\mathbb{A}_{k(t)}^1$

5 Properties of Schemes and Morphisms

(See example sheet 2 for more properties)

Definition 5.1. A scheme X is *integral* if for every open $U \subseteq X$, $\mathcal{O}_X(U)$ is an integral domain

A scheme X is *irreducible* if it is irreducible as a topological space (ie it is not

the union of two proper closed subsets) (the opposite is reducible)

A scheme X is *reduced* if for every open $U \subseteq X$, $\mathcal{O}_X(U)$ is reduced (the opposite is non-reduced).

Proposition 5.1. *A scheme X is integral if and only if it is irreducible and reduced*

Proof. On ES2 □

Definition 5.2. A morphism $f: X \rightarrow Y$ is *locally of finite type* if there is a covering of Y by open affines $V_i = \text{Spec } B_i$ such that $f^{-1}(V_i)$ can be covered by open affines $U_{ij} = \text{Spec } A_{ij}$ such that $f|_{U_{ij}}: U_{ij} \rightarrow V_i$ inducing $B_i \rightarrow A_{ij}$ makes A_{ij} a finite B_i algebra

We say f is of *finite type* if the cover $\{U_{ij}\}$ of $f^{-1}(V_i)$ is finite for all i

Example 5.1. If $Y = \text{Spec } k$, then $X \rightarrow Y$ is locally of finite type if X can be covered by open affines $\text{Spec } A_i$ where A_i is a finitely generated k -algebra. Thus $A_i = k[x_1, \dots, x_n]/I$, so $\text{Spec } A_i$ is a closed subscheme of $\mathbb{A}_k^n$

Definition 5.3. Let k be an algebraically closed field. Then a *variety* over k is a scheme X over $\text{Spec } k$ such that X is integral and $X \rightarrow \text{Spec } k$ is finite type.

Remark. This still allows the two-origin line.

5.1 Separated and Proper Morphisms

Definition 5.4. Let $f: X \rightarrow Y$ be a morphism of schemes and $\Delta: X \rightarrow X \times_Y X$ the diagonal morphism induced by

$$\begin{array}{ccccc}
 X & & & & \\
 & \searrow^{1_X} & & \searrow^{1_X} & \\
 & X \times_Y X & \xrightarrow{\quad} & X & \\
 & \downarrow & & \downarrow f & \\
 & X & \xrightarrow{\quad f \quad} & Y &
 \end{array}$$

We say f is *separated* if Δ is a closed immersion

Remark. Hausdorffness of a topological space X is equivalent to the diagonal in the product topology of $X \times X$ being closed. The line with two origins is not separated over $\text{Spec } k$, as the diagonal only includes two of the four origins in the product.

Definition 5.5. A scheme X is *locally Noetherian* if there is an open cover $\{U_i = \text{Spec } A_i\}$ with each A_i Noetherian. X is *Noetherian* if there is a finite such cover

Theorem 5.1. (*Valuative criterion for separatedness*) Let $f: X \rightarrow Y$ be a morphism with X Noetherian. Then X is separated if and only if the following

condition holds:

For any field k and any valuation ring R of k (ie R is an integral domain with field of fractions k and for all $x \in k \setminus \{0\}$, either $x \in R$ or $x^{-1} \in R$). Let $T = \text{Spec } R, U = \text{Spec } k$ and $i: U \rightarrow T$ the morphism induced by the inclusion. Given a diagram of the solid lines

$$\begin{array}{ccc} U & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow f \\ T & \longrightarrow & Y \end{array}$$

there is at most one morphism $T \rightarrow X$ making the diagram commute.

Proof. No Proof □

Example 5.2. (Sample application) If $f: X \rightarrow Y$ is separated and $g: Y \rightarrow Z$ is separated, so is $g \circ f: X \rightarrow Z$

Proof. We draw

$$\begin{array}{ccccc} U & \longrightarrow & X & & \\ \downarrow & \nearrow & \downarrow f & & \\ & & Y & & \\ \downarrow & \nearrow & \downarrow g & & \\ T & \longrightarrow & Z & & \end{array}$$

If we have two maps along the dotted line, we use the map $U \rightarrow X \rightarrow Y$ to get at most one map $T \rightarrow Y$ making the diagram commute, and hence we get at most one map $T \rightarrow X$ making the diagram commute. □

Definition 5.6. A morphism $f: X \rightarrow Y$ is *proper* if it is separated, of finite type, and universally closed (ie for any morphism $Y' \rightarrow Y$, the morphism $X \times_Y Y' \rightarrow Y'$ is a closed map)

Remark. In topology, $f: X \rightarrow Y$ is proper if the inverse image of a compact set is compact.

Example 5.3. $\mathbb{A}_k^1 \rightarrow \text{Spec } k$ is not proper. It is separated (any morphism of affine schemes is, as we just move the valuative criterion diagram into the category of rings), it is of finite type, and it is closed, but not universally closed. If we take $Y' = \mathbb{A}_k^1$, then we get

$$\begin{array}{ccccc} V(xy - 1) & \hookrightarrow & \text{Spec } k[x, y] & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ D(x) & \hookrightarrow & \text{Spec } k[y] & \longrightarrow & \text{Spec } k \end{array}$$

contradiction as $D(x)$ is not closed.

Theorem 5.2. Let $f: X \rightarrow Y$ be a morphism of finite type with X Noetherian. Then f is proper if the following holds:
For any field k and any valuation ring R of k (ie R is an integral domain with field of fractions k and for all $x \in k \setminus \{0\}$, either $x \in R$ or $x^{-1} \in R$). Let $T = \text{Spec } R, U = \text{Spec } k$ and $i: U \rightarrow T$ the morphism induced by the inclusion. Given a diagram of the solid lines

$$\begin{array}{ccc} U & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow f \\ T & \longrightarrow & Y \end{array}$$

there is exactly one morphism $T \rightarrow X$ making the diagram commute.

Example 5.4. If $\mathbb{P}_A^n = \text{Proj } A[x_0, \dots, x_n]$ with $\deg x_i = 1$, then there is a morphism $\mathbb{P}_A^n \rightarrow \text{Spec } A$ which is proper

6 Sheaves of $\mathcal{O}_X$ -modules

Definition 6.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Then a *sheaf of $\mathcal{O}_X$ -modules* is a sheaf of abelian groups $\mathcal{F}$ on X with $\mathcal{F}(U)$ having the structure of an $\mathcal{O}_X(U)$ -module, compatible with restriction, ie for $V \subseteq U, s \in \mathcal{O}_X(U), m \in \mathcal{F}(V)$ then $(sm)|_V = s|_V \cdot m|_V$

Definition 6.2. A *morphism of sheaves of $\mathcal{O}_X$ modules*, $\phi: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves of abelian groups such that $\phi_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$ -module homomorphism

Remark. We can define kernels, cokernels, etc and these inherit $\mathcal{O}_X$ module structures.

Remark. We denote by $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \{\phi: \mathcal{F} \rightarrow \mathcal{G}, \mathcal{O}_X\text{-module homomorphism}\}$, which is an $\mathcal{O}_X(X)$ module and we have a sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ defined by $U \mapsto \text{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ which is a sheaf of $\mathcal{O}_X$ module. We can also define $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Given $f: X \rightarrow Y$ a morphism of ringed spaces, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$ modules and $\mathcal{G}$ a sheaf of $\mathcal{O}_Y$ -modules. Then

1. $f_*\mathcal{F}$ is a sheaf of $f_*\mathcal{O}_X$ modules
2. Via $f^\#: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$, we see $f_*\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$ modules
3. $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$ module, but not an $\mathcal{O}_X$ module
4. The morphism of sheaves $f^\#: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ gives a map $f^\#: f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ by adjunction (ES1) and hence we can define $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, an $\mathcal{O}_X$ module.

Definition 6.3. A sheaf of $\mathcal{O}_X$ modules is *free* if it is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X$ for some index set I . If $|I| = r < \infty$, then we say it is of *rank* r .

A sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules is *locally free* if there is an open covering $\{U_i\}$ of X such that $\mathcal{F}|_{U_i}$ is free for all indices i . We say $\mathcal{F}$ is a *line bundle* or an *invertible sheaf* if $\mathcal{F}$ is locally free of rank 1

Example 6.1. Let $X = \text{Spec } A$ and let M be an A -module. For any $\mathfrak{p} \in \text{Spec } A$ we have a localisation $M_{\mathfrak{p}}$ which is an $A_{\mathfrak{p}}$ -module. We define a sheaf $\tilde{M}$ via

$$\tilde{M}(U) = \left\{ s: U \rightarrow \prod_{\mathfrak{p} \in U} M_{\mathfrak{p}} \left| \begin{array}{l} (1) \ s(\mathfrak{p}) \in M_{\mathfrak{p}} \\ (2) \ \forall p \in U, \exists \text{ a neighbourhood } p \in V, a \in M, f \in R \\ \text{such that } \forall q \in V, f \notin \mathfrak{q}, s(\mathfrak{q}) = a/f \end{array} \right. \right\}$$

Proposition 6.1.

1. $(\tilde{M})_{\mathfrak{p}} = M_{\mathfrak{p}}$
2. $\Gamma(D(f), \tilde{M}) = M_f$ and so $\Gamma(X, \tilde{M}) = M$

Proof. Same as before □

Definition 6.4. Let X be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$ modules. We say $\mathcal{F}$ is *quasi-coherent* if there is an affine covering $\{U_i = \text{Spec } A_i\}$ and A_i modules M_i such that $\mathcal{F}|_{U_i} \cong \tilde{M}_i$. We say $\mathcal{F}$ is *coherent* if the M_i can be taken to be finitely generated M_i modules.

Example 6.2. A locally free sheaf is always quasi coherent since $\bigoplus \mathcal{O}_{\text{Spec } A} = \widetilde{\bigoplus_{i \in I} A}$. It is coherent if the rank is finite everywhere

Remark. Kernels, cokernels, images and tensor products of quasicoherent sheaves are quasicoherent (this follows from statements like $\ker(\tilde{M} \rightarrow \tilde{N}) = \widetilde{\ker(M \rightarrow N)}$)

Remark. Note that if $\mathcal{L}$ is a line bundle on X with trivialising cover $\{U_i\}$, then on $U_i \cap U_j$, $\mathcal{O}_{U_i}|_{U_i \cap U_j} \cong \mathcal{L}|_{U_i \cap U_j} \cong \mathcal{O}_{U_j}|_{U_j \cap U_i}$, and composing these we get an automorphism of $\mathcal{O}_{U_i \cap U_j}$, called ϕ_{ij} . Giving such an automorphism is equivalent to giving $g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$ which is invertible. We denote by $\mathcal{O}_X^*$ the sheaf of abelian groups where $\mathcal{O}_X^*(U)$ is the units of $\mathcal{O}_X(U)$. Thus $g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$. Note that on $U_i \cap U_j \cap U_k$, we have $g_{ik} = g_{ij}g_{jk}$

Given a morphism $f: Y \rightarrow X$ of ringed spaces, how do we define $f^*\mathcal{L} = f^{-1}\mathcal{L} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y$, the pullback of a line bundle? Let $\{U_i\}$ be a trivialising cover with transition maps g_{ij} . Let $Y_i = f^{-1}(U_i)$ and f_i be the restriction of f to Y_i . Then $f_i^*(\mathcal{L}|_{U_i}) \cong f^*(\mathcal{O}_{U_i}) = \mathcal{O}_{Y_i}$. So Y_i forms a trivialising cover for $f^*\mathcal{L}$ and the transition functions are $f^\#(g_{ij})$.

Now let $\mathcal{L}_1, \mathcal{L}_2$ be two line bundles on X and let $\{U_i\}$ trivialise both. Let g_{ij}, h_{ij} be the respective transition functions. Then

$$(\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2)|_{U_i} = \mathcal{L}_1|_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{L}_2|_{U_i} = \mathcal{O}_{U_i}$$

as both are free on U_i . The transition functions are $g_{ij}h_{ij}$.

If we write $\mathcal{L}_1^\vee = \text{Hom}(\mathcal{L}_1, \mathcal{O}_X)$, then $\mathcal{L}_1^\vee$ is also a line bundle, with the same trivialising cover as $\mathcal{L}_1$. The transition functions are g_{ij}^{-1} , so in conclusion $\mathcal{L}_1^\vee \otimes_{\mathcal{O}_X} \mathcal{L}_1$ has transition maps 1, and hence is isomorphic to $\mathcal{O}_X$.

Definition 6.5. Let $(X, \mathcal{O}_X)$ be a ringed space. We denote by $\text{Pic}(X)$, the *Picard group* of X the set of isomorphism classes of line bundles in X , with multiplication the tensor product, and inversion being dualising. The identity is $\mathcal{O}_X$ and this is a group.

We care about line bundles as they tell us about morphisms into $\mathbb{P}^n$

6.1 Morphisms to Projective Space

We work over a field k , which we fix now, and work in Sch/k . We write $\mathbb{P}_k^n = \text{Proj } k[x_0, \dots, x_n]$ with the standard grading and we let F be the contravariant functor

$$T \in \text{ob}(\text{Sch}/k) \mapsto \left\{ \begin{array}{l} \text{Surjections } \mathcal{O}_T^{\oplus n} \rightarrow \mathcal{L} \\ \mathcal{L} \text{ a line bundle over } T \end{array} \right\} / \cong$$

where two surjections are isomorphic if there is an isomorphism of line bundles commuting with the surjection. Given $f: T_1 \rightarrow T_2$ we get a map of sets $F(T_2) \rightarrow F(T_1)$ by sending

$$(\mathcal{O}_{T_2}^{\oplus(n+1)} \xrightarrow{\phi} \mathcal{L}) \mapsto (f^* \mathcal{O}_{T_2}^{\oplus(n+1)} \cong \mathcal{O}_{T_1}^{\oplus(n+1)} \xrightarrow{f^* \phi} f^* \mathcal{L})$$

. The map on the right hand side is a surjection as $(f^{-1}\mathcal{F})_p = i_p^{-1}f^{-1}\mathcal{F} = (f \circ i_p)^{-1}\mathcal{F} = \mathcal{F}_{f(p)}$. If $M_1 \rightarrow M_2$ is a surjective map of A modules, then so is $M_1 \otimes_A N \rightarrow M_2 \otimes_A N$, and hence the surjection passes to f^* .

Theorem 6.1. F can be represented by $\mathbb{P}_k^n$.

Proof. If this holds, then there is a universal object, ie $1_{\mathbb{P}_k^n} \in \text{Hom}(\mathbb{P}_k^n, \mathbb{P}_k^n) = h_{\mathbb{P}_k^n}(\mathbb{P}_k^n) \cong F(\mathbb{P}_k^n)$, hence we have a canonical $\mathcal{O}_{\mathbb{P}_k^n}^{\oplus(n+1)} \rightarrow \mathcal{L}$ for some $\mathcal{L}$. Following the proof of Yoneda, given $f: X \rightarrow \mathbb{P}_k^n$ and the natural transformation $T: h_{\mathbb{P}_k^n} \rightarrow F$ we get a commutative diagram

$$\begin{array}{ccc} 1_{\mathbb{P}_k^n} & \xrightarrow{\quad} & (\mathcal{O}_{\mathbb{P}_k^n}^{\oplus(n+1)} \xrightarrow{\phi} \mathcal{L}) \\ \downarrow & & \downarrow \\ h_{\mathbb{P}_k^n}(\mathbb{P}_k^n) & \xrightarrow{T(\mathbb{P}_k^n)} & F(\mathbb{P}_k^n) \\ \downarrow & & \downarrow F(f) \\ h_{\mathbb{P}_k^n}(X) & \xrightarrow{T(X)} & F(X) \\ \downarrow & & \downarrow \\ f & \xrightarrow{\quad} & (\mathcal{O}_X^{\oplus(n+1)} \xrightarrow{f^* \phi} f^* \mathcal{L}) \end{array}$$

so in conclusion the element of $F(X)$ corresponding to a morphism $f: X \rightarrow \mathbb{P}^n$ is $(\mathcal{O}_X^{\oplus(n+1)} \xrightarrow{f^*\phi} f^*\mathcal{L})$ where $(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \xrightarrow{\phi} \mathcal{L})$ is the universal object. We aim to construct this universal object.

Now $\mathbb{P}^n$ has a cover $\{D_+(x_i) \mid 0 \leq i \leq n\}$ where $\mathbb{P}^n = \text{Proj } k[x_0, \dots, x_n]$. This is a cover as if $\mathfrak{p} \in \text{Proj } k[x_0, \dots, x_n]$ and $\mathfrak{p} \notin D_+(x_i)$ for any i , then every $x_i \in \mathfrak{p}$, so the irrelevant ideal is in $\mathfrak{p}$ contradiction. We take this cover to be the trivialising cover and we set $g_{ij} \in \mathcal{O}_{\mathbb{P}^n}^*(U_i \cap U_j) = x_i/x_j$, well defined as $U_i \cap U_j = D_+(x_i x_j) = \text{Spec } S_{(x_i x_j)}$ where $S = k[x_0, \dots, x_n]$, so $x_j/x_i, x_i/x_j \in S_{(x_i x_j)}$. On $U_i \cap U_j \cap U_k$, $g_{ik} = x_i/x_k = (x_i/x_j)(x_j/x_k) = g_{ij}g_{jk}$. This defines a line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. We have a map $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)$ where the former has basis $e_0, \dots, e_n$, defined on $D_+(x_i)$ by $e_j \mapsto x_j/x_i \in \mathcal{O}_{\mathbb{P}^n}(U_i)$. This is well defined as on $U_i \cap U_k$, we have $x_j/x_i = (x_k/x_i)(x_j/x_k)$. We need to verify this map is surjective. This is because on U_i , $e_i \mapsto x_i/x_i = 1$ so $f e_i \mapsto f$. It remains to show that given $(\mathcal{L}, \phi) \in F(X)$, there is a unique $f: X \rightarrow \mathbb{P}^n$ such that $(L, \phi) \cong (\mathcal{O}_{\mathbb{P}^n}(1), F(f))$. This gives a natural transformation $F \rightarrow h_{\mathbb{P}^n}$ with inverse given by pullback and we will be done. To construct the morphism, let s_i be $\phi(e_i) \in \mathcal{L}(X)$. Define $Z_i = \{x \in X \mid (s_i)_x \in \mathfrak{m}_x \mathcal{L}_x\}$ where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is the maximal ideal. We claim this set is closed. We can check this on an open cover $\{V_i\}$, so it suffices to check for $\text{Spec } A \subseteq X$ open with $\mathcal{L}|_{\text{Spec } A} \cong \mathcal{O}_{\text{Spec } A}$. Now $s_i|_{\text{Spec } A} = s_i \in A$ which induces $s_i/1 \in A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec } A$, so the condition $s_i/1 \in \mathfrak{m}_{\mathfrak{p}} A_{\mathfrak{p}}$ is equivalent to $s_i \in \mathfrak{m}_{\mathfrak{p}} A_{\mathfrak{p}} \cap A = \mathfrak{p}$, hence $Z_i \cap \text{Spec } A = V(s_i)$ is closed in $\text{Spec } A$. Let $V_i = X \setminus Z_i$ which is open and note $\mathcal{L}|_{V_i} \cong \mathcal{O}_{V_i}$ with the reverse isomorphism being $1 \mapsto s_i$ and the forwards being $s \mapsto s/s_i$. We define $f_i: V_i \rightarrow D_+(x_i) \subseteq \mathbb{P}^n$. It suffices to give a map $k[x_0/x_i, \dots, x_n/x_i] = S_{(x_i)} \rightarrow \Gamma(V_i, \mathcal{O}_X)$ given by $x_j/x_i \mapsto s_j/s_i$. On $V_i \cap V_j$ we need to check $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$. Now $(f_i^\#)|_{V_i \cap V_j}: S_{(x_i x_j)}$ acts by $x_k/x_i \mapsto s_k/s_i$ and so $x_k/x_j = (x_k/x_i)(x_j/x_i)^{-1}$ maps to s_k/s_j . A similar argument shows that f_j acts the same and so we can glue the f_i to get a morphism $f: \bigcup V_i \rightarrow \mathbb{P}^n$. Finally note that $\{V_i\}$ is a cover of X by surjectivity of ϕ as if $x \notin \bigcup V_i$, then $\phi(e_i) \in \mathfrak{m}_x \mathcal{L}_x$ for all i and hence we don't have a surjective map of sheaves. This gives $f: X \rightarrow \mathbb{P}^n$. Then $f^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{L}$ as the transition map x_i/x_j pulls back to s_i/s_j . We can think of ϕ being given by $s_0, \dots, s_n \in \Gamma(T, \mathcal{L})$ with $s_i = \phi(e_i)$. Similarly the universal object on $\mathbb{P}^n$ gives sections $s_0, \dots, s_n \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ with x_i the image of e_i . On U_i , x_j corresponds to $x_j/x_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. Let's assume that given $g: T \rightarrow \mathbb{P}^n$ and an isomorphism of sheaves as satisfied by f . We want to show $g = f$. The existence of the isomorphism tells us that $s_i \in \mathfrak{m}_p \mathcal{L}_p$ if and only if $x_i \in \mathfrak{m}_{g(p)} \mathcal{O}_{\mathbb{P}^n}(1)_{g(p)}$, so V_i , where s_i doesn't vanish, is $g^{-1}(U_i)$. Thus we need to show $g|_{V_i} = f_i$, but $g^\#(x_j/x_i) = s_j/s_i$, so we are done. $\square$

Remark. Morally if we were in the world of varieties, think of $f: X \rightarrow \mathbb{P}^n$ as given by $x \mapsto (s_0(x): \dots : s_n(x))$ - this makes sense as the ratios of sections do as long as $s_j(x) \neq 0$

7 Divisors and the Picard Group

We will define two types of divisor. Weil divisors will be linear combinations of codimension 1 subvarieties, and Cartier divisors will be subschemes defined locally by single equations.

Definition 7.1. The *dimension* of a topological space X is the length n of the longest chain $X_0 = \emptyset \subsetneq X_1 \subsetneq \dots \subsetneq X_n = X$ of irreducible closed subsets of X

Definition 7.2. The *Krull dimension* of a ring A is the dimension of $\text{Spec } A$, or equivalently the length of the longest chain of prime ideals $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n$

Example 7.1. $\dim \mathbb{A}_k^n = n$ (this is non-trivial, may be done in commutative algebra)

Definition 7.3. If $Z \subseteq X$ is irreducible closed, then the *codimension*, $\text{codim}(Z, X)$ is the length of the longest closed irreducible chain starting with Z

Remark. This notion of dimension can be non-intuitive even for Noetherian affine schemes - we may not have $\text{codim}(Z, X) = \dim X - \dim Z$. However if B is a domain and a finitely generated k -algebra, then this holds for $\text{Spec } B$ (in particular, this holds for varieties) Also if $\eta \in Z$ is the generic point of Z (the unique point with $\overline{\{\eta\}} = Z$, then $\text{codim}(Z, X) = \dim \mathcal{O}_{X, \eta}$ (easy from definitions, at least in the affine case)

Proposition 7.1. If X is a Noetherian scheme, then X is a Noetherian topological space (every decreasing set of closed sets stabilises). Every closed subset of a Noetherian topological space has a decomposition into a finite number of closed irreducible subsets

Proof. Trivial □

7.1 Weil Divisors

For discussing Weil divisors, we work under the following assumption:

Remark. (Assumption (*)) X is a Noetherian integral scheme over $\text{Spec } k$ which is *regular in codimension 1*, ie whenever $x \in X$ has $\dim \mathcal{O}_{X, x} = 1$, the ring $\mathcal{O}_{X, x}$ is regular, ie $\dim_{\mathcal{O}_{X, x}/\mathfrak{m}_x} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X, x}$

Standard commutative algebra (A+M, Chapter 9) tell us that if A is a regular Noetherian local domain of dimension 1, then A is a DVR (Discrete Valuation Ring, ie if $K = \text{Frac } A$ then there is a group homomorphism $\nu: K^* \rightarrow \mathbb{Z}$ such that $A = \{x \in K \mid \nu(x) \geq 0\}$ and $\mathfrak{m}_A = \{x \in K \mid \nu(x) \geq 1\}$, then $\nu(a) = k > 0$ if and only if $a \in \mathfrak{m}_A^k/\mathfrak{m}_A^{k+1}$)

Example 7.2.

1. $A = k[x]_{(x)}$, with $\nu(f/g)$ being the unique highest power of x dividing f , minus that of g

2. For $A = \mathbb{Z}_{(p)}$, you can do the same with powers of p instead of x .

Definition 7.4. Suppose assumption $(*)$ holds. A *prime divisor* on X is a closed subvariety (integral closed subscheme) of codimension 1. Let $\text{Div}(X)$ be the free abelian group generated by the prime divisors.

Remark. Let $K(X)$ denote the function field of X and recall that if $\text{Spec } A$ is a non-empty open subset of X , $\text{Frac } A = K(X)$. If Y is a prime divisor of X , let η be the generic point of Y . Then $\dim \mathcal{O}_{X,\eta} = 1$ so we have a valuation ν_Y on $K(X) \setminus \{0\} \rightarrow \mathbb{Z}$ such that $\mathcal{O}_{X,\eta} = \{x \in K(X)^* \mid \nu_Y(x) \geq 0\}$.

Lemma 7.1. *If X satisfies $(*)$ and $f \in K(X)^*$ then $\nu_Y(f) = 0$ for all but a finite number of prime divisors Y .*

Proof. We can find an open affine $U \subseteq X$, $U = \text{Spec } A$ so that $f \in \Gamma(U, \mathcal{O}_X)$. Taking any open affine $\text{Spec } B$ we can write $f = a/s$ for $a, s \in B$. Passing to $D(s) = \text{Spec } B_s$, $f \in B_s$ so we can take $A = B_s$. Let $Z = X \setminus U$. This is a closed subset of a Noetherian topological space, hence Z can be decomposed into a finite number of irreducible components, ie Z only contains a finite number of prime divisors of X . So we can replace X with $U = \text{Spec } A$. Now $\nu_Y(f) \geq 0$ for all prime divisors $Y \subseteq U$ as f is regular, and $\nu_Y(f) > 0$ if and only if $f \in \mathfrak{m}_\eta$, so $f \in A \cap \mathfrak{m}_\eta = \mathfrak{p}$, the prime ideal, so $\eta \in V((f))$ which is a proper closed subset. Repeat the before argument, and we are done. $\square$

Definition 7.5. Let X satisfy $(*)$ and $f \in K(X)^*$. The *divisor* of f is $(f) = \sum_{Y \subseteq X \text{ prime divisor}} \nu_Y(f) \cdot Y \in \text{Div}(X)$. We say $D, D' \in \text{Div } X$ are *linearly equivalent* if there is $f \in K(X)^*$ with $(f) = D - D'$. If $D \sim 0$, ie $D = (f)$, we say D is a *principal divisor*.

Remark. $(\cdot): K(X)^* \rightarrow \text{Div } X$ is a group homomorphism as every ν_Y is a group homomorphism. Also nothing above requires X to be defined over $\text{Spec } k$.

Definition 7.6. The (*divisor*) *class group* $\text{Cl}(X)$ is $\text{Div } X / \sim$, where $\sim$ is linear equivalence.

Proposition 7.2. *If A is an integrally closed Noetherian, then $A = \bigcap_{\mathfrak{p} \subseteq A, \text{ of ht } 1} A_{\mathfrak{p}} \subseteq \text{Frac } A$*

Proof. (Proof is Matsumura Theorem 38 p124) $\square$

Theorem 7.2. *Let A be a Noetherian integral domain. Then A is a UFD if and only if $X = \text{Spec } A$ is normal (integrally closed in its field of fractions) and $\text{Cl}(X) = 0$*

Proof. In the forwards direction, if A is a UFD, then A is integrally closed in its field of fractions (see Numbers and Sets), and A is a UFD if and only if every height 1 prime is principal. Let $Y \subseteq X$ be a prime divisor, which therefore corresponds to a height 1 prime $\mathfrak{p} \subseteq A$, so $\mathfrak{p} = (f)$. Then $(f) = 1 \cdot Y$, so $Y \sim 0$, so $\text{Cl}(X) = 0$.

In the other direction, suppose X is normal with trivial class group and $\mathfrak{p} \subseteq A$

is a prime of height 1. Then the corresponding prime divisor has $1 \cdot Y = (f)$ for some $f \in K(X)^*$. Since $\nu_Y(f) = 1$, $f \in A_{\mathfrak{p}} = \mathcal{O}_{X,\eta}$. If $\mathfrak{p}'$ is any other height 1 prime, then $\nu_{Y(\mathfrak{p}')} (f) = 0$ so $f \in A_{\mathfrak{p}'}$ is a unit. Thus $f \in A$ and $f \in \mathfrak{m}_{\mathfrak{p}} \subseteq A_{\mathfrak{p}}$ since $\nu_Y(f) = 1$, so $f \in A \cap \mathfrak{m}_{\mathfrak{p}} = \mathfrak{p}$. To show f generates $\mathfrak{p}$, let $g \in \mathfrak{p}$. Then $\nu_Y(g) \geq 1$ and $\nu_{Y'}(g) \geq 0$. So $\nu_{Y'}(g/f) = \nu_{Y'}(g) - \nu_{Y'}(f) \geq 0$, so $g/f \in A$ and $g = f(g/f) \in (f)$, so we are done. $\square$

Example 7.3. $\text{Cl}(\mathbb{A}_k^n) = 0$ as $k[x_1, \dots, x_n]$ is a UFD.

Proposition 7.3. *If X satisfies $(*)$, $Z \subseteq X$ is a proper closed subset, and $U = X \setminus Z$, then*

1. *There is a surjective homomorphism $\text{Cl}(X) \rightarrow \text{Cl}(U)$ by sending Y_i to $Y_i \cap U$, with the empty set treated as 0.*
2. *If $\text{codim}(Z, X) \geq 2$, the map is an isomorphism.*
3. *There is an exact sequence*

$$\mathbb{Z} \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 0$$

with $1 \mapsto 1 \cdot Z$ if Z is a prime divisor.

Proof.

1. If Y is a prime divisor on X , then $Y \cap U$ is empty, or a prime divisor in U . If $f \in K(X)^* = K(U)^*$ and $(f) = \sum a_i Y_i$, then $(f)_U = \sum a_i (Y_i \cap U)$. We thus obtain the desired map. To see that it is surjective, if Y is a prime divisor on U , its closure is a prime divisor in X .
2. $\text{Div } X$ and $\text{Cl } X$ only depend on codimension 1 subvarieties, and these are the same for X, U .
3. We know that $\ker(\text{Cl } X \rightarrow \text{Cl } U)$ consists of divisors supported on Z . Multiples of Z are the only such, so these form the kernel.

$\square$

Proposition 7.4. $\text{Cl } \mathbb{P}_k^n = \mathbb{Z}$ generated by the class of a hyperplane $H = V((x_i))$

Proof. H is a prime divisor by dimension theory. Now we have $\mathbb{P}^n \setminus H = D_+(x_i) = \mathbb{A}_k^n$ and $\text{Cl } \mathbb{A}_k^n = 0$. Therefore we have a surjective map $\mathbb{Z} \rightarrow \text{Cl } \mathbb{P}_k^n$, and so $\text{Cl } \mathbb{P}_k^n$ is generated by H , but we don't know if some $nH \sim 0$. The field of fractions $K(\mathbb{P}_k^n) = \{f/g \mid f, g \text{ homogeneous of same degree}\}$, so if there is $n \neq 0$ with $nH \sim 0$, then there are f, g such that $(f/g) = nH$, but $\deg(f/g) = 0$ and $\deg nH > 0$, contradiction. $\square$

Remark.

1. When X is a curve, we studied this in II Algebraic Geometry.
2. We call these divisors Weil divisors

7.2 Cartier Divisors

Let X be a scheme. We define the *sheaf of rational functions*, $\mathcal{K}_X$ on X to be the sheaf associated to the presheaf $U \mapsto S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ where $S(U)$ is the subset of elements whose stalks in $\mathcal{O}_{X,x}$ are non-zero-divisors for each $x \in U$. If X is integral, this is just the constant sheaf $U \rightarrow K(X)$. Let $\mathcal{K}_X$ be the sheaf of invertible elements of $\mathcal{K}_X$. We have an inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{K}_X^*$.

Definition 7.7. A *Cartier divisor* on a scheme X is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$. A Cartier divisor is *principal* if it is in the image of the homomorphism $\Gamma(X, \mathcal{K}_X^*) \rightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Two divisors are *equivalent* if they differ by a principal divisor, and the *Cartier class group* $\text{CaCl } X$ is the quotient of $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ by the principal divisors.

Remark. A section of $\mathcal{K}_X^*/\mathcal{O}_X^*$ can be locally lifted to a section of $\mathcal{K}_X^*$, so we can represent a section as a ‘compatible’ collection of liftings $\{(U_i, f_i)\}$ over an open cover. The compatibility is given by, on $U_i \cap U_j$, $\frac{f_i|_{U_i \cap U_j}}{f_j|_{U_i \cap U_j}} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$

7.2.1 Relation with Weil Divisors

We assume X satisfies (*). There is a homomorphism $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \rightarrow \text{Div } X$ which descends to $\text{CaCl } X \rightarrow \text{Cl } X$ given by the following: Given $\{(U_i, f_i)\}$ and a prime divisor Y on X , choose U_i such that $U_i \cap Y \neq \emptyset$ and set $n_Y = \nu_Y(f_i)$. If $U_j \cap Y \neq \emptyset$ then $U_i \cap U_j \cap Y \neq \emptyset$ and $\nu_Y(f_j) = \nu_Y(f_i(f_j/f_i)) = \nu_Y(f_i) + \nu_Y(f_j/f_i) = \nu_Y(f_i)$, so n_Y is well defined and we define $\psi(\{(U_i, f_i)\}) = \sum n_Y Y$. This is a group homomorphism, and a principal Cartier divisor is of the form $\{(X, f)\}$, so this descends to class groups.

Proposition 7.5. *If X satisfies (*) and all local rings $\mathcal{O}_{X,x}$ are UFDs, then the map on divisors is an isomorphism.*

Proof. We need to construct an inverse map $\text{Div } X \rightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Let $x \in X$. We get a morphism $j_x: \text{Spec } \mathcal{O}_{X,x} \rightarrow X$ (If $x \in \text{Spec } A \subseteq X$, then this is the localisation map. Now let $D \in \text{Div } X$ and pull back D along j_x . In more detail, if $Y \subseteq X$ is a prime divisor that doesn’t contain x , then $j_x^{-1}(Y) = \emptyset$, otherwise if $\text{Spec } A$ is an affine open containing x , then $Y \cap \text{Spec } A = V(q)$ for a prime ideal q , and then $q \cap \mathcal{O}_{X,x}$ is a prime of height 1. So we get a prime divisor in $\text{Spec } \mathcal{O}_{X,x}$, and this assembles to a map $\text{Div } X \rightarrow \text{Div Spec } \mathcal{O}_{X,x}$, say $D \rightarrow D_x$. Since height 1 primes are principal, $D_x = (f_x)$ where $f_x \in \text{Frac } \mathcal{O}_{X,x} = K(X)$. Now consider (f_x) as a divisor on X . Observe that D and (f_x) differ only on primes that don’t contain x . Let U_x be the complement of the support of the divisors where (f_x) and D disagree, so they agree on U_x . Note $x \in U_x$ so it is non-empty. Repeat this for all $x \in X$ to get a cover $\{U_x\}$ of X , and $\{f_x\} \subseteq K(X)$. We claim $\{(U_x, f_x)\}$ gives a Cartier divisor. We need f_x/f_y to be a unit on $U_x \cap U_y$. This function defines the zero divisor on $U_x \cap U_y$ by definition, so $f_x/f_y \in \mathcal{O}_{X,p}$ for all p of height 1 in $U_x \cap U_y$. Cover $U_x \cap U_y$ by opens of the form $\text{Spec } A$ and note $f_x/f_y \in A_{\mathfrak{p}}^*$ for any height 1 prime $\mathfrak{p}$. Now A_q are UFDs

for every q prime, and hence are integrally closed. Then $A = \bigcap_{\mathfrak{p} \subseteq A, \text{ height } 1} A_{\mathfrak{p}}$, so $f_x/f_y \in A^*$, so they glue to a section of $\mathcal{K}_X^*/\mathcal{O}_X^*$ $\square$

Remark. A deep theorem of Serre states that a regular local ring is a UFD. More generally if the hypothesis holds, we say X is *locally factorial*.

7.2.2 Cartier Divisors and Line Bundles

Definition 7.8. Let D be a Cartier divisor represented by $\{(U_i, f_i)\}$. Define $\mathcal{O}_X(D)$ to be the subsheaf of $\mathcal{K}_X$ generated by f_i^{-1} on U_i as an $\mathcal{O}_X$ module

Remark. This is a line bundle.

$$\mathcal{O}_X|_{U_i \cap U_j} \longrightarrow \mathcal{O}_X(D)|_{U_i \cap U_j} \longleftarrow \mathcal{O}_X|_{U_i \cap U_j}$$

$$1 \longmapsto f_i | f_j \longleftarrow 1$$

so the transition functions are f_i/f_j

Remark. If D_1, D_2 are Cartier divisors represented by $\{(U_i, f_i)\}$ and $\{(U_i, g_i)\}$ then $D_1 - D_2$ is represented by $\{(U_i, f_i/g_i)\}$. By checking transition maps, $\mathcal{O}_X(D_1 - D_2)$ has the same transition functions (and hence the same line bundle) as $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^\vee$, so we have proved that $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \rightarrow \text{Pic} X$ is a homomorphism.

Lemma 7.3. $D_1 \sim D_2$ if and only if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$

Proof. It suffices to check that D is principal if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X$. In the forwards direction, if D is principal, then D is represented by $\{(X, f)\}$ for $f \in \Gamma(X, \mathcal{K}_X^*)$, so $\mathcal{O}_X(D) = \mathcal{O}_X \cdot f^{-1} \subseteq \mathcal{K}_X$, which is clearly isomorphic to $\mathcal{O}_X$.

In the backwards direction, if $\mathcal{O}_X(D) \cong \mathcal{O}_X$, then let $1 \in \Gamma(X, \mathcal{O}_X) \mapsto f \in \Gamma(X, \mathcal{O}_X(D)) \subseteq \Gamma(X, \mathcal{K}_X)$. Then $\{(X, f^{-1})\}$ represents $D = \{(U_i, g_i)\}$ as f^{-1} and g_i differ by a factor of an invertible function on U_i , so D is principal. $\square$

Corollary 7.3.1. On any scheme X , $D \mapsto \mathcal{O}_X(D)$ defines an injective homomorphism $\text{CaCl}(X) \rightarrow \text{Pic} X$

Proof. Trivial $\square$

Proposition 7.6. If X is integral, then the above injection is an isomorphism.

Proof. We have to show every invertible sheaf is isomorphic to a subsheaf of $\mathcal{K}_X$, which is the constant sheaf $U \mapsto K(X)$ (as X is integral). On any open $U \subseteq X$ with $\mathcal{L}|_U \cong \mathcal{O}_U$ we have $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X)|_U \cong \mathcal{K}_X|_U$. Thus there is an open covering $\{U_i\}$ such that $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X)|_{U_i} \cong \mathcal{K}_X|_{U_i}$. But then $\mathcal{F} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is the constant sheaf $U \mapsto K(X)$. Indeed if $U \subseteq X$ is non-empty open, then $\mathcal{F}(U \cap U_i)$ is identified with $K(X)$ canonically by identifying with the stalk at the generic point, so by the sheaf axioms, $\mathcal{F}(U) = K(X)$ also, so the morphism

$\mathcal{L} \rightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ by $s \mapsto s \otimes 1$ demonstrates $\mathcal{L}$ as a subsheaf of $\mathcal{K}_X$. Thus $\mathcal{L}$ comes from a Cartier divisor D constructed from the trivialising cover $\mathcal{O}_{U_i} \cong \mathcal{L}|_{U_i} \hookrightarrow \mathcal{K}_X$ via $1 \mapsto s_i \mapsto g_i^{-1}$. The corresponding Cartier divisor is then $\{(U_i, g_i)\}$. $\square$

7.3 Effective Divisors

Definition 7.9. A Weil divisor $\sum a_i X_i$ is *effective* if every $a_i \geq 0$

Definition 7.10. A Cartier divisor is *effective* if it can be represented by some $\{(U_i, f_i)\}$ with $f_i \in \Gamma(U_i, \mathcal{O}_X)$

If $\mathcal{L}$ is a line bundle, $s \in \Gamma(X, \mathcal{L})$, and $\{U_i\}$ is a trivialising cover for $\mathcal{L}$ with ϕ_i the trivialisations, then we get a Cartier divisor $(s)_0 = \{(U_i, \phi_i(s|_{U_i}))\}$. This is a Cartier divisor, as on overlaps these differ by the transition function, which is invertible.

Theorem 7.4. Let $X \subseteq \mathbb{P}_k^n$ be a closed subscheme and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$ modules, then $\Gamma(X, \mathcal{F})$ is a finite dimensional k -vector space

Proof. Hartshorne II.5.19 $\square$

Theorem 7.5. Let $X \subseteq \mathbb{P}_k^n$ be an integral closed subscheme with k algebraically closed. Then $\Gamma(X, \mathcal{O}_X) = k$

Proof. Hartshorne I.3.4 $\square$

Theorem 7.6. Let $X \subseteq \mathbb{P}_k^n$ be an integral closed subscheme with k algebraically closed, D_0 a Cartier divisor on X and $\mathcal{L} \cong \mathcal{O}_X(D_0)$. Then

1. For $0 \neq s \in \Gamma(X, \mathcal{L})$, $(s)_0$ is effective and $(s)_0 \sim D_0$.
2. Every effective divisor $D \sim D_0$ satisfies $D = (s)_0$ for some $s \in \Gamma(X, \mathcal{L})$
3. Two sections $s, s' \in \Gamma(X, \mathcal{L})$ have $(s)_0 = (s')_0$ if and only if there is $\lambda \in k^*$ with $s = \lambda s'$

Proof.

1. $\mathcal{O}_X(D_0) \subseteq \mathcal{K}_X$, so s corresponds to some $f \in K(X)$. If D_0 is represented by $\{(U_i, f_i)\}$, then $\mathcal{O}_X(D_0)$ is trivialised by $\phi_i: \mathcal{O}_X(D_0)|_{U_i} \rightarrow \mathcal{O}_{U_i}$ via $t \mapsto t f_i$, so $D = (s)_0 = \{(U_i, f f_i)\} = D_0 + (f)$ and is effective by construction
2. Let D be effective, $D = D_0 + (f)$ for some $f \in K(X)^*$. Writing $D = \{(U_i, g_i)\}$ and $D_0 = \{(U_i, f_i)\}$ and $g_i = f f_i$ with $g_i \in \mathcal{O}_X(U_i)$. Then $\phi_i^{-1}(g_i) = f$, so $f \in \Gamma(X, \mathcal{O}_X(D_0)) \subseteq K(X)$. Then $(f)_0 = D$.
3. If $(s)_0 = (s')_0$ then $(s)_0 = D_0 + (f)$ and $(s')_0 = D_0 + (f')$ so $(f) = (f')$, so $(f/f') = 0$ and so $f/f' \in \Gamma(X, \mathcal{O}_X^*) = k^*$ since X is a closed subscheme of $\mathbb{P}_k^n$ with k algebraically closed. The reverse is clear.

□

Example 7.4. $\mathbb{P}_k^n$ satisfies all the necessary hypotheses to conclude

$$\mathbb{Z} = \mathrm{Cl} \mathbb{P}_k^n \cong \mathrm{CaCl} \mathbb{P}_k^n \cong \mathrm{Pic} \mathbb{P}_k^n$$

Now $s \in \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(1))$ corresponds to $1 \cdot H$ a Weil divisor with H a hyperplane. More generally we write $\mathcal{O}_{\mathbb{P}_k^n}(d) \cong \mathcal{O}_{\mathbb{P}_k^n}(1)^{\otimes d}$, and $\mathcal{O}_{\mathbb{P}_k^n}(-d)$ for the dual

We summarise the correspondences we have derived so far, and define some new ones

Line Bundles	Linear Systems
Line Bundle $\mathcal{L}$	$D \in \mathrm{CaCl}$ such that $\mathcal{L} \cong \mathcal{O}_X(D)$
$s \in \Gamma(X, \mathcal{L})$	Effective divisor $(s)_0 \sim D$
For X integral closed subscheme of $\mathbb{P}_k^n$, $\mathbb{P}(\Gamma(X, \mathcal{L})) = (\Gamma(X, \mathcal{L}) \setminus \{0\})/k^*$	A <i>complete linear system</i> $ D = \{D' \text{ effective Cartier divisor with } D' \sim D\}$
If $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ generate $\mathcal{L}$ ie for each $x \in X$, there is some i with $s_i \notin \mathfrak{m}_x \mathcal{L}_x$, we say $\mathcal{L}$ is <i>generated by global sections</i> (Equivalently, $\exists \mathcal{O}_X^{\oplus(n+1)} \rightarrow \mathcal{L}$)	$\mathcal{D} \subseteq D $ a linear subspace called a <i>linear system</i> (think of as the projectivisation of the subspace of $\Gamma(X, \Gamma)$ spanned by $s_1, \dots, s_n$) $\mathcal{D}$ is <i>basepoint free</i> if for all $x \in X$, there is $D \in \mathcal{D}$ such that $x \notin \mathrm{supp} D'$ (ie $s_1, \dots, s_n$ generate $\mathcal{L}$)
Sections of $\mathcal{L}$ inducing a closed immersion $X \rightarrow \mathbb{P}^n$ (<i>very ample</i>)	$ D $ inducing a closed immersion, we say D is <i>very ample</i>
$\mathcal{L}$ is <i>ample</i> if some $\mathcal{L}^{\otimes n}$ is very ample	D is <i>ample</i> if some nD is very ample

Remark. There are good geometric criteria for very ampleness (see example sheet). There are numerical criteria for ampleness.

Morally, controlling the dimension of $\Gamma(X, \mathcal{L})$ can be very useful

8 Cohomology of Sheaves

Given an exact sequence of sheaves $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ on X , applying the global section functor does not retain exactness (it is only left exact). There are things called right derived functors of $\Gamma(X, \cdot)$, which we write $H^i(X, \cdot)$ for $i \geq 0$ (for construction, see Homological Algebra next term).

Remark. We may also work with a ringed space $(X, \mathcal{O}_X)$ and work in the category of $\mathcal{O}_X$ modules. All of the constructions of the Homological algebra work here, and we get $H_{\mathcal{O}_X}^i(X, \mathcal{F}) = H^i(X, \mathcal{F})$

8.1 Čech Cohomology

We want a more practical theory of cohomology for computation. For this we define Čech cohomology. Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X , and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of X . We choose a well-order on I and define $U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$. We set

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \Gamma(U_{i_0 \dots i_p}, \mathcal{F})$$

If we let $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ be $(\alpha_{i_0 \dots i_p})_{i_0 < \dots < i_p}$, we define the coboundary map $d: C^p(\mathcal{U}, \mathcal{F}) \rightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(d\alpha)_{i_0 < \dots < i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \dots \hat{i}_k \dots i_{p+1}}|_{U_{i_0 \dots i_p}}$$

It is an exercise to check $d^2 = 0$ (this happens as usual).

We define $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(C^\bullet(\mathcal{U}, \mathcal{F})) = \ker d: C^p \rightarrow C^{p+1} / \text{im } d: C^{p-1} \rightarrow C^p$.

Example 8.1.

1. Let $X = S^1$ and $\mathcal{F}$ the constant sheaf with stalk $\mathbb{Z}$. We take the covering by opens $U = S^1 \setminus (1, 0)$ and $V = S^1 \setminus (0, 1)$. Then $C^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U) \times \mathcal{F}(V) = \mathbb{Z}^2$, $C^1(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U \cap V) = \mathbb{Z}^2$ and all further groups are 0. The differential $d: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ is $d(a, b) = (b - a, b - a)$. Then $\check{H}^0(\mathcal{U}, \mathcal{F}) = \ker d = \mathbb{Z}(1, 1)$ and $\check{H}^1(\mathcal{U}, \mathcal{F}) = \text{coker } d = \mathbb{Z}^2 / \mathbb{Z}(1, 1) \cong \mathbb{Z}$.
2. Let $X = \mathbb{P}_k^1$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(-2)$. Let $U_i = D_+(x_i)$. The transition map for $\mathcal{O}_{\mathbb{P}^1}(-2)$ from U_0 to U_1 is x_1^2/x_0^2 . Now

$$\begin{aligned} C^0(\mathcal{U}, \mathcal{F}) &= \Gamma(U_0, \mathcal{O}_{\mathbb{P}^1}(-2)) \times \Gamma(U_1, \mathcal{O}_{\mathbb{P}^1}(-2)) \\ &= k[x_1/x_0] \times k[x_0/x_1] \end{aligned}$$

by using trivialisations. Similarly

$$\begin{aligned} C^1(\mathcal{U}, \mathcal{F}) &= \Gamma(U_0 \cap U_1, \mathcal{O}_{\mathbb{P}^1}(-2)) \\ &= k[x_1/x_0]_{x_1/x_0} \end{aligned}$$

using the trivialisation on U_0 . We compute $d: C^0 \rightarrow C^1$.

$$d(f(x_1/x_0), g(x_0/x_1)) = g(x_0/x_1)(x_0^2/x_1^2) - f(x_1/x_0)$$

$\ker d = 0$ for degree reasons, so $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = \check{H}^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$. $\check{H}^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(-2)) = \text{coker } d = k\{x_0/x_1\}$, as single powers of x_0/x_1 are the only things not in the image of d .

Theorem 8.1. *If X is a Noetherian scheme with an open affine cover $\mathcal{U} = \{U_i\}_{i \in I}$ with the property that all the finite intersections are also affine. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$ for all quasicoherent $\mathcal{F}$.*

Proof. No Proof □

Remark. If $X \rightarrow S$ is a separated morphism with S affine, then any affine open cover of X satisfies the hypothesis of the above theorem

Remark. It is also true that $H^i(\text{Spec } A, \mathcal{F}) = 0$ for all $i > 0$ if $\mathcal{F}$ is quasicoherent.

Theorem 8.2. *Let X be a Noetherian topological space of dimension n , $\mathcal{F}$ a sheaf of abelian groups on X . Then $H^i(X, \mathcal{F}) = 0$ for $i > n$*

Proof. Hartshorne III.2 □

8.2 Computation of Cohomology of $\mathbb{P}_k^r$

Fix a field k , $X = \mathbb{P}_k^r$. Every line bundle on X is of the form $\mathcal{O}_{\mathbb{P}_k^r}(m) = \mathcal{O}_X(mH)$ for $m \in \mathbb{Z}$ and H a hyperspace.

Definition 8.1. A *perfect pairing* of k -vector spaces is a bilinear map $\langle \cdot, \cdot \rangle: V \times W \rightarrow k$ such that the induced map $V \rightarrow W^*$ is an isomorphism.

Theorem 8.3. *Let $S = k[x_0, \dots, x_r]$. Then*

1. $S \cong \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$ as graded modules.
2. $H^i(X, \mathcal{O}_X(m)) = 0$ for $0 < i < r$
3. *There is a perfect pairing $H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n - r - 1)) \rightarrow H^r(X, \mathcal{O}_X(-r - 1)) \cong k$*

Proof. We will calculate using the standard cover $\mathcal{U} = \{U_i\}_{0 \leq i \leq r}$ with $U_i = D_+(x_i)$. We will compute $\check{H}^i(\mathcal{U}, \mathcal{F})$ for $\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(m)$. The transition map for $\mathcal{O}_X(1)$ from U_i to U_j is x_i/x_j , so the transition map for $\mathcal{O}_X(m) = x_i^m/x_j^m$. For $J \subseteq \{0, \dots, r\}$, on $U_J = \bigcap_{j \in J} U_j$, we identify $\Gamma(U_J, \mathcal{O}_X(m))$ with the space spanned by Laurant monomials of degree m where the power of x_i is negative implies $i \in J$. We do this as follows. Given such a monomial M , using the trivialisation on U_i , M/x_i^m is degree 0, and hence represents a section of $\mathcal{O}_{U_i}$. These are sent to each other under the transition maps. Thus we can identify $\Gamma(U_J, \mathcal{F})$ with S_{x_J} where $x_J = \prod_{i \in J} x_i$. So the Čech complex for $\mathcal{F}$ with respect to $\mathcal{U} = \{U_i\}$ is

$$\prod_{i_0=0}^r S_{x_{i_0}} \rightarrow \prod_{0 \leq i_0 \leq i_1 \leq r} S_{x_{i_0} x_{i_1}} \rightarrow \dots \rightarrow S_{x_0 \dots x_r}$$

Now $H^0(\mathbb{P}_k^r, \mathcal{F}) = \ker d_0$ and all restriction maps are inclusions of S -submodules of $S_{x_0 \dots x_r}$, so we can think of $\ker d_0$ as $\bigcap_{i=0}^r S_{x_{i_0}} = S$, so $H^0(\mathbb{P}^r, \mathcal{F}) = S$. Now consider $d_{r-1}: \prod_{k=0}^r S_{x_0 \dots \hat{x}_k \dots x_r} \rightarrow S_{x_0 \dots x_r}$. The codomain is the k -vector space with basis $\{x_0^{i_0} \dots x_r^{i_r} \mid i_j \in \mathbb{Z}\}$. The cokernel of d_{r-1} is therefore $\{x_0^{i_0} \dots x_r^{i_r} \mid i_j \in \mathbb{Z}_{<0}\}$. This is the basis for $H^r(\mathbb{P}^r, \mathcal{F})$, and in particular, $H^r(\mathbb{P}^r, \mathcal{O}(-r-1))$ has basis $(x_0 \dots x_r)^{-1}$. We want a perfect pairing

$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \rightarrow H^r(X, \mathcal{O}_X(-r-1))$. If $m < 0$, then $H^0(X, \mathcal{O}_X(n)) = H^r(X, \mathcal{O}_X(-n-r-1)) = 0$. If $n \geq 0$, then we have a basis of degree n monomials with positive powers for $H^0(X, \mathcal{O}_X(n))$, and we know a basis for $H^r(X, \mathcal{O}_X(-n-r-1))$. The pairing is given on this basis by multiplication, interpreted as 0 if it wouldn't land in $H^r(\mathbb{P}^r, \mathcal{O}(-r-1))$, and this is perfect.

It remains to show that $H^i(\mathbb{P}_k^r, \mathcal{F}) = 0$ for $0 < i < r$. We induct on r . The base case is $r = 1$, and this is trivial. For the inductive step, we assume it for $r - 1$. We localise $C^\bullet(\mathcal{U}, \mathcal{F})$ at x_r . This replaces $\Gamma(U_I, \mathcal{F})$ with $\Gamma(U_I \cap U_r, \mathcal{F})$. This $C^\bullet(\mathcal{U}, \mathcal{F})_{x_r}$ is the Čech complex computing $H^*(U_r, \mathcal{F}|_{U_r})$ using the open covering $\{U_0 \cap U_r, \dots, U_{r-1} \cap U_r, U_r\}$. As $U_r \cong \mathbb{A}_k^r$, $H^i(U_r, \mathcal{F}) = 0$ for all $i > 0$. Localisation preserves exact sequences, and so $H^i(\mathbb{P}^r, \mathcal{F})_{x_r} = H^i(U_r, \mathcal{F}|_{U_r}) = 0$. This means that every element of $H^i(\mathbb{P}^r, \mathcal{F})$ is annihilated by some element of x_r . Let $H = V(x_r) \cong \mathbb{P}^{r-1}$. This is a closed subscheme of $\mathbb{P}^r$ defined by $H = \text{Proj } S/(x_r)$. In particular we have a surjection $\mathcal{O}_{\mathbb{P}^r} \rightarrow i_* \mathcal{O}_H$ where $i: H \hookrightarrow \mathbb{P}^r$ is the inclusion. The kernel of this surjection, $\mathcal{I}$, is the *ideal sheaf* defining H . Since H is a Cartier divisor, it is locally given by a single equation, so $\mathcal{I}$ is in fact a line bundle. Explicitly, on $U_i = D_+(x_i) = \text{Spec } S_{(x_i)}$, the ideal defining H is x_r/x_i . Hence the transition map between U_i and U_j is x_i/x_j , so $\mathcal{I} = \mathcal{O}(-1)$. So we have an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^r}(-1) \xrightarrow{\cdot x_r} \mathcal{O}_{\mathbb{P}^r} \rightarrow \mathcal{O}_H \rightarrow 0$$

Tensoring with $\mathcal{O}_{\mathbb{P}^r}(m)$ gives an exact sequence (as it is locally free, hence flat)

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^r}(m-1) \xrightarrow{\cdot x_r} \mathcal{O}_{\mathbb{P}^r}(m) \rightarrow \mathcal{O}_H(m) \rightarrow 0$$

Also $\mathcal{O}_H \otimes \mathcal{O}_{\mathbb{P}^r}(m) \cong \mathcal{O}_H(m)$ which we can see by restricting transition maps. Taking the direct sum over all m , we get

$$0 \rightarrow \mathcal{F}(-1) \rightarrow \mathcal{F} \rightarrow \mathcal{F}_H \rightarrow 0$$

so we get the long exact sequence

$$\dots \rightarrow H^i(\mathbb{P}^r, \mathcal{F}(-1)) \rightarrow H^i(\mathbb{P}^r, \mathcal{F}) \rightarrow H^i(\mathbb{P}^r, \mathcal{F}_H) \rightarrow H^{i+1}(\mathbb{P}^r, \mathcal{F}(-1)) \rightarrow \dots$$

The induction hypothesis tells us that $H^i(H, \mathcal{F}_H) = 0$ for $0 < i < r - 1$. Note that this is isomorphic to the 3rd term since the Čech complexes are the same. So if $1 < i < r - 1$, we get

$$0 \rightarrow H^i(\mathbb{P}^r, \mathcal{F}(-1)) \rightarrow H^r(\mathbb{P}^r, \mathcal{F}) \rightarrow 0$$

where the map is multiplication by x_r , and hence that is an isomorphism, but we know every element of $H^i(\mathbb{P}^r, \mathcal{F})$ is annihilated by some power of x_r , so $H^i(\mathbb{P}^r, \mathcal{F}) = 0$. For $i = 1$, we have

$$0 \rightarrow H^0(\mathbb{P}^r, \mathcal{F}(-1)) \rightarrow H^0(\mathbb{P}^r, \mathcal{F}) \rightarrow H^0(H, \mathcal{F}_H) \rightarrow H^1(\mathbb{P}^r, \mathcal{F}(-1)) \rightarrow H^1(\mathbb{P}^r, \mathcal{F})$$

This sequence is

$$0 \rightarrow S(-1) \rightarrow S \rightarrow S/(x_r) \rightarrow H^1(\mathbb{P}^r, \mathcal{F}(-1)) \rightarrow H^1(\mathbb{P}^r, \mathcal{F})$$

where $S(-1)$ is S with shifted grading, so exactness gives that $\cdot x_r$ is injective on $H^1(\mathbb{P}^r, \mathcal{F}(-1))$. We know it is surjective, so it is an isomorphism, and we conclude as before. For $i = r - 1$, we have

$$H^{r-1}(H, \mathcal{F}_H) \rightarrow H^r(\mathbb{P}^r, \mathcal{F}(-1)) \rightarrow H^r(\mathbb{P}^r, \mathcal{F}) \rightarrow H^r(H, \mathcal{F}_H) = 0$$

Now $H^r(\mathbb{P}^r, \mathcal{F})$ has basis $\{x_0^{a_0} \dots x_r^{a_r} \mid a_0, \dots, a_r \leq -1\}$, so the kernel of multiplication by x_r is then $\{x_0^{a_0} \dots x_{r-1}^{a_{r-1}} x_r^{-1} \mid a_0, \dots, a_{r-1} \leq -1\}$, but this is the basis of $H^{r-1}(H, \mathcal{F}_H)$, so the map $H^{r-1}(H, \mathcal{F}_H) \rightarrow H^r(\mathbb{P}^r, \mathcal{F}(-1))$ is injective. Now we look further and we get

$$0 = H^{r-2}(H, \mathcal{F}_H) \rightarrow H^{r-1}(\mathbb{P}^r, \mathcal{F}(-1)) \rightarrow H^{r-1}(\mathbb{P}^r, \mathcal{F}) \rightarrow 0$$

so we conclude as before $\square$

8.3 Computational Tools

We have two big computational tools for cohomology, namely Serre duality and Riemann Roch.

8.3.1 Normal and Conormal Sheaves

If X is a scheme and $i: Z \rightarrow X$ is a closed subscheme, then we have the ideal sheaf $\mathcal{I}_{Z/X} = \ker(\mathcal{O}_X \rightarrow i_*\mathcal{O}_Z)$. On example sheet 4, we see $\mathcal{I}_{Z/X}$ is a coherent sheaf if X is Noetherian

Definition 8.2. We define the *conormal sheaf* by $N_{Z/X}^\vee = \mathcal{I}_{Z/X}/\mathcal{I}_{Z/X}^2$ where the latter is the sheafification of the pointwise square presheaf

Remark. This is an $\mathcal{O}_Z$ module as $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}_{Z/X}$ and $\mathcal{I}_{Z/X}/\mathcal{I}_{Z/X}^2$ is an $\mathcal{O}_X/\mathcal{I}_{Z/X}$ module

It is a fact that if X, Z are non-singular (ie all local rings are regular), then $N_{Z/X}^\vee$ is a locally free sheaf on Z of rank $\text{codim}(Z, X)$ (assuming X, Z are connected)

Definition 8.3. In this case, we define the *normal bundle* of Z in X by $N_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(N_{Z/X}^\vee, \mathcal{O}_Z)$

Definition 8.4. Suppose $f: X \rightarrow Y$ is a separated morphism, so that $\Delta: X \rightarrow X \times_Y X$ is a closed immersion, then the *sheaf of differentials* is defined to be $\Omega_{X/Y} = \Delta^*(N_{X/X \times_Y X}^\vee)$

Example 8.2. Suppose $Y = \text{Spec } k$ and X is a non-singular variety over $\text{Spec } k$. Then $\Omega_{X/Y}$ is locally free of rank $\dim X$

Definition 8.5. In the above case, we define the *canonical bundle* of X , $\omega_X = \bigwedge^{\dim X} \Omega_{X/Y}$, which has transition functions $\det g_{ij}$ where g_{ij} are the transition functions of $\Omega_{X/Y}$. ω_X corresponds to the *canonical divisor* $K_X \in \text{CaCl } X$.

8.3.2 Serre Duality

Theorem 8.4. *Let X be a non-singular projective variety. Then for any locally free sheaf $\mathcal{F}$ on X , we have*

$$H^i(X, \mathcal{F}^\vee \otimes \omega_X) \cong H^{n-i}(X, \mathcal{F})^\vee$$

where $n = \dim X$

Remark. $\omega_{\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}(-r-1)$, so we have shown Serre duality for line bundles on $\mathbb{P}^r$

In general if X is a projective variety over $\text{Spec } k$, then $H^i(X, \mathcal{F})$ is a finite dimensional vector space over k if $\mathcal{F}$ is coherent.

Definition 8.6. Define the *Euler characteristic* $\chi(\mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i \dim_k H^i(X, \mathcal{F})$

Remark. If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is short exact, then $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$ by the long exact sequence for cohomology.

8.3.3 Riemann Roch

We want to say $\chi(\mathcal{F})$ is a topological invariant Suppose $\dim X = 1$, let X be a non-singular projective variety over an algebraically closed field, and let $p \in X$ be a non-generic point, hence a prime divisor. Then we have

$$0 \rightarrow \mathcal{I}_{p/X} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_p = k \rightarrow 0$$

and we can tensor with $\mathcal{L}$ to get

$$0 \rightarrow \mathcal{L}(-p) \rightarrow \mathcal{L} \rightarrow \mathcal{O}_p \rightarrow 0$$

hence

$$\chi(\mathcal{L}) = \chi(\mathcal{L}(-p)) + \chi(\mathcal{O}_p) = \chi(\mathcal{L}(-p)) + 1$$

. Thus if $D \in \text{Div } X$ and $D = \sum a_i P_i$ we define the *degree* to be $\deg D = \sum a_i$. Then we use the above equation to see $\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \deg D$

Definition 8.7. The *genus* of X is $g = \dim H^1(X, \mathcal{O}_X)$

Theorem 8.5. (*Riemann Roch*) $\dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \mathcal{O}_X(-D) \otimes \omega_X) = \deg D + 1 - g$

Proof.

$$\begin{aligned} \deg D + \chi(\mathcal{O}_X) &= \chi(\mathcal{O}_X(D)) \\ &= \dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) \\ &= \dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \mathcal{O}_X(-D) \otimes \omega_X) \end{aligned}$$

but $\chi(\mathcal{O}_X) = 1 - g$, so we are done. $\square$

Remark. $g = \dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \omega_X)$ (which was the definition in II AG)

Remark.

1. As $\chi(\omega_X) = \dim H^0(X, \omega_X) - \dim H^1(X, \omega_X) = g - 1$, Riemann Roch tells us that $g - 1 = \chi(\omega_X) = \deg K_X + 1 - g$, so $\deg K_X = 2g - 2$
2. If $\deg D < 0$, then $H^0(X, \mathcal{O}_X(D)) = 0$ since there are no effective divisors of negative degree. Thus if $\deg D > 2g - 2$ then $H^0(X, \omega_X \otimes \mathcal{O}_X(-D)) = 0$ since $\deg(K_X - D) < 0$, so $\dim H^0(X, \mathcal{O}_X(D)) = \deg D + 1 - g$
3. A linear system $|D|$ is basepoint free if and only if $\dim |D - p| = \dim |D| - 1$ for all $p \in X$. Thus if $\deg D \geq 2g$, $|D|$ is basepoint free
4. $|D|$ is very ample if $\dim |D - p - q| = \dim |D| - 2$ for all $p, q \in X$, so $|D|$ is very ample if $\deg D \geq 2g + 1$

Example 8.3.

1. If $g = 0$ then $|p|$ is very ample, and of degree 1, giving a closed immersion $f: X \rightarrow \mathbb{P}^1$, hence f is an isomorphism
2. If $g = 1$, suppose $p_0 \in X$. Then $3p_0$ is very ample, and is dimension 2, giving an embedding $f: X \rightarrow \mathbb{P}^2$ as a curve of degree 3. We look at the degree 0 divisors. If $D \in \text{Div } X$, then $D \sim P - P_0$ for some $P \in X$. To see this, we consider $D + P_0$. We have $\dim H^0(X, \mathcal{O}_X(D + P_0)) = \deg D + 1 - g = 1$, so there is a unique effective divisor $P \sim D + P_0$ and we are done. Thus we have an exact sequence

$$0 \rightarrow \text{Cl}^0 X \rightarrow \text{Cl } X \xrightarrow{\deg} \mathbb{Z} \rightarrow 0$$

we have a bijection between $X(k)$ and $\text{Cl}^0 X$ by $P \rightarrow P - P_0$. In genus g , $\text{Cl}^0 X$ can be identified with $J(k)$, a g -dimensional abelian variety