

Amenability

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January 2025

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1 Introduction

Remark. Notation: $G \curvearrowright X$ for G acts on X from the left. No proofs will be given in this section.

Definition 1.1. $G \curvearrowright X$ is **paradoxical** if there are $A_1, \dots, A_n \subset X$ and $g_1, \dots, g_n, h_1, \dots, h_n \in G$ such that

$$\begin{aligned}
X &= A_1 \sqcup \dots \sqcup A_n \sqcup B_1 \sqcup \dots \sqcup B_n \\
&= g_1 A_1 \sqcup \dots \sqcup g_n A_n \\
&= h_1 B_1 \sqcup \dots \sqcup h_n B_n
\end{aligned}$$

The data of these A_i, B_i, g_i, h_i are called a **paradoxical decomposition**

Theorem 1.1. (Hausdorff) $SO(3) \curvearrowright S^2$ is paradoxical

Theorem 1.2. (Banach) $SO(2) \curvearrowright S^1$ is not paradoxical

To understand this, von Neumann introduced the following

Definition 1.2. $G \curvearrowright X$ is **amenable** if there is a G -invariant finitely additive probability measure on $\mathcal{P}(X)$

Remark. If $G \curvearrowright X$ is amenable then the paradox gives you $\mu(\text{paradox})$ which shows $1 = 1 + 1$ so amenable groups are not paradoxical. Therefore the action of $SO(3)$ on S^2 isn't paradoxical. This is because of the existence of non-lebesgue measurable sets

Theorem 1.3. (Von Neumann, 1929). *All actions of abelian groups are amenable. In particular, $SO(2) \curvearrowright S^1$ is amenable*

Remark. But $SO(3)$ contains a free groups so we will see it isn't amenable

Theorem 1.4. (Tarski, 1938) *An action is amenable if and only if it is not paradoxical*

Definition 1.3. $G \curvearrowright X$ has **almost invariant sets** if for all finite subsets $S \subset G$ and for every $\epsilon > 0$, there is a finite $A \subset X$ such that $|sA \triangle A| < \epsilon|A|$ for all $s \in S$

Theorem 1.5. (1955) *Having an almost invariant set is equivalent to amenability*

Theorem 1.6. (Day, 1957) *$G \curvearrowright X$ is amenable if and only if given a affine action of G on a convex compact subset K with a G -map $X \rightarrow K$ there is a fixed point in K of the G map*

Definition 1.4. If $G \curvearrowright G$ by left translation is amenable, then we say G is **amenable**

Remark. A couple remarks on the limits of the course are in order

1. We will only talk about discrete groups here
2. There are other notions of an amenable action

1.1 Rigidity Properties

We look at rigidity properties of amenable groups

1. All actions on the circle are "essentially" actions by rotations (Mangulis)
2. Having a geometric action of G on a CAT(0) space forces every amenable subgroup to be virtually abelian (Adams-Bellman)
3. Every uniformly bounded representation on a hilbert space is conjugate to a unitary representation. Anything that is "almost unitary" is close to a unitary representation (Dixmier, Ulam)

This makes amenable groups central to dynamical group theory

1.2 Vanishing Properties

The following invariants vanish for amenable groups

1. Bounded Cohomology (Johnson)
2. (almost) l^∞ cohomology (Gersten)
3. l^2 Betti numbers (Cheegev, Gromov)
4. rank gradient (Leckenby)

This gives us a lot of homological information for homological invariants

Theorem 1.7. *Finite and Abelian groups are amenable. Subgroups, Quotients, Extensions and Directed Unions of amenable groups are amenable. In particular virtually solvable groups are amenable*

Definition 1.5. Groups obtained this way are **elementary amenable**

Corollary 1.7.1. *Groups with non-abelian free subgroups are not amenable*

Remark. Two obvious problems arise from this

1. Von-Neumann-Day problem: Does every non-amenable G contain F_2
2. Is every amenable G elementary?

Neither of these are true. (Olshansky, Grigorchuk, 1980). They are true in some special cases, like linear groups (The Tits alternative)

2 Definition and Basics

Definition 2.1. A **finitely additive probability measure** on X is a map $\mu: \mathcal{P}(X) \rightarrow [0, 1]$ such that

1. $\mu(X) = 1$
2. $\mu(A \sqcup B) = \mu(A) + \mu(B)$

Example 2.1. The following are examples of finitely additive probability measures

1. Dirac mass, $\delta_X(A) = 1_{\{x\}}$
2. $\alpha \in l^1(X)$ ie $\alpha: X \rightarrow \mathbb{R}$ and $\sum |\alpha_x| < \infty$ such that $\alpha_x \in [0, 1]$ and $\sum \alpha_x = 1$. We get a measure by $\alpha(A) = \sum_{x \in A} \alpha_x$. These are called probability distributions
3. Convex combinations of finitely additive probability measures are also finitely additive probability measures

Definition 2.2. Given $f: X \rightarrow Y$ and μ is a finitely additive probability measure on X , we can define such a measure on Y , $f_*\mu$ the **pushforward** by $f_*\mu(A) = \mu(f^{-1}(A))$

Definition 2.3. If $G \curvearrowright X$ then $G \curvearrowright$ finitely additive probability measures on X by pushforward, ie $g_*\mu(A) = \mu(g^{-1}(A))$. μ is **G invariant** if $g_*\mu = \mu$ for all $g \in G$. If such a measure exists, $G \curvearrowright X$ is **amenable**

Example 2.2. $G \curvearrowright X$, $Y \subset X$ is a finite G invariant set then the action is amenable, with invariant measure $\mu(A) = \frac{|Y \cap A|}{|Y|} = \frac{1}{|Y|} \sum_{y \in Y} \delta_y$

Exercise 2.1. If $G \curvearrowright X$ has all orbits infinite, then every invariant μ gives measure 0 to finite sets.

$G \curvearrowright l^1(X)$ by $(g\alpha)_x = \alpha_{g^{-1}x}$ has no non-zero fixed point so there is no invariant probability distribution

Definition 2.4. If $G \curvearrowright G$ by left translation is amenable, then G is **amenable**

Example 2.3. Finite groups are amenable.

Proposition 2.1. If $G \curvearrowright X, Y$ is amenable on X and we have $f: X \rightarrow Y$ which is G -equivariant then $G \curvearrowright Y$ is also amenable

Proof. Suppose μ on X is invariant. $f_*\mu$ is invariant on Y as $f_*\mu(gA) = \mu(f^{-1}(gA)) = \mu(gf^{-1}(A)) = \mu(f^{-1}(A)) = f_*\mu(A)$ $\square$

Corollary 2.0.1. A group is amenable if and only if each of its actions is

Proof. We get a map $G \rightarrow X$ by $g \mapsto gx_0$ if $G \curvearrowright X$, and the result follows from above. $\square$

Corollary 2.0.2. Quotients of amenable groups are amenable

Proof. As every action of G/N is also an action of G $\square$

Exercise 2.2. This is all equivalent for working with the right action

2.1 Free groups

The free group of rank 2, F_2 has basis $\{a, b\}$ and elements are reduced words on $\{a, a^{-1}, b, b^{-1}\}$ with concatenation. We can similarly define F_n, F_∞ where ∞ is countable by default. We have the following universal property that every map $\{a, b\} \rightarrow G$ extends uniquely to a homomorphism $F_2 \rightarrow G$ and similarly for F_n

Corollary 2.0.3. Every group is a quotient of a free group

Proof. Trivial $\square$

Theorem 2.1. (Nielsen-Schrier) Subgroups of free groups are free and all ranks can happen (eg $F_{10} < F_2$)

Proof. No Proof $\square$

Remark. Rank 1 is just $\mathbb{Z}$ and is very different (and will be forgotten).

Theorem 2.2. F_2 is not amenable

Proof. Suppose μ is a measure on F_2 . Decompose $F_2 = \{1\} \sqcup a* \sqcup a^{-1}* \sqcup b* \sqcup b^{-1}* \sqcup \dots$ where each term is "words starting with".

Now $a^{-1}a* = \{1\} \sqcup a* \sqcup b* \sqcup b^{-1}* \sqcup \dots$ as words can't start aa^{-1} . Applying measures, as it is finitely additive and F_2 invariant, we get $\mu(\{1\}) + \mu(b*) + \mu(b^{-1}*) = 0$ so each term is 0. Similarly $\mu(a*), \mu(a^{-1}*) = 0$ so $\mu(F_2) = 0$ contradiction $\square$

Later on, free subgroups will be important.

Exercise 2.3. SO_3 has an F_2 subgroup (Use ping-pong). This will be used for the Hausdorff-Banach-Tarski paradox

3 Means

3.1 Some Functional Analysis

Nets are a trick to pretend every space is first countable

Definition 3.1. A **net** is a map $X_\bullet: I \rightarrow X$ where I is a directed set and X is a space

Example 3.1. $I =$ set of neighbourhoods of $x \in X$ with reverse inclusion
 $I = \mathbb{N}$ is a sequence

Definition 3.2. We have notions of

1. **Convergence:** $x_0 \rightarrow x$ if for all neighbourhoods $U \ni x$ there is an i_0 such that for all $i \geq i_0, x_i \in U$
2. **Accumulation:** x is an **accumulation point** of $(x_i)_{i \in I}$ if for all neighbourhoods $U \ni x$ and for all $i_0 \in I$ there is $i \geq i_0$ such that $x_i \in U$

Theorem 3.1. $f: X \rightarrow Y$ is continuous if and only if $f(x_i) \rightarrow f(x)$ whenever $x_i \rightarrow x$

There is a tricky notion of **subnet**

Definition 3.3. A net $(s_a)_{a \in A}$ is a **subnet** of a net $(x_i)_{i \in I}$ if there is a cofinal, order preserving map $h: A \rightarrow I$ such that $s_a = x_{h(a)}$

Theorem 3.2. x is an accumulation point of $(x_i)_{i \in I}$ if there is a subnet (s_a) of (x_i) such that $s_a \rightarrow x$.

A space is compact if and only if every net has an accumulation point.

More, similar characterisations of topological properties in terms of nets can be found elsewhere, and may be used in these notes

3.2 Normed Vector Spaces

Definition 3.4. A **normed vector space**, $(V, \|\cdot\|)$ is a vector space V with a norm $\|\cdot\|$ which satisfies

1. $\|v\| = 0$ if and only if $v = 0$
2. if $\alpha \in \mathbb{R}, v \in V$ then $\|\alpha v\| = |\alpha|\|v\|$
3. $\|v + w\| \leq \|v\| + \|w\|$ for $v, w \in V$

This induces a metric by $d(x, y) = \|x - y\|$ and if it is complete with this metric, V is **Banach**

Example 3.2. Let $p \in [1, \infty)$ and X a set.

Let $l^p(X) = \{f: X \rightarrow \mathbb{R} \mid \|f\|_p = (\sum |f|^p)^{1/p} < \infty\}$. $(l^p(X), \|\cdot\|_p)$ is Banach. Further $l^\infty(X) = \{f: X \rightarrow \mathbb{R} \mid \|f\|_\infty = \sup_{x \in X} |f(x)| < \infty\}$ has $(l^\infty(X), \|\cdot\|_\infty)$ Banach

Theorem 3.3. For V, W normed vector spaces, $T: V \rightarrow W$ is continuous if and only if it is bounded ie there is C such that $\|T(v)\|_W \leq C\|v\|_V$ for $v \in V$. We can define a norm on the space $L(V, W)$ called the **operator norm** via $\|T\|_{\text{op}} = \sup_{\|v\| \leq 1} \|T(v)\|$

m

Proof. No Proof □

Definition 3.5. $L(V, \mathbb{R})$ is called the **dual** of V , denoted V^* . Elements of V^* are called functionals

Exercise 3.1. There is a pairing $\langle \cdot, \cdot \rangle: l^\infty(X) \times l^1(X) \rightarrow \mathbb{R}$ such that

$$(F, f) \mapsto (x \mapsto \sum_{x \in X} F(x)f(x))$$

which induces maps $l^\infty(X) \rightarrow l^1(X)^*$, $l^1(X) \rightarrow l^\infty(X)^*$ where the first is an isomorphism

Remark. We have $V \not\cong V^{**}$ in general, but there is a map $\text{ev}: V \rightarrow V^{**}$ where $\text{ev}_v(\lambda) = \lambda(v)$. The map $l^1(X) \rightarrow l^\infty(X)^*$ of the preceding exercise is an instance

3.3 Weak-* Topology

Definition 3.6. The **Weak-* topology** (aka **pointwise topology**) on V^* is the coarsest topology such that all evaluation maps $e_v: V^* \rightarrow \mathbb{R}$ for $v \in V$ are continuous. In terms of nets, $\lambda_i \xrightarrow{*} \lambda$ (**converges in the weak-* topology**) if and only if $\lambda_i \rightarrow \lambda(v)$ for all $v \in V$.

Exercise 3.2. $V^* \rightarrow \prod_{v \in V} \mathbb{R}$ by $\lambda \mapsto (\lambda(v))_{v \in V}$ is a homeomorphism onto its image, for the weak-* topology

Remark. The weak-* topology is therefore (from the exercise) Hausdorff so limits of nets are unique.

From the exercise and Tychenoff's theorem we get

Theorem 3.4. (*Banach-Alaoglu*) The unit ball in V^* , $\{\lambda \in V^* \mid \|\lambda\|_{\text{op}} \leq 1\}$, is weak-* compact

If $G \curvearrowright V$ (always meaning linearly and isometrically), we get an induced $G \curvearrowright V^*$ by $(g\lambda)(v) = \lambda(g^{-1}v)$

3.4 Means

Definition 3.7. A mean on $l^\infty(X)$ is an element $m \in l^\infty(X)^*$ such that

1. $m(1_X) = 1$ (hence constants map to themselves)
2. $\|m\|_{\text{op}} = 1$ ($|m(f)| \leq \|f\|_\infty$)
3. If $f \geq 0$ then $m \geq 0$

We should think of this as an "averaging operator"

Example 3.3. $f \in l^1(X) \mapsto l^\infty(X)^*$ maps to a mean if and only if $f \geq 0$ and $\sum_{x \in X} f(x) = 1$.

Compare this to probability distributions.

Exercise 3.3. 1. Any two of the properties defining a mean imply the third

2. Means are weak-* closed (if $m_i \xrightarrow{*} m$ and m_i are means, then m is a mean)
3. We can strengthen $|f| \leq \|f\|_\infty$ to $m(f) \in [\inf f, \sup f]$

Recall that we have

Theorem 3.5. (*Riesz Representation theorem*) If K is a compact hausdorff space and $I \in C(K)^*$ is a mean on $C(K)$, we get a Radon probability measure on K , and the correspondence is bijective. The reverse direction is defined by

$$\mu \mapsto I(f) = \int f d\mu$$

Proof. No Proof □

Proposition 3.1. Similarly, there is a bijection between means on $l^\infty(X)$ and finitely additive probability measures on X

Proof. $m \mapsto \mu$ where $\mu(A) = m(1_A)$ is the forward direction
Fix μ . Define $D \subset l^\infty(X) = \text{span of } 1_A, A \subset X = \{f \in l^\infty(X) \mid \text{im}(f) \text{ is finite}\}$.
Then D is dense in $l^\infty(X)$ so we can define, for $f \in D$ written uniquely as $f = \sum c_i 1_{A_i}$ for disjoint A_i , and c_i distinct non-zero, so we can define a mean by $m(f) = \sum c_i \mu(A_i)$ and extend uniquely by continuity to define a mean on all $l^\infty(X)^*$ as it is 1-lipschitz $\square$

Exercise 3.4. Guided meditation on $l^\infty(K)^*$ vs $C(K)^*$.

Denote by $\mathcal{M}(X)$ the space of means with the weak-* topology, which is isomorphic to the space of finitely additive probability measures with the weak-* topology.

For means, $m_i \xrightarrow{*} m$ if and only if $m_i(f) \rightarrow m(f)$ for all $f \in l^\infty(X)$

For measures, $\mu_i \xrightarrow{*} \mu$ if and only if $\mu_i(A) \rightarrow \mu(A)$ for all $A \subset X$ (as D is dense in $l^\infty(X)$)

Proposition 3.2. $\mathcal{M}(X)$ is convex and compact

Proof. For convexity, this was previously proven

$\mathcal{M} \subset l^\infty(X)^*$ is closed (previous exercise) and contained in the unit ball, so it is compact $\square$

If $G \curvearrowright X$ then $G \curvearrowright$ finitely additive probability measures on X by push-forward, and on $l^\infty(X)$ by $(gf)(x) = f(g^{-1}x)$ hence on $l^\infty(X)^*$ which preserves $\mathcal{M}(X)$.

You can check these actions are the same

Corollary 3.5.1. $G \curvearrowright X$ is amenable if there is a G invariant mean $m \in \mathcal{M}(X)$

3.5 Amenability of $\mathbb{Z}$

Theorem 3.6. $\mathbb{Z}$ is amenable

Proof. Let $\mathbb{Z} = \langle t \rangle \curvearrowright \{\dots, -1, 0, 1, \dots\} = \mathbb{Z}$ and take $n \geq 1$. Consider the finitely additive probability measure $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_i$. Then we consider and compute $\|t_*\mu_n - \mu_n\| = \|\frac{1}{n}(\delta_1 - \delta_{n+1})\| \leq \frac{2}{n}$. So $|t_*\mu_n(f) - \mu_n(f)| \leq \frac{2}{n}\|f\|_\infty \rightarrow 0$ for all $f \in l^\infty(X)$ and hence $t_*\mu_n - \mu_n \xrightarrow{*} 0$.

Let μ be an accumulation point of $(\mu_n)_{n \in \mathbb{N}}$. Let $(\mu_i)_{i \in I}$ be a subnet such that $\mu_i \rightarrow \mu$. Then $t_*\mu_i - \mu_i \xrightarrow{*} 0$ and so $t_*\mu - \mu = 0$ hence $t_*\mu = \mu$ $\square$

4 Fixed Point Theorems

4.1 Some more functional analysis

Definition 4.1. A **locally convex vector space** is a (Hausdorff) topological vector space such that 0 has a neighbourhood basis of compact convex sets

Definition 4.2. A **seminorm** on V is a map $p: V \rightarrow \mathbb{R}$ such that the triangle inequality, $p(v + w) \leq p(v) + p(w)$, holds and $p(\alpha v) = |\alpha|p(v)$ for all $v, w \in V$ and $\alpha \in \mathbb{R}$

Theorem 4.1. V is locally convex if and only if the topology is induced by a family of seminorms

Proof. No Proof □

Example 4.1. We have two major examples

1. Normed vector spaces are locally convex (with the family of seminorms being just the norm)
2. V^* with the weak-* topology (with family of seminorms $\lambda \mapsto |\lambda(v)|$)

Theorem 4.2. (Hyperplane separation theorem) Let $A, B \subset \mathbb{R}^n$ be disjoint convex compact sets. Then there is a functional $\lambda \in (\mathbb{R}^n)^*$ such that $\max_{a \in A} \lambda(a) < \min_{b \in B} \lambda(b)$

Proof. Trivial □

Theorem 4.3. (Hahn-Banach) Let V be locally convex and $A, B \subset V$ disjoint which are convex and compact. Then there is $\lambda \in V^*$ such that $\max_{a \in A} \lambda(a) < \min_{b \in B} \lambda(b)$

Proof. No proof. □

Corollary 4.3.1. V^* separates points in V , ie if $x \neq y \in V$ then there is $\lambda \in V^*$ with $\lambda(x) \neq \lambda(y)$

Proof. Trivial □

Remark. In this course, a convex compact set = A convex compact subset of a locally convex vector space. If K is such a set and $G \curvearrowright K$ we assume it acts by affine homeomorphisms. In this case, K is a compact convex G -set.

Example 4.2. If $G \curvearrowright X$, then $\mathcal{M}(X)$ is a convex compact G -set

4.2 The Barycentre Theorem

Theorem 4.4. Let K be a convex compact set and let $m \in \mathcal{M}(K)$. Then there is a unique $x \in K$, called the **barycentre** such that for every continuous affine $\lambda: K \rightarrow \mathbb{R}$, $\lambda(x) = m(\lambda)$. It suffices to prove it for means I on $C(K)$ as every l^∞ mean restricts to one

Proof. Uniqueness: Hahn-Banach corollary, as $\lambda(x) = \lambda(y)$ for any barycentres x, y

Existence: If we have $\lambda: K \rightarrow \mathbb{R}$ continuous and affine, we let the set K_λ be defined by $K_\lambda = \{x \in K \mid \lambda(x) = m(\lambda)\}$. We need to show that $\bigcap K_\lambda \neq \emptyset$. The sets K_λ is closed as λ is continuous, hence it is also compact. So it is enough to

show that given $\lambda_1, \dots, \lambda_n$, $\bigcap_{i=1}^n K_{\lambda_i} \neq \emptyset$. In other words, define $\Lambda: K \rightarrow \mathbb{R}^n$ by $x \mapsto (\lambda_1(x), \dots, \lambda_n(x))$. We need $\Lambda(K) \ni y = (m(\lambda_1), \dots, m(\lambda_n))$. Suppose that $y \notin \Lambda(K)$. Then as $\Lambda(K)$ is convex and compact, we may apply the Hyperplane Separation theorem to get $\phi \in (\mathbb{R}^n)^*$ such that $\phi(y) > \max_{x \in K} \phi(\Lambda(x))$. Write $\phi = (\phi_1, \dots, \phi_n)$ in the dual basis. We have

$$\max_K \phi \circ \Lambda < \phi(y) = \sum_i \phi_i(m(\lambda_i)) = m(\sum_i \phi_i \lambda_i) = m(\phi \circ \Lambda) \leq \max_K \phi \circ \Lambda$$

with the second equality following as m is linear, hence contradiction. $\square$

4.3 The Fixed Point Theorem

Theorem 4.5. *$G \curvearrowright X$ is amenable if and only if all compact convex G -sets K with a G -map $X \rightarrow K$ have a fixed point*

Proof. Exercise. $\square$

Corollary 4.5.1. *G is amenable if and only if every compact convex G set K has a fixed point*

Proof. Reverse direction: Apply the theorem to $G \curvearrowright \mathcal{M}(G)$

Forwards direction: Let $G \curvearrowright K$ and let $m \in \mathcal{M}(K)$ be G -invariant. Then $x = \text{bar}(m)$ (the barycentre) is fixed by g . $\square$

4.4 Dynamical Criterion

Theorem 4.6. *G is amenable if and only if for every action of $G \curvearrowright Y$ where K is a compact hausdorff space and G acts by homeomorphisms, there exists an invariant Radon probability measure on K*

Proof. Exercise: formulate for $G \curvearrowright X$ for sets X .

The second condition is equivalent to "For every $G \curvearrowright Y$ which is compact and hausdorff there is a G -invariant mean on $C(Y)$." We claim this is equivalent to "Every $G \curvearrowright K$ convex compact has a fixed point".

In the backwards direction, we apply it to $G \curvearrowright$ means on $C(Y)$.

In the forwards direction, suppose $G \curvearrowright K$ convex compact so there is a G -invariant mean I on $C(K)$. Then $\text{bar}(I)$ is a G -fixed point.

Indeed, $g(\text{bar}(I)) = \text{bar}(g_* I) = \text{bar}(I)$ as

$$\lambda(g \text{bar}(I)) = (g^{-1} \lambda)(\text{bar}(I)) = I(g^{-1} \lambda) = g_* I(\lambda)$$

so $g \text{bar}(I)$ fits the definition of $\text{bar}(g_* I)$ $\square$

5 The Class of Amenable Groups

5.1 From Amenable Actions to Amenable Groups

Proposition 5.1. *Suppose $G \curvearrowright X$ is an amenable action with amenable stabilisers. Then G is amenable*

Proof. We use the fixed point criterion for $G \curvearrowright X$ and $G \curvearrowright G, G_x$ for $x \in X$. Let $G \curvearrowright K$ be convex compact. Let $R \subset X$ be a set of orbit representatives. For $r \in R$, there is a point $f(r) \in K$ which is fixed by G_r . If $x = gr$ then set $f(x) = gf(r)$. This is well defined as if $gr = hr$ then $h = g \cdot g_r$ for some $g_r \in G_r$ so $hf(r) = gg_rf(r) = gf(r)$. So we define $f: X \rightarrow K$ which is G -equivariant in this way. By the amenability of $G \curvearrowright X$ there is a G -fixed point in K . $\square$

5.2 Permanence Properties for Groups

We have seen that amenability passes to quotients.

Corollary 5.0.1. *Subgroups of amenable groups are amenable*

Proof. Apply the previous proposition to $H \curvearrowright G$ $\square$

Corollary 5.0.2. *If $F_2 \leq G$ then G is not amenable*

Proof. Trivial $\square$

Example 5.1. $SL_2(\mathbb{Z}), SO(3)$ are non-amenable

Corollary 5.0.3. *Extensions of amenable groups are amenable*

Proof. Let $N \leq G$ be normal such that $N, G/N$ are amenable. Then G is amenable by the proposition on $G \curvearrowright G/N$ $\square$

Exercise 5.1. Virtually amenable groups are amenable

If G is amenable then G has a bi-invariant measure

Definition 5.1. Let $(H_i)_{i \in I}$ be subgroups of G . G is the **directed union** of the H_i if $G = \bigcup_I H_i$ and for all i, j there is a k such that $H_j \cup H_i \subset H_k$

Corollary 5.0.4. *A directed union of amenable groups is amenable*

Proof. Let $G \curvearrowright K$ be convex compact. Let $K_i = \{x \in K \text{ fixed by } H_i\} \neq \emptyset$ as H_i is amenable. It is also closed. To finish, we need to show $\bigcap_{i \in I} K_i \neq \emptyset$. It suffices to show the finite intersection property by compactness. As it is a directed union, the finite union of the corresponding H_{i_k} is in some H_j and hence the intersection contains K_j so is non-empty $\square$

Exercise 5.2. Locally amenable groups are amenable (A group is locally P if every finitely generated subgroup is P)

Direct sums of amenable groups are amenable

Corollary 5.0.5. *Abelian groups are amenable*

Corollary 5.0.6. *As $\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$ are amenable (as the latter are finite), so are all finitely generated abelian groups. Hence every abelian group is locally amenable, hence amenable*

Exercise 5.3. $SO(2) \curvearrowright S^1$ is not paradoxical

5.3 Von Neumann-Day Problem

We have proven the backwards direction of each of these, so the following questions arise:

1. If a group is non-amenable, do we necessarily have $G > F_2$?
2. G is **elementary amenable** if it can be obtained from abelian and finite groups via extensions and directed unions. Is every amenable group elementary amenable?

5.4 An Embedding Theorem

The goal of this subsection is to prove that every countable amenable group embeds in a 2-generated amenable group. There is

Theorem 5.1. (*Higman-Neumann-Neumann*) *Every countable group embeds in a 2-generated group*

but this is useless as the original construction doesn't preserve amenability, so we won't be concerned by it.

Definition 5.2. We define the **wreath product** of groups G, H by the semidirect product $G \wr H = G^H \rtimes H$ where H acts by left multiplication on the indexing set of the factors

We prove the theorem above with a much more useful construction

Proof. Let $G = \{g_1, g_2, \dots\}$ and let $Z = \langle z \rangle \cong \mathbb{Z}$ and define $P = G \wr Z$. Define $f_i: Z \rightarrow G$ by $z^t \mapsto g_i$ if $t \geq 0$ and e_G otherwise. Now $f_i \in G^Z < P$. Consider $c_i = f_i z f_i^{-1} z^{-1} = f_i \cdot (z_* f_i^{-1})$. Now $z_* f_i^{-1}(z^t) = f_i^{-1}(z^{t-1}) = g_i^{-1}$ for $t \geq 1$ and e_G otherwise and so $c_i(z^t) = g_i$ for $t = 0$, and e_G otherwise. Now define $D = \langle d \rangle \cong \mathbb{Z}$. Denote by d_i the element d^{2^i-1} . If $i \neq j$ then $d_i d_j \neq d_k, e_D$ for all $i \neq j \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Let $Q = P \wr D$ and let $q: D \rightarrow P$ by $q(e_D) = z$ and $q(d_i^{-1}) = f_i$ or the identity otherwise. Let $k_i = [(d_i)_* q, q] \in \langle q, d \rangle$. Now $k_i(e_D) = c_i$, $k_i(d_j^{-1}) = e_D$ and $k_i(x) = e_D$ for all other x . Therefore $\langle d, q \rangle > \langle k_i \mid i \in \mathbb{Z} \rangle \cong \langle c_i \mid i \in \mathbb{Z} \rangle \cong G$ $\square$

Theorem 5.2. *Every countable amenable group embeds in a 2-generated group*

Proof. With the same notation as before, we want to show that if G is amenable, then so is $H = \langle q, d \rangle$. Let $N = \langle \langle q \rangle \rangle < H$. Then $H/N \cong \mathbb{Z}$ is amenable so it suffices to show that N is amenable. Write $N_k = \langle \langle (d_i)_* q \mid |i| \leq k \rangle \rangle$. N is the directed union of the N_k so it suffices to prove that N_k are all amenable. In fact it suffices to prove that $[N_k, N_k]$ is amenable as abelian groups are amenable and so are extensions of amenable groups. Now $q(d^n) = e_P$ unless $n = 1 - 2^i$ for some $i \geq 0$, similarly $d_*^j q(d_n) = q(d^{n-j}) = 1_P$ unless $n = j + 1 - 2^i$. Suppose that $(d_*^j q)(d^n) \neq e_P \neq (d_*^{j'} q)(d^n)$. Then $j + 1 - 2^i = j' + 1 - 2^{i'}$ so $2^i - 2^{i'} = j' - j \leq 2k$ and so $|n| \leq 5k + 1$. So for $|n| > 5k + 1$, no two generators

of N_k have non-trivial coordinate at n and so $[N_k, N_k]$ has coordinate e_P for such n and so $[N_k, N_k] \hookrightarrow \prod_{|n| \leq 5k+1} P$ with image only including $f_i, z \in P$. By extensions and subgroups, it suffices to show that $\langle f_i, z \mid i \in \mathbb{Z} \rangle \leq P$ is amenable. Now again by extensions, it suffices to show that $\langle \langle f_i \mid i \in \mathbb{Z} \rangle \rangle$ is amenable. This is $f \in G^{\mathbb{Z}}$ such that f is eventually constant to the right and identity to the left. This has a tail map into G and G is amenable, so it suffices to show that the kernel is amenable, but this sits in a direct sum of copies of G so we are done $\square$

5.5 Elementary Amenability

Everything here is an exercise.

1. Every countable elementary amenable group embeds in a 2-generated elementary amenable group
2. There is a notion of elementary amenability class, which is an ordinal defining the number of steps it takes to get to the group

Theorem 5.3. (*Olskanski-Osm*) *The ordinals arising as elementary amenability classes are precisely $\{\alpha + 1 \mid \alpha \leq \omega_1\} \cup \{0\}$*

6 Almost Invariance

In this section we will introduce the Følner and Reiter R_1 properties and we will prove that the Følner property implies the Reiter R_1 property which implies amenability

Remark. $A \subset_f B$ means A is a finite subset of B

6.1 Almost invariant sets

Definition 6.1. Let $G \supset X$, $S \subset_f G$ and $\epsilon > 0$. We say $A \subset_f X$ is (S, ϵ) -invariant if for every $s \in S$, $|sA \Delta A| < \epsilon|A|$.

We say $G \supset X$ has **almost invariant sets/Følner's property** if there are (S, ϵ) -invariant sets for all $S \subset_f G, \epsilon > 0$

Exercise 6.1. The Følner property is equivalent to the following:

For all $S \subset_f G, \epsilon > 0$, there is $A \subset_f X$ such that $|SA| < (1 + \epsilon)|A|$

Exercise 6.2. 1. $\mathbb{Z}^n \supset \mathbb{Z}^n$ is Følner

2. $F_2 \supset F_2$ is not Følner

Exercise 6.3. If $G = \langle S_0 \rangle$ for S_0 finite, it suffices to find (S_0, ϵ) -invariant sets

To clarify the negation,

Proposition 6.1. *$G \supset X$ is non-Følner if and only if there is $S \subset_f G$ such that for all $A \subset_f X$, $|SA| \geq 2|A|$*

Proof. Non-Følner is equivalent to the existence of $S \subset_f G, \epsilon > 0$ such that for all $A \subset_f X$, $|SA| \geq (1 + \epsilon)|A|$. Now

$$|S^n A| = |SS^{n-1} A| \geq (1 + \epsilon)|S^{n-1} A| \geq \cdots \geq (1 + \epsilon)^n |A|$$

. Taking n sufficiently large, the result follows with $S = S^n$ $\square$

6.2 Subexponential Growth

Definition 6.2. G has **subexponential growth** if $\lim_{n \rightarrow \infty} |S^n|^{1/n} = 1$ for all $S \subset_f G$

Exercise 6.4. 1. This limit exists and is at least 1

2. If G is finitely generated, it suffices to check for a finite generating set

Exercise 6.5. If $(a_n)_{n \in \mathbb{N}}$ is a positive sequence, then $\liminf \frac{a_n}{a_{n-1}} \leq \liminf (a_n)^{1/n}$

Proposition 6.2. If G has subexponential growth then $G \curvearrowright G$ is Følner

Proof. Let $S \subset_f G$. Now $1 \leq \liminf \frac{|S^n|}{|S^{n-1}|} \leq \liminf |S^n|^{1/n} = 1$ so for all $\epsilon > 0$ there is some $n \in \mathbb{N}$ such that $|S^n| = |SS^{n-1}| \leq |S^{n-1}|(1 + \epsilon)$ so S^{n-1} is (S, ϵ) -invariant $\square$

Exercise 6.6. There are amenable groups of exponential growth

6.3 Almost Invariant Vectors

Definition 6.3. Let V be a normed vector space and $G \curvearrowright V$ with $S \subset_f G$, $\epsilon > 0$. $v \in V$ is (S, ϵ) -**invariant** if $\|s \cdot v - v\| < \epsilon \|v\|$ for all $s \in S$. The strict inequality forces $v \neq 0$. $G \curvearrowright V$ has **almost invariant vectors** if for all $S \subset_f G, \epsilon > 0$ there is some $v \in V$ which is (S, ϵ) -invariant.

Definition 6.4. $G \curvearrowright X$ has **Reiter's property R_1** if $G \curvearrowright l^1(X)$ has almost invariant vectors

Exercise 6.7. If we have Reiter's property R_1 we can choose $v \in l^1(X)$ to be a probability distribution, ie $v \geq 0$ and $\|v\|_1 = 1$

Lemma 6.1. If $G \curvearrowright X$ is Følner then it is R_1

Proof. Let $S \subset_f G$ and $\epsilon > 0$. Then there is $A \subset_f X$ which is (S, ϵ) -invariant. Then $1_A \in l^1(X)$ is (S, ϵ) -invariant $\square$

Proposition 6.3. If $G \curvearrowright X$ is R_1 then it is amenable

Proof. By exercise 6.7, for all $S \subset_f X, \epsilon > 0$ there is some $m_{S, \epsilon} \in \mathcal{M}(X) \cap l^1(X)$ which is (S, ϵ) -invariant. Define $(S, \epsilon) \leq (S', \epsilon')$ if $S \subset S'$ and $\epsilon \geq \epsilon'$ then this is a directed set. In this case (S', ϵ') -invariant implies (S, ϵ) -invariant. Therefore $m_{S, \epsilon}$ is a net, so as $\mathcal{M}(X)$ is compact we have an accumulation point, call it $m \in \mathcal{M}(X)$. We claim $g_* m = m$ for all $g \in G$. For all $g \in G$ we eventually have $\|g_* m_{S, \epsilon} - m_{S, \epsilon}\| < \epsilon$, as soon as $g \in S$ so $g_* m_{S, \epsilon} - m_{S, \epsilon} \xrightarrow{*} 0$ and hence $g_* m - m = 0$ $\square$

We have proven that Subexponential implies Følner implies Reiter R_1 implies amenable.

6.4 All the Reiter Properties

Definition 6.5. $G \curvearrowright X$ has R_p if $G \curvearrowright l^p(X)$ has almost invariant vectors

Exercise 6.8. Understand R_∞ (not interesting)

We will focus on $1 \leq p < \infty$. $p = 2$ is an important case (as we act on a Hilbert space)

Definition 6.6. If we have an inner product space $(V, (\cdot, \cdot))$, we can define a norm $\|v\|^2 = (v, v)$ and if V is Banach with this norm, it is a **Hilbert space**

Some special properties are:

1. Parallelogram law: $(V, \|\cdot\|)$ is an inner product space if and only if for all $v, w \in V$, $\|v + w\|^2 + \|v - w\|^2 = 2(\|w\|^2 + \|v\|^2)$
2. Cauchy-Schwartz: For all $v, w \in V$, $|(v, w)| \leq \|v\| \cdot \|w\|$

Example 6.1. $l^2(X)$ with the inner product $(f, g) = \sum_{x \in X} f(x)g(x)$ is a Hilbert space. $l^p(X)$ is a Hilbert space if and only if $p = 2$. In $l^2(X)$, Cauchy Schwartz takes the form

$$\left| \sum_{x \in X} f(x)g(x) \right| \leq \left| \sum_{x \in X} f(x)^2 \right|^{1/2} \left| \sum_{x \in X} g(x)^2 \right|^{1/2}$$

Theorem 6.2. All R_p for $1 \leq p < \infty$ are equivalent

Proof. We need the following exercise

Exercise 6.9. For all $a, b \geq 0, t \geq 1$ we have $|a - b|^t \leq |a^t - b^t|$

We will show $R_p \implies R_q$ for $q \geq p$ and that $R_{2p} \implies R_p$. We are then done.

1. $R_p \implies R_q$ for $q \geq p$: Let $G \curvearrowright X$ be R_p and fix $S \subset_f G, \epsilon > 0$. Set $t = \frac{q}{p} \geq 1$. There is some $v \in l^p(X)$ such that $\|sv - v\|_p < \epsilon^t \|v\|_p$ for all $s \in S$. Set $w = |v|^{p/q}$ coordinate-wise. Note that $\|v\|_p^p = \|w\|_q^q$ so $w \in l^q(X)$.

We will show that w is (S, ϵ) -invariant.

$$\begin{aligned}
\|sw - w\|_q^q &= \sum_{x \in X} |w(s^{-1}x) - w(x)|^q \\
&= \sum_{x \in X} (|v(s^{-1}x)|^{p/q} - |v(x)|^{p/q})^p \\
&\leq \sum_{x \in X} ||v(s^{-1}x)| - |v(x)||^p \\
&\leq \sum_{x \in X} |v(s^{-1}x) - v(x)|^p \\
&= \|sv - v\|_p^p < \epsilon^q \|v\|_p^p \\
&= \epsilon^q \|w\|_q^q
\end{aligned}$$

So we are done

2. $R_{2p} \implies R_p$: Suppose $G \curvearrowright X$ is R_{2p} and fix $\epsilon > 0, S \subset_f G$. There is $v \in l^{2p}(X)$ such that $\|sv - v\|_{2p} < \frac{\epsilon}{2} \|v\|_{2p}$ for all $s \in S$. Define $w = v^2$ and note that $\|v\|_{2p}^{2p} = \|w\|_p^p$ so $w \in l^p(X)$. We will show that w is (S, ϵ) -invariant.

$$\begin{aligned}
\|sw - w\|_p^p &= \sum_{x \in X} |w(s^{-1}x) - w(x)|^p \\
&= \sum_{x \in X} |v(s^{-1}x) - v(x)|^p \\
&\leq \sum_{x \in X} (|v(s^{-1}x) - v(x)| |v(s^{-1}x) + v(x)|)^p \\
&\leq \left(\sum_{x \in X} |v(s^{-1}x) - v(x)|^{2p} \right)^{1/2} \left(\sum_{x \in X} |v(s^{-1}x) + v(x)|^{2p} \right)^{1/2} \\
&= \|sv - v\|_{2p}^p \|sv + v\|_{2p}^p \\
&\leq \left(\frac{\epsilon}{2} \|v\|_{2p} \right)^p (2\|v\|_{2p})^p \\
&= \epsilon^p \|w\|_p^p
\end{aligned}$$

where between the 3rd and 4th lines we use Cauchy Schwartz

So we are done. $\square$

6.5 Kazhdan's Property (T)

Definition 6.7. A group G has **property (T)** if every action of G on a Hilbert space with almost invariant vectors has a non-zero invariant vector

Proposition 6.4. Suppose G has (T). Then every amenable action has finite orbit. In particular, if G has (T) and is amenable then G is finite

Proof. Assume that amenable implies R_2 (which we will prove later), let $G \curvearrowright X$ be amenable. Then $G \curvearrowright l^2(X)$ has almost invariant vectors. Then $G \curvearrowright l^2(X)$ has a non-zero invariant vector f . There is some x such that $f(x) \neq 0$, so $|G \cdot x| < \infty$ as otherwise we would have an infinite sum of $f(x)$, which is not square summable. $\square$

Theorem 6.3. (Kazhdan) *If G has (T) then G is finitely generated*

Proof. Consider $X = \bigsqcup_{S \subset_f G} G/\langle S \rangle$. Then every $S \subset_f G$ has a fixed point in X , so in particular, $G \curvearrowright X$ is Følner and hence is R_2 . By the proposition, $G \curvearrowright X$ has a finite orbit and so there is some $S \subset_f G$ such that $G/\langle S \rangle$ is finite, and choosing coset representatives, along with S , we are done. $\square$

Other applications of the proposition are:

Theorem 6.4. (Margulis) *If G is a high rank simple Lie group and $\Gamma \leq G$ is a lattice then any non-trivial normal subgroup of Γ is finite or finite index*

Proof. No Proof $\square$

Theorem 6.5. (Arzhantseva) *If all hyperbolic groups are residually amenable, then all hyperbolic groups are residually finite*

Proof. No Proof $\square$

7 Paradoxes

We recall from the introduction,

Definition 7.1. $G \curvearrowright X$ is **paradoxical** if there are $A_1, \dots, A_n \subset X$ and $g_1, \dots, g_n, h_1, \dots, h_n \in G$ such that

$$\begin{aligned} X &= A_1 \sqcup \dots \sqcup A_n \sqcup B_1 \sqcup \dots \sqcup B_n \\ &= g_1 A_1 \sqcup \dots \sqcup g_n A_n \\ &= h_1 B_1 \sqcup \dots \sqcup h_n B_n \end{aligned}$$

The data of these A_i, B_i, g_i, h_i are called a **paradoxical decomposition**

Lemma 7.1. *Paradoxical implies non-amenable*

Proof. Suppose A_i, B_j, g_i, h_j is a paradoxical decomposition and μ is a G in-

variant finitely additive probability measure. But then

$$\begin{aligned}
1 &= \mu(X) \\
&= \mu(\coprod A_i \coprod B_j) \\
&= \sum \mu(A_i) + \sum \mu(B_j) \\
&= \sum \mu(g_i A_i) + \sum \mu(h_j B_j) \\
&= \mu(\coprod g_i A_i) + \mu(\coprod h_j B_j) \\
&= \mu(X) + \mu(X) \\
&= 2
\end{aligned}$$

□

So we have Følner implies Reiter R_1 (which is equivalent to Reiter R_p for $1 \leq p < \infty$) implies amenable implies non-paradoxical

Theorem 7.2. *Non-Følner implies paradoxical, and so all the above properties are equivalent*

Proof. Deferred

□

7.1 Piecewise- G maps

Throughout, fix $G \curvearrowright X$

Definition 7.2. If $A, B \subset X$, a map $f: A \rightarrow B$ is **piecewise- G** if $A = \coprod A_i$ and there are $g_i \in G$ such that $f(a) = g_i a$ for $a \in A_i$

Remark. $G \curvearrowright X$ is paradoxical if and only if $X = A \sqcup B$ and there are piecewise- G bijections $A \rightarrow X$ and $B \rightarrow X$

Proposition 7.1. *$f: A \rightarrow B$ is piecewise- G if and only if there is $S \subset_f G$ such that $f(a) \in S \cdot a$ for all $a \in A$*

Proof. In the forward direction, $f(a) = g_i a$ for $a \in A_i$ so set $S = \{g_1, \dots, g_n\}$
In the backwards direction let $S = \{g_1, \dots, g_n\}$ and define

$$A_i = \left\{ a \in A \setminus \bigcup_{j=1}^{i-1} A_j \mid f(a) = g_i(a) \right\}$$

□

Proposition 7.2. *Properties of piecewise- G maps:*

1. If $f: A \rightarrow B$ is piecewise- G , and $C \subset A$, then $f|_C$ is piecewise G
2. A composition of piecewise- G maps is piecewise G

3. If $f: A \rightarrow B$ is a piecewise- G bijection then f^{-1} is piecewise G
4. A disjoint union of piecewise- G maps is piecewise- G
5. If there is a piecewise- G surjection from $A \rightarrow B$ then there is a piecewise- G injection $B \rightarrow A$

Proof. Trivial □

Proposition 7.3. *If there are piecewise- G injections $A \xrightarrow{\phi} B \xrightarrow{\psi} A$ then there is a piecewise- G bijection $A \rightarrow B$*

Proof. Let

$$C = \bigcup_{i=1}^{\infty} (\psi\phi)^n(A \setminus \psi(B))$$

. We claim $x \in \phi(C)$ if and only if $\psi(x) \in C$. If we assume the claim then $\phi: C \rightarrow \phi(C)$ is a piecewise- G bijection and $\psi: B \setminus \phi(C) \rightarrow A \setminus C$ is a piecewise- G bijection and so we are done.

Suppose $x \in \phi(C)$. Then for some $n > 0$, $x \in \phi(\psi\phi)^n(A \setminus \psi(B))$ so

$$\psi(x) \in (\psi\phi)^{n+1}(A \setminus \psi(B)) \subset C$$

In the other direction if $\psi(x) \in C$ then for some $n \geq 0$,

$$\psi(x) \in (\psi\phi)^{n+1}(A \setminus \psi(B))$$

so

$$\psi(x) \in \psi\phi((\psi\phi)^n(A \setminus \psi(B))) \subset \psi\phi(C)$$

so the result follows as ψ is injective □

Corollary 7.2.1. *$G \sqsubset X$ is paradoxical if and only if there are piecewise- G injections $\sigma_{\pm}: X \rightarrow X$ with disjoint image*

Proof. Recall that G is paradoxical if and only if there are $\sigma_{\pm}$ as above with $X = \sigma_{-}(X) \sqcup \sigma_{+}(X) = X$ so the forwards direction is trivial. In the backwards direction let $A = \sigma_{-}(X)$, $B = X \setminus A \supset \sigma_{+}(X)$. Now $\sigma_{-}: X \rightarrow A$ is a piecewise- G bijection and $X \xrightarrow{\sigma_{+}} B \hookrightarrow X$ so there is a piecewise- G bijection $X \rightarrow B$ and $X = A \sqcup B$ □

Definition 7.3. $A \sim_G B$ are **equidecomposable** if there is a piecewise- G bijection $A \rightarrow B$ and $A <_G B$ is **subequidecomposable** if there is a piecewise- G injection $A \rightarrow B$

So the proposition says $A <_G B <_G A$ implies $A \sim_G B$, so $<_G$ is a preorder and $\sim_G$ is the induced equivalence relation

Definition 7.4. $A \stackrel{k}{\sim}_G B$ if there is an equidecomposition into k pieces, with k minimal.

The **Tarski number** of G is the minimal k such that $G \stackrel{l}{\sim}_G A, G \stackrel{k-l}{\sim}_G B$ and $A \sqcup B \subset G$. This is denoted by t_n

Exercise 7.1. $t_n(G) \geq 4$ with equality if and only if $F_2 \leq G$

7.2 Matchings

We wish to have tools to build piecewise- G injections. These will come from graph theory

Definition 7.5. If X, Y are sets, a **matching problem** is a map $\Sigma: X \rightarrow P(Y)$ and a **matching** for Σ is an injective map $\sigma: X \rightarrow Y$ such that $\sigma(x) \in \Sigma(x)$

Theorem 7.3. (*Hall's Marriage Lemma*)

Let X, Y be finite and $\Sigma: X \rightarrow P(Y)$ a matching problem. Then a matching exists if and only if for all $A \subset X$, $|\Sigma(A)| \geq |A|$

Proof. The forwards direction is clear as σ is an injection from $A \hookrightarrow \Sigma(A)$. For the backwards direction, we use induction. It is clear for $|X| = 1$. Suppose $|X| > 1$. There are two cases

1. For all proper subsets, $\emptyset \neq Z \subset X$, $|\Sigma(Z)| > |Z|$. Choose some $x_0 \in X$ and define $\sigma(x_0)$ arbitrarily. Now consider $\Sigma|_{X \setminus \{x_0\}}$ and we are done by induction
2. There is some proper $\emptyset \neq Z \subset X$ such that $|\Sigma(Z)| = |Z|$. By induction there is a matching $\sigma: Z \rightarrow \Sigma(Z)$. Now consider $\Sigma: X \setminus Z \rightarrow Y \setminus \sigma(Z)$ and apply induction.

□

Theorem 7.4. (*Infinitary marriage lemma*) Let X, Y be possibly infinite sets and $\Sigma: X \rightarrow P_f(Y)$ be a matching problem where $P_f(Y)$ is the finite subsets of Y . Then there is a matching if and only if for all $A \subset_f X$, $|\Sigma(A)| \geq |A|$

Proof. The forward direction is as before. For the backwards direction, let $K = \prod_{x \in X} \Sigma(x) \subset Y^X$ with the prodiscrete topology. By Tychonoff, K is compact. We need to show there is some injective $\sigma \in K$. Note that σ is injective if and only if $\sigma|_A$ is injective for all $A \subset_f X$, so if we denote the set $K_A = \{\sigma \in K \mid \sigma|_A \text{ is injective}\}$. This is closed, and we want to show that $\bigcap_{A \subset_f X} K_A \neq \emptyset$. By compactness it suffices to show that finite intersections are non-empty, but $\bigcap K_{A_i} \supset K_{\bigcup A_i}$ so we just need to show that $K_A \neq \emptyset$ for all $A \subset_f X$, and we are done by the Hall marriage lemma, extending the σ_A we get to X arbitrarily □

Exercise 7.2. Show that this fails if $\Sigma(x)$ is not necessarily finite

Corollary 7.4.1. (*Infinitary bigamist lemma*) Let $\Sigma: X \rightarrow P_f(Y)$ be a matching problem. Then there are two matchings with disjoint image if and only if for all $A \subset_f X$, $|\Sigma(A)| \geq 2|A|$

Proof. The forwards direction is clear. For the other direction, set $\tilde{X} = X \times \{\pm\}$ and define $\tilde{\Sigma}: \tilde{X} \rightarrow P_f(Y)$ by $\tilde{\Sigma}(x, \pm) = \Sigma(x)$. This satisfies the hypothesis of the infinitary marriage lemma, so we are done, by matching x with the elements matched with $(x, \pm)$ □

7.3 Proof of Tarski's Theorem

Theorem 7.5. *If $G \curvearrowright X$ is non-Følner, then it is paradoxical*

Proof. Recall that $G \curvearrowright X$ is non-Følner if and only if there is an $S \subset_f G$ such that for all $A \subset_f X$, $|SA| \geq 2|A|$. Define a matching problem by $\Sigma: X \rightarrow X$ such that $\Sigma(x) = S \cdot x$. The bigamist lemma applies by the definition of non-Følner, so there are matchings $\sigma_+, \sigma_-: X \rightarrow X$ with disjoint images. We know $\sigma_{\pm}(x) \in \Sigma(x) = S \cdot x$, and hence these are piecewise- G injections $\square$

7.4 The Classical Paradoxes

Corollary 7.5.1. *(Hausdorff) $SO(3) \curvearrowright S^2$ is paradoxical*

Proof. We need to show that $SO(3) \curvearrowright S^2$ is non-amenable. Note that $SO(3)$ is non-amenable as a group as it contains a free subgroup, and all the stabilisers of the actions are isomorphic to $SO(2)$ which is abelian and hence amenable. Therefore if $SO(3) \curvearrowright S^2$ were amenable, $SO(3)$ would be amenable, contradiction $\square$

Theorem 7.6. *(Banach-Tarski) If $A, C \subset \mathbb{R}^3$ are bounded with non-empty interior, then $A \sim_G C$ for $G = \text{Isom}^+(\mathbb{R}^3) = \mathbb{R}^3 \rtimes SO(3)$.*

Proof. Let $\dot{B} = B(0, 1) \setminus \{0\} \subset \mathbb{R}^3$. Note that $SO(3) \curvearrowright \dot{B}$ is paradoxical. The theorem follows from

Proposition 7.4. *If $A \subset \mathbb{R}^3, G = \text{Isom}^+(\mathbb{R}^3)$. If A is bounded then $A <_G \dot{B}$ and if the interior of A is not empty, then $\dot{B} <_G A$*

Proof. For the first claim, let $SO(3) \curvearrowright \dot{B}$. We can decompose $\dot{B} = X \sqcup Y$ such that $X \sim_G \dot{B} \sim_G Y$. Also $\dot{B} \sim_G \dot{B} + v$ for all $v \in \mathbb{R}^3$. So if we pick $w \in \mathbb{R}^3$ such that $\|w\| \geq 2$, then $\dot{B} \sim_G \dot{B} \sqcup (\dot{B} + w)$. By induction, if $\|w_i - w_j\| \geq 2$ for $i \neq j$ then $\dot{B} \sim_G \coprod_{i=1}^n (\dot{B} + w_i)$. Let $A \subset \mathbb{R}^3$ be bounded. Then there are $v_1, \dots, v_n$ such that $A \subset \bigcup_{i=1}^n (\dot{B} + v_i)$. Choose $w_1, \dots, w_n$ such that $\|w_i - w_j\| \geq 2$. Then there is a piecewise- G surjection $\coprod_{i=1}^n (\dot{B} + w_i) \rightarrow \bigcup_{i=1}^n (\dot{B} + v_i)$. So $A <_G \bigcup_{i=1}^n \dot{B} + v_i <_G \coprod_{i=1}^n (\dot{B} + w_i) \sim_G \dot{B}$.

Note that we never used the radius of $\dot{B}$ so $A <_G \dot{B}(0, \epsilon)$ for all $\epsilon > 0$. For the second claim, suppose $v \in \text{int}(A)$ and let ϵ be such that $v + \epsilon\dot{B} \subset A$. Now $\dot{B} <_G \epsilon\dot{B} \sim_G v + \epsilon\dot{B} <_G A$ so we are done $\square$

$\square$

Exercise 7.3. If $A \sim_G \dot{B}$ then A must be bounded but need not have non-empty interior

Definition 7.6. For $k, l \geq 1$, $kA = A \times \{1, \dots, k\} = \coprod_{i=1}^k A$ and lB is defined similarly. A map $f: kA \rightarrow lB$ is piecewise- G if there is $S \subset_f G$ such that $f(a, i) \in S \cdot a \times \{1, \dots, l\}$. We can then define $kA <_G lB$ and then $kA \sim_G lB$

Theorem 7.7. *If $kA <_G kB$ then $A <_G B$*

Proof. Suppose $f: kA \rightarrow kB$ is a piecewise- G injection. For $a \in A$ there are k elements in kA corresponding to a , and their images have labels $\Sigma(a) \subset B$ and $|\Sigma(a)| \leq k$. $\Sigma: A \rightarrow \mathcal{P}_f(B)$ is a matching problem. We are left to show that $|\Sigma(F)| \geq |F|$ for all $F \subset_f A$, then conclude by Hall. Every $a \in A$ is adjacent to k edges and every $b \in B$ is adjacent to at most k edges. Now consider the graph restricted to $F, \Sigma(F)$ and call its edges E . We know that $k|F| = |E| \leq k|\Sigma(F)|$ and so the result follows $\square$

Theorem 7.8. *If $A \subset S^2$ with non-empty interior, then $A \sim_{SO(3)} S^2$*

Proof. It suffices to show that $S^2 <_{SO(3)} A$ as $A <_G S^2$ trivially. Because $A^\circ \neq \emptyset$ and S^2 is compact, finitely many translates of A cover S^2 . Hence $S^2 <_{SO(3)} kA$ for some k . By the Hausdorff paradox, $S^2 \sim_{SO(3)} 2S^2$ so by induction, $S^2 \sim kS^2$ and so $kS^2 <_{SO(3)} kA$. The result follows from the previous theorem $\square$

Exercise 7.4. Generalise this theorem to all higher dimensions

7.5 The Banach-Ruziewicz Problem

It is fairly easy to show

Proposition 7.5. *The Lebesgue measure λ on S^{n-1} is the unique countably additive $SO(n)$ -invariant measure defined on Lebesgue sets $\mathcal{L}(S^{n-1})$*

Proof. Trivial $\square$

The Banach Ruziewicz problem is "is it the only finitely-additive one?". Banach proved that it is no for $n = 2$ (as $SO(2)$ is amenable). Margulis-Sullivan proved the problem for $n \geq 5$ (using property (T)) and Drinfeld proved it for $n = 3, 4$

Proposition 7.6. *Let μ be as in the problem for $n \geq 3$. Then μ is absolutely continuous with respect to λ (ie $\lambda(A) = 0$ implies $\mu(A) = 0$)*

Remark. The problem becomes "is the lebesgue integral the only mean on $l^\infty(S^{n-1})$ "

Proof. Let $D \subset S^{n-1}$ be a disk and suppose there are $g_1, \dots, g_n \in SO(3)$ such that $g_i(D)$ are disjoint. Then $\mu(D) \leq \frac{1}{n}$ by finite additivity. So if $\text{diam } D_i \rightarrow 0$ then $\mu(D_i) \rightarrow 0$. We know that $S^{n-1} \sim_{SO(n)} D_i$. Take A with $\lambda(A) = 0$. There is $A_i \subset D_i$ such that $A \sim_{SO(n)} A_i$. Hence as every set of measure 0 is measurable, we can apply μ to get $\mu(A) = \mu(A_i) \leq \mu(D_i) \rightarrow 0$ and we are done. $\square$

8 Random Walks

In this section Γ will be a connected d -regular graph with an origin e , and G will be a group. $(l^2(V(\Gamma)), \|\cdot\|_2)$ will be abbreviated as $(l^2(\Gamma), \|\cdot\|)$. We will also use $\|\cdot\|$ for the operator norm of $H: l^2(\Gamma) \rightarrow l^2(\Gamma)$. we write $x \sim y$ for x is a neighbour of y

8.1 Definitions and Basics

Definition 8.1. The **(simple) random walk** on Γ is the stochastic process with random variables $X_0, X_1, X_2, \dots$ where $X_0 = e$ and X_{i+1} is a vertex chosen uniformly among the d neighbours of X_i

Definition 8.2. We define the **return probability**, $r_n = \mathbb{P}[X_n = e] \in [0, 1]$

Exercise 8.1. If Γ is a tree, then $r_n = 0$ for all odd n

Definition 8.3. We define the **Markov operator** to be $M: l^2(\Gamma) \rightarrow l^2(\Gamma)$ given by $Mf(x) = \frac{1}{d} \sum_{y \sim x} f(y)$

Lemma 8.1. 1. $\|M\| \leq 1$

2. M is positive

Proof. 1. Colour the edges of Γ with colours $1, \dots, d$ so that no vertex is adjacent to two edges of the same colour. Define $M_i: l^2(\Gamma) \rightarrow l^2(\Gamma)$ by $M_i f(x) = f(y)$ for $y \sim x$ with colour i . Now $\|M_i\| = 1$ as it is induced from a permutation, and $M = \frac{1}{d} \sum_i M_i$ so the result follows from the triangle inequality

2. Trivial by definition □

Definition 8.4. Let V be a Hilbert space. $M: V \rightarrow V$ is **self-adjoint** if $\langle Mv, w \rangle = \langle v, Mw \rangle$

Exercise 8.2. On l^2 , $\|M\| = \sup_{\|v\| \leq 1} \langle Mv, v \rangle$ for self-adjoint M

Lemma 8.2. The Markov operator is self adjoint

Proof. Let $f, g \in l^2(\Gamma)$ and let a_{xy} be the indicator of the adjacency relation.

$$\begin{aligned} \langle f, Mg \rangle &= \sum_{x \in V(\Gamma)} f(x) Mg(x) \\ &= \sum_{x \in V(\Gamma)} f(x) \left(\frac{1}{d} \sum_{y \sim x} g(y) \right) \\ &= \sum_{x, y \in V(\Gamma)} \frac{1}{d} f(x) g(y) a_{xy} \end{aligned}$$

. But the same holds for $\langle MF, g \rangle$, so the result follows □

8.2 The Norm of the Markov Operator

We consider $\rho^{-1} = \limsup_{n \rightarrow \infty} (r_n)^{1/n} \leq 1$. This is strictly smaller than 1 if and only if r_n decays exponentially

Lemma 8.3. ρ is the radius of convergence of the return generating function $R(z) = \sum_{n \geq 0} r_n z^n$

Proof. See IA analysis □

Theorem 8.4. $\|M\| = \rho^{-1}$

Proof. Denote by $\delta_x \in l^2(\Gamma)$ the dirac delta, and we know $\langle \delta_x, f \rangle = f(x)$.

Claim 1: The probability that $X_n = x$ is $\langle \delta_x, M^n \delta_e \rangle = M^n \delta_e(x)$. Hence in particular $r_n = \langle \delta_e, M^n \delta_e \rangle$

Proof. We prove the claim by induction on n . If $n = 0$ the inner product is $\langle \delta_x, \delta_e \rangle = \delta_e(x)$.

For the inductive step,

$$\begin{aligned} \mathbb{P}[X_{n+1} = x] &= \sum_{y \sim x} \mathbb{P}[X_n = y] \cdot \frac{1}{d} \\ &= \sum_{y \sim x} \frac{1}{d} M^n \delta_e(y) \\ &= M^{n+1} \delta_e(x) \end{aligned}$$

and we are done □

We know

$$\begin{aligned} r_n &= \langle \delta_e, M^n \delta_e \rangle \\ &\leq \|\delta_e\| \cdot \|M^n \delta_e\| \\ &\leq \|M^n\| \\ &\leq \|M\|^n \end{aligned}$$

and so $\|M\| \geq r_n^{1/n}$, and taking the lim sup, we have $\|M\| \geq \rho^{-1}$.

Now we need a subsequence of $r_n^{1/n}$ which converges to $\|M\|$. We will show that $r_{2^k}^{1/2^k}$ works.

Claim 2: If $f \in l^2(\Gamma)$, $\|f\| \leq 1$ then

$$\|Mf\| \leq \|M^2 f\|^{1/2} \leq \dots \leq \|M^{2^k} f\|^{1/2^k}$$

Proof. If H is self adjoint then

$$\|Hf\|^2 = \langle Hf, Hf \rangle = \langle f, H^2 f \rangle \leq \|H^2 f\|$$

and so $\|Hf\| \leq \|H^2 f\|^{1/2}$ and we are done by induction □

Now

$$\begin{aligned} r_{2n} &= \langle \delta_e, M^{2n} \delta_e \rangle \\ &= \langle M^n \delta_e, M^n \delta_e \rangle \\ &= \|M^n\|^2 \end{aligned}$$

So

$$\begin{aligned} (r_{2^k})^{2^{-k}} &= \|M^{2^{k-1}} \delta_e\|^{2^{1-k}} \\ &\leq \|M^{2^k} \delta_e\|^{2^{-k}} \\ &= (r_{2^{k-1}})^{2^{k-1}} \end{aligned}$$

and so $(r_{2^k})^{2^{-k}} \rightarrow \mu \leq \|M\|$. We want to show $\mu \geq \|M\|$. Take $x \in V(\Gamma)$. As Γ is connected, there is some $n_x \in \mathbb{N}$ such that $\mathbb{P}[X_{n_x} = x] = \beta_x > 0$. Also we know $\beta_x = \langle \delta_x, M^{n_x} \delta_e \rangle$

Claim 3: $\|M^{2^k} \delta_x\| \leq \beta_x^{-1} \mu^{2^k}$

Proof. Note that $0 \leq \delta_x \leq \beta_x^{-1} M^{n_x} \delta_e$. At x we get 1 on both sides. At $y \neq x$ we get $0 \leq$ some probability times a non-negative number.

As M is positive, we get $0 \leq M^{2^k} \delta_x \leq \beta_x^{-1} M^{2^k+n_x} \delta_e$. Applying the l_2 norm, we get

$$\|M^{2^k} \delta_x\| \leq \beta_x^{-1} \|M\|^{n_x} \|M^{2^k} \delta_e\| \leq \beta_x^{-1} \|M^{2^k} \delta_e\| = \beta_x^{-1} \sqrt{r_{2^{k+1}}} \leq \beta_x^{-1} \mu^{2^k}$$

□

Now let $f \in l^2(\Gamma)$ have $\|f\| \leq 1$. We need to show $\|Mf\| \leq \mu$. By density, we may assume f has finite support S . Therefore $f = \sum_{x \in S} \alpha_x \delta_x$, $\alpha_x \in \mathbb{R}$. Let $\beta = \min_{x \in S} \beta_x > 0$. Then for all $x \in S$, we know $\|M^{2^k} \delta_x\| \leq \beta^{-1} \mu^{2^k}$ and so

$$\begin{aligned} \|Mf\| &\leq \|M^{2^k} f\|^{2^{-k}} \\ &= \left\| \sum_{x \in S} \alpha_x M^{2^k} \delta_x \right\|^{2^{-k}} \\ &\leq \left(\sum_{x \in S} |\alpha_x| \beta^{-1} \mu^{2^k} \right)^{2^{-k}} \\ &= \mu \left(\beta^{-1} \sum_{x \in S} |\alpha_x| \right)^{2^{-k}} \\ &\rightarrow \mu \end{aligned}$$

□

Our aim now is to show that $G \curvearrowright X$ is amenable if and only if $\|M_S\| = 1$, where M_S is the Markov operator associated with the random walk on the graph

we will associate with $G \curvearrowright X$.

We recall that $G \curvearrowright X$ has the R_2 property if $G \curvearrowright l^2(X)$ has almost invariant vectors, ie for all $S \subseteq_f G$, $\epsilon > 0$ there is $f \in l^2(X)$ such that $\|sf - f\| < \epsilon\|f\|$ for all $s \in S$

8.3 Cayley Graphs and Schreier Graphs

We assume $S \subseteq_f G$ is symmetric (ie if $s \in S$ then $s^{-1} \in S$)

Definition 8.5. Let $S \subseteq_f G$. The **Cayley graph** of G with respect to S is $\Gamma(G, S)$ having vertices corresponding to the elements of G and edges between g and sg for all $s \in S, g \in G$. This is non-directed as S is symmetric, and is $|S|$ regular

For a Cayley graph, we have

$$Mf(x) = \frac{1}{|S|} \sum_{s \in S} f(sx) = \frac{1}{|S|} \sum_{s \in S} s^{-1}f(x) = \frac{1}{|S|} \sum_{s \in S} sf(x)$$

Example 8.1. If we take $G = \mathbb{Z}^2$ and $S = \{(\pm 1, 0), (0, \pm 1)\}$. This has Cayley graph the 2d grid with integer coordinates

If $G = F_2$, $S = \{a^{\pm 1}, b^{\pm 1}\}$, the Cayley graph is the 4-regular tree

If $G = \mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a \rangle * \langle 1 \rangle$, $S = \{a^{\pm 1}, 1\}$, we get the 3-regular tree

Definition 8.6. Let $G \curvearrowright X$. We define the **Schreier graph**, $\Gamma(G \curvearrowright X, S)$ to have vertices corresponding to points of X and edges between x, sx for every $x \in X, s \in S$

We again have $Mf = \frac{1}{|S|} \sum_{s \in S} sf$ by the same methods

8.4 Kesten's Characterisation

Theorem 8.5. Given $S \subseteq_f G$ which is symmetric and M_S the Markov operator associated to the simple random walk on the graph $\Gamma(G \curvearrowright X, S)$ starting at $x_0 \in X$ then $G \curvearrowright X$ is amenable if and only if $\|M_S\| = 1$ for every such S

Proof. We prove that $G \curvearrowright X$ has the R_2 property if and only if $\|M_S\| = 1$ for every symmetric $S \subseteq_f G$.

We may consider S symmetric on the left hand side as if $\bar{S} = S \cup S^{-1}$, then $(\bar{S}, \epsilon)$ -invariance implies (S, ϵ) -invariance

In the forwards direction, assume $G \curvearrowright X$ satisfies R_2 and let $\epsilon > 0$. Then there is $f \in l^2(X)$ that is (S, ϵ) -invariant, ie $\|sf - f\| < \epsilon\|f\|$ for all $s \in S$. By summing, we see

$$\left\| \sum_{s \in S} sf - f \right\| \leq \sum_{s \in S} \|sf - f\| < |S|\epsilon \cdot \|f\|$$

The first term is equal to

$$\begin{aligned} ||(|S|f - \sum_{s \in S} sf)|| &\geq |S| \cdot ||f|| - ||\sum_{s \in S} sf|| \\ &= |S|(||f|| - ||M_S f||) \end{aligned}$$

We have

$$|S| \cdot ||f|| - |S|\epsilon \cdot ||f|| < |S| \cdot ||M_S f||$$

and so get $(1 - \epsilon)||f|| < ||M_S f||$. As ϵ was arbitrary, $||M_S|| \geq 1$. We already know $||M_S|| \leq 1$, and so $||M_S|| = 1$

In the backwards direction, assume that $||M_S|| = 1$, $\epsilon > 0$ and we want to show that there is some $f \in l^2(X)$ that is (S, ϵ) -invariant.

We know that $||M_S|| = \sup_{||f||=1} \langle M_S f, f \rangle$. We set $\delta = \frac{\epsilon^2}{2|S|}$. We know there is some $f \in l^2(X)$ such that

$$1 - \delta < \langle M_S f, f \rangle = \frac{1}{|S|} \sum_{s \in S} \langle sf, f \rangle$$

Recall that for inner products we have

$$||v - w||^2 = \langle v - w, v - w \rangle = ||v||^2 + ||w||^2 - 2\langle v, w \rangle$$

If we take $v = sf, w = f$ then

$$||sf - f||^2 = 2 - 2\langle sf, f \rangle$$

We deduce that $\langle sf, f \rangle = 1 - \frac{||sf - f||^2}{2}$. Therefore

$$\begin{aligned} \frac{1}{|S|} \sum_{s \in S} \langle sf, f \rangle &= \frac{1}{|S|} \sum_{s \in S} \left(1 - \frac{||sf - f||^2}{2} \right) \\ &= 1 - \frac{1}{|S|} \sum_{s \in S} \frac{||sf - f||^2}{2} \end{aligned}$$

We know that

$$1 - \delta < 1 - \frac{1}{|S|} \sum_{s \in S} \frac{||sf - f||^2}{2}$$

and hence

$$\delta > \frac{1}{|S|} \sum_{s \in S} \frac{||sf - f||^2}{2}$$

and so by the definition of δ ,

$$\epsilon^2 > \sum_{s \in S} ||sf - f||^2$$

So in particular, $\epsilon^2 > ||sf - f||^2$ for all S , and taking square roots we are done $\square$

Exercise 8.3. Show that $G \circ X$ is amenable if and only if $\|M_S\| = 1$ for a generating set S

Exercise 8.4. $\|M_S\| = 1$ if and only if, for all $\epsilon > 0$, there is some $f \in \ell^2(X)$ such that

$$\left\| \sum_{s \in S} sf \right\| \geq |S|(1 - \epsilon) \cdot \|f\|$$

8.5 Random Walks on Trees

We consider the ‘opposite’ case to amenability: trees. Recall that trees are the Cayley graphs of free groups, which are intuitively as far as possible from being amenable. It would stand to reason that these are also the ones furthest from having a Markov operator of norm 1. This also makes sense in the interpretation of $\|M\|$ as the rate of exponential decay of r_n , as the d -regular tree has the smallest probability of returning to the origin, as it has no loops. More precisely:

Lemma 8.6. *Let Γ be a connected d -regular graph and T the d -regular tree, with Markov operators M_Γ, M_T . Then $\|M_T\| \leq \|M_\Gamma\|$*

Proof. T is the unique simply connected d -regular graph, so is the universal covering space of Γ . Denote the covering map by $\pi: T \rightarrow \Gamma$ which maps to origins to each other. Then π maps the simple random walk on T to that on Γ , so $r_n^T \leq r_n^\Gamma$. This means that

$$\|M_T\| = \limsup (r_n^T)^{1/n} \leq \limsup (r_n^\Gamma)^{1/n} = \|M_\Gamma\|$$

□

Theorem 8.7. *Let $d \geq 3$ and suppose T is the d -regular tree with Markov operator M . Then*

$$\|M\| = \frac{2\sqrt{d-1}}{d}$$

(and actually this characterises trees, but we will not prove this)

Proof. Let $R(z) = \sum_{n=1}^{\infty} r_n z^n$ and let ρ be its radius of convergence. We know $\|M\| = \rho^{-1}$ so we need to show $\rho = \frac{d}{2\sqrt{d-1}}$. We define q_n to be the probability of returning to e for the first time, at time n , and set $q_0 = 0$.

Let $Q(z) = \sum_{n=1}^{\infty} q_n z^n$. Now $r_n = \sum_{i=0}^n q_i r_{n-i}$ so $R(z) = 1 + Q(z)R(z)$, or in other words

$$R(z) = \frac{1}{1 - Q(z)}$$

Fix $y \sim e$ and let t_n be the probability that the simple random walk starting at y arrives at e for the first time after n steps. Now $q_n = \frac{1}{d} \sum_{y \sim e} t_{n-1} = t_{n-1}$ so if we set $T(z) = \sum_{n=0}^{\infty} t_n z^n$, then $Q(z) = zT(z)$ and so

$$R(z) = \frac{1}{1 - zT(z)}$$

Define $T_2(z)$ with some y' a distance 2 away from e . Then

$$(t_2)_n = \sum_{i+j=n} t_i t_j$$

so $T_2(z) = T(z)^2$. By considering the neighbourhood of distance 2,

$$T(z) = \frac{1}{d}z + \frac{d-1}{d}zT(z)^2$$

and so

$$T(z) = \frac{d \pm \sqrt{d^2 - 4(d-1)z^2}}{2(d-1)z}$$

The denominator is 0 if and only if z is, and so this makes sense if $z \neq 0$ and we are in the domain of convergence of T . We consider the sign. For $z \in (0, 1]$, $T(z)$ is the probability that $Y_n = e$ for some n , where Y_n is the non-simple random walk where at each step, you move to a neighbour with probability z , and stop with probability $1 - z$, so $T(z) \in [0, 1]$. But the positive sign gives a value larger than 1, so it is the negative sign, and

$$T(z) = \frac{d - \sqrt{d^2 - 4(d-1)z^2}}{2(d-1)z}$$

We know $z = 0$ is a zero of the denominator of order 1, and of the numerator of order 2, so setting $T(0) = 0$ (ie $t_0 = 0$), this converges on an open around 0. We also know this is analytic outside where the value in the square root is 0, so the radius of convergence is $\frac{d}{2\sqrt{d-1}}$. But

$$R(z) = \frac{1}{1 - zT(z)}$$

so as this is analytic where both $1 - zT(z) \neq 0$, and $T(z)$ is analytic, we deduce that

$$\begin{aligned} 1 &= zT(z) \\ \sqrt{d^2 - 4(d-1)z} &= 2 - d \\ z^2 &= 1 \end{aligned}$$

so this never happens in the range of convergence of T , and so R has the correct radius of convergence $\square$

9 Tilings and the Rank Gradient

9.1 Tiles on Groups

Remark. We say $A_1 \dots A_k \stackrel{!}{=} B$ if every $b \in B$ is uniquely writable in the form $a_1 \dots a_k$ for $a_i \in A_i$

Example 9.1. If $H \leq G$ and L is a left transversal, then $G \stackrel{!}{=} LH$

Definition 9.1. $T \subseteq_f G$ is a **tile** of G if there is some $X \subseteq G$ such that $G \stackrel{!}{=} XT$

G is **monotileable** if every $S \subseteq_f G$ is contained in a tile

Example 9.2. $\{1, \dots, n\}^d \subseteq \mathbb{Z}^d$ is a tile, so $\mathbb{Z}^d$ is monotileable

A question arises: Is every group monotileable?

Theorem 9.1. *The class of monotileable groups is closed under*

1. *Extensions*

2. *Directed unions*

3. *Fully-Residually (ie G such that for all $S \subseteq_f G$, there is a map $\pi: G \rightarrow Q$ such that $\pi|_S$ is injective and Q is monotileable)*

Proof. If $S \subseteq_f G$ and $\pi: G \rightarrow Q$ is as above, let $S_Q = \pi(S)$ and T_Q be a tile of Q containing S_Q . Then $Q \stackrel{!}{=} X_Q T_Q$. Lift X_Q to $X \subseteq G$ and T_Q to $T \subseteq G$. Then $G \stackrel{!}{=} (NX)T$ for $N = \ker \pi$, so T is a tile. The other two cases are easy $\square$

Corollary 9.1.1. *Elementary amenable groups are monotileable*

Proof. Trivial $\square$

Corollary 9.1.2. *Residually finite implies monotileable*

Proof. Trivial $\square$

9.2 Følner tiles

We have the following fundamental result in Ergodic theory

Theorem 9.2. *(Rokhlin Lemma) Take X a probability space and let $T: X \rightarrow X$ be an aperiodic measure preserving. Then for every $n \geq 1, \epsilon > 0$ there is $S \subseteq X$ such that $S, TS, T^2S, \dots, T^n S$ (a **Rokhlin tower**) are disjoint and their complement has measure at most ϵ*

Proof. No Proof $\square$

Ornstein-Weiss realised that the key is that $1, \dots, n$ is a tile of $\mathbb{Z}$ which is almost invariant

Definition 9.2. Let G be an amenable group. We say G has **Følner tiles** if for all $S \subseteq_f G$ and $\epsilon > 0$ there is a tile $T \subseteq_f G$ which is (S, ϵ) invariant

Do all (monotilable) amenable groups have Følner tiles? This is still open. Ornstein Weiss get around this by introducing ϵ -quasitiles

Theorem 9.3. *Directed unions (or extensions) of groups with Følner tiles have Følner tiles*

Hence elementary amenable groups have Følner tiles

Proof. Trivial $\square$

9.3 Residually Finite Amenable Groups

Theorem 9.4. (Weiss) *Countable residually finite amenable groups have Følner tiles. More precisely, let $S \subseteq_f G$ and $\epsilon > 0$ and $G = G_0 > G_1 > \dots$ be a residual chain of G (ie G_n is normal in G of finite index with $\bigcap G_n = \{1\}$). Then for all sufficiently large n , there is a set of transversals T_n for G_n that are (S, ϵ) -invariant*

Proof. (Sketch) We need to show that if Q is a ‘large enough’ finite quotient of G , then there is an (S, ϵ) -invariant transversal.

Choose δ very small and n such that $(1 - \delta/2)^n < \epsilon/100$. Further choose $S \subseteq F_1 \subseteq_f G$ such that F_1 is $(S \cdot S^{-1}, \delta)$ -invariant and inductively choose F_i such that $F_{i-1} \subseteq F_i \subseteq G$ which is $(F_{i-1} \cdot F_{i-1}^{-1}, \delta)$ -invariant for $i \leq n$. There is some $\pi: G \rightarrow Q$ where Q is finite such that $\pi|_{F_n}$ is injective. Now $\{\pi(F_n)q \mid q \in Q\}$ is an even covering, so we can find a subcollection $\{\pi(F_n)u \mid u \in U\}$ which is δ -disjoint (ie the intersections are at most size δ of each $\pi(F_n)$) and cover a $\delta/2$ portion of Q . Now put these $F_n\pi^{-1}(u)$ in the transversal. Cover the rest similarly with F_{n-1} and repeat for all F_i .

Repeating this, we get $1 - (1 - \delta/2)^n < \epsilon/100$ of Q is covered. Complete the set of transversals arbitrarily. Each of the balls F_i is almost invariant under S , and so the transversal is (as $\epsilon/100$ is small) $\square$

9.4 The Rank Gradient

Definition 9.3. The **rank** of G is $\text{rk } G$, the minimal size of a generating set. We define $\text{rk } 1 = 0$

Example 9.3. $\text{rk } \mathbb{Z}^n = n$

$\text{rk } \Gamma \geq \text{rk } \Lambda$ if Γ surjects onto Λ (and hence $\text{rk } F_n = n$)

Definition 9.4. Let G be a finitely generated residually finite group and let $\mathcal{G} = (G = G_0 > G_1 > \dots)$ a residual chain. The **rank gradient** is

$$\text{rg}(G, \mathcal{G}) = \lim_{n \rightarrow \infty} \frac{\text{rk}(G_n) - 1}{[G : G_n]}$$

Proposition 9.1. (Nielsen Schreier) *If $H \leq G$ has finite index, then*

$$\text{rk}(H) - 1 \leq [G : H](\text{rk}(G) - 1)$$

with equality if G is free

Proof. Trivial $\square$

Corollary 9.4.1. $\text{rg}(F_n, \mathcal{G}) = n - 1$ for all residual chains $\mathcal{G}$

Proof. Trivial $\square$

Remark. It is open in general if $\text{rg}(G, \mathcal{G})$ depends on $\mathcal{G}$

Lemma 9.5. *Let $S \subseteq G$ be a generating set and $H \leq G$ finite index with transversal T including 1. Then if $s \in S, t \in T$, $st \stackrel{!}{=} \tau_{st}h_{st}$ where $h_{st} \in H, \tau_{st} \in T$. We have $H = \langle h_{st} \mid s \in S, t \in T \rangle$*

Proof. Let $h \in H$ and write $h = s_1 \dots s_n$. Now $s_n = s_n \cdot 1 = \tau_{s_n 1}h_{s_n 1}$, so $h = s_1 \dots s_{n-1} \tau_{s_n 1}h_{s_n 1}$. Repeating, we get $h = \tau h_1 \dots h_n$ with h_i in the given generating set. But then $\tau \in H$, so $\tau = 1$ and we are done $\square$

Theorem 9.6. (Lackenby) *Let G be finitely generated and residually finite. Then $\text{rg}(G, \mathcal{G}) = 0$ for all normal residual chains $\mathcal{G}$*

Proof. Fix $S \subseteq_f G$ a generating set of size $\text{rk } G$. For sufficiently large n , by Weiss, there is a transversal T for G_n such that $|ST| < (1 + \epsilon)|T|$. Assume that $1 \in T$. Then $|ST \setminus T| < \epsilon|T|$. If $st \in T$ then $h_{st} = 1, \tau_{st} = st$ so st contributes a non-identity element to $\{h_{st} \mid s \in S, t \in T\}$ only if $st \in ST \setminus T$. So

$$\text{rk}(G_n) \leq |\{h_{st} \mid s \in S, t \in T\}| \leq |ST \setminus T| < \epsilon|T|$$

Hence

$$\frac{\text{rk}(G_n)}{[G : G_n]} = \frac{\text{rk}(G_n)}{|T|} \leq \epsilon$$

and we are done by taking limits. $\square$

Definition 9.5. If p is a prime, we define $\text{rk}_p(G) = \dim_{\mathbb{F}_p} H(G; \mathbb{F}_p) = \text{rk } G/[G, G]G^p$. We also define

$$\text{rg}_p(G, \mathcal{G}) = \lim \frac{\text{rk}_p(G_n) - 1}{[G : G_n]}$$

.

Remark. Note that $\text{rk } G \geq \text{rk}_p(G)$, so $\text{rg}(G, \mathcal{G}) \geq \text{rg}_p(G, \mathcal{G})$

Corollary 9.6.1. *With the assumptions of the theorem above, $\text{rg}_p(G, \mathcal{G}) = 0$*

Proof. Trivial $\square$

10 Non-Amenable Groups with no Free Subgroups

Fix a prime p throughout

Theorem 10.1. *There is a finitely generated residually finite p -torsion non-amenable group*

10.1 p -Power Deficiency

Definition 10.1. Let $f \in F(X)$. Define $\nu_p(f) = \max n \geq 0 \{n \mid f = g^{p^n}\}$. Let $\langle X \mid V \rangle$ be a presentation with X finite. Its **p -power deficiency** is

$$\text{def}_p \langle X \mid R \rangle = |X| - 1 - \sum_{r \in R} p^{-\nu_p(r)}$$

The **p -power deficiency** of a group is $\text{def}_p(G) = \sup \text{def}_p \langle X \mid R \rangle$ where the supremum is taken over $G = \langle X \mid R \rangle$

Proposition 10.1. *There is a finitely generated p -torsion group G with deficiency $\text{def}_p G > 0$*

Proof. Suppose $|X| \geq 2$. Then enumerate $F(X)$ as $f_1, f_2, \dots$ and choose $k_1, k_2, \dots \in \mathbb{N}$. Let $R = \{f_i^{p^{k_i}} \mid i \in \mathbb{N}\}$. Note $\nu_p(f_i^{p^{k_i}}) \geq k_i$. Now $\text{def}_p \langle X \mid R \rangle = |X| - 1 - \sum_{r \in R} p^{-\nu_p(r)} \geq 1 - \sum_{i \geq 1} p^{-k_i} \geq 0$ for large k_i . Then if $G = \langle X \mid R \rangle$, $g \in G$ is the image of some $f_i \in F(X)$, and hence g is p -torsion as $f_i^{p^{k_i}} \in R$ $\square$

10.2 def_p vs rg_p

Lemma 10.2. *For a group G , $\text{def}_p G \leq \text{rk}_p(G) - 1$*

Proof. Suppose $G = \langle X \mid R \rangle$. If $R_0 \subseteq R$ is the set of relations with $\nu_p = 0$. Then

$$\text{rk}_p G = \text{rk}_p \langle X \mid R \rangle = \text{rk}_p \langle X \mid R_0 \rangle \geq |X| - |R_0|$$

where the last inequality follows from linear algebra over $\mathbb{F}_p$. Therefore $\text{def}_p \langle X \mid R \rangle \leq |X| - 1 - |R_0| \leq \text{rk}_p G - 1$. Taking the supremum over all presentations, we are done $\square$

Theorem 10.3. *Let $H \leq G$ be normal of p -power index. Then*

$$\text{def}_p H \geq [G : H] \text{def}_p G$$

Proof. It is enough to prove it for H with index p by induction. We want to show that $\text{def}_p H \geq p \cdot \text{def}_p G$ in this case. Write $G = \langle X \mid R \rangle$. We have

$$\begin{array}{ccc} N & \xrightarrow{p} & F \\ \downarrow & & \downarrow \\ H & \xrightarrow{p} & G \end{array}$$

By Nielsen Schreier, there is some $Y \subseteq F$ generating N such that

$$|Y| - 1 = p \cdot (|X| - 1)$$

We need some $S \subseteq F$ such that $\langle\langle S \rangle\rangle_N = \langle\langle R \rangle\rangle_F$ and

$$\text{def}_p \langle Y \mid S \rangle \geq p \cdot \text{def}_p \langle X \mid R \rangle$$

If $f \in F$, denote by ${}^N f$ its N -conjugates and ${}^F f$ its F conjugates. Let $\nu_{p,N}(f) = \max\{n \mid f = g^{p^n}, g \in N\}$. Now

$$\nu_{p,F}(f) \geq \nu_{p,N}(f) \geq \nu_{p,F} - 1$$

as if $g \in F$ then $g^p \in N$.

We claim that if $f \in N$, one of the following holds:

1. ${}^N f = {}^F f$
2. There are $f_1, \dots, f_p \in {}^F f$ such that ${}^F f = \coprod_{i=1}^p {}^N f_i$, and in this case we have $\nu_{p,N}(f_i) = \nu_{p,F}(f)$

Proof. Let $N \triangleleft {}^F f$. Then either N has 1 or p orbits as F acts transitively. Everything except the very last claim is trivial. Having p orbits is equivalent to $C_F(f) \leq N$, which implies that $f = g^{p^n}$, then g commutes with f so $g \in N$ so the last claim follows $\square$

We build S as follows. For $r \in R \subseteq N$

1. If ${}^N r = {}^F r$, add r to S
2. Else, ${}^F r = \coprod_{i=1}^p {}^N r_i$ so add all the r_i to S

Then

1. $\nu_{p,N}(r) \geq \nu_{p,F}(r) - 1$, so $p^{-\nu_{p,N}(r)} \leq p \cdot p^{-\nu_{p,F}(r)}$
2. $\nu_{p,N}(r_i) = \nu_{p,F}(r)$ so $\sum_{i=1}^p p^{-\nu_{p,N}(r_i)} = p \cdot p^{-\nu_{p,N}(r)}$

Now

$$\begin{aligned} \text{def}_p \langle Y \mid S \rangle &= |Y| - 1 - \sum_{s \in S} p^{-\nu_{p,N}(s)} \\ &\geq p \left(|X| - 1 - \sum_{r \in R} p^{-\nu_{p,N}(r)} \right) \\ &= p \cdot \text{def}_p \langle X \mid R \rangle \end{aligned}$$

So the result follows $\square$

Corollary 10.3.1. *Let G be a finitely generated group and Q its largest residually finite quotient (ie the quotient by the intersection of all finite index subgroups). Let $\mathcal{Q}$ be a normal residual chain of Q such that every subgroup in $\mathcal{Q}$ has p -power index. Then $\text{rg}_p(Q, \mathcal{Q}) \geq \text{def}_p G$*

Proof. We have a short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

and we say $\mathcal{Q} = (Q = Q_0 \geq Q_1 \geq \dots)$ and lift this to $\mathcal{G} = (G = G_0 \geq G_1 \geq \dots)$. Note that $[G : G_i] = [Q : Q_i]$. Now $\ker(G_i \rightarrow H_1(G, \mathbb{F}_p))$ has finite index in G ,

so N is contained in it. Therefore $G_i \rightarrow H_1(G_i, \mathbb{F}_p)$ factors through Q_i . We see that $\text{rk}_p G_i = \text{rk}_p Q_i$, and so

$$\frac{\text{rk}_p Q_i - 1}{[Q : Q_i]} = \frac{\text{rk}_p G_i - 1}{[G : G_i]} \geq \frac{\text{def}_p(G_i)}{[G : G_i]} \geq \text{def}_p(G)$$

□

Proof. (Of theorem 10.1) Let G be a finitely generated p -torsion group with $\text{def}_p G > 0$ and let Q be its maximal residually finite quotient. Then Q is also p -torsion, and so every finite index subgroup has p -power index. So for every normal residual chain $\mathcal{Q}$, $\text{rg}_p(Q, \mathcal{Q}) \geq \text{def}_p G \geq 0$ so Q is non-amenable. □

Exercise 10.1. Any G with positive $\text{def}_p G$ has a quotient Q as above, and so is non-amenable