

Algebraic Topology

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1 Introduction

In this course, a map is always a continuous function.

Definition 1.1. Two maps $f_0, f_1: X \rightarrow Y$ are *homotopic* if there is a map $F: X \times [0, 1] \rightarrow Y$ such that $F(y, 0) = f_0(y)$ and $F(y, 1) = f_1(y)$ for all $y \in Y$. Then we write $f_0 \simeq f_1$.

Exercise 1.1. This is an equivalence relation on maps.

Definition 1.2. A map $f: X \rightarrow Y$ is a *homotopy equivalence* if there is a map $g: Y \rightarrow X$ such that $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$. We say g is a *homotopy inverse* of f (this is unique up to homotopy). We write $X \simeq Y$ (X is *homotopy equivalent* to Y) if such an f exists.

Example 1.1.

1. If $f: X \rightarrow Y$ is a homeomorphism, it is a homotopy equivalence (with $g = f^{-1}$ the homotopy inverse, with $f \circ f^{-1} \simeq \text{id}_Y$ via the homotopy $F(y, t) = y$).
2. Consider $i: \{0\} \rightarrow \mathbb{R}^n$. Let $r: \mathbb{R}^n \rightarrow \{0\}$. We have $r \circ i = \text{id}_{\{0\}}$. On the other hand, $i \circ r(x) = 0$ for all $x \in \mathbb{R}^n$. We define a homotopy $F(x, t) = tx$. This is a homotopy from $i \circ r$ to $\text{id}_{\mathbb{R}^n}$, so i is a homotopy equivalence, and $\mathbb{R}^n$ is homotopy equivalent to a point.
3. Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere. Let the inclusion of S^n into $\mathbb{R}^{n+1} \setminus \{0\}$ be i . We define $r: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow S^n$ by $x \mapsto \frac{x}{\|x\|}$. We know $r \circ i = \text{id}_{S^n}$. We compute that $i \circ r(x) = \frac{x}{\|x\|}$. Let $F(x, t) = (1 - t)\frac{x}{\|x\|} + tx$. This is continuous as $x \neq 0$, and it is clear $F(x, t)$ is never 0, so i is a homotopy equivalence.

Definition 1.3. A space which is homotopy equivalent to a point is called *contractible*.

To show $\mathbb{R}^n \not\cong \mathbb{R}^m$ for $n \neq m$, we can't show they're not homotopy equivalent. However we can show that, after removing a point, they are no longer homotopy equivalent (as we will later show that S^n and S^m are not homotopy equivalent), so they are not homeomorphic.

2 Homology and Cohomology

2.1 Chain Complexes

Definition 2.1. A *chain complex* is a sequence of abelian groups and homomorphisms between them of the form

$$\dots \longrightarrow C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

such that $d_i \circ d_{i+1} = 0$ for all i . We write $(C_\bullet, d)$ for the complex. A *cochain complex* is a sequence of abelian groups and homomorphisms between them of the form

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \longrightarrow \dots$$

such that $d^{i+1} \circ d^i = 0$ for all i . We write $(C^\bullet, d)$ for the complex. The *homology* of a chain complex is

$$H_i(C_\bullet, d) = \ker(d_i: C_i \rightarrow C_{i-1}) / \operatorname{im}(d_{i+1}: C_{i+1} \rightarrow C_i)$$

We call elements of the kernel *cycles* and of the image *boundaries*. The *cohomology* of a cochain complex is

$$H^i(C^\bullet, d) = \ker(d^i: C^i \rightarrow C^{i+1}) / \operatorname{im}(d^{i-1}: C^{i-1} \rightarrow C^i)$$

We call elements of the kernel *cocycles* and of the image *coboundaries*.

2.2 Singular (co)chains

Definition 2.2. The *standard n -simplex* is the space

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1 \right\}$$

The i th *face* of Δ^n is $\Delta_i^n = \Delta^n \cap \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i = 0\}$. The map $\delta_i: \Delta_{n-1} \rightarrow \Delta_n$ sends $(t_0, \dots, t_{n-1}) \rightarrow (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$. This is a homeomorphism onto Δ_i^n .

Definition 2.3. A *singular n -simplex* in X is a map $\tau: \Delta^n \rightarrow X$, and we define $C_n(X)$ to be the free abelian group on the set of singular n -simplices (ie, sums of the form $\sum h_\sigma \sigma$ where $h_\sigma \in \mathbb{Z}$, σ is a singular n -simplex in X , and $h_\sigma \neq 0$ for only finitely many σ).

Definition 2.4. We define $d_n: C_n(X) \rightarrow C_{n-1}(X)$ via $\sigma \mapsto \sum_{i=0}^n (-1)^i (\sigma \circ \delta_i)$ on the basis.

Lemma 2.1. The maps $\delta_j \circ \delta_i, \delta_i \circ \delta_{j-1}: \Delta^{n-2} \rightarrow \Delta^n$ for $i < j$ are equal.

Proof. Both maps put 0 in the i, j th positions, and distribute the remaining terms of Δ^{n-2} monotonically. $\square$

Corollary 2.1.1. The map $d_{n-1} \circ d_n: C_n(X) \rightarrow C_{n-2}(X)$ is zero.

Proof. As $C_n(X)$ is free, it suffices to check on a basis.

$$\begin{aligned} d_{n-1} \circ d_n &= \sum_{i=0}^{n-1} (-1)^i \left(\sum_{j=0}^n (-1)^j \sigma \circ \delta_j \right) \circ \delta_i \\ &= \sum_{i < j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i + \sum_{i \geq j} (-1)^{i+j} \sigma \circ \delta_j \circ \delta_i \\ &= \sum_{i < j} (-1)^{i+j} \sigma \circ \delta_i \circ \delta_{j-1} + \sum_{i \leq j} (-1)^{i+j} \sigma \circ \delta_i \circ \delta_j \\ &= 0 \end{aligned}$$

$\square$

Therefore $(C_\bullet(X), d)$ is a chain complex.

Definition 2.5. The n th homology of X is $H_n(X) = H_n(C_\bullet(X), d)$.

Definition 2.6. We define $C^n(X) = \text{Hom}(C_n(X), \mathbb{Z})$ (as abelian group homomorphisms), and $d^n: C^n \rightarrow C^{n+1}$ is the adjoint of d_n , ie $d^n(\phi)(\sigma_{n+1}) = \phi \circ d_{n+1}(\sigma_{n+1})$.

Remark 2.1. It is immediate that

$$d^{n+1} \circ d^n(\phi)(\sigma_{n+2}) = \phi(d_{n+1}d_{n+2}(\sigma_{n+2})) = \phi(0) = 0$$

so $d^{n+1} \circ d^n(\phi) = 0$, so $d^{n+1} \circ d^n = 0$. Hence $(C^\bullet(X), d)$ is a cochain complex.

Definition 2.7. The cohomology of X is $H^n(X) = H^n(C^\bullet(X), d)$

Definition 2.8. If $(C_\bullet, d^C)$ and $(D_\bullet, d^D)$ are chain complexes, a chain map $f_\bullet: (C_\bullet, d^C) \rightarrow (D_\bullet, d^D)$ is a sequence of homomorphisms $f_n: C_n \rightarrow D_n$ such that $f_n \circ d_{n+1}^C = d_{n+1}^D \circ f_{n+1}$, ie

$$\begin{array}{ccc} C_n & \xrightarrow{d_n^C} & C_{n-1} \\ \downarrow f_n & & \downarrow f_{n-1} \\ D_n & \xrightarrow{d_n^D} & D_{n-1} \end{array}$$

commutes for all n

Lemma 2.2. *If $f_\bullet: (C_\bullet, d^C) \rightarrow (D_\bullet, d^D)$ is a chain map, we get a well-defined homomorphism on homology $f_*: H_n(C_\bullet, d^C) \rightarrow H_n(D_\bullet, d^D)$ via $[x] \mapsto [f(x)]$*

Proof. We need to prove that this does not depend on representative of x , and that $f(x)$ is a cycle if x is. We compute

$$d_n^D f_n(x) = f_{n+1} d_n^C(x) = f_{n+1}(0) = 0$$

Furthermore if $[x] = [x']$ then $x' = x + d_{n+1}^C(y)$, so

$$f(x') = f(x) + f d_{n+1}^C(y) = f(x) + d_{n+1}^D f(y)$$

and hence $[f(x)] = [f(x')]$, so the map is well defined and we are done. $\square$

Remark 2.2. As differentials commute with chain maps, we will now frequently denote every differential as d .

Remark 2.3. If $f: X \rightarrow Y$ is a map then we get $f_\bullet: (C_\bullet(X), d) \rightarrow (C_\bullet(Y), d)$ via $f_\bullet(\sigma) = f \circ \sigma$. This is a chain map as differentials are defined by precomposing, this is defined by postcomposing, and function composition is associative. This further induces a map on homology as above, which we denote as f_* .

This satisfies $(g \circ f)_* = g_* \circ f_*$, and $(\text{id}_X)_* = \text{id}_{H_n(X)}$.

Similarly applying $\text{Hom}(\cdot, \mathbb{Z})$ gives $f^\bullet: C^\bullet(Y) \rightarrow C^\bullet(X)$, by $\phi \mapsto \phi \circ f$, and this induces a map f^* on cohomology, such that $(f \circ g)^* = g^* \circ f^*$.

Lemma 2.3. *If $f: X \rightarrow Y$ is a homeomorphism, then $f_*: H_n(X) \rightarrow H_n(Y)$ is an isomorphism (and the same holds on cohomology).*

Proof. Let $g = f^{-1}$. Then $f_* \circ g_* = (f \circ g)_* = (\text{id}_X)_* = \text{id}_{H_n(X)}$ and similarly for $g_* \circ f_*$, so g_* is a two sided inverse for f_* , so it is an isomorphism. $\square$

Remark 2.4. Soon we will prove homotopic maps induce the same maps on (co)homology, and an identical proof will show that homotopy equivalent spaces have isomorphic homology.

What do homology groups ‘measure’ (philosophically)?

Now for one dimensional simplices, cycles are, in a sense, ‘loops’ and only the image of the loop matters as we can send in a 2-simplex which gives a linear homotopy between two with the same image. If we are homotopic rel the end-points, the same proof works. If these loops are the boundary of a simplex, then they can be ‘filled in’, so homology is in a sense those loops that can’t be filled in

Example 2.1. We compute the homology of a point. This will turn out to be $H_n(*) = \mathbb{Z}$ if $n = 0$, and 0 otherwise. There is only one map $\sigma_n: \Delta^n \rightarrow *$, so the chain complex $(C_\bullet(*), d)$ is

$$\dots \longrightarrow \mathbb{Z} \xrightarrow{d} \mathbb{Z} \xrightarrow{d} \mathbb{Z} \xrightarrow{d} \mathbb{Z}$$

. Now $d(\sigma_n) = \sum_{i=0}^n (-1)^i \sigma_n \circ \delta_i = \begin{cases} \sigma_{n-1} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$ So alternating maps are the identity and the zero map, and the result follows. The computation of the cohomology of the point is similar.

If X is a disjoint union of path connected spaces, $X = \coprod_{\alpha \in A} X_\alpha$, then any map from a Δ^n lies in one of the X_α , and so does its boundary. Therefore the homology breaks up as the direct sum of the homology of the X_α , $H_n(X) = \bigoplus_{\alpha \in A} H_n(X_\alpha)$

Example 2.2. Let X be path connected and non-empty. The map $\varepsilon: C_0(X) \rightarrow \mathbb{Z}$ defined by $\sum n_\sigma \sigma \mapsto \sum n_\sigma$ satisfies $\varepsilon d(\tau) = \varepsilon(\tau \circ \delta_0 - \tau \circ \delta_1) = 0$. Therefore ε induces a surjective map $\varepsilon: H_0(X) \rightarrow \mathbb{Z}$. Let $\sigma_0: \Delta^0 \rightarrow X$ be a point of X . As X is path connected, for any other $\sigma: \Delta^0 \rightarrow X$, there is $\tau_\sigma: \Delta^1 \cong [0, 1] \rightarrow X$ which is a path from σ_0 to σ . Suppose $\varepsilon(\sum_\sigma n_\sigma \sigma) = 0$, ie $\sum n_\sigma = 0$. Consider $\sum n_\sigma \tau_\sigma \in C_1(X)$. This has boundary $\sum n_\sigma (\sigma - \sigma_0)$, which is equal to $\sum n_\sigma \sigma + (\sum n_\sigma) \sigma_0 = \sum n_\sigma \sigma$, hence $\sum n_\sigma \sigma$ is a boundary, and ε is an isomorphism. We therefore know $H_0(X)$ for every space X , combining the above remark.

3 Computing (Co)Homology

We will state 4 theorems without proof, use them to compute (co)homology, then prove them in the next section. They will be stated for homology, but hold similarly for cohomology.

Theorem 3.1. (*Homotopy invariance*): If $f \simeq g: X \rightarrow Y$, then $f_* = g_*: H_n(X) \rightarrow H_n(Y)$

Remark 3.1. Therefore if X and Y are homotopy equivalent via $f: X \rightarrow Y$, then $f_*: H_n(X) \rightarrow H_n(Y)$ is an isomorphism

Definition 3.1. Homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ are *exact at B* if $\text{Im}(f) = \ker(g)$. We say a longer sequence

$$\dots \xrightarrow{f_{i+3}} A_{i+2} \xrightarrow{f_{i+2}} A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{f_{i-1}} \dots$$

is *exact* if it is exact at each A_i .

Theorem 3.2. (*Mayer-Vietoris Theorem*) Let $X = A \cup B$ is the union of two open subsets. We have maps

$$\begin{array}{ccc} A \cap B & \xrightarrow{i_A} & A \\ i_B \downarrow & & \downarrow j_A \\ B & \xrightarrow{j_B} & X \end{array}$$

by inclusion. Then there exist homomorphisms $\partial_{MV}: H_i(X) \rightarrow H_{i-1}(A \cap B)$ such that

$$\begin{array}{ccccc}
& & & & \dots \\
& & \swarrow & & \\
H_{i+1}(X) & \xleftarrow{\partial_{MV}} & H_i(A \cap B) & \xrightarrow{(i_A, i_B)_*} & H_i(A) \oplus H_i(B) \\
& \swarrow & \searrow & & \\
H_i(X) & \xleftarrow{(j_A - j_B)_*} & \dots & \xrightarrow{(j_A - j_B)_*} & H_0(A \cap B) \\
& \swarrow & \searrow & & \\
H_0(A) \oplus H_0(B) & \xleftarrow{(j_A - j_B)_*} & H_0(X) & \xrightarrow{(i_A, i_B)_*} & 0
\end{array}$$

is exact. Furthermore it is natural in the sense that if $Y = U \cup V$ similarly, and we have $f: X \rightarrow Y$ such that $f(A) \subseteq U, f(B) \subseteq V$, then there is a natural map of Mayer-Vietoris sequences

Remark 3.2. We will construct ∂_{MV} when we prove the theorem, but if $[x] = [a + b]$ with $a \in C_n(A), b \in C_n(B)$, then $\partial_{MV}([x]) = [d(a)] = [-d(b)]$

If $A \subseteq X$ and $i: A \rightarrow X$, then $i_\bullet: C_\bullet(A) \rightarrow C_\bullet(X)$ is injective, and we define $C_\bullet(X, A) = C_\bullet(X)/i_\bullet C_\bullet(A)$. One can check that $d[x] = [d(x)]$ defines a differential on $C_\bullet(X, A)$, and $q_\bullet: C_\bullet(X) \rightarrow C_\bullet(X, A)$

Definition 3.2. We call the homology of the above complex $C_\bullet(X, A)$ the *relative homology* of X with respect to A , and denote it $H_\bullet(X, A)$

Theorem 3.3. (Exact sequence of relative homology) There is a homomorphism $\partial: H_n(X, A) \rightarrow H_{n-1}(A)$, and

$$\begin{array}{ccccc}
& & & & \dots \\
& & \swarrow & & \\
H_{i+1}(X, A) & \xleftarrow{\partial} & H_i(A) & \xrightarrow{(i_A)_*} & H_i(X) \\
& \swarrow & \searrow & & \\
H_i(X, A) & \xleftarrow{q_*} & \dots & \xrightarrow{(i_A)_*} & H_0(A) \\
& \swarrow & \searrow & & \\
H_0(X) & \xleftarrow{q_*} & H_0(X, A) & \xrightarrow{(i_A)_*} & 0
\end{array}$$

is exact. If $f: X \rightarrow Y$ has $f(A) \subseteq B$, we get a natural map of exact sequences from the one for (X, A) to the one for (Y, B) . In particular, such an f induces a map on relative homology, $f_*: H(X, A) \rightarrow H(Y, B)$ by $[[x]] \mapsto [[f_\bullet x]]$

Theorem 3.4. (Excision Theorem) Let (X, A) be spaces with $A \subseteq X$, and let $Z \subseteq A$ such that $\bar{Z} \subseteq A^\circ$. The map of pairs $(X - Z, A - Z) \rightarrow (X, A)$ induces an isomorphism $H_n(X - Z, A - Z) \cong H_n(X, A)$

Remark 3.3. Homotopy invariance in the A variable means that the $\bar{Z} \subseteq A^\circ$ condition can usually be forgotten.

3.1 Homology of Spheres

Example 3.1. Cover the circle $S^1 \subseteq \mathbb{C}$ by $A = \{z \in S^1 \mid \frac{-3\pi}{4} < \arg z < \frac{3\pi}{4}\}$ and $B = \{z \in S^1 \mid \frac{-\pi}{4} > \arg z \text{ or } \arg z > \frac{\pi}{4}\}$. $B \cong A \cong (0, 1) \simeq *$, and $A \cap B \cong (0, 1) \sqcup (0, 1) \simeq * \sqcup *$, so by Mayer-Vietoris we get

$$\begin{array}{ccccccc}
 & & 0 & & 0 \oplus 0 & & \\
 & & \parallel & & \parallel & & \\
 \dots & \longrightarrow & H_1(A \cap B) & \longrightarrow & H_1(A) \oplus H_1(B) & \longrightarrow & H_1(S^1) \\
 & & & & \swarrow & & \\
 & & H_0(A \cap B) & \longrightarrow & H_0(A) \oplus H_0(B) & \longrightarrow & H_0(S^1) \\
 & & \parallel & & \parallel & & \\
 & & \mathbb{Z}\{p, q\} & \xrightarrow{f} & \mathbb{Z}\{*\} \oplus \mathbb{Z}\{*\} & &
 \end{array}$$

The map f is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and its kernel is $\mathbb{Z}\{p - q\}$, so exactness at $H_0(A \cap B)$ says that $\text{im}(H_1(S^1) \rightarrow H_0(A \cap B)) \cong \mathbb{Z}$ and exactness at $H_1(S^1)$ says that $\ker(H_1(S^1) \rightarrow H_0(A \cap B)) = 0$, so $H_1(S^1) \xrightarrow{\partial_{MV}} \mathbb{Z}\{p - q\}$ is an isomorphism. We also get $H_0(S^1) \cong \mathbb{Z}\{p, q\}/(p + q) \cong \mathbb{Z}$. In any higher homology, we get $0 \rightarrow H_n(S^1) \rightarrow 0$, so $H^n(S^1) = 0$ for $n > 1$.

Theorem 3.5. For $n \geq 1$, $H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise} \end{cases}$

Proof. By the above example this holds for $n = 1$, so suppose $n \geq 2$ and proceed by induction. Cover S^n by the complement of the north pole for A and the complement of the south pole for B . Then $A, B \cong \mathbb{R} \simeq *$. $A \cap B = S^n \setminus \{N, S\} \cong S^{n-1} \times (0, 1) \simeq S^{n-1}$.

By Mayer Vietoris we get

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \parallel & & & & \\
 \dots & \longrightarrow & H_i(S^{n-1}) & \longrightarrow & H_i(*) \oplus H_i(*) & \longrightarrow & H_i(S^n) \\
 & & & & \swarrow & & \\
 & & H_{i-1}(S^{n-1}) & \longrightarrow & H_{i-1}(*) \oplus H_{i-1}(*) & \longrightarrow & H_{i-1}(S^n) \longrightarrow \dots \\
 & & \parallel & & & & \\
 & & 0 & & & &
 \end{array}$$

For $i \geq 2$. By exactness, $H_i(S_i) \cong H_{i-1}(S^{n-1})$ so in particular we get the result for $i \geq 2$ by induction. In the lower degrees we get

$$0 \longrightarrow H_1(S^n) \longrightarrow H_0(S^{n-1}) \longrightarrow H_0(*) \oplus H_0(*) \longrightarrow H_0(S^n) \longrightarrow 0$$

ie

$$0 \longrightarrow H_1(S^n) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(S^n) \longrightarrow 0$$

so as $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ is injective, we get $0 \rightarrow H_1(S^n) \rightarrow 0$ so $H_1(S^n) = 0$, and similarly we get $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow H_0(S^n) \rightarrow 0$, and so (as we know the maps are as in the S^1 case) $H_0(S^n) = \mathbb{Z}$. $\square$

Corollary 3.5.1. *If $n \neq m$, then $S^n \not\cong S^m$*

Proof. As they have different $H_n(\bullet)$ $\square$

Corollary 3.5.2. $\mathbb{R}^n \not\cong \mathbb{R}^m$ of $m \neq n$

Proof. See argument in introduction for how to derive from above corollary $\square$

Definition 3.3. If $f: S^n \rightarrow S^n$ is a map, then $f_*: H_n(S^n) \rightarrow H_n(S^n)$ is a map $f_*: \mathbb{Z} \rightarrow \mathbb{Z}$, so is multiplication by some $n \in \mathbb{Z}$. We call this n the *degree* of f .

Remark 3.4. You must pick the same isomorphism $H^n(S^n)$ on both sides, otherwise this is only defined up to a sign

Remark 3.5. All of the following are immediate.

1. $\deg \text{id}_{S^n} = 1$
2. If $f: S^n \rightarrow S^n$ is not surjective, then $\deg f = 0$, as $f \simeq \text{const}$ (as f factors through $S^n \setminus \{*\} \cong \mathbb{R}^n \simeq *$)
3. If $f \simeq g$, then $\deg f = \deg g$

Corollary 3.5.3. (*Brouwer Fixed Point Theorem*) *Any $f: D^n \rightarrow D^n$ has a fixed point.*

Proof. Suppose not. Then $f(x) \neq x$ for any $x \in X$. Then $f(x), x$ lie on a line. Let $r(x)$ be the intersection of the ray from $f(x)$ to x . This is continuous and is the identity on S^{n-1} , so $r: D^n \rightarrow S^{n-1}$ is a retraction. Consider $S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$. The composition is the identity, but applying H_{n-1} we get

$$\begin{array}{ccccc} H_{n-1}(S^{n-1}) & \longrightarrow & H_{n-1}(D^n) & \longrightarrow & H_{n-1}(S^{n-1}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & \mathbb{Z} \\ & \searrow & & \nearrow & \\ & & 1 & & \end{array}$$

where all the vertical maps are isomorphisms, contradiction. $\square$

Proposition 3.1. *A reflection $r: S^n \rightarrow S^n$ has $\deg r = -1$*

Proof. We begin with $n = 1$. By symmetry, r is reflection in the y -axis. We cover S^1 by two arcs of length just over π , A, B which are symmetric with respect to r , and choose p, q in different connected components of $A \cap B$ which are sent to each other by reflection. We get a map of Mayer-Vietoris sequences where every square commutes

$$\begin{array}{ccccccc}
& \mathbb{Z}\{p-q\} & & \mathbb{Z}\{p, q\} & & \mathbb{Z} \oplus \mathbb{Z} & \\
& \parallel & & \parallel & & \parallel & \\
0 & \longrightarrow & H_1(S^1) & \longrightarrow & H_0(A \cap B) & \longrightarrow & H_0(A) \oplus H_0(B) \\
& & \downarrow r_* & & \downarrow r_* & & \downarrow r_* \\
0 & \longrightarrow & H_1(S^1) & \longrightarrow & H_0(A \cap B) & \longrightarrow & H_0(A) \oplus H_0(B) \\
& & \parallel & & \parallel & & \parallel \\
& \mathbb{Z}\{p-q\} & & \mathbb{Z}\{p, q\} & & \mathbb{Z} \oplus \mathbb{Z} &
\end{array}$$

Then $\partial r_*(p-q) = r_*\partial(p-q) = r_*(p-q) = q-p = \partial(-(p-q))$ so $r_*(p-q) = -(p-q)$. For a reflection $r: S^n \rightarrow S^n$, we choose the reflection to preserve the north and south poles, so it restricts to a reflection on the equator $r': S^{n-1} \rightarrow S^{n-1}$, so using the cover $A = S^n \setminus \{N\}, B = S^n \setminus \{S\}$. We have $\partial: H_n(S^n) \rightarrow H_{n-1}(S^{n-1})$ is an isomorphism, so by naturality, the map induced on S^n and on the equator are the same, so r has degree -1 by induction. $\square$

Corollary 3.5.4. *The antipodal map $f: S^n \rightarrow S^n, x \mapsto -x$ has degree $(-1)^{n+1}$*

Proof. It is the composition of $n+1$ reflections, and degrees multiply when you compose. $\square$

Corollary 3.5.5. *(Hairy ball theorem) S^n has a nowhere 0 vector field if and only if n is odd*

Remark 3.6. For S^n , a vector field is a map $v: S^n \rightarrow \mathbb{R}^{n+1}$ selecting a vector which is tangent to the sphere at that point, ie $\langle x, v(x) \rangle = 0$.

Proof. For odd n , we exploit that $\mathbb{R}^{n+1}$ is a complex vector space, and we can set $v(x_1, y_1, \dots, x_n, y_n) = (-y_1, x_1, \dots, -y_n, x_n)$ (ie multiplication by i). If v is nowhere 0, consider the homotopy of length π $H: [0, \pi] \rightarrow S^n$ by $(t, x) \mapsto (\cos t)x + (\sin t)\frac{v(x)}{x}$. Then $H(0, x) = x$ and $H(\pi, x) = -x$ and this is a well defined homotopy. So $1 = \deg \text{id} = \deg(-\text{id}) = (-1)^{n+1}$. So n is odd. $\square$

Example 3.2. Let K be the Klein bottle, thought of as the usual pasting diagram. We take B to be a cylinder from the middle of the bottle, and A be a cylinder which slightly overlaps, and crosses the other pasted edge. Then $K = A \cup B$. B is homotopy equivalent to a circle, which we call $b \in H_1(B)$, as is A , whose circle we call $a \in H_1(A)$. $A \cap B$ is homotopy equivalent to two

cylinders, with circles $s, t \in H_1(A \cap B)$ around their cores. Writing down the Mayer Vietoris sequence and substituting in what we know,

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_2(K) & \longrightarrow & \mathbb{Z}\{s\} \oplus \mathbb{Z}\{t\} & \xrightarrow{s \mapsto b-a, t \mapsto b+a} & \mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\} \\
& & & & \searrow & & \searrow \\
& & H_1(K) & \longrightarrow & \mathbb{Z}\{*\} \oplus \mathbb{Z}\{*\} & \xrightarrow{* \mapsto (*,*)} & \mathbb{Z}\{*\} \oplus \mathbb{Z}\{*\} \\
& & & & \searrow & & \searrow \\
& & H_0(K) & \longrightarrow & 0 & &
\end{array}$$

so the map $H_1(A \cap B) \rightarrow H_1(A) \oplus H_1(B)$ is injective (as it has determinant -2), so $H_2(K) = 0$. The cokernel of that map is $\mathbb{Z}\{a, b\} / \langle a+b, b-a \rangle = \mathbb{Z}\{a\} / \langle 2a \rangle = \mathbb{Z}/2\mathbb{Z}$. We therefore get an exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow H_1(K) \longrightarrow \mathbb{Z} \longrightarrow 0$$

as the kernel of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (ie the map $H_0(A \cap B) \rightarrow H_0(A) \oplus H_0(B)$) is $\mathbb{Z}$.

As $\mathbb{Z}$ is free (hence projective) this short exact sequence splits, and hence $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (and $H_0(K) = \mathbb{Z}$, as the pasting diagram gives it as the image of a connected space, hence it is connected).

If we computed the cohomology, we would find $H^0(K) = \mathbb{Z}, H^1(K) = \mathbb{Z}, H^2(K) = \mathbb{Z}/2\mathbb{Z}$. These are so similar because of the universal coefficient theorem (of which we will do a weaker version later)

3.2 Local Degrees

Lemma 3.6. *Let M be a d -dimensional manifold (a Hausdorff, locally- $\mathbb{R}^2$ space) and $x \in M$. Then $H_i(M, M \setminus \{x\}) = \begin{cases} \mathbb{Z} & i = d \\ 0 & i \neq d \end{cases}$*

Proof. Let U be an open neighbourhood of x homeomorphic to $\mathbb{R}^d$ with $x \mapsto 0 \in \mathbb{R}^d$. Then $Z = M \setminus U$ is closed and lies in the open set $M \setminus \{x\}$, so since $Z, \bar{Z} \subseteq \text{int } M \setminus \{x\}$, we have $H_i(M, M \setminus \{x\}) \cong H_i(U, U \setminus \{x\})$ by excision. We get $H_i(U, U \setminus \{x\}) \cong H_i(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \cong H_{i-1}(\mathbb{R}^d \setminus \{0\})$ where the latter is from the exact sequence of relative homology, for $d > 1$, and $d = 1$ is similar $\square$

Definition 3.4. Suppose $f: S^d \rightarrow S^d$ is such that there is a $y \in S^d$ with $f^{-1}(y) = \{x_1, \dots, x_k\}$. Let U_i be disjoint open neighbourhoods and V a neighbourhood of y such that $f(U_i) \subseteq V$. The map $f|_U: H_d(U, U \setminus \{x_i\}) \rightarrow H_d(V, V \setminus \{y\})$ is multiplication by an integer (from above). The *local degree* of f at x_i is this integer

Theorem 3.7. $\deg f = \sum_{i=1}^k (\deg f)|_{x_i}$

Proof. We draw the following diagram (left) and trace the image of 1 from the top left (see right)

$$\begin{array}{ccc}
H_d(S^d) & \xrightarrow{f_*} & H^d(S^d) \\
\downarrow \cong & & \downarrow \cong \\
H_d(S^d, S^d \setminus \{x_1, \dots, x_k\}) & \xrightarrow{f_*} & H_d(S^d, S^d \setminus \{y\}) \\
\uparrow \cong & & \uparrow \cong \\
H_d(\coprod U_i, \coprod U_i \setminus \{x_i\}) & \xrightarrow{\coprod f_{i*}} & H_d(V, V \setminus \{y\}) \\
\parallel & & \\
\oplus H_d(U_i, U_i \setminus \{x_i\}) & &
\end{array}
\quad
\begin{array}{ccc}
1 & \xrightarrow{\quad} & \deg f \\
\downarrow & & \parallel \\
(1, \dots, 1) & \longrightarrow & \sum (\deg f)|_{x_i}
\end{array}$$

□

Example 3.3. Consider $f(z) = z^n: S^1 \rightarrow S^1$ for $n \in \mathbb{Z}$. For $n = -1$ this is a reflection so $\deg f = -1$, so, composing with $z \mapsto z^{-1}$ we may assume $n > 0$. Let $y = 1$, and x_i the n th roots of unity. Choose a neighbourhood of length $\frac{\pi}{n}$ around each x_i . Each U_i is sent homeomorphically to U_0 so $(\deg f)|_{x_i} \in \pm 1$. We exploit that multiplication by x_i is homotopic to the identity and commutes with f . We draw this diagram. Everything commutes, so as the routes from the top to the bottom are multiplication by $(\deg f)|_{x_i}$ and $(\deg f)|_{x_j}$, they are the same

$$\begin{array}{ccc}
H_1(S^1) & \xrightarrow{\cdot x_i} & H_1(S^1) \\
\cong \downarrow & & \downarrow \cong \\
H_1(S^1, S^1 \setminus \{x_j\}) & \xrightarrow{\cdot x_i} & H_1(S^1, S^1 \setminus \{x_{j+i}\}) \\
\downarrow \cong & & \downarrow \cong \\
H_1(U_j, U_j \setminus \{x_j\}) & \xrightarrow{\cdot x_i} & H_1(U_{i+j}, U_{i+j} \setminus \{x_{i+j}\}) \\
& \searrow f_* & \swarrow f_* \\
& H_1(V, V \setminus \{1\}) & \\
& \downarrow \cong & \\
& H_1(S^1, S^1 \setminus \{1\}) & \\
& \uparrow \cong & \\
& H_1(S^1) &
\end{array}$$

We now need to show this common value is 1. We do this by finding a homotopy from id_{S^1} to some $g: S^1 \rightarrow S^1$ such that $g|_{U_0} = f|_{U_0}$ (trivial)

4 Proving That We Can Compute (Co)Homology

4.1 Relative Homology Exact Sequence

For $A \subseteq X$, $C_\bullet(X, A) = C_\bullet(X)/C_\bullet(A)$ with the induced differential. The long exact sequence for $H_*(A), H_*(X), H_*(X, A)$ comes from a general homological algebra fact:

Theorem 4.1. *Let $0 \rightarrow A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{q} C_\bullet \rightarrow 0$ be a short exact sequence of chain complexes (ie every $0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0$ is exact). Then there is a homomorphism $\partial: H_k(C_\bullet) \rightarrow H_{k-1}(A_\bullet)$ such that*

$$\dots \xrightarrow{\partial} H_k(A) \xrightarrow{i_*} H_k(B) \xrightarrow{q_*} H_k(C) \xrightarrow{\partial} H_{k-1}(A) \xrightarrow{i_*} \dots$$

is exact

Proof. We draw

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_i & \xrightarrow{i} & B_i & \xrightarrow{q} & C_i & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & A_{i-1} & \xrightarrow{i} & B_{i-1} & \xrightarrow{q} & C_{i-1} & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & A_{i-2} & \xrightarrow{i} & B_{i-2} & \xrightarrow{q} & C_{i-2} & \longrightarrow & 0 \end{array}$$

and want to define $\partial: H_i(C_\bullet) \rightarrow H_{i-1}(A_\bullet)$. Let $[c] \in H_i(C_\bullet)$. We may pull this back along q to some $b \in B_i$. Now by commutativity, $qd(b) = 0$, so $d(b) = i(a)$ for some $a \in A_{i-1}$. Again by commutativity, $d(a) = 0$ as i_{i-2} is injective and $dd(b) = 0$. We send $[c] \mapsto [a]$. We need to prove this is well defined. Let $b' \in B_i$ be another lift of c , and so we get $d(b')$ and a' similarly. We want to show $a - a'$ is a boundary. Now $b - b' \mapsto 0$ under q , so we have $z \in A_i$ with $i(z) = b - b'$. Now $id(z) = di(z) = d(b - b') = d(b') - d(b) = i(a' - a)$ so $a' - a = d(z)$ and we are done.

We do one case of exactness, at $H_i(C_\bullet)$. We want to compute $\partial q([b])$. $q([b]) = [q(b)]$ and we need to choose a lift of $q(b)$. We choose b , but $d(b) = 0$ so $\partial q([b]) = 0$. We now show $\ker \partial \subseteq \text{im } q$. Let $[c] \in \ker \partial$. Then there is a, b, z with $q(b) = c$ and $i(a) = d(b)$ and $a = d(z)$. Consider $i(z)$. $di(z) = i(a) = d(b)$, so $b - i(z)$ is a cycle. $q(b - i(z)) = c$ by definition, so we are done. $\square$

We apply this to the tautological short exact sequence

$$0 \rightarrow C_\bullet(A) \rightarrow C_\bullet(X) \rightarrow C_\bullet(X, A) \rightarrow 0$$

and get the exact sequence for relative homology.

Theorem 4.2. (5-lemma) Consider a commutative diagram

$$\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E \\
\downarrow l & & \downarrow m & & \downarrow n & & \downarrow p & & \downarrow q \\
A' & \xrightarrow{r} & B' & \xrightarrow{s} & C' & \xrightarrow{t} & D' & \xrightarrow{u} & E'
\end{array}$$

with exact rows, m, p isomorphisms, l an epimorphism and q monic. Then n is an isomorphism

Proof. Surjective: Suppose $c' \in C'$. Then letting $d' = l(c')$, we get $d = p^{-1}(d') \in D$ with $d, d' \mapsto 0$. Therefore there is $c \in C$ with $h(c) = d$. We get $n(c) \in C'$. then $t(c' - n(c)) = 0$, so there is $b' \in B'$ with $s(b') = c' - n(c)$, so there is $b \in B$ with $m(b) = b'$. Then we get $g(b) \in C$ which maps to $c' - n(c)$ under n , so $c' = n(c + g(b))$, ie $c' \in \text{im } n$.

Injective is similar and an exercise to the reader. (It will use injectivity of q) $\square$

Proposition 4.1. Let $f: (X, A) \rightarrow (Y, B)$ be a map of pairs. If any two of the following maps are isomorphisms, then all three are: $(f|_A)_*, f_*: H_\bullet(X) \rightarrow H_\bullet(Y), f_*: H_\bullet(X, A) \rightarrow H_\bullet(Y, B)$

Proof. Consider the map of long exact seequences

$$\begin{array}{ccccccccc}
H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) \\
f_* \downarrow & & f_* \downarrow & & f_* \downarrow & & f_* \downarrow & & f_* \downarrow \\
H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y, B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(Y)
\end{array}$$

By the 5-lemma we are done (after rotating for the other two cases) $\square$

4.2 Homotopy Invariance

Definition 4.1. A chain homotopy between $f_\bullet, g_\bullet: C_\bullet \rightarrow D_\bullet$ is a collection of homotopies $h_n: C_n \rightarrow D_{n+1}$ such that $g_n - f_n = dh_n + h_{n-1}d$

Lemma 4.3. If $f_\bullet, g_\bullet: C_\bullet \rightarrow D_\bullet$ are chain homotopic, then $f_* = g_*$ on homology

Proof. Let $[c] \in H_\bullet(C_\bullet)$. Then $f_n([c]) = [g_n(c) - dh_n(c) - h_{n+1}d(c)] = [g_n(c)]$ as $d(c) = 0$, so we are done. $\square$

If $H: [0, 1] \times X \rightarrow Y$ is a homotopy from f to g , we need to show $f_\bullet, g_\bullet$ are chain homotopic. Suppose we have $\sigma: \Delta^n \rightarrow X$, then we have $[0, 1] \times \Delta^n \xrightarrow{\text{id} \times \sigma} [0, 1] \times X \xrightarrow{H} Y$ is a homotopy from $f \circ \sigma$ to $g \circ \sigma$, but the first term isn't a simplicial complex. Suppose there are chains $p_n \in C_{n+1}([0, 1] \times \Delta^n)$ with ends i_j such that $dp_n = i_1 - i_0 - \sum_{j=1}^n (-1)^j ([0, 1] \times \delta_i)(P_{n-1})$. Let $H: [0, 1] \times X \rightarrow Y$

be a homotopy from f to g . Let $H_n: C_n(X) \rightarrow C_{n+1}(Y)$ be $(\sigma: \Delta^n \rightarrow X) \mapsto (H \circ ([0, 1] \times \sigma))_\bullet(p_n)$. We calculate

$$\begin{aligned} dH_n(\sigma) &= (H \circ ([0, 1] \times \sigma))_\bullet(dp_n) \\ &= (H \circ ([0, 1] \times \sigma))_\bullet \left(i_1 - i_0 - \sum_{j=1}^n (-1)^j ([0, 1] \times \delta_j)(P_{n-1}) \right) \\ &= g_\bullet \sigma - f_\bullet \sigma - \sum_{j=1}^n (-1)^j ([0, 1] \times (H \circ ([0, 1] \times \sigma) \delta_j)(P_{n-1})) \\ &= g_\bullet \sigma - f_\bullet \sigma - H_{n-1}(d\sigma) \end{aligned}$$

so we have our desired chain homotopy. We only need to provide the desired P_n objects. Now $[0, 1] \times \Delta^n$ is convex, so let $v_0, \dots, v_n$ be the vertices of $\{0\} \times \Delta^n$ and $w_0, \dots, w_n$ be the vertices of $\{1\} \times \Delta^n$. For $x_0, \dots, x_p$ a sequence of the v_i, w_j , we denote by $[x_0, \dots, x_p]$ the p -simplex given by $t_0, \dots, t_p \mapsto \sum t_i x_i$. Let $P_n = \sum_{i=0}^n (-1)^i [v_0, \dots, v_i, w_i, \dots, w_n]$. Then

$$\begin{aligned} dP_n &= \sum_{i=0}^n (-1)^i \left(\sum_{j \leq i} (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots] \right. \\ &\quad \left. + \sum_{j \geq i} (-1)^{j+1} [v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots] \right) \\ &= i_1 - i_0 + \sum_{j < i} (-1)^{i+j} [v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots] \\ &\quad + \sum_{j > i} (-1)^{i+j+1} [v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots] \end{aligned}$$

and staring long enough, the latter is what we want and homotopy invariance follows.

Similarly there are notions of cochain homotopies (exercise: define this) and the whole argument dualises to give homotopy invariance for cohomology (exercise if you hate yourself: write this out explicitly)

4.3 Mayer Vietoris and Excision

Let $\mathcal{U}$ be a set of subspaces of X indexed by I

Definition 4.2. Let $C_n^{\mathcal{U}}(X)$ be the free abelian group of singular simplices in X which land wholly in some (not-necessarily-unique) $U_\alpha \in \mathcal{U}$. This is a subchaincomplex of $C_\bullet(X)$ with homology $H_\bullet^{\mathcal{U}}(X)$

Theorem 4.4. (*Small Simplices Theorem*) *If the interiors of the $U_\alpha \in \mathcal{U}$ cover X , then the natural map $H_n^{\mathcal{U}}(X) \rightarrow H_n(X)$ is an isomorphism.*

Proof. No Proof (see handout) □

Proof. (Mayer Vietoris Proof) Let $X = A \cup B$ with A, B open in X and set $\mathcal{U} = \{A, B\}$ and let $C_\bullet^{\mathcal{U}}(X) = C_\bullet(A + B)$. The natural map $C_i(A) \oplus C_i(B) \xrightarrow{j_A - j_B}$

$C_\bullet(A + B)$ is surjective. Its kernel is (x, y) such that $j_A(x) = j_B(y)$, ie chains in A and B , so we get

$$0 \longrightarrow C_\bullet(A \cap B) \longrightarrow C_i(A) \oplus C_i(B) \longrightarrow C_\bullet(A + B) \longrightarrow 0$$

so we get a long exact sequence on homology, using that $H_\bullet(C_\bullet(A + B)) = H_\bullet^{\mathcal{U}}(X) \cong H_\bullet(X)$, so we get the Mayer Vietoris sequence $\square$

Proof. (Excision proof) Let $X \supseteq A \supseteq Z$ with $\bar{Z} \subseteq A^\circ$. Let $B = X - Z$. Then A° and B° cover. Let $\mathcal{U} = \{A, B\}$ so that $C_\bullet(A + B) = C_\bullet^{\mathcal{U}}(X) \rightarrow C_\bullet(X)$ is a homology isomorphism. Consider

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_\bullet(A) & \longrightarrow & C_\bullet(A + B) & \longrightarrow & C_\bullet(A + B)/C_\bullet(A) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_\bullet(A) & \longrightarrow & C_\bullet(X) & \longrightarrow & C_\bullet(X, A) \longrightarrow 0 \end{array}$$

as the middle and first maps are homology isomorphisms, by the 5-lemma, so is the last one. Also $C_\bullet(B, B \cap A) = C_\bullet(B)/C_\bullet(A \cap B) \cong C_\bullet(A + B)/C_\bullet(A)$, so $C_\bullet(X - Z, A - Z) = C_\bullet(B, B \cap A) \cong C_\bullet(X, A)$, so we are done. $\square$

For a space X with basepoint x_0 , the *reduced homology* is $\tilde{H}_n(X) = H_n(X, \{x_0\})$. A pair (X, A) is *good* if there is an open subset $U \subseteq X$ such that U deformation retracts to A .

Theorem 4.5. *If (X, A) is good, then $H_n(X, A) \rightarrow H_n(X/A, A/A) = \tilde{H}_n(X/A)$ is an isomorphism.*

Proof. We have the diagram

$$\begin{array}{ccccc} H_n(X, A) & \longrightarrow & H_n(X, U) & \longleftarrow & H_n(X \setminus A, U \setminus A) \\ \downarrow & & \downarrow & & \downarrow \\ H_n(X/A, A/A) & \longrightarrow & H_n(X/A, U/A) & \longleftarrow & H_n((X/A) \setminus (A/A), (U/A) \setminus (A/A)) \end{array}$$

The third column map is trivially an isomorphism, the two maps between the second and third columns are isomorphisms by excision, and the 5-lemma (on the relative homology exact sequence, as $A \simeq U$) gives $H_n(X, A) \cong H_n(X, U)$. The deformation retraction of U onto A descends to one of U/A onto A/A , so we get $H_n(X/A, A/A) \cong H_n(X/A, U/A)$ similarly, and the result follows as every map in the diagram is known to be an isomorphism except the one we want. $\square$

5 Cell Complexes, Cellular Homology and Coefficients

5.1 Identifications and Generators

S^n can show up in two ways, as a subset of $\mathbb{R}^{n+1}$, or as $D^n/\partial D^n$. We want preferred generators $u_n \in H_n(S^n)$. We choose $u_0 = [1] - [-1] \in H_0(S^0)$. We

calculate

$$0 \rightarrow H_1(D^1, S^0) \xrightarrow{\partial} H_0(S^0) \rightarrow H_0(D^1)$$

We know $u_0 \mapsto 0$, and this corresponds to an element of $H_1(D^1, S^0) \cong \tilde{H}^1(D^1/S^0 = S^1)$, which we call u_1 . In higher degrees, we get $0 \rightarrow H_n(D^n, S^{n-1}) \rightarrow H_{n-1}(S^{n-1}) \rightarrow 0$ from the exact sequence of relative homology, where the second term is just $\tilde{H}_n(S^n)$, and we choose u_n to be the preimage of u_{n-1} . To be explicit, we let $u'_1: \Delta^1 \rightarrow D^1$ by $(t_0, t_1) \mapsto 2t_0 - 1$. Then $[u'_1] \in H_1(D^1, S^0)$ with boundary u_0 so u_1 is $\Delta \rightarrow D^1 \rightarrow D^1/S^0 \cong S^1$

5.2 Cell Complexes

Definition 5.1. A *cell complex* X is a space constructed inductively via

1. X^0 is a discrete space
2. X^n is obtained from X^{n-1} by choosing a collection of maps $\{\phi_\alpha: S^{n-1} \rightarrow X^{n-1}\}$ and setting $X^n = (X^{n-1} \sqcup \coprod_{\alpha \in A} D_\alpha^n) / \sim$ where $x \in \partial D_\alpha^n \sim \phi_\alpha(x) \in X^{n-1}$. The image of $D_\alpha^n \setminus \partial D_\alpha^n$ is called C_α , an *open cell*. These partition the space
3. We set $X = \bigcup X^n$ with the topology where A is open if every $A \cap X^n \subseteq X^n$ is open

If $X = X^n$ for some n , we say X is *finite dimensional*. If also X^0 and A_n are finite sets for all n , we say X is a *finite cell complex*. For each open cell e_α , we call $\Phi_\alpha: D_\alpha^n \hookrightarrow X^{n-1} \sqcup \coprod_{\alpha \in A} D_\alpha^n \rightarrow X^n \rightarrow X$ the *characteristic map* for e_α . A *subcomplex* A of X is a subspace which is a union of open cells e_α such that $\bar{e}_\alpha \subseteq A$ when $e_\alpha \subseteq A$.

X^n is the *n-skeleton* of X

Lemma 5.1. If $A \subseteq X$ is a subcomplex, then (X, A) is good (there is a neighbourhood U of A which deformation retracts to A)

Proof. Example sheet 2 □

Lemma 5.2. Let X be a cell complex.

1. $H_i(X^n, X^{n-1}) = \begin{cases} 0 & i \neq n \\ \mathbb{Z}^{\#n\text{-cells}} & i = n \end{cases}$
2. $H_i(X^n) = 0$ for $i > n$
3. $H_i(X^n) \rightarrow H_i(X)$ is an isomorphism for $n > i$

Proof. 1. As $X^{n-1} \subseteq X^n$ is a subcomplex, $H_i(X^n, X^{n-1}) \cong \tilde{H}^i(X^n/X^{n-1})$, but $X^n/X^{n-1} \cong \bigvee_{\alpha \in A_n} S_\alpha^n$ (where we identify the basepoints of the S_α^n)

and nothing else). Also $\bigvee_{\alpha \in A_n} S_\alpha^n = \coprod_{\alpha \in A_n} S_\alpha^n / \coprod_{\alpha \in A_n} \{*\}$. We get

$$H_i(\coprod_{\alpha \in A_n} \{*\}) \longrightarrow H_i(\coprod_{\alpha \in A_n} S_\alpha^n) \longrightarrow H_i(\bigvee_{\alpha \in A_n} S_\alpha^n)$$

$$\begin{cases} \bigoplus \mathbb{Z} & i = 0 \\ 0 & \text{else} \end{cases} \longrightarrow \begin{cases} \bigoplus \mathbb{Z} & i = 0, n \\ 0 & \text{else} \end{cases} \longrightarrow \begin{cases} \bigoplus \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

where the first two columns are isomorphic, so we are done by observation.

2. We have

$$H_i(X^{n-1}) \longrightarrow H_i(X^n) \longrightarrow H_i(X^n, X^{n-1}) \longrightarrow H_{i-1}(X^{n-1})$$

so by induction on n , and part (i), the result follows

3. We will do the finite dimensional case, the general case is on example sheet 2. Since $X = X^m$, the m case is done. Assume it holds for $i < n \leq m$, we have

$$H_{i+1}(X^{n+1}, X^n) \longrightarrow H_i(X^n) \longrightarrow H_i(X^{n+1}) \longrightarrow H_i(X^{n+1}, X^n)$$

so by part (i), $H_i(X^n) \cong H_i(X^{n+1})$ and we are done by induction. $\square$

5.3 Cellular Homology

Definition 5.2. Given a cell complex X , we let $C_n^{cell}(X) = H_n(X^n, X^{n-1})$, the free abelian group on the n -cells of X , with $d_n^{cell}: C_n^{cell}(X) \rightarrow C_{n-1}^{cell}(X)$ given by $H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}) \xrightarrow{q} H_{n-1}(X^{n-1}, X^{n-2})$. We define $H_n^{cell}(X)$ to be the homology of this chain complex.

Remark 5.1. This is a cell complex as $d_{n-1}^{cell} \circ d_n^{cell}$ is given by $q \circ \partial \circ q \circ \partial$, but $\partial \circ q = 0$ by the long exact sequence of relative homology, so $d_{n-1}^{cell} \circ d_n^{cell} = 0$

Theorem 5.3. $H_n^{cell}(X) \cong H_n(X)$

Proof. We draw (with all isomorphisms from Lemma 5.2)

$$\begin{array}{ccccccc} & & & & H_n(X^{n+1}, X^n) = 0 & & \\ & & & & \nearrow & & \\ & H_n(X^{n-1}) = 0 & & H_n(X^{n+1}) & & & \\ & \searrow & \nearrow i & \searrow q & & & \\ & H_n(X^n) & & & & & \\ \nearrow \partial & & & & & & \\ H_{n+1}(X^{n+1}, X^n) & \xrightarrow{d_{n+1}^{cell}} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n^{cell}} & H_{n-1}(X^{n-1}, X^{n-2}) & & \\ & & \searrow \partial & \nearrow q & & & \\ & & H_{n-1}(X^{n-1}) & & & & \\ & & \nearrow & & & & \\ & & H_{n-1}(X^{n-2}) = 0 & & & & \end{array}$$

Now $H_n(X) \cong H_n(X^{n+1}) \cong H_n(X^n)/\text{im } \partial$ by exactness at $H_n(X^n)$. As $q: H_n(X^n) \rightarrow H_n(X^n, X^{n-1})$ is injective, we can write this as $q_* H_n(X^n)/\text{im}(q_* \circ \partial = d_{n+1}^{cell}) = \ker(\partial: H_n(X^n, X^{n-1}) \rightarrow H_{n-1}(X^{n-1}))/\text{im}(d_{n+1}^{cell})$. Now this is $\ker(q_* \partial = d_n^{cell})/\text{im}(d_{n+1}^{cell})$ as q is injective so we are done $\square$

This has many immediate applications

Corollary 5.3.1.

1. If X has finitely many n -cells, $H_n(X)$ is a finitely generated abelian group (as it is a subquotient of $C_n^{cell}(X)$). From this proof, we even get that $H_n(X)$ is generated by at most number-of- n -cells elements.
2. If X has a cell structure with no n -cells then $H_n(X) = 0$ and $H_{n-1}(X)$ is torsion free (as the chain complex looks like $0 \rightarrow C_{n-1}^{cell}(X) \rightarrow \dots$, so $H_{n-1}(X) \leq C_{n-1}^{cell}(X)$ which is torsion free)
3. If $H_n(X) \neq 0$ then any cell structure on X has an n -cell
4. If X has a cell structure with only even dimensional cells, then $H_n(X) \cong C_n^{cell}(X) = \mathbb{Z}^{\#n\text{-cells}}$

Proof. Trivial $\square$

We can similarly define *cellular cohomology* with $C_{cell}^n(X) = H^n(X^n, X^{n-1})$ and $d_{cell}^n = \partial \circ q^*$. This defines a cochain complex and $C_{cell}^\bullet \cong \text{Hom}(C_\bullet^{cell}, \mathbb{Z})$ as cochain complexes.

To use cellular homology to calculate, we need to determine $d_n^{cell}: \bigoplus_{\alpha \in I_n} \mathbb{Z}\{e_\alpha\} \rightarrow \bigoplus_{\beta \in I_{n-1}} \mathbb{Z}\{e_\beta\}$. This is just a matrix, and we wish to compute the components $d_{\alpha\beta} \in \mathbb{Z}$

Lemma 5.4. $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \xrightarrow{\phi_\alpha} X^{n-1} \rightarrow X^{n-1}/X^{n-2} \cong \bigvee_{\beta \in I_{n-1}} S_\beta^{n-1} \rightarrow S_\beta^{n-1}$

Proof. This diagram commutes

$$\begin{array}{ccccccc}
 1 & \xrightarrow{\quad} & 1 & \xrightarrow{\quad} & \text{deg}(\text{map given in lemma}) & & \\
 & & & & & & \\
 u_\alpha & & H_n(D_\alpha^n, \partial D_\alpha^n) & \xrightarrow{\partial} & H_{n-1}(\partial D_\alpha^n) & & H_{n-1}(S_\beta^{n-1}) \\
 \downarrow & & \downarrow (\Phi_\alpha)_* & & \downarrow (\phi_\alpha)_* & & \uparrow \text{collapse} \\
 e_\alpha & \xleftarrow{\quad} & H_n(X^n, X^{n-1}) & \xrightarrow{\partial} & H_{n-1}(X^{n-1}) & \xrightarrow{q_*} & H_{n-1}(X^{n-1}/X^{n-2}) \\
 & & \searrow d_n^{cell} & & \downarrow q_* & & \downarrow \cong \\
 & & & & H_{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{\text{exc}} & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \\
 & & & & \searrow & & \\
 & & & & \Sigma d_{\alpha\beta} e_\beta & \xrightarrow{\quad} &
 \end{array}$$

$\square$

Example 5.1. Klein Bottle. The usual pasting diagram gives the cell structure. $K^0 = \{v\}$, $K^1 = S^1 \vee S^1$, $K^2 = K$. Therefore $C_{\bullet}^{cell}(K)$ is

$$0 \longrightarrow \mathbb{Z}\{\alpha\} \longrightarrow \mathbb{Z}\{a, b\} \longrightarrow \mathbb{Z}\{v\} \longrightarrow 0$$

where $d_1^{cell}(a) = d_1^{cell}(b) = 0$, $d_2^{cell}(\alpha) = 2b$ so $H_0^{cell}(K) \cong \mathbb{Z}$, $H_1^{cell}(K) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$, $H_2^{cell}(K) = 0$.

Dually $C_{cell}^{\bullet}$ is

$$0 \longleftarrow \mathbb{Z}\{\alpha^*\} \xleftarrow{(0,2)} \mathbb{Z}\{a^*, b^*\} \xleftarrow{(0,0)} \mathbb{Z}\{v^*\} \longleftarrow 0$$

so $H_{cell}^0(K) = \mathbb{Z}$, $H_{cell}^1(K) = \mathbb{Z}$, $H_{cell}^2(K) = \mathbb{Z}/2\mathbb{Z}$

Example 5.2. We compute the (co)homology of

$$\begin{aligned} \mathbb{RP}^n &= S^n / \{x \sim -x\} \\ &= D^n / \{x \in \partial D^n \sim -x \in \partial D^n\} \\ &= \mathbb{RP}^{n-1} \cup_{\phi} D^n \end{aligned}$$

where $\phi: \partial D^n \rightarrow \mathbb{RP}^{n-1}$ via the quotient. We conclude that $\mathbb{RP}^n$ has a cell structure with one cell in each dimension $0, \dots, n$. Let e_i be the i -dimensional cell. How do we calculate the differential? We consider the attaching map $q: \partial D^n \xrightarrow{\phi} \mathbb{RP}^{n-1} \rightarrow \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} = S^{n-1}$ where ϕ is 2-1. This sends the upper and lower open hemispheres homeomorphically onto $S^{n-1} \setminus \{*\}$, and sends the equator to $\{*\}$. Also the antipodal map interchanges the upper and lower hemispheres, so

$$\deg q = 1 + (-1)^n = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

So our chain complex $C_{\bullet}^{cell}(\mathbb{RP}^n)$ is

$$0 \longrightarrow \mathbb{Z}e_n \xrightarrow{2 \text{ or } 0} \mathbb{Z}e_{n-1} \xrightarrow{2 \text{ or } 0} \dots \xrightarrow{0} \mathbb{Z}e_2 \xrightarrow{2} \mathbb{Z}e_1 \xrightarrow{0} \mathbb{Z}e_0$$

so

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2\mathbb{Z} & i \text{ odd}, 0 < i < n \\ \mathbb{Z} & i = n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

and $C_{cell}^{\bullet}(\mathbb{RP}^n)$ is

$$0 \longleftarrow \mathbb{Z}e_n \xleftarrow{2 \text{ or } 0} \mathbb{Z}e_{n-1} \xleftarrow{2 \text{ or } 0} \dots \xleftarrow{0} \mathbb{Z}e_2 \xleftarrow{2} \mathbb{Z}e_1 \xleftarrow{0} \mathbb{Z}e_0$$

and so

$$H^i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2\mathbb{Z} & i \text{ even}, 0 < i \leq n \\ \mathbb{Z} & i = n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Example 5.3. Let $\phi: S^{n-1} \rightarrow X$ and $Y = X \cup_{\phi} D^n$. Then $H_{\bullet}(Y, X)$ is isomorphic to

$$\tilde{H}_{\bullet}(Y/X = S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$

by excision, so we get a long exact sequence

$$0 \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow \mathbb{Z} \xrightarrow{\partial} H_{n-1}(X) \longrightarrow H_{n-1}(Y) \longrightarrow 0$$

$$1 \longmapsto \phi_*(u_{n-1})$$

and in each of the cases $\phi_*(u_{n-1})$ is 0, has finite order, has infinite order, we can see the effect on homology

5.4 (Co)Homology with Coefficients

For an abelian group A , let $C_{\bullet}(X; A) = C_{\bullet}(X) \otimes_{\mathbb{Z}} A$ with differential $d \otimes 1_A$, and we denote $H_i(X; A) = H_i(C_{\bullet}(X; A))$. We also define $H_i^{cell}(X; A) = H_i(C_{\bullet}^{cell}(X) \otimes_{\mathbb{Z}} A)$ (and this is isomorphic to $H_i(X; A)$). Similarly we can define $C^{\bullet}(X, A) = \text{Hom}_{\mathbb{Z}}(C_{\bullet}(X), A)$ to get a cochain complex with cohomology $H^i(X; A)$ and we can similarly define the cellular version. We will mainly take $A = \mathbb{Z}/n\mathbb{Z}, \mathbb{Q}$.

Example 5.4. In $C_{\bullet}^{cell}(\mathbb{RP}^n)$ all differentials are 0, 2 so tensoring by $\mathbb{Z}/2\mathbb{Z}$ these are all 0, so $H_i(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ in all degrees $0 \leq i \leq n$ and the cohomology is the same. On the other hand, $2: \mathbb{Q} \rightarrow \mathbb{Q}$ is an isomorphism, so

$$H_i(\mathbb{RP}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0 \text{ or } i = n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

5.5 Euler Characteristic

If X is a finite cell complex, then $\chi(X) = \sum_{i=0}^{\infty} (-1)^i \#i\text{-cells of } X$ is the *Euler characteristic* of X . Let $\chi_{\mathbb{Z}}(X) = \sum_{i=0}^{\infty} (-1)^i \text{rank } H_i(X)$ and generally for any field we can define $\chi_{\mathbb{F}}(X) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{F}} H_i(X; \mathbb{F})$

Theorem 5.5. $\chi = \chi_{\mathbb{Z}} = \chi_{\mathbb{F}}$

Proof. Let $C_i = C_i^{cell}(X)$, $Z_i = \ker d$ and $B_i = \text{im } d$. Then there are exact sequences

$$0 \longrightarrow Z_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0$$

and

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i(X) \longrightarrow 0$$

so $\text{rank } H_i(X) = \text{rank}(Z_i) - \text{rank}(B_i)$, so

$$\begin{aligned}\chi_{\mathbb{Z}}(X) &= \sum_{i=0}^{\infty} (-1)^i (\text{rank } Z_i - \text{rank } B_i) \\ &= \sum_{i=-1}^{\infty} (-1)^i (\text{rank } Z_i - (\text{rank } C_{i+1} - \text{rank } Z_{i+1})) \\ &= \sum_{i=-1}^{\infty} (-1)^{i+1} (\text{rank } C_{i+1}) \\ &= \chi(X)\end{aligned}$$

and the same proof works for $\mathbb{F}$. □

Remark 5.2. By the same proof, $\chi(X) = \sum_{i=0}^{\infty} (-1)^i H^i(X)$

Example 5.5. 1. $\chi(*) = 1$

2. $\chi(S^n) = 1 + (-1)^n$

3. $\chi(\mathbb{CP}^n) = n + 1$

4. $\chi(\Sigma_g) = 2 - 2g$

5. $\chi(\mathbb{RP}^n) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$

6 The Cup Product and the Universal Coefficient Theorem

6.1 The Cup Product

Definition 6.1. Let R be a ring, and suppose $\phi \in C^k(X; R)$, $\psi \in C^l(X; R)$. Then we define their *cup product*, $\phi \smile \psi \in C^{k+l}(X; R)$ is defined by

$$(\phi \smile \psi)(\sigma: \Delta^{k+l} \rightarrow X) = \phi(\sigma|_{[e_0, \dots, e_k]}) \psi(\sigma|_{[e_k, \dots, e_{k+l}]}) \in R$$

where $\sigma|_{[x_0, \dots, x_i]}$ is σ restricted to its first $i + 1$ coordinates, and $e_0, \dots, e_{k+l} \in \Delta^{k+l}$ are the vertices in standard order

Lemma 6.1. $d(\phi \smile \psi) = (d\phi) \smile \psi + (-1)^{\deg \phi} \phi \smile (d\psi)$

Proof. We compute:

$$\begin{aligned}((d\phi) \smile \psi)(\sigma) &= (d\phi)(\sigma|_{[e_0, \dots, e_{k+1}]}) \psi(\sigma|_{[e_{k+1}, \dots, e_{k+l+1}]}) \\ &= \phi \left(\sum_{i=0}^{k+1} (-1)^i \sigma|_{[e_0 \dots \hat{e}_i \dots e_{k+1}]} \right) \psi(\sigma|_{[e_{k+1}, \dots, e_{k+l+1}]})\end{aligned}$$

$$\begin{aligned}
(\phi \smile (d\psi))(\sigma) &= (d\phi)(\sigma|_{[e_0, \dots, e_k]})\psi(\sigma|_{[e_k, \dots, e_{k+l+1}]}) \\
&= \phi(\sigma|_{[e_0, \dots, e_k]})\psi\left(\sum_{i=k}^{l+k+1} (-1)^{i-k} \sigma|_{[e_k, \dots, \hat{e}_i, \dots, e_{k+l+1}]}\right)
\end{aligned}$$

Adding these together, the $i = k + 1$ term of the first expansion cancels with the $i = k$ term of the second. The rest is

$$(d(\phi \smile \psi))(\sigma) = (\phi \smile \psi) \sum_{i=0}^{l+k+1} (-1)^i \sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_{l+k+1}]}$$

so we are done $\square$

Corollary 6.1.1. *There is a well defined map $\cdot \smile \cdot : H^k(X; R) \times H^l(X; R) \rightarrow H^{k+l}(X; R)$ by $([\phi], [\psi]) \mapsto [\phi \smile \psi]$*

Proof. If ϕ, ψ are cocycles, then $d(\phi \smile \psi) = (d\phi) \smile \psi + (-1)^k \phi \smile (d\psi) = 0$, so $\phi \smile \psi$ represents a cohomology class.

To prove it is well defined, if $\phi = \phi' + d\chi$, then

$$\begin{aligned}
\phi \smile \psi &= \phi' \smile \psi + (d\chi) \smile \psi \\
&= \phi' \smile \psi + d(\chi \smile \psi) \pm \chi \smile (d\psi) \\
&= \phi' \smile \psi + d(\chi \smile \psi)
\end{aligned}$$

so we are done as they differ by a boundary $\square$

Remark 6.1.

1. The class $1 \in C^0(X; R)$ which sends every 0-simplex to $1 \in R$ is a two sided identity for the cup product
2. The cup product is associative on cochains, and hence on cohomology
3. If $f: X \rightarrow Y$ is a map and $\phi, \psi \in C^\bullet(Y; R)$, then $f^*(\phi \smile \psi) = f^*\phi \smile f^*\psi$ by observation, and so $f^*: H^*(Y; R) \rightarrow H^*(X; R)$ is a ring homomorphism
4. For spaces X, Y with projections π_X, π_Y from their product, the *cross product* is $\cdot \times \cdot : H^k(X; R) \times H^l(Y; R) \rightarrow H^{k+l}(X \times Y; R)$ by $(a, b) \mapsto \pi_X^*a \smile \pi_Y^*b$. Conversely, using $\Delta: X \rightarrow X \times X$ by $x \mapsto (x, x)$, we can define the cup product in terms of the cross product by

$$H^k(X; R) \times H^l(X; R) \xrightarrow{\times} H^{k+l}(X \times X; R) \xrightarrow{\Delta^*} H^{k+l}(X; R)$$

5. The cup product is bilinear

Proposition 6.1. *If R is commutative, the cup product satisfies*

$$[\phi \smile \psi] = (-1)^{\deg \phi \deg \psi} [\psi \smile \phi]$$

(Note that this is not true on cochains)

Proof. Let $\rho: C_n(X) \rightarrow C_n(X)$ be given by $\sigma \mapsto (-1)^{n(n+1)/2} \sigma|_{[e_n \dots e_0]} = \varepsilon_n \sigma|_{[e_n \dots e_0]}$, where ε_n is the sign of the permutation reversing the $n+1$ vertices.

We claim that $\rho_\bullet$ is a chain map which is chain homotopic to the identity. If we assume this claim, then

$$\begin{aligned} (\rho^* \phi \smile \rho^* \psi)(\sigma) &= \phi(\varepsilon_k \sigma|_{[e_k \dots e_0]}) \psi(\varepsilon_l \sigma|_{[e_{k+l} \dots e_k]}) \\ (\rho^*(\psi \smile \phi))(\sigma) &= \varepsilon_{k+l} \psi(\sigma|_{[e_{k+l}, \dots, e_k]}) \phi(\sigma|_{[e_k, \dots, e_0]}) \end{aligned}$$

so $\varepsilon_k \varepsilon_l (\rho^* \phi \smile \rho^* \psi) = \varepsilon_{k+l} (\rho^*(\psi \smile \phi))$. Observing that $\varepsilon_{k+l} = (-1)^{kl} \varepsilon_k \varepsilon_l$, we have $\rho^* \phi \smile \rho^* \psi = (-1)^{kl} \rho^*(\psi \smile \phi)$. As $\rho_\bullet$ is chain homotopic to the identity, ρ^* acts as the identity on cohomology, so

$$\begin{aligned} [\phi] \smile [\psi] &= [\rho^* \phi] \smile [\rho^* \psi] \\ &= [\rho^* \phi \smile \rho^* \psi] \\ &= (-1)^{kl} [\rho^*(\psi \smile \phi)] \\ &= (-1)^{kl} [\psi \smile \phi] \\ &= (-1)^{kl} [\psi] \smile [\phi] \end{aligned}$$

Now to prove the claim.

We compute

$$\begin{aligned} d\rho(\sigma) &= \varepsilon_n \sum_{i=0}^n (-1)^i \sigma|_{[v_n \dots \hat{v}_{n-i} \dots v_0]} \\ \rho(d\sigma) &= \rho \left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0 \dots \hat{v}_i \dots v_n]} \right) \\ &= \varepsilon_{n-1} \sum_{i=0}^n (-1)^i \sigma|_{[v_n \dots \hat{v}_i \dots v_0]} \\ &= \varepsilon_{n-1} \sum_{i=0}^n (-1)^{n-i} \sigma|_{[v_n \dots \hat{v}_{n-i} \dots v_0]} \\ &= \varepsilon_n \sum_{i=0}^n (-1)^i \sigma|_{[v_n \dots \hat{v}_{n-i} \dots v_0]} \end{aligned}$$

so $\rho_\bullet$ is a chain map.

To construct the chain homotopy, we consider prism-like elements

$$P_n = \sum_i (-1)^i \varepsilon_{n-i} [v_0 \dots v_i, w_n \dots w_i] \in C_{n+1}([0, 1] \times \Delta^n)$$

. We let $\pi: [0, 1] \times \Delta^n \rightarrow \Delta^n$ be the projection. We define $F_n: C_n(X) \rightarrow$

$C_{n+1}(X)$ by $\sigma \mapsto (\sigma \circ \pi)_\bullet(P_n)$. We compute

$$\begin{aligned} dF(\sigma) &= (\sigma \circ \pi)_\bullet(dP_n) \\ &= (\sigma \circ \pi)_\bullet\left(\sum_i \sum_{j \leq i} (-1)^j (-1)^i \varepsilon_{n-i}[v_0 \dots \hat{v}_j \dots v_i, w_i \dots w_n]\right) \\ &\quad + \sum_{j \geq i} (-1)^{n+i+1-j} (-1)^i \varepsilon_{n-i}[v_0 \dots v_i, w_i \dots w_j \dots w_n] \end{aligned}$$

The terms with $j = i$ cancel, leaving $(\sigma \circ \pi)_\bullet(\varepsilon_n[w_n \dots w_0] - [v_0 \dots v_n]) = \rho(\sigma) - \sigma$ and the remaining terms with $j \neq i$ give $-F(d\sigma)$, so $dF + Fd = \rho - \text{id}$ $\square$

Remark 6.2. There is also a relative cup product,

$$H^k(X, A; R) \otimes H^l(X; R) \rightarrow H^{k+l}(X, A; R)$$

given by the usual formula on cochains, because $\phi \in C^k(X, A; R)$ is exactly a $\phi: C_k(X) \rightarrow R$ that vanishes on chains in A , but then the cup product of this with any cochain in X vanishes on chains in A . Via $q^*: H^*(X, A; R) \rightarrow H^*(X; R)$ we get a cup product on relative cohomology proper, but the first map is stronger. It says $H^*(X, A; R)$ is a right $H^*(X; R)$ -module

Example 6.1. Suppose X has

i	0	1	2	3	4	5	6
$H^i(X; \mathbb{Z})$	$\mathbb{Z}\{1\}$	0	0	$\mathbb{Z}\{x\}$	0	0	$\mathbb{Z}\{y\}$

Then $x \smile x = Ay$ for some $A \in \mathbb{Z}$. But $x \smile x = (-1)^{3 \cdot 3} x \smile x = -(x \smile x)$, so $A = 0$ and $x \smile x = 0$.

Theorem 6.2. (Künneth) Let R be a commutative ring and let Y be a space such that every $H^i(Y; R)$ are free R -modules. Then the cross product map

$$\bigoplus_{i+j=n} H^i(X; R) \otimes H^j(Y; R) \xrightarrow{\times} H^n(X \times Y; R)$$

is an isomorphism for every finite cell complex X .

Proof. We proceed by induction on the number of cells of X , and the dimension of X . The map is an isomorphism for $X = \{*\}$. If $X = X' \cup_f D^n$ for $f: S^{n-1} \rightarrow X'$, let $F^n(\cdot) = \sum_{i+j=n} H^i(\cdot; R) \otimes_R H^j(Y; R)$ and similarly for pairs. We get

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & F^n(X) & \longrightarrow & F^n(X') & \longrightarrow & F^{n+1}(X, X') & \longrightarrow & F^{n+1}(X) & \longrightarrow & F^{n+1}(X') & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H^n(X \times Y; R) & \longrightarrow & H^n(X' \times Y; R) & \longrightarrow & H^n(X \times Y, X' \times Y; R) & \longrightarrow & H^{n+1}(X \times Y; R) & \longrightarrow & H^{n+1}(X' \times Y; R) & \longrightarrow & \dots \end{array}$$

which has exact rows as $H^*(Y; R)$ is flat and the bottom row is just the exact sequence of relative cohomology, and every vertical map is the cross product. Every square commutes (most are trivial, the second is an exercise). By induction,

we may assume the second and final vertical maps are isomorphisms, so by the 5-lemma it suffices to show that the middle vertical map is an isomorphism. Now $F^*(D^n, \partial D^n) \cong F^*(X, X') \rightarrow H^*(X \times Y, X' \times Y; R) \cong H^*(D^n \times Y, \partial D^n \times Y; R)$, by excision on the left and right, as we have a good pair on both sides. We look at the map of long exact sequences as above for $X = D^n, X' = \partial D^n$. The left-most map is an isomorphism by homotopy invariance, and the case of a point, and the second map is an isomorphism by induction on dimension, so the middle map is an isomorphism as required. $\square$

Remark 6.3. The full Künneth theorem drops the requirement of freeness, and tells us that the cokernel is a certain Tor module.

Remark 6.4.

1. It is convenient to write $H^*(X; R) \otimes H^*(Y; R)$ for the graded R -module which in degree n is $\bigoplus_{i+j=n} H^i(X; R) \otimes H^j(Y; R)$, and then the theorem is that the cross product from $H^*(X; R) \otimes H^*(Y; R)$ to $H^*(X \times Y; R)$ is an isomorphism of graded commutative rings
2. If R is a field, all R modules are free, so the Künneth theorem always applies.

Example 6.2. If we take coefficients in a field $\mathbb{F}$, then $H^i(Y; \mathbb{F})$ is always free so we get

$$H^*(X \times Y; \mathbb{F}) \cong H^*(X; \mathbb{F}) \otimes_{\mathbb{F}} H^*(Y; \mathbb{F})$$

as graded commutative rings

Example 6.3. We know

$$H^*(S^1, \mathbb{Z}) = \begin{cases} \mathbb{Z}\{1\} & * = 0 \\ \mathbb{Z}\{x\} & * = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $1 \smile x = x, x \smile x = 0$, so $H^*(S^1, \mathbb{Z}) = \mathbb{Z}[x]/(x^2) = \Lambda[x]$. These are free $\mathbb{Z}$ -modules, so $H^*(T^n, \mathbb{Z}) = H^*(S^1, \mathbb{Z})^{\otimes n} = \Lambda[x_1, \dots, x_n]$ which is $\mathbb{Z}\langle x_1, \dots, x_n \rangle / (x_i^2, x_i x_j + x_j x_i)$ where $x_i = \pi_i^*(x)$

Example 6.4. Any $f: S^n \rightarrow T^n$ for $n \geq 2$ has $f^*: H^n(T^n, \mathbb{Z}) = \mathbb{Z} \rightarrow H^n(S^n, \mathbb{Z})$ be zero where the generator of the former is $x_1 x_2 \dots x_n$, but the image of this is a cup product of n elements in $H^1(S^n)$, which must therefore be 0.

6.2 Universal Coefficient Theorem

Theorem 6.3. (*Universal Coefficient Theorem*) Let R be a PID and M be an R -module. Then there is a natural map

$$H_*(X; R) \otimes_R M \rightarrow H_*(X; M)$$

which is an isomorphism if $H_*(X; R)$ are all free

Proof. We write $C_n = C_n(X; R)$ and $Z_n \subseteq C_n$ for the cycles, and B_n for the boundaries. We have a short exact sequence

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

which gives a short exact sequence of chain complexes

$$0 \longrightarrow (Z_\bullet, 0) \longrightarrow (C_\bullet, d) \longrightarrow (B_{\bullet-1}, 0) \longrightarrow 0$$

As R is a PID, $B_{n-1} \subseteq C_n$ is free, so there is a splitting $s: B_{n-1} \rightarrow C_n$ of the first sequence, so the first sequence is isomorphic to

$$0 \longrightarrow Z_n \longrightarrow Z_n \oplus B_{n-1} \longrightarrow B_{n-1} \longrightarrow 0$$

As the tensor product preserves direct sums, tensoring with M preserves this exact sequence, so we get a short exact sequence

$$0 \longrightarrow (Z_\bullet \otimes_R M, 0) \longrightarrow (C_\bullet \otimes_R M, d) \longrightarrow (B_{\bullet-1} \otimes_R M, 0) \longrightarrow 0$$

The associated long exact sequence on homology is

$$\begin{array}{ccccccc} \dots & \longrightarrow & B_n \otimes_R M & \xrightarrow{\partial} & Z_n \otimes_R M & \longrightarrow & H_n(X; M) \\ & & & & \searrow & & \\ & & & & B_{n-1} \otimes_R M & \xrightarrow{\partial} & Z_{n-1} \otimes_R M \longrightarrow \dots \end{array}$$

We compute that ∂ is just the inclusion. We also have a short exact sequence

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n(X; R) \longrightarrow 0$$

which splits by the freeness of the homology. This remains exact on tensoring with M ,

$$0 \longrightarrow B_n \otimes_R M \longrightarrow Z_n \otimes_R M \longrightarrow H_n(X; R) \otimes_R M \longrightarrow 0$$

The first map is the same as ∂ , and so comparing the sequences the result follows. $\square$

Theorem 6.4. (*Universal Coefficient Theorem for Cohomology*) Let R be a PID and M be an R -module. Then there is a natural map

$$H^*(X; M) \rightarrow \text{Hom}(H_*(X; R), M)$$

which is an isomorphism if $H_*(X; R)$ are all free

Proof. Same form of argument, using

$$C^n(X; M) = \text{Hom}(C_n, M) = \text{Hom}_R(C_n(X; R), M)$$

(hom-tensor adjunction) $\square$

7 Vector bundles

Definition 7.1. A *vector bundle* of dimension d on X is a map $\pi: E \rightarrow X$, with the structure of a real d -dimensional vector space on each fibre $e_x = \pi^{-1}(x)$, subject to the *local triviality* condition: Every $x \in X$ has a neighbourhood $U \subseteq X$ such that there is a homeomorphism $\phi: \pi^{-1}(U) \rightarrow \mathbb{R}^d \times U$ such that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times \mathbb{R}^d \\ & \searrow \pi \quad \swarrow \pi_U & \\ & U & \end{array}$$

commutes and is a linear isomorphism on each fibre

Remark 7.1. We can do the same for $\mathbb{C}$ vector bundles

Definition 7.2. A *section* of a vector bundle $\pi: E \rightarrow X$ is a map $s: X \rightarrow E$ such that $\pi \circ s = 1_X$, ie continuously choosing some $s(x) \in E_x$. The *zero section* is $s_0: X \rightarrow E$ given by $s_0(x) = 0_x \in E_x$

Remark 7.2. Some operations on vector bundles:

1. If $\pi: E \rightarrow X$ is a vector bundle and $f: Y \rightarrow X$ is a map, then $f^*E = \{(y, e) \in Y \times E \mid f(y) = \pi(e)\} \rightarrow Y$ via $(y, e) \mapsto y$ is a vector bundle with $(f^*E)_y = E_{f(y)}$ as vector spaces. This is the *pullback* of $\pi: E \rightarrow X$ along f .
2. If $\pi: E \rightarrow X$ and $p: F \rightarrow X$ are vector bundles, then $E \oplus F = \{(e, f) \in E \times F \mid \pi(e) = p(f)\} \rightarrow X$ via $(e, f) \mapsto \pi(e) = p(f)$. This has $(E \oplus F)_x = E_x \oplus F_x$ as vector spaces. This is called the *Whitney sum* of vector bundles
3. Similarly, we can define a tensor product, using common local trivialisations to give the topology.
4. If $\pi: E \rightarrow X$ is a vector bundle and $F \subseteq E$ is a subspace such that for each $x \in X$, there is a neighbourhood U and a homeomorphism $\phi: E|_U \rightarrow U \times \mathbb{R}^d$ over U taking $F|_U$ to $U \times \mathbb{R}^k$. In this case, F is a *subbundle* of E .
5. We can form $E/F \rightarrow X$ fibrewise and glue to get a *quotient vector bundle*.

Example 7.1.

1. Let $X = \text{Gr}_k(\mathbb{R}^n)$, the set of k -dimensional subspaces. We topologise this as a quotient of $GL_n(\mathbb{R}) \rightarrow \text{Gr}_k(\mathbb{R}^n)$ by action on $\mathbb{R}^k$. We let $E = \{(V, v) \in X \times \mathbb{R}^n \mid v \in V\} \xrightarrow{\pi} X = \text{Gr}_k(\mathbb{R}^n)$. Then $\pi^{-1}(\{V\}) = V$ is a vector space of dimension k . We write $E = \gamma_{k,n}^{\mathbb{R}}$. To prove local triviality, we have $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$. For $V \in \text{Gr}_k(\mathbb{R}^n)$, let $U = \{W \in \text{Gr}_k(\mathbb{R}^n) \mid W \cap V^\perp = 0\}$, an open neighbourhood of V . Let $\phi: E|_U \rightarrow U \times V$ be defined by $(W, w) \mapsto (W, \pi_V(w))$ where π_V is the orthogonal projection onto V . If $W \in U$ then $W \hookrightarrow \mathbb{R}^n \xrightarrow{\pi_V} V$ is an isomorphism, as they have the same dimension, and the map is linear with kernel $W \cap V^\perp = 0$, so ϕ is a bijection, which is a homeomorphism.

2. We can similarly make a complex Grassmanian, in particular $\gamma_{1,n+1}^{\mathbb{R}} \rightarrow \text{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n$, and $\gamma_{1,n+1}^{\mathbb{C}} \rightarrow \text{Gr}_1(\mathbb{C}^{n+1}) = \mathbb{CP}^n$
3. If X is a smooth manifold, TX is a vector bundle. If $Y^k \subseteq X^n$ is a submanifold, then $TY \subseteq i^*TX$ is a subbundle. The *normal bundle* of Y in X is $\nu_{Y \subseteq X} = i^*TX/TY \rightarrow Y$, and also

Theorem 7.1. (*Tubular Neighbourhood Theorem*) *If $Y^k \subseteq X^n$ is a submanifold, there is an open neighbourhood $Y \subseteq U_Y$ in X and a homeomorphism $\nu_{Y \subseteq X} \rightarrow U_Y$ and the zero section corresponds to Y*

Proof. No Proof □

Proposition 7.1. *If X is compact Hausdorff and $\{U_\alpha\}$ is an open cover then there are functions $\lambda_\alpha: X \rightarrow \mathbb{R}_{\geq 0}$ with*

1. $\text{supp } \lambda_\alpha \subseteq U_\alpha$
2. *Each point has a neighbourhood which intersects finitely many of the supports*
3. $\sum \lambda_\alpha(x) = 1$ for all $x \in X$

Proof. No Proof □

Lemma 7.2. *Let $\pi: E \rightarrow X$ be a vector bundle on a compact Hausdorff space X .*

1. *E admits a continuously varying inner product.*
2. *There is another vector bundle $F \xrightarrow{p} X$ such that $E \oplus F = X \times \mathbb{R}^N$*

Proof. Let $\{U_\alpha\}$ be a trivialising cover for $\pi: E \rightarrow X$ and ϕ_α be the trivialisation. Under this homeomorphism, the inner product on $\mathbb{R}^n$ gives an inner product on each fibre. Let $\{\lambda_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$ and define $\langle v, w \rangle = \sum \lambda_\alpha(\pi(v)) \langle v, w \rangle_\alpha$ for v, w in the same fibre (note that any undefined terms are annihilated by λ_α). Axiom 2 of partitions of unity show this is continuous, and it is positive definite and symmetric as it is a positive linear combination of inner products.

For the second part it suffices to show that E is a subbundle of some $X \times \mathbb{R}^N$, then take $F = E^\perp$ (this is a vector bundle by ES3). Let $\{U_\alpha\}$ be a finite trivialising cover of size n . Let $f_\alpha: E|_{U_\alpha} \rightarrow U_\alpha \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ giving $f: E \rightarrow X \times \mathbb{R}^{nd}$ by $v \mapsto (\pi(v), \sum \lambda_\alpha(\pi(v)) f_\alpha(v))$. This is a linear injection on each fibre and one checks this is a subbundle □

Example 7.2. Let $\mathcal{U} = [0, 1] \times \mathbb{R} / \{(1, x) \sim (0, -x)\}$. This has a map to S^1 by projection onto the first coordinate. This is locally trivial (remove any point from the circle). Suppose there was an isomorphism $\mathcal{U} \rightarrow S^1 \times \mathbb{R}$ over S^1 as vector bundles. The bundle $S^1 \times \mathbb{R}$ has a nowhere zero section $t \mapsto (t, 1)$. This must map to a nowhere-zero section of $\mathcal{U}$. By IVT we get a contradiction, as this corresponds to a nowhere-zero function from $[0, 1] \rightarrow \mathbb{R}$ with $f(1) = -f(0)$.

7.1 Orientations and the Thom Isomorphism

Let R be a commutative ring, and $\pi: E \rightarrow B$ be a vector bundle. $E^\#$ is $E \setminus s_0(B)$ the subspace of non-zero vectors. If π has d -dimensional fibres, then

$$H^i(E_x, E_x^\#; R) \cong \begin{cases} 0 & i \neq d \\ \mathbb{Z} & i = d \end{cases} \quad \text{as } E_x \cong \mathbb{R}^d$$

Definition 7.3. A *local orientation* for E at $x \in B$ is a choice $\varepsilon_x \in H^d(E_x, E_x^\#; R)$ of R -module generator. An *R -orientation* of E is a choice of local orientation $\{\varepsilon_x\}_{x \in B}$ at each point such that for each $x \in B$ there is a neighbourhood U such that for any $y \in U$, under the linear isomorphisms

$$\begin{array}{ccccc} E_x & & & & \\ & \searrow & & & \\ & E|_U & \xrightarrow{\phi} & U \times \mathbb{R}^d & \xrightarrow{\pi} \mathbb{R}^d \\ & \nearrow & & & \\ E_y & & & & \end{array}$$

$\varepsilon_x, \varepsilon_y$ correspond to the same element of $H^d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R)$

Lemma 7.3.

1. If $R = \mathbb{F}_2$, then all bundles are $\mathbb{F}_2$ -oriented
2. If $\{U_\alpha\}_{\alpha \in I}$ is a trivialising cover of $\pi: E \rightarrow B$ and for each $\alpha, \beta \in I$ the homomorphism $U_\alpha \cap U_\beta \times \mathbb{R}^d \xrightarrow{\phi_\alpha} E|_{U_\alpha \cap U_\beta} \xrightarrow{\phi_\beta} U_\alpha \cap U_\beta \times \mathbb{R}^d$ is orientation preserving on each fibre in the sense of linear algebra, then $\pi: E \rightarrow B$ has an R -orientation for any ring R

Proof.

1. There is only one possible choice of generator for $\mathbb{F}_2$.
2. Let $u \in H^d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R)$ be a generator and for $x \in U_\alpha \subseteq B$ choose ε_x to correspond to u under $E_x \hookrightarrow E|_{U_\alpha} \xrightarrow{\phi_\alpha} U_\alpha \times \mathbb{R}^d \rightarrow \mathbb{R}^d$. The collection $\{\varepsilon_x\}_{x \in B}$ is compatible as if $x \in U_\beta$, then the analogue of the above isomorphism using ϕ_β differs by postcomposition with a linear isomorphism $\mathbb{R}^d \rightarrow \mathbb{R}^d$ in $GL_d^+(\mathbb{R})$. Scaling gives a deformation retraction of $GL_d^+(\mathbb{R})$ to $SL_d(\mathbb{R})$, and this group is path-connected, so the isomorphism is isotopic to the identity, and so it induces the identity on H^d , so we would get the same ε_x using ϕ_β , but then as all the $y \in U_\alpha$ are compatible with each other and with every other element of the open cover they lie in, we are done.

□

7.2 Thom isomorphism theorem

Theorem 7.4. (*Thom isomorphism theorem*) Let $\pi: E \rightarrow B$ be a d -dimensional vector bundle, $\{\varepsilon_x\}_{x \in B}$ an R -orientation. Then

1. $H^i(E, E^\#; R) = 0$ for $i < d$
2. There is a unique $u_E \in H^d(E, E^\#; R)$ restricting to ε_x on each fibre
3. $\Phi: H^i(B; R) \xrightarrow{\pi^*} H^i(E; R) \xrightarrow{\smile u_E} H^{i+d}(E, E^\#; R)$ is an isomorphism

Proof. Suppose for simplicity that there is a finite trivialising cover $\{U_\alpha\}$ for $\pi: E \rightarrow B$. We proceed by induction on the size of the cover. If the cover has size 1, then a relative form of the Künneth gives

$$H^i(U_\alpha; R) \otimes_R H^d(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R) \rightarrow H^*(U_\alpha \times \mathbb{R}^d, U_\alpha \times (\mathbb{R}^d \setminus \{0\}); R)$$

is an isomorphism so using ϕ_α , we can get $H^*(E|_{U_\alpha}, E|_{U_\alpha}^\#; R)$ and we can take $u_E|_{U_\alpha}$ to be the class corresponding to $1 \otimes u_d$, we get exactly part (iii), which implies (i), and uniqueness is clear as $H^d(E|_{U_\alpha}, E|_{U_\alpha}^\#; R) \cong H^0(U; R) \otimes R\{u_d\}$ and $1 \in H^0(U_\alpha; R)$ is the unique class restricting to $1 \in H^0(\{x\}; R)$ for all $x \in U_\alpha$

For the inductive step, let $U, V \subseteq B$ and suppose the three claims hold for $E|_U, E|_V$. Then (a relative version of) M-V gives

$$\begin{array}{c} \dots \longrightarrow H^{d-1}(E|_{U \cap V}, E|_{U \cap V}^\#; R) \longrightarrow H^d(E|_{U \cup V}, E|_{U \cup V}^\#; R) \\ \quad \quad \quad \nwarrow \\ H^d(E|_U, E|_U^\#; R) \oplus H^d(E|_V, E|_V^\#; R) \longrightarrow H^d(E|_{U \cap V}, E|_{U \cap V}^\#; R) \longrightarrow \dots \end{array}$$

For part (i), the portion in degree $< d$ shows that $H^d(E|_{U \cup V}, E|_{U \cup V}^\#; R) = 0$ for $i < d$. For part (ii), $(j_U^* - j_V^*)(u_E|_U - u_E|_V) = 0$ by uniqueness on the intersection. As the first shown term is 0, by exactness there is unique $u_E|_{U \cup V}$ which maps to $u_E|_U, u_E|_V$. For (iii) consider the map of exact sequences given by $\Phi, \alpha \mapsto \pi^* \alpha \smile u_E$. It follows from the 5 lemma that it is an isomorphism on the $U \cup V$ term, so we are done. $\square$

Remark 7.3. Use transfinite induction for infinite cover (consider the collection of subspaces for which it's true for U and all its subspaces - this set is non-empty closed under union, then Zorn)

Remark 7.4. Why does the square in the long exact sequence involving ∂ commute?

If ψ is a cocycle, then $C^i(X; R) \rightarrow C^{i+j}(X; R)$ cupping on the right with ψ is a cochain map (as the $\pm$ term vanishes).

Using this, the MV sequence becomes a module of right $H^*(U \cup V; R)$ modules, via restriction of scalars, and similarly for the LES for X, A)

Definition 7.4. We call u_E the *Thom class* of $\pi: E \rightarrow B$, and its image under $H^d(E, E^\#; R) \xrightarrow{q^*} H^d(E; R) \xrightarrow{s_0^*} H^d(B; R)$ is $e(E) \in H^d(B; R)$, called the *Euler class*

Remark 7.5. If $f: A \rightarrow B$ is a map, then we have $f^*E \xrightarrow{\pi'} A$ and $\hat{f}: f^*E \rightarrow E$. This gives isomorphisms $(f^*E)_x \rightarrow E_{f(x)}$, and so we can pull back an orientation on E to an orientation on f^*E . Doing so, we get $f^*(u_E) = u_{f^*E}$ by uniqueness. We deduce that $e(f^*E) = f^*(e(E))$.

Theorem 7.5. *If there is a section $s: B \rightarrow E$ which is nowhere-zero, then $e(E) = 0$.*

Proof. Note $s: B \rightarrow E^\# \subseteq E$. We look at the long exact sequence for the pair $(E, E^\#)$. This commutes, so $e(E) = 0$

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & H^d(E, E^\#; R) & \longrightarrow & H^d(E; R) & \longrightarrow & H^d(E^\#; R) \longrightarrow \dots \\
 & & & & \downarrow & & \nearrow \\
 & & & & 0 & & \\
 u_E & \longleftarrow & & \xleftarrow{s_0^* = s^*} & & \xrightarrow{s^*} & 0 \\
 \downarrow & & & & \downarrow & & \nearrow \\
 & & & & H^d(B; R) & & \\
 & & & \nearrow & & \searrow & \\
 & & & e(E) & & &
 \end{array}$$

□

Corollary 7.5.1. *If $e(E) \neq 0 \in H^d(B; R)$, then any section $s: B \rightarrow E$ must be 0 somewhere.*

Proof. Trivial

□

Lemma 7.6. *If d is odd, then $2e(E) = 0$*

Proof. There is a bundle isomorphism $E \rightarrow E$ given by $\phi: v \mapsto -v$. Then $\phi^*(u_E) = -u_E$ as ϕ does d reflections on each fibre and d is odd, applying s_0^* gives $e(E) = -e(E)$, so the result follows. □

7.3 Gysin Sequence

Let $\pi: E \rightarrow B$ be an R -oriented vector bundle with d -dimensional fibres. Choose an inner product on it, supposing B is compact Hausdorff and set $\mathbb{S}(E) = \{e \in E \mid (e, e) = 1\} \xrightarrow{p} B$ (with p via $e \mapsto \pi(e)$). Now $\mathbb{S}(E) \rightarrow E^\#$ is a homotopy equivalence by the usual argument. The long exact sequence for $(E, E^\#)$ is (omitting the notation for the coefficients, which are in R)

$$\begin{array}{ccccccc}
 H^{i+d}(E, E^\#) & \xrightarrow{q^*} & H^{i+d}(E) & \xrightarrow{i^*} & H^{i+d}(E^\#) & \xrightarrow{\partial} & H^{i+d+1}(E, E^\#) \\
 \uparrow \Phi & & \uparrow \pi^* & \downarrow s_0^* & \downarrow j^* & & \uparrow \Phi \\
 H^i(B) & \xrightarrow{\sim e(E)} & H^{i+d}(B) & \xrightarrow{p^*} & H^{i+d}(\mathbb{S}(E)) & \xrightarrow{p!} & H^{i+1}(B)
 \end{array}$$

where the vertical maps are isomorphisms, where we are defining $p_!$, we call it the *Gysin map*, and the whole bottom row is the *Gysin sequence* for $\pi: E \rightarrow B$. This is a sequence of $H^*(B)$ -modules.

Example 7.3. Let $L = \gamma_{1,n+1}^{\mathbb{C}} \rightarrow \mathbb{CP}^n$ be the tautological bundle. By definition, this is a subbundle of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, so it has a canonical Hermitian inner product, and

$$\mathbb{S}(L) = \{(V, v) \in \mathbb{CP}^{n+1} \times \mathbb{C}^{n+1} \mid v \in V, \|v\| = 1\}$$

and the map sending $(V, v) \rightarrow v$ gives a homeomorphism to S^{2n+1} , so the Gysin sequence is, with $y = e(L)$,

$$\dots \rightarrow H^{i+1}(S^{2n+1}) \rightarrow H^i(\mathbb{CP}^n) \rightarrow H^{i+2}(\mathbb{CP}^n) \rightarrow H^i(S^{2n+1}) \rightarrow \dots$$

so for $0 < i + 1 < 2n + 1$, the cup product with y is injective, and for $0 < i + 2 < 2n + 1$ it is surjective, and so $\cdot \smile y: H^i(\mathbb{CP}^n) \rightarrow H^{i+2}(\mathbb{CP}^n)$ is an isomorphism for $i \leq 2n - 2$, and so $H^{2i}(\mathbb{CP}^n) \cong \mathbb{Z}\{y^i\}$ for $i \leq 2n$, and then we must have $y^{n+1} = 0$ as the ring it lands in is 0. So $H^*(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}[y]/(y^{n+1})$, and $y = e(L)$. (We used that $\mathbb{C}$ -vector bundles are $\mathbb{Z}$ -oriented.

Similarly, $H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[x]/(x^{n+1})$ as any vector bundle is $\mathbb{Z}/2\mathbb{Z}$ -orientable, where x has degree 1.

8 Poincaré Duality

We aim to prove that, for a d -dimensional compact R -oriented manifold, $H^i(M; R) \cong H_{d-i}(M; R)$ (orientation isn't defined for manifolds yet, but we will define it). To do this, we define compactly supported cohomology, which will be equivalent to cohomology for compact manifolds, so that we can induct on the size of an open cover.

8.1 Compactly-Supported Cohomology

Definition 8.1. If $\phi \in C^n(X)$, we say it has *support* in $S \subseteq X$ if $\phi(\sigma) = 0$ for all $\sigma: \Delta^n \rightarrow X \setminus S$.

Remark 8.1. If ϕ is supported in S , then so is $d\phi$.

Definition 8.2. Let $C_c^\bullet(X) \subseteq C^\bullet(X)$ be the chain complex of those ϕ supported in some compact set.

Remark 8.2. If ϕ is supported in K and ψ is supported in L , then $\phi + \psi$ is supported in $K \cup L$.

An equivalent view is that $C_c^\bullet(X) = \bigcup_{K \text{ compact}} C^\bullet(X, X \setminus K) \subseteq C^\bullet(X)$.

Definition 8.3. We call $H_c^*(X) = H^*(C_c^\bullet(X))$ the *compactly supported cohomology* of X .

Definition 8.4. A *directed set* is a partially ordered set $(I, \leq)$ such that for all $a, b \in I$ there is $c \in I$ with $a, b \leq c$. A *direct system* of abelian groups is a set of abelian groups $\{G_a\}_{a \in I}$ and homomorphisms $\rho_{ab}: G_a \rightarrow G_b$ for $a \leq b$ such that $\rho_{aa} = 1_{G_a}$ and $\rho_{ac} = \rho_{bc} \circ \rho_{ab}$ for $a \leq b \leq c$. The *direct limit* of a direct system is

$$\varinjlim_I G_a = \bigoplus_{a \in I} G_a / \langle x - \rho_{ab}(x) \mid x \in G_a, b \geq a \rangle$$

Remark 8.3. If $J \subseteq I$ is *cofinal* (ie if $a \in I$, there is $b \in J$ with $a \leq b$) then $\varinjlim_J G_a \cong \varinjlim_I G_a$

Remark 8.4. Any element can be represented in a single G_a by pushing to an upper bound, and elements $x, y \in G_a$ are equal in the direct limit if there is some $b \geq a$ such that in G_b , x and y become equal. Using this, one can prove that taking limits of directed systems is exact

Remark 8.5. For any space X , $(K(X), \subseteq)$, the set of compact subsets of X , is a directed set, and $K \mapsto C^n(X, X \setminus K)$, $K \mapsto H^n(X, X \setminus K)$ are directed systems

Lemma 8.1.

$$H_c^n(X) \cong \varinjlim_{K(X)} H^n(X, X \setminus K)$$

Proof. $C_c^\bullet(X) = \varinjlim_{K(X)} C^\bullet(X, X \setminus K)$, and taking homology/cohomology commutes with direct limits (on example sheet) $\square$

Lemma 8.2. $H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$

Proof. A compact $K \subseteq \mathbb{R}^n$ is contained in some $B_N = B_{\mathbb{R}}(0, N)$, so

$$H_c^*(\mathbb{R}^n) = \varinjlim_{K(X)} H^*(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong \varinjlim_{n \in \mathbb{N}} H^*(\mathbb{R}^n, \mathbb{R}^n \setminus B_n)$$

but

$$\begin{array}{ccc} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_n) & \longrightarrow & H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_{n+1}) \\ \uparrow \partial & & \uparrow \partial \\ \tilde{H}^{i-1}(\mathbb{R}^n \setminus B_n) & \longrightarrow & \tilde{H}^{i-1}(\mathbb{R}^n \setminus B_{n+1}) \end{array}$$

commutes, and every map except the vertical one is an isomorphism (as $\tilde{H}^*(\mathbb{R}^n)$ vanishes, and the bottom map is induced by a homotopy equivalence), so the direct system is only isomorphisms, so $H^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong H_c^*(\mathbb{R}^n)$, and we've computed the former to be what we desire. $\square$

Remark 8.6. Compactly supported cohomology is not homotopy invariant, see the above example, and compute the result for a point in your head. In fact a general map $f: X \rightarrow Y$ does not even induce a map on compactly supported cohomology (but it does work for proper maps - we need the preimage of compact sets to be compact. Homotopy invariance does hold if everything in sight (both maps, and the homotopy between them) are proper).

Remark 8.7. If $U \subseteq X$ is open and ϕ is a compactly supported cochain on U with support in $K \subseteq U$, then excision gives an isomorphism $H^n(X, X \setminus K) \rightarrow H^n(U, U \setminus K)$ (with some niceness conditions like Hausdorff), so we get a map

$$\begin{aligned} H_c^n(U) &= \varinjlim_{K(U)} H^n(U, U \setminus K) \\ &\cong \varinjlim_{K(U)} H^n(X, X \setminus K) \\ &\rightarrow \varinjlim_{K(X)} H^n(X, X \setminus K) \\ &= H_c^n(X) \end{aligned}$$

This is called extension by 0.

Example 8.1. If $U \subseteq \mathbb{R}^n$ is an open ball, then $i^*: H_c^*(U) \cong H_c^*(\mathbb{R}^n)$

Remark 8.8. For $K \subseteq X$ we write $H^n(X \mid K)$ for $H^n(X, X \setminus K)$. By excision this only depends on a neighbourhood of K . We call this the *local cohomology* of K in X .

Proposition 8.1. *Suppose $K, L \subseteq X$ are compact subsets of a Hausdorff space. Then there is a Mayer-Vietoris sequence*

$$\begin{array}{ccccc} \dots & \longrightarrow & H^n(X \mid K \cap L) & \xrightarrow[i+j]{(i, -j)} & H^n(X \mid K) \oplus H^n(X \mid L) \\ & & & \nwarrow & \\ & & H^n(X \mid K \cup L) & \longrightarrow & H^{n+1}(X \mid K \cap L) \longrightarrow \dots \end{array}$$

Proof. Cover $X \setminus (K \cap L)$ by $U = \{X \setminus L, X \setminus K\}$. We get

$$\begin{array}{ccccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \rightarrow \mathrm{Hom}(C_{\bullet}(X)/C_{\bullet}^{\mathcal{U}}(X \setminus (K \cap L)), \mathbb{Z}) \rightarrow C^{\bullet}(X, X \setminus K) \oplus C^{\bullet}(X, X \setminus L) \rightarrow C^{\bullet}(X, X \setminus (K \cup L)) \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow C^{\bullet}(X) \xrightarrow{(i, -j)} C^{\bullet}(X) \oplus C^{\bullet}(X) \xrightarrow{i+j} C^{\bullet}(X) \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow \mathrm{Hom}(C_{\bullet}^{\mathcal{U}}(X \setminus (K \cap L)), \mathbb{Z}) \longrightarrow C^{\bullet}(X \setminus K) \oplus C^{\bullet}(X \setminus L) \longrightarrow C^{\bullet}(X \setminus (K \cup L)) \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

This commutes, and the columns are short exact. The middle and bottom are short exact by the proof of usual MV. A diagram chase implies the top row is short exact. The left column and the fact that $C^\bullet(X \setminus (K \cap L)) \rightarrow \text{Hom}(C_\bullet^{\mathcal{U}}(X \setminus (K \cap L)), \mathbb{Z})$ is an isomorphism on H^* , and the 5-lemma implies that $C^\bullet(X, X \setminus (K \cap L)) \rightarrow \text{Hom}(C_\bullet(X)/C_\bullet^{\mathcal{U}}(X \setminus (K \cap L)), \mathbb{Z})$ is an isomorphism on H^* too, so the long exact sequence for the top row is the claimed M-V sequence. $\square$

Corollary 8.2.1. *Let X be a manifold covered by open sets A, B . Then there is a Mayer-Vietoris sequence for compactly supported cohomology using A, B*

Proof. For $K \subseteq A, L \subseteq B$ compact, using that

$$\begin{aligned} H^*(X | K) &\cong H^*(A | K) \\ H^*(X | L) &\cong H^*(B | L) \\ H^*(X | K \cap L) &\cong H^*(A \cap B | K \cap L) \end{aligned}$$

by excision, the LES from the above proposition gives

$$\begin{array}{ccc} \dots \longrightarrow H^n(A \cap B | K \cap L) & \xrightarrow{(i, -j)} & H^n(A | K) \oplus H^n(B | L) \\ & \searrow i+j & \\ H^n(X | K \cup L) & \longrightarrow & H^{n+1}(A \cap B | K \cap L) \longrightarrow \dots \end{array}$$

Taking direct limits over $(K, L) \in K(A) \times K(B)$ we get the desired sequence, as taking direct limits is exact. $\square$

8.2 (Co)homology of Manifolds

Definition 8.5. If M is a d -dimensional manifold and $x \in M$, then the *local homology* is $H_i(M | x; R) = H_i(M, M \setminus \{x\}; R) \cong H_i(\mathbb{R}^d, \mathbb{R}^d \setminus \{x\}; R)$. A *local R -orientation* of M at x is an R -module generator $\varepsilon_x \in H_d(M | x; R)$. An *R -orientation* of M is a collection of local orientations $\{\varepsilon_x | x \in M\}$ such that if $\phi: \mathbb{R}^d \rightarrow U \subseteq M$ and $p, q \in D^d \subseteq \mathbb{R}^d$, then $\varepsilon_{\phi(p)}, \varepsilon_{\phi(q)}$ correspond under the isomorphisms

$$\begin{aligned} H_d(M | \phi(p); R) &\cong H_d(\mathbb{R}^d | p; R) \\ &\cong H_d(\mathbb{R}^d | D^d; R) \\ &\cong H_d(\mathbb{R}^d | q; R) \\ &\cong H_d(M | \phi(q); R) \end{aligned}$$

ie there is some $\varepsilon_{\phi(D^d)}$ restricting to $\varepsilon_{\phi(p)}, \varepsilon_{\phi(q)}$

Lemma 8.3. 1. For $R = \mathbb{F}_2$, every manifold is $\mathbb{F}_2$ -oriented

2. If $\{\phi_\alpha: \mathbb{R}^d \rightarrow U_\alpha \subseteq M\}$ is a cover such that for each $x \in U_\alpha \cap U_\beta$, the isomorphism

$$\begin{array}{ccccccc} H_d(\mathbb{R}^d | 0) & \longrightarrow & H_d(\mathbb{R}^d | \phi_\alpha^{-1}(x)) & \longrightarrow & H_d(U_\alpha | x) & \longleftarrow & H_d(M | x) \\ & & & & & & \downarrow \\ H_d(\mathbb{R}^d | 0) & \longrightarrow & H_d(\mathbb{R}^d | \phi_\beta^{-1}(x)) & \longrightarrow & H_d(U_\beta | x) & \longleftarrow & H_d(M | x) \end{array}$$

is the identity (every map shown is an isomorphism), then M is $\mathbb{Z}$ -orientable

Proof. 1. There is only ever one possible generator.

2. If $x \in U_\alpha$. define $\varepsilon_x^\alpha \in H_d(M \mid x; R)$ to correspond to the standard generator of $H_d(\mathbb{R}^d \mid 0)$ under the top row. If $y \in U_\alpha$ as well, $\varepsilon_x^\alpha, \varepsilon_y^\alpha$ are clearly compatible. We have defined many local orientations at x (one for each $U_\alpha \ni x$), and the isomorphism being the identity shows these are the same.

□

Theorem 8.4. *Let $A \subseteq M$ be compact, M an R -oriented d -manifold*

1. *There is a unique $\varepsilon_A \in H_d(M \mid A; R)$ restricting to ε_x for all $x \in A$*
2. *$H_i(M \mid A; R) = 0$ for $i > d$*

Proof. We omit R from notation, and say a set is *good* if it satisfies the conclusion.

Step 1: If $A, B, A \cap B$ are good, then so is $A \cup B$.

Consider the MV sequence

$$0 \longrightarrow H_d(M \mid A \cup B) \longrightarrow H_d(M \mid A) \oplus H_d(M \mid B) \longrightarrow H_d(M \mid A \cap B) \longrightarrow \dots$$

We have classes $\varepsilon_A, \varepsilon_B, \varepsilon_{A \cap B}$ by goodness, and $\varepsilon_A, -\varepsilon_B$ map to $\varepsilon_{A \cap B}$ by uniqueness. So we may pull $\varepsilon_A + \varepsilon_B$ back to a $\varepsilon_{A \cup B}$ which is unique. Further to the left we get zeros in 2/3 of the groups, so $H_i(M \mid A \cup B) = 0$ for $i > d$

Step 2: Convex subsets of $\mathbb{R}^d$ are good (ie sets which are convex in some chart). If $A \subseteq \mathbb{R}^d$ is convex, then by radial scaling, $\mathbb{R}^d \setminus A \simeq \mathbb{R}^d \setminus \{p\}$ for all $p \in A$. So we get $H_i(\mathbb{R}^d \mid A) \cong H_i(\mathbb{R}^d \mid p)$. We have an ε_p , so we define ε_A to be the element corresponding to it. This does not depend on p using $p, q \in A \subseteq D^d$ and the definition of $\varepsilon_p, \varepsilon_q$ being compatible. This also shows the second part.

Step 3: All subsets of $\mathbb{R}^d$ are good

As A is compact, for any open neighbourhood U of A , we can find finitely many closed balls $B_1, \dots, B_n$ such that $U \supseteq B = \bigcup B_i \supseteq B^\circ \subseteq A$. By induction on n using step 1, 2, all such B are good. This implies the set of good neighbourhoods, $\mathcal{G}$ of A is non-empty. If $B, B' \in \mathcal{G}$, then there is a good B'' such that $A \subseteq B'' \subseteq B^\circ \cap B'^\circ \subseteq B \cap B'$, so $\mathcal{G}$ is directed with respect to $\supseteq$. We also have the directed system indexed by this via $B \mapsto H_i(M \mid B)$. We claim the induced map $\varinjlim_{\mathcal{G}} H_i(M \mid B) \rightarrow H_i(M \mid A)$ is an isomorphism. To prove it is surjective, let $[[c]] \in H^i(M \mid A)$. Then $dc \in C_{i-1}(M \setminus A)$, but dc is a finite sum of simplices, so in fact it lies in some compact $K \subseteq M \setminus A$, ie $A \subseteq M \setminus K$ which is open, so we can find some good B such that $A \subseteq B^\circ \subseteq B \subseteq M \setminus K$. Therefore $dc \in C_{i-1}(M \setminus B)$, so $[[c]] \in H_i(M \mid B)$, and so $[[c]]$ is in the image. To prove it is injective is similar and omitted.

The second point is therefore immediate for A . Any of the classes ε_B map to some ε_A , and uniqueness follows from the fact it is an isomorphism.

Step 4: Any compact subset of M is good.

A compact $A \subseteq M$ can be written as a finite union of A_i which lie in charts of M . Apply 1. □

Corollary 8.4.1. *If M is a compact R -oriented d -manifold, there is a unique $[M] = \varepsilon_M \in H_d(M; R)$ restricting to ε_x at every point*

Proof. Trivial □

Definition 8.6. This is the *fundamental class* of M

Remark 8.9. We take on faith that complex things are orientable

Example 8.2. If we take $\mathbb{CP}^k \hookrightarrow \mathbb{CP}^n$, we get $i_*[\mathbb{CP}^k] \in H_{2k}(\mathbb{CP}^n, \mathbb{Z})$. We will see that all the $[\mathbb{CP}^i]$ form a basis of $H_*(\mathbb{CP}^n, \mathbb{Z})$

8.3 Poincaré Duality

Theorem 8.5. *Let M be a d -dimensional R -oriented manifold. Then there is an isomorphism $D = D_M: H_c^k(M; R) \rightarrow H_{d-k}(M; R)$*

Proof. Deferred □

Definition 8.7. The *cap product* is $\cdot \cap \cdot: C^k(X; R) \times C_l(X; R) \rightarrow C_{l-k}(X; R)$ via $(\phi, \sigma) \mapsto \sigma|_{[v_0 \dots v_{l-k}]} \phi(\sigma|_{[v_{l-k} \dots v_l]})$

Lemma 8.6. $d(\phi \cap \sigma) = \phi \cap d\sigma - (-1)^{l-k} d\phi \cap \sigma$

Proof.

$$\begin{aligned} \phi \cap d\sigma &= \phi \cap \sum_{i=0}^l (-1)^i \sigma|_{[v_0 \dots \hat{v}_i \dots v_l]} \\ &= \sum_{i=0}^{l-k} (-1)^i \sigma|_{[v_0 \dots \hat{v}_i \dots v_{l-k}]} \phi(\sigma|_{[v_{l-k} \dots v_l]}) \\ &\quad + \sum_{i=0}^{l-k} (-1)^i \sigma|_{[v_0 \dots v_{l-k-1}]} \phi(\sigma|_{[v_{l-k-1} \dots \hat{v}_j \dots v_l]}) \\ &= d(\phi \cap \sigma) + (-1)^{l-k} (d\phi) \cap \sigma \end{aligned}$$

□

Remark 8.10. This ensures that we get a well defined product on passing to (co)homology. One checks that the same formula gives a map $H^k(X; R) \times H_l(X, A; R) \rightarrow H_{l-k}(X, A; R)$ and also $H^k(X, A; R) \times H_l(X, A; R) \rightarrow H_{l-k}(X; R)$

Lemma 8.7. *If $f: X \rightarrow Y$ is a map, $x \in H_k(X; R)$, $y \in H^l(Y; R)$ then $f_*((f^*y) \cap x) = y \cap f_*(x) \in H_*(Y; R)$, ie f_* is a map of $H^*(Y; R)$ modules*

Proof. Let $y = [\phi]$, $x = [\sum a_\sigma \sigma]$. Then

$$\begin{aligned} f_\bullet \left(\phi \cap \sum a_\sigma \sigma \right) &= \sum a_\sigma f_\bullet(\sigma|_{[v_0 \dots v_{k-l}]} \phi(f\sigma|_{[v_{k-l} \dots v_k]})) \\ &= \sum a_\sigma f\sigma|_{[v_0 \dots v_{k-l}]} \phi(f\sigma|_{[v_{k-l} \dots v_k]}) \\ &= \phi \cap \left(\sum a_\sigma f_\bullet \sigma \right) \\ &= \phi \cap f_\bullet x \end{aligned}$$

□

If M is compact, we have $[M] = \varepsilon_M \in H_d(M; R)$ and the map in the Poincaré duality theorem is $D_M = \cdot \cap M$. In general if $K \subseteq L \subseteq M$ are compact and we have the map $i: (M, M \setminus L) \rightarrow (M, M \setminus K)$, we have $i_*(\varepsilon_L) = \varepsilon_K$. Then, as in the lemma, we have

$$\begin{array}{ccc} H^k(M \mid K; R) & \xrightarrow{i^*} & H^k(M \mid L; R) \\ \cdot \cap \varepsilon_K \downarrow & & \downarrow \cdot \cap \varepsilon_L \\ H_{d-k}(M; R) & \xrightarrow{\text{id}} & H_{d-k}(M; R) \end{array}$$

so the maps $\cdot \cap \varepsilon_K$ assemble to $\varinjlim_{K \in \mathcal{K}(M)} H^k(M \mid K; R) \xrightarrow{D_M} H_{d-k}(M; R)$

Proof. (Of Poincaré duality) We say M is good if it satisfies the conclusion of Poincaré duality.

Step 1: If $M = U \cup V$, and $U, V, U \cap V$ are all good, then M is good. For this, we claim the following diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_c^k(U \cap V) & \xrightarrow{i_U + i_V} & H_c^k(U) \oplus H_c^k(V) & \xrightarrow{j_U - j_V} & H_c^k(U \cup V) \longrightarrow \dots \\ & & \downarrow D_{U \cap V} & & \downarrow D_U \oplus D_V & & \downarrow D_{U \cup V} \\ \dots & \longrightarrow & H_{d-k}(U \cap V) & \longrightarrow & H_{d-k}(U) \oplus H_{d-k}(V) & \longrightarrow & H_{d-k}(U \cup V) \longrightarrow \dots \end{array}$$

commutes (almost). The top row is a direct limit for $K \subseteq U, L \subseteq V$, so it suffices to check it commutes for those. For the left two squares this is immediate, and the rightmost one commutes up to a sign, then apply the 5-lemma (replacing the last vertical map with a negative).

Step 2: If $U_1 \subseteq U_2 \subseteq \dots \subseteq \bigcup U_i = M$ and all the U_i are open and good, then M is good. For this, $H_c^k(M; R) = \varinjlim_{i \in \mathbb{N}} H_c^k(U_i; R)$ as any compact set is in some U_j . The same analysis works for $H_{d-k}(M; R)$. Then we use that direct limits preserve isomorphisms.

Step 3: $\mathbb{R}^d$ is good. We know that

$$H_c^i(\mathbb{R}^d; R) \cong H^i(\mathbb{R}^d \mid 0; R) \cong \text{Hom}(H_i(\mathbb{R}^d \mid 0; R), R)$$

by UCT. The map $H_c^d(\mathbb{R}^d; R) \rightarrow H_0(\mathbb{R}^d; R) \xrightarrow{\varepsilon} R$ is given by $[f] \mapsto f(\varepsilon_0)$, so it is an isomorphism.

Step 4: Any open subset $U \subseteq \mathbb{R}^d$ is good. This is immediate for a finite union of balls by 1, 3. Any U is a countable union of balls, and we can write such as a nested union of finite unions of balls, so we are done by 2.

Step 5: If M has a countable cover by copies of $\mathbb{R}^d$, it is good. The proof is identical to 4.

Step 6: All M are good. Zorn's lemma. □

Corollary 8.7.1. *If M is a compact d -dimensional R -oriented manifold, then $\cdot \cap [M]: H^k(M; R) \rightarrow H_{d-k}(M; R)$ is an isomorphism.*

Proof. Trivial □

Corollary 8.7.2. *If M is a compact odd dimensional manifold, then $\chi(M) = 0$*

Proof. Every manifold is $\mathbb{F}_2$ oriented, hence $H_i(M; \mathbb{F}_2) \cong H^{2n+1-i}(M; \mathbb{F}_2) \cong H_{2n+1-i}(M; \mathbb{F}_2)$ by Poincaré duality and UCT, so

$$\begin{aligned}\chi(M) &= \sum_{i=0}^{2n+1} (-1)^i \dim_{\mathbb{F}_2} H_i(M; \mathbb{F}_2) \\ &= 0\end{aligned}$$

□

For $\alpha \in C_{k+l}(X; R)$, $\phi \in C^k(X; R)$, $\psi \in C^l(X; R)$, we have

$$\psi(\phi \cap \sigma) = \psi(\sigma|_{[v_0 \dots v_l]}) \phi(\sigma|_{v_l \dots v_{k+l}}) = (\psi \smile \phi)(\sigma)$$

and linearity implies the same result for α . Hence we have

$$C^k(X; R) \otimes C_l(X; R) \xrightarrow{\cdot \cap} C_0(X; R) \xrightarrow{\varepsilon} R$$

sending $(\phi, \alpha) \rightarrow \phi(\alpha)$. This induces the map $H^k(X; R) \xrightarrow{h} \text{Hom}(H_k(X; R), R)$ in the UCT. This is an isomorphism if $H_*(X; R)$ is free. We consider the following diagram

$$\begin{array}{ccc} H^l(X; R) & \xrightarrow{h} & \text{Hom}_R(H_l(X; R), R) \\ \downarrow \cdot \smile \phi & & \downarrow (\phi \cap \cdot)^* \\ H_{k+l}(X; R) & \xrightarrow{h} & \text{Hom}_R(H_{k+l}(X; R), R) \end{array}$$

which commutes by the above observations, so in the case of UCT, the cup product and cap product determine each other. Poincaré duality puts constraints on the cup product in this way.

We consider the bilinear form

$$\langle \cdot, \cdot \rangle: H^k(M; R) \times H^{d-k}(M; R) \xrightarrow{\cdot \cap} H^d(M; R) \xrightarrow{\cdot \cap [M]} H_0(M; R) \xrightarrow{\varepsilon} R$$

Theorem 8.8. *If R is a field, or more generally if $H_*(M; R)$ is free, this pairing is non-degenerate. The induced map $H^k(M; R) \rightarrow H_{d-k}(M; R)^*$ is an isomorphism*

Proof. We have

$$H^k(M; R) \xrightarrow{h} \text{Hom}_R(H_k(M; R), R) \xrightarrow{D_M^*} \text{Hom}_R(H^{d-k}(M; R), R)$$

$$[\phi] \longmapsto ([\psi] \mapsto ([\phi] \smile [\psi])([M]))$$

This is the map adjoint to $\langle \cdot, \cdot \rangle$, and it is an isomorphism, so we are done. □

Example 8.3. We know $H^*(\mathbb{CP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & \text{otherwise} \end{cases}$ by eg. cellular cohomology. We know $\mathbb{CP}^n$ is $\mathbb{Z}$ -oriented by example sheets 3, 4. Suppose we know $H^*(\mathbb{CP}^{n-1}, \mathbb{Z}) = \mathbb{Z}[x]/(x^n)$ with $|x| = 2$. As $\mathbb{CP}^{n-1} \rightarrow \mathbb{CP}^n$ is the inclusion of a cellular $2n - 2$ skeleton, it is an isomorphism on H^* for $* \leq 2n - 4$. Let $x \in H^2(\mathbb{CP}^n, \mathbb{Z}) \xrightarrow{i^*} H^2(\mathbb{CP}^{n-1}, \mathbb{Z})$. This maps to x , so in degrees below $2n - 2$, $H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$. To conclude we need to know $x^n \in H^{2n}(\mathbb{CP}^n, \mathbb{Z})$ is a generator. We use that

$$H^2(\mathbb{CP}^n, \mathbb{Z}) \otimes H^{2n-2}(\mathbb{CP}^n, \mathbb{Z}) \rightarrow H^{2n}(\mathbb{CP}^n, \mathbb{Z}) \rightarrow H_0(\mathbb{CP}^n; \mathbb{Z}) \rightarrow \mathbb{Z}$$

is non-degenerate, so it is (± 1) , ie $x \smile x^{n-1}$ is a generator of $H^{2n}(\mathbb{CP}^n; \mathbb{Z})$.

Example 8.4. $H^*(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z})$ is similar

8.4 Applications

8.4.1 Middle Cohomology

If M is a compact $\mathbb{Z}$ -oriented manifold of dimension $2n$, then we get

$$H^n(M; \mathbb{Z}) \otimes H^n(M; \mathbb{Z}) \rightarrow H^{2n}(M; \mathbb{Z}) \rightarrow H_0(M; \mathbb{Z}) \rightarrow \mathbb{Z}$$

which is non-degenerate. It satisfies

$$\langle \alpha, \beta \rangle = (\alpha \smile \beta)([M]) = (-1)^{n^2} (\beta \smile \alpha)([M]) = (-1)^n \langle \beta, \alpha \rangle$$

This is symmetric if n is even, and skew-symmetric if n is odd.

If n is even, so that the form is symmetric. After tensoring with $\mathbb{R}$ we get a symmetric, hence diagonalisable, matrix representing the form. We define the *signature* of M to be the signature of this symmetric form, and denote it $\text{sign } M$. It is a fact that if M is the boundary of a compact $\mathbb{Z}$ -oriented $2n + 1$ -manifold W , then $\text{sign } M = 0$, and more generally, if the union of M, M' form the boundary of a manifold (they are *cobordant*), they have the same signature.

Example 8.5. $\mathbb{CP}^2$ is 4-dimensional, and has cohomology $\mathbb{Z}[x]/(x^3)$, so $\langle \cdot, \cdot \rangle$ is the matrix 1, so $\text{sign } \mathbb{CP}^2 = 1 \neq 0$, so $\mathbb{CP}^2$ is not the boundary of any $\mathbb{Z}$ -oriented manifold.

Let $\overline{\mathbb{CP}^2}$ be $\mathbb{CP}^2$ with reversed orientation. Then the matrix for its cup product is -1 , so its sign is -1 . Therefore it is not orientation-preserving diffeomorphic, and is also not oriented cobordant to $\mathbb{CP}^2$

8.4.2 Degree

If $f: M \rightarrow N$ is a map between compact $\mathbb{Z}$ -oriented manifolds of the same dimension d . Then $f_*[M] \in H_d(N; \mathbb{Z}) = \mathbb{Z}\{[N]\}$. We define $\deg(f)$ by $f_*[M] = (\deg f)[N]$.

Corollary 8.8.1. *If $\mathbb{F}$ is a field and $\deg f \neq 0 \in \mathbb{F}$, then $f^*: H^*(N; \mathbb{F}) \rightarrow H^*(M; \mathbb{F})$ is injective*

Proof. Let $\alpha \in H^k(N; \mathbb{F})$ be non-zero. Then there is $\beta \in H^{d-k}(N; \mathbb{F})$ such that $(\alpha \smile \beta)([M]) = 1$.

$$\begin{aligned} \deg f &= (\alpha \smile \beta)((\deg f)[M]) \\ &= (\alpha \smile \beta)(f_*[N]) \\ &= f^*(\alpha \smile \beta)([N]) \\ &= (f^*(\alpha) \smile f^*(\beta))[N] \end{aligned}$$

so $f^*\alpha \neq 0$ and we are done. $\square$

8.4.3 Submanifolds

Let M be a smooth compact R -oriented d -manifold, and N be a smooth compact R -oriented n -submanifold, and i the inclusion.

We have $[N] \in H_n(N; R)$, so we can form $i_*[N] \in H_n(M; R)$. We also call this $[N]$. Choosing an inner product on TM , we have $i^*TM = TN \oplus \nu_{N \subseteq M}$. The tubular neighbourhood theorem gives a neighbourhood U of N homeomorphic to $\nu_{N \subseteq M}$. The R -orientation of M restricts to one on U . So

$$\begin{aligned} H^i(\nu_{N \subseteq M}, \nu_{N \subseteq M}^\#; R) &\cong H_c^i(\nu_{N \subseteq M}; R) \\ &\cong H_c^i(U; R) \\ &\cong H_{d-i}(U; R) \\ &\cong H_{d-i}(N; R) \end{aligned}$$

Under these isomorphisms, $[N]$ corresponds to a class $u_{\nu_{N \subseteq M}} \in H^{d-n}(\nu_{N \subseteq M}, \nu_{N \subseteq M}^\#; R)$. This is a Thom class. Therefore $\nu_{N \subseteq M}$ is R -oriented.

We also have

$$\begin{aligned} H_n(N; R) &\cong H_n(U; R) \\ &\cong H_c^{d-n}(U; R) \\ &\rightarrow H_c^{d-n}(M; R) \\ &\cong H_n(M; R) \end{aligned}$$

and this commutes with the map $H_n(U; R) \rightarrow H_n(M; R)$. These send $[N] \rightarrow D_M^{-1}([N])$ and also $[N]$ maps to the extension by 0 of the above Thom class. The Poincaré dual of a submanifold is the extension by 0 of its normal Thom class.

Definition 8.8. $N, W \subseteq M$ intersect *transversely* if for each $x \in N \cap W$, $T_x M = T_x N + T_x W$.

Example 8.6. In the above case, $N \cap W$ is a submanifold, and $(\nu_{N \cap W \subseteq M})_x = (\nu_{N \subseteq M})_x \oplus (\nu_{W \subseteq M})_x$. This is because the normal bundle for $N \cap W$ is the

sum of the pullbacks of those for N, W . We get $i_N^*(u_{\nu_{N \subseteq M}}) \smile i_W^*(u_{\nu_{W \subseteq M}}) \in H^*(u_{\nu_{N \cap W \subseteq M}}, u_{\nu_{N \cap W \subseteq M}}^\#; R)$ and this can be checked to be a Thom class. Using tubular neighbourhood theorem again, $D_M(D_M^{-1}([N]) \smile D_M^{-1}([W])) = [N \cap W]$ (where $N \cap W$ gets its orientation from the normal bundle discussion above)

We use the above example a couple times.

Example 8.7. Let $y_k = [\mathbb{CP}^k] \in H_{2k}(\mathbb{CP}^n, \mathbb{Z})$, which are generators. We embed $\mathbb{CP}^{k_1}$ in the first k_1 coordinates, and $\mathbb{CP}^{k_2}$ in the last k_2 . Then $\mathbb{CP}^{k_1} \cap \mathbb{CP}^{k_2} = \mathbb{CP}^{k_1+k_2-n}$, and this is a transverse intersection. All of this is done over $\mathbb{C}$, so all the orientations are canonical, so we have that the intersection product satisfies $y_{k_1} \cdot y_{k_2} = y_{k_1+k_2-n}$, which is Poincaré dual to $H^{2n-2k_1} \otimes H^{2n-2k_2} \rightarrow H^{4n-2k_1-2k_2}$ is an isomorphism when possible.

Example 8.8. We consider the closed surface of genus 2, with homology generators $\alpha_1, \alpha_2, \beta_1, \beta_2$ where α_i, β_i are the generators of the torus-without-disk around one of the holes. Then $\alpha_i \cap \beta_i = *$, so we choose orientations so $\alpha_i \cdot \beta_i = 1$, and $\alpha_1 \cdot \alpha_2 = 0$ etc. If we define the Poincaré duals $D(\alpha_i) = a_i, D(\beta_i) = b_i$, this dualises to $a_i \smile b_j = \delta_{ij}[\text{generator of } H^2]$, all other products 0

8.5 The diagonal and the Euler Characteristic

Let M be a compact d -dimensional $\mathbb{Z}$ -oriented (hence $\mathbb{Q}$ -oriented) manifold. We have $\langle \cdot, \cdot \rangle: H^*(M; \mathbb{Q}) \otimes H^*(M; \mathbb{Q}) \rightarrow \mathbb{Q}$ via $a \otimes b \mapsto (a \smile b)[M]$ (or 0 if the sum of degrees of a, b is not d). Let $\{a_i\}_{i \in I}$ be a homogeneous basis for $H^*(M; \mathbb{Q})$. Let $\{b_i\}_{i \in I}$ be the dual basis to $\{a_i\}_{i \in I}$. We have the diagonal map $\Delta: M \rightarrow M \times M$ landing in a compact $2d$ -dimensional $\mathbb{Z}$ -oriented manifold. We define $\delta = D^{-1}\Delta_*[M] \in H^d(M \times M, \mathbb{Q})$

Theorem 8.9. $\delta = \sum_{i \in I} (-1)^{|a_i|} a_i \otimes b_i \in \bigoplus_j H^j(M; \mathbb{Q}) \otimes H^{d-j}(M; \mathbb{Q})$

Proof. We know $\delta = \sum c_{ij} a_i \otimes b_j$ for some $c_{ij} \in \mathbb{Q}$. Now

$$\begin{aligned} ((b_k \otimes a_l) \smile \delta)([M \times M]) &= \sum_{i,j} c_{ij} (-1)^{|a_i||a_l|} ((b_k \smile a_i) \otimes (a_l \smile b_j))[M \times M] \\ &= \sum_{i,j} c_{ij} (-1)^{|a_i||a_l|} (-1)^{|a_i||b_k|} ((a_i \smile b_k)[M] \cdot (a_l \smile b_j)[M]) \\ &= c_{k,l} (-1)^{|a_k|(|a_l|+|b_k|)} \end{aligned}$$

But also

$$\begin{aligned} ((b_k \otimes a_l) \smile \delta)([M \times M]) &= (b_k \otimes a_l)(\delta \cap [M \otimes M]) \\ &= (b_k \otimes a_l)(\Delta_*[M]) \\ &= \Delta^*(b_k \otimes a_l)[M] \\ &= (b_k \smile a_l)[M] \\ &= (-1)^{|a_l||b_k|} \delta_{kl} \end{aligned}$$

so

$$c_{kl} = \delta_{kl}(-1)^{|a_k|^2} = \delta_{kl}(-1)^{|a_k|}$$

as desired $\square$

Corollary 8.9.1. $(\Delta^*(\delta))[M] = \chi(M)$

Proof. We know

$$\Delta^*(\delta) = \sum_{i \in I} (-1)^{|a_i|} (a_i \smile b_i)$$

so evaluating on $[M]$ gives $\sum_{i \in I} (-1)^{|a_i|} = \chi(M)$ $\square$

If M is in addition a smooth manifold, we consider $D\Delta: TM \rightarrow T(M \times M) = TM \oplus TM$. This is also the diagonal map, so $\nu_{\Delta(M) \subseteq M \oplus M} \cong TM$ as $TM \oplus TM / \Delta(TM) \cong TM$. If we take a tubular neighbourhood $\Delta \subseteq U \subseteq M \times M$, then we can form $\delta_U = D_U^{-1} \Delta_*[M] \in H_c^d(U)$, and its extension by 0 is δ . Under $U \cong \nu_{\Delta M \subseteq M \times M}$, $\delta_U \in H_c^d(\nu_{\Delta M \subseteq M \times M}) \cong H^d(\nu_{\Delta M \subseteq M \times M}, \nu_{\Delta M \subseteq M \times M}^\#)$ can be checked to be a Thom class for $\nu_{\Delta M \subseteq M \times M}$. Thus $\Delta^* \delta = e(\nu_{\Delta M \subseteq M \times M}) = e(TM)$. Therefore $e(TM)[M] = \chi(M)$.

Corollary 8.9.2. *If M has a nowhere-zero vector field, then $\chi(M) = 0$*

Proof. We saw that if $TM \cong E \oplus \mathbb{R}$, then $e(TM) = 0$. $\square$

More generally, $D_M(e(TM)) \in H_0(M)$ is given by $s^{-1}(0)$ for $s: M \rightarrow TM$ transverse to s_0 (counted with orientations). We saw that $e(E \oplus F) = e(E) \smile e(F)$ because $H^*(E \oplus F, (E \oplus F)^\#) \ni \pi_1^* u_E \smile \pi_2^* u_F$ is a Thom class for $E \oplus F$

Corollary 8.9.3. $TS^{2n} \rightarrow S^{2n}$ has no proper subbundles

Proof. If E is a subbundle, take an inner product on TS^{2n} and get a complementary bundle F . Then $e(TS^{2n}) = e(E) \smile e(F)$, and $e(E), e(F) = 0$ so $e(TS^{2n}) = 0$. But we know $\chi(S^{2n}) \neq 0$, contradiction. $\square$

8.6 Lefschetz Fixed Point Theorem

Theorem 8.10. *Let M be a compact $\mathbb{Z}$ -oriented (hence $\mathbb{Q}$ -oriented) d -dimensional manifold, and suppose $f: M \rightarrow M$ is a smooth function such that the graph of f intersects Δ_M transversely (hence the intersection is 0 dimensional, so can be counted). Then $\sum_{x \in \text{Fix } f} \text{sign det}(I - D_x f) = \sum (-1)^i \text{tr}(f^*: H^i(M; \mathbb{Q}) \rightarrow H^i(M; \mathbb{Q}))$*

Proof. The strategy is to calculate $\varepsilon([\Gamma_f] \cdot [\Delta(M)])$ in two different ways. Algebraically,

$$\begin{aligned} (D_{M \times M}^{-1}([\Gamma_f]) \smile D_{M \times M}^{-1}(\Delta(M)))[M \times M] &= (-1)^d (D_{M \times M}^{-1}(\Delta(M)) \smile D_{M \times M}^{-1}([\Gamma_f]))[M \times M] \\ &= (-1)^d (D_{M \times M}^{-1}(\Delta(M)))([\Gamma_f]) \\ &= (-1)^d \delta((\text{id} \times f)_*[M]) \\ &= (-1)^d (\text{id} \times f)^*(\delta)[M] \end{aligned}$$

We know that $\delta = \sum (-1)^{|a_i|} a_i \otimes b_i$ where a_i is a basis for cohomology and b_i its dual basis. Therefore

$$(\text{id} \times f)^*(\delta) = \sum (-1)^{|a_i|} a_i \smile f^* b_i$$

We write $f^* b_i = \sum c_{ij} b_j$. Continuing with the above calculation, we get

$$\begin{aligned} (-1)^d (\text{id} \times f)^*(\delta)[M] &= (-1)^d \sum (-1)^{|a_i|} c_{ij} (a_i \smile b_j)[M] \\ &= (-1)^d \sum (-1)^{|a_i|} c_{ii} \\ &= \sum (-1)^{|b_i|} c_{ii} \end{aligned}$$

which is the right hand side. Geometrically, Γ_f and $\Delta(M)$ are transverse, with intersection at the fixed points. The orientation on the intersection is given by comparing the orientations on $D_x(\text{id} \times f)(TM) \oplus D_x(\text{id} \times \text{id})(TM)T_{(x,x)}(M \times M) = T_x M \otimes T_x M$, that is, the sign of

$$\det \begin{pmatrix} I & I \\ D_x f & I \end{pmatrix} = \det \begin{pmatrix} I - D_x f & 0 \\ D_x f & I \end{pmatrix} = \det(I - D_x f)$$

which gives the left hand side. $\square$

Example 8.9. Let $f: \mathbb{CP}^{2n} \rightarrow \mathbb{CP}^{2n}$. We claim this always has a fixed point. We know $f^*(x) = \lambda x$ where $H^2(\mathbb{CP}^2; \mathbb{Z}) = \langle x \rangle$ for some $\lambda \in \mathbb{Z}$, so $f^*(x^i) = \lambda^i x^i$, and so $\sum (-1)^i \text{Tr}(H^i(f)) = 1 + \lambda^2 + \dots + \lambda^{2n}$. If $\text{Fix } f = \emptyset$, then Γ_f, Δ are transverse, so λ is an integer root of the above polynomial, but no such integer exists (it would have to be ± 1 by rational root test).

Example 8.10. Let $f: \Sigma_g \rightarrow \Sigma_g$ have order p where p is an odd prime. Then $f_*[\Sigma_g] = \mu[\Sigma_g]$, so iterating p times, $[\Sigma_g] = \mu^p[\Sigma_g]$ and $\mu \in \mathbb{Z}$, so $\mu = 1$. If $x \in \Sigma_g$ is a fixed point, then we have an order p map $D_x f: T_x \Sigma_g \rightarrow T_x \Sigma_g$ is a linear map of order p , so is a rotation, say by θ . You can check $\det(I - D_x f) = 2 - 2\cos\theta > 0$ if $f \neq \text{id}$, so LFPT gives $|\text{Fix } f| = 2 - \text{Tr}(H^1(f))$. As p is prime, $\text{Fix } f = \text{Fix } f^2 = \dots = \text{Fix } f^{p-1}$, so if A is the matrix of $H^1(f)$ in some basis, then $\text{Tr } A = \text{Tr } A^2 = \dots = \text{Tr } A^{p-1}$. Now over $\mathbb{C}$ we can upper-triangularise A , and as A has order p , the diagonal is all p th roots of unity and $\sum \lambda_i = \sum \lambda_i^k$ for $k \leq p-1$, so $\lambda_1, \dots, \lambda_{2g} = 1, \dots, 1, (\zeta, \zeta^2, \dots, \zeta^{p-1}), \dots$ with m 1s and n copies of each other root of unity. So $\text{Tr } A = m - n$, and $2g = m + (p-1)n$. So $|\text{Fix } f| = 2 + n - m$, and this is $2 - 2g = \chi(\Sigma_g) \bmod p$

Example 8.11. For $g = 2$, we need $n, m \geq 0$ to satisfy $4 = m + (p-1)n$ and $2 + n - m \geq 0$, so $(m, n, p, \text{Fix } f) = (0, 1, 5, 3), (2, 1, 3, 1), (0, 2, 3, 4)$ so C_p does not act on Σ_2 for $p \geq 7$