

Analysis of Functions

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1 Review of Measure Theory

1. Consider a measure space $(E, \mathcal{E}, \mu)$. E is a set. $\mathcal{E}$ is a σ -field (-algebra) of subsets of E , and $\mu: \mathcal{E} \rightarrow [0, \infty]$ a measure
2. If A is a collection of subsets of E then the σ -field $\sigma(A) = \bigcap_{\mathcal{E} \supset A, \mathcal{E} \text{ a } \sigma \text{ field}} \mathcal{E}$ is the σ -algebra generated by A .
If E is a topological space, $\mathcal{A} = \{A \subset E, A \text{ open}\}$ generates the Borel σ -field $\mathcal{B}(E) = \sigma(\mathcal{A})$.

If $E = \mathbb{R}^n$ with the product topology, we write $\mathcal{B} = \mathcal{B}(\mathbb{R}^n)$.
On $\mathcal{B}$ there is a unique measure $\mu = dx$ such that

$$\mu((a_1, b_1) \times \cdots \times (a_n, b_n)) = (b_1 - a_1) \cdots (b_n - a_n)$$

for all $a_i < b_i$. The volume of $A \in \mathcal{B}$ is denoted by $|A|$.

Note μ is translation invariant, ie $\mu(A + z) = \mu(A)$ for all $A \in \mathcal{B}, z \in \mathbb{R}^n$.
Moreover μ is inner regular, ie $\mu(A) = \sup\{\mu(K) \mid K \subset A, K \text{ compact}\}$.
Further recall that μ extends to the μ completion of $\mathcal{B}$ which is defined by $\mathcal{M}_\mu = \{B \cup A \mid B \in \mathcal{B}, A \subset N, \mu(N) = 0\}$. These are the lebesgue measurable sets

3. Let $(E, \mathcal{E}), (F, \mathcal{F})$ be measure spaces.
 $f: E \rightarrow F$ is measurable if $f^{-1}(A) \in \mathcal{E}$ for all $A \in \mathcal{F}$.
 If $(F, \mathcal{F}) = (\mathbb{R}, \mathcal{B})$ or $(F, \mathcal{F}) = (\mathbb{C}, \mathcal{B})$ then we say f is borel and this extends to maps possibly taking values $\pm\infty$ if $f^{-1}(\{\pm\infty\}) \in \mathcal{E}$.
 If such f takes values in $[0, \infty]$, we write $f \geq 0$
4. A function $f: (E, \mathcal{E}, \mu) \rightarrow \mathbb{R}$ is **simple** if it is of the form $f = \sum_{i=1}^N a_i 1_{A_i}$ for $a_i \in \mathbb{R}, A_i \in \mathcal{E}, N \in \mathbb{N}$.
 For such f we define $\mu(f) = \int_E f d\mu = \sum_{i=1}^N a_i \mu(A_i)$. For $f \geq 0$ measurable we define its μ -integral as $\mu(f) = \sup\{\mu(g) \mid 0 \leq g \leq f \text{ simple}\}$. This may be ∞ .
 For general $f: E \rightarrow \mathbb{R}$ measurable, f is integrable if $\mu(|f|) = \int_E |f| d\mu < \infty$ in which case we define $\int_E f d\mu = \int_E f^+ d\mu + \int_E f^- d\mu$

Theorem 1.1. (MCT) Let $f_n \geq 0$ measurable monotonically converging to f almost everywhere then $\int_E f_n d\mu$ converges monotonically to $\int_E f d\mu$

Proof. See Probability and Measure □

Theorem 1.2. (DCT) Let f_n measurable such that f_n converges almost everywhere to f and such that $|f_n| \leq g$ almost everywhere for some integrable g . Then f is integrable and $\int_E f_n d\mu \rightarrow \int_E f d\mu$

Proof. See Probability and Measure □

1.1 L^p -spaces and their approximation

For $f: (E, \mathcal{E}, \mu) \rightarrow \mathbb{R}$ measurable we define $\|f\|_{L^p} = (\int_E |f|^p d\mu)^{1/p}, 1 \leq p \leq \infty$ and $\|f\|_{L^\infty} = \text{ess. sup } |f| = \inf\{\lambda > 0 \mid |f| \leq \lambda \text{ a.e.}\}$.

$L^p(E, \mu) = \{f: E \rightarrow \mathbb{R} \text{ measurable} \mid \|f\|_{L^p} < \infty\}$

We write L^p for $L^p(\mathbb{R}^n, dx)$ or when no confusion will arise.

Theorem 1.3. (Riesz-Fischer) Identifying $f \sim g$ for $f = g$ almost everywhere, the spaces $L^p, 1 \leq p \leq \infty$ are Banach spaces

Proof. See Probability and Measure □

Recall the spaces $C^k(\mathbb{R}^n)$ consisting of all functions on $\mathbb{R}^n$ with continuous partial derivatives of all orders up to k and $C^\infty = \bigcap_{k>0} C^k(\mathbb{R}^n)$. Further denote by $C_c(\mathbb{R}^n)$, $C_c^k(\mathbb{R}^n)$, $C_c^\infty(\mathbb{R}^n)$ the linear subspaces of compactly supported functions. These spaces consist of bounded integrable functions.

Remark. $C_c^\infty(\mathbb{R}^n)$ is non-empty by $\psi = e^{\frac{-1}{1-|x|^2}}$ for $|x| < 1$ and 0 otherwise

Theorem 1.4. *The space $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, dx)$ for $1 \leq p < \infty$*

Proof. Proof deferred □

Remark. False for $p = \infty$

We admit the following lemma from Probability and Measure

Lemma 1.5. *$C_c(\mathbb{R}^n)$ is dense in L^p for $1 \leq p < \infty$*

Proof. See Probability and Measure □

We recall that the convolution of f, g (when defined) is given by

$$f * g(x) = \int_{\mathbb{R}^n} g(x-y)f(y)dy, \text{ for } x \in \mathbb{R}^n$$

Basic properties from probability and measure include

Lemma 1.6. *Let $f, g, h \in C_c(\mathbb{R}^n)$, then*

1. $f * g = g * f$
2. $(f * g) * h = f * (g * h)$
3. $\int_{\mathbb{R}^n} f * g(x)dx = \int_{\mathbb{R}^n} f(x)dx \cdot \int_{\mathbb{R}^n} g(x)dx$

Proof. See Probability and Measure □

Recall that a multi-index $\alpha \in \mathbb{Z}_+^n$ is written $\alpha = (\alpha_1, \dots, \alpha_n)$ with order $|\alpha| = \alpha_1 + \dots + \alpha_n$ and we set $\alpha! = \alpha_1! \dots \alpha_n!$.

For $x \in \mathbb{R}^n$, we write $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ so that we get partial differential operators $\mathcal{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{dx_1^{\alpha_1} \dots dx_n^{\alpha_n}}$. $\mathcal{D}_i$ is still defined as usual.

Theorem 1.7. *Let $f \in L_{loc}^1$ (ie $f1_K \in L^1$ for any $K \subset \mathbb{R}^n$ compact) and $g \in C_c^k(\mathbb{R}^n)$. Then $f * g \in C^k(\mathbb{R}^n)$ and $\forall 0 \leq |\alpha| \leq k$ we have*

$$\mathcal{D}^\alpha(f * g) = f * (\mathcal{D}^\alpha g)$$

Proof. Recall the translation operator $\tau_z(h) = h(\cdot - z)$ for $z \in \mathbb{R}^n$. Then for $u \in \mathbb{R}^n$,

$$\tau_u(f * g) = \int_{\mathbb{R}^n} g(x-u-y)f(y)dy$$

Now $\tau_u(g(x-y)) \leq \|g\|_\infty 1_K$ for all $|u| \leq 1$ where K is compact and depends on x, g but not u , so that $\|g\|_\infty 1_K |f|$ gives an integrable upper bound for the integrand. Since $g(x-u-y) \rightarrow g(x-y)$ as $u \rightarrow 0$ for all y , by DCT, we conclude $\tau_u(f * g) \rightarrow f * g(x)$ so $f * g$ is continuous ($k = 0$ is true). For $k = 1$, we define difference operators for all basis vectors e_i and let $h \in \mathbb{R}^1$.

$$\Delta_h^i g(x) = \frac{g(z + he_i) - g(z)}{h} \rightarrow \mathcal{D}_i g(z)$$

as $g \in C^1$. We can write $\Delta_h^1(f * g)(x) = \int_{\mathbb{R}^n} (\Delta_h^i g(x-y)) f(y) dy$. Now we have the pointwise limit of the inside of the integral. We use the mean value theorem along the line segment connecting $x-y$ and $x-y+he_i$ so we can bound $|\Delta_h^i g(x-y)| \leq \|\mathcal{D}_i g\|_\infty 1_K$ for some compact $K \subset \mathbb{R}^n$ depending only on x, y for $|h| \leq 1$. Thus as before the $k = 1$ result follows, with the expression being continuous by $k = 0$. The total result follows by trivial induction $\square$

Proposition 1.1. (*Continuity of Translation in L^p*). Let $1 \leq p < \infty$ and τ_z as before. Then $\|\tau_z f - f\|_{L^p} \rightarrow 0$ as $z \rightarrow 0$ for any $f \in L^p$

Proof. The result holds for $g \in C_c(\mathbb{R}^n)$ since

$$\|\tau_z g - g\|_{L^p}^p = \int_{\mathbb{R}^n} |g(x-z) - g(x)|^p dx$$

and this is bounded by $|2g|_\infty^p 1_K$ for some compact K depending only on g for $z \leq 1$ so we can apply DCT

Since $C_c(\mathbb{R}^n)$ is dense in L^p , we can choose for any $f \in L^p, \epsilon > 0$ some $f \in C_c$ such that $\|f - g\|_{L^p} < \epsilon/3$.

Now $\|\tau_z f - \tau_z g\|_{L^p} = \|\tau_z(f - g)\|_{L^p} = \|f - g\|_{L^p} < \epsilon/3$ so we are done as

$$\|\tau_z f - f\|_{L^p} \leq \|\tau_z f - \tau_z g\|_{L^p} + \|\tau_z g - g\|_{L^p} + \|f - g\|_{L^p} < \epsilon$$

for sufficiently small z , so we are done $\square$

Theorem 1.8. (*Minkowski's inequality for integrals*) Let $F: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ measurable and either non-negative or $dx \otimes dx$ integrable. Then

$$\left\| \int_{\mathbb{R}^n} F(x, \cdot) dx \right\|_{L^p} \leq \int_{\mathbb{R}^n} \|F(x, \cdot)\|_{L^p} dx$$

Proof. Example sheet $\square$

Theorem 1.9. (*Mollification/Approximate identity*) Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be non-negative with $\int_{\mathbb{R}^n} \phi(x) dx = 1$ and define $\pi_\epsilon = \epsilon^{-n} \phi(\frac{\cdot}{\epsilon}), \epsilon > 0$.

Then for $1 \leq p < \infty$ and $f \in L^p$, we have

$$\|f - \phi_\epsilon * f\|_{L^p} \rightarrow 0$$

as $\epsilon \rightarrow 0$

Proof. $|\phi_\epsilon * f(x) - f(x)| = |\int_{\mathbb{R}^n} f(x-y)\epsilon^{-n}\phi(\frac{y}{\epsilon})dy - f(x)| = |\int_{\mathbb{R}^n} (f(x-\epsilon u) - f(x))\phi(u)du| \leq \int_{\mathbb{R}^n} |(f(x-\epsilon u) - f(x))\phi(u)|du.$

Applying Minkowski's inequality for integrals, we get

$$\|\phi_\epsilon * f - f\|_{L^p} \leq \int_{\mathbb{R}^n} \|\tau_{\epsilon u} f - f\|_{L^p} \phi(u) du$$

. By dominated convergence (Using $2\|f\|_{L^p}$) we are done. $\square$

In particular, since $C_c(\mathbb{R}^n)$ is dense in L^p and since $\{\phi_\epsilon * f \mid f \in C_c(\mathbb{R}^n)\} \subset C_c^\infty(\mathbb{R}^n)$, we have proven 1.4.

1.2 Lebesgue's Differentiation Theorem

We recall the fundamental theorem of calculus. For $f: \mathbb{R} \rightarrow \mathbb{R}$ continuous, $\lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} (f(y) - f(x))dy = 0$ for all $x \in \mathbb{R}$.

In the Lebesgue theory, we have:

Theorem 1.10. *Let $f \in L^1(\mathbb{R}^n)$, $B_r(x) = \{y \in \mathbb{R}^n \mid |y - x| < r\}$ then*

$$\lim_{r \rightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| dy = 0$$

for almost all $x \in \mathbb{R}^n$

Remark. The set of points $A = \{x \mid \text{Lebesgue Differentiation holds}\}$ are called **Lebesgue points** of f and $\mu(A^c) = 0$

For the proof we introduce

Definition 1.1. For $f \in L^1$, $x \in \mathbb{R}^n$, set $Mf(x)$, the **Hardy-Littlewood maximal function** to be $Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy$

Lemma 1.11. *For $f \in L^1$, Mf maps $\mathbb{R}^n \rightarrow \mathbb{R}$, is borel and*

$$|\{x \mid Mf(x) > \lambda\}| \leq \frac{3^n}{\lambda} \|f\|_{L^1}$$

for all $\lambda > 0$

Proof. Define $A_\lambda = \{x \mid Mf(x) > \lambda\}$. If $x \in A_\lambda$, there is r_x such that $\frac{1}{|B_{r_x}(x)|} \int_{B_{r_x}(x)} |f(y)| dy > \lambda$. If $x_n \in A_\lambda$ has $x_n \rightarrow x$ in $\mathbb{R}^n$ then we have $\frac{1}{|B_{r_{x_n}}(x_n)|} \int_{B_{r_{x_n}}(x_n)} |f(y)| dy \leq \lambda$ by definition.

The left hand side is $\frac{1}{|B_{r_{x_n}}(x_n)|} \int_{\mathbb{R}^n} 1_{B_{r_{x_n}}(x_n)} |f(y)| dy \xrightarrow{\text{DCT}} \frac{1}{|B_{r_x}(x)|} \int_{\mathbb{R}^n} 1_{B_{r_x}(x)} |f(y)| dy$ contradiction, so A_λ is open for all λ , so Mf is measurable

To prove the bound on Mf we use inner regularity of the lebesgue measure and take an arbitrary $K \subset A_\lambda$ compact (if there is no such set, $\mu(A_\lambda) = 0$ so done). K admits the open cover $\{B_{r_x}(x) \mid x \in A_\lambda\}$ so by compactness we get a finite subcover $B_1, \dots, B_n$ of such balls. By Wiener's covering lemma (Example sheet

1), we can find a further subcollection of disjoint such $B_1, \dots, B_k$ such that $|K| \leq 3^n \sum_{i=1}^k |B_i| = \frac{3^n}{\lambda} \sum_{i=1}^k \lambda |B_i| \leq \frac{3^n}{\lambda} \sum_{i=1}^k \int_{B_i} |f(y)| dy \leq \frac{3^n}{\lambda} \|f(y)\|_{L^1}$. Taking limits over compact K , we are done by inner regularity. Finally $\{x \mid Mf(x) = \infty\} \subset \{x \mid Mf(x) > \lambda\}$ for all λ , so it has 0 measure $\square$

Proof. Proof of 1.10.

Consider $\overline{A_\lambda} = \{x \mid \lim_{r \rightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| dy > 2\lambda\}$.

For $\epsilon > 0$ arbitrary, take $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_{L^1} < \epsilon$. Then

$$\begin{aligned} & \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| dy \\ & \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - g(y)| dy + \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y) - g(x)| dy + |g(x) - f(x)| \\ & = a + b + c \end{aligned}$$

For b , since g is uniformly continuous, $|g(y) - g(x)| < \epsilon'$ for ϵ' arbitrary if r is small enough, so $\lim_{r \rightarrow 0} b = 0$ hence if $x \in \overline{A_\lambda}$ we must have $a > \lambda$ or $c > \lambda$.

For c , by Markov's inequality,

$$|\{x \mid |f(x) - g(x)| > \lambda\}| \leq \frac{\|f - g\|_{L^1}}{\lambda} \leq \frac{\epsilon}{\lambda}$$

For a we have, from the Hardy-Littlewood maximal inequality, that

$$|\{x \mid a > \lambda\}| \leq |\{x \mid M[f - g](x) > \lambda\}| \leq \frac{3^n}{\lambda} \|f - g\|_{L^1} \leq \frac{3^n}{\lambda} \epsilon$$

Hence $|\overline{A_\lambda}| \leq \frac{3^n + 2}{\lambda} \epsilon$ hence is 0 as ϵ arbitrary.

The set where the Lebesgue Differentiation Principle doesn't hold is contained in the countable union of the $\overline{A_{1/n}}$ hence has measure 0. $\square$

Remark. In particular, for $f \in L^1(\mathbb{R})$, $\lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(y) dy = f(x)$ almost everywhere

Theorem 1.12. (Egorov) Let $E \in \mathcal{B}(\mathbb{R}^n)$, $|E| < \infty$. Suppose $f_j: E \rightarrow \mathbb{R}$ measurable such that $f_j \rightarrow f$ almost everywhere on E . Then for all $\epsilon > 0$ there is A_ϵ such that $|E \setminus A_\epsilon| < \epsilon$ and $f_j \rightarrow f$ uniformly on A_ϵ

Proof. By discarding a set of measure 0, we may assume $f_j \rightarrow f$ pointwise on E . Define

$$E_k^m = \left\{ x \mid |f_j(x) - f(x)| < \frac{1}{m}, \forall j > k \right\}$$

Clearly E_k^m is increasing in k , so monotonically converges to $\bigcup_{k=1}^\infty E_k^m = E$ since $|f_j(x) - f(x)| < \frac{1}{m}$ eventually by hypothesis.

Thus $|E \setminus E_k^m| \rightarrow 0$ as $k \rightarrow \infty$ by countable additivity. We pick a subsequence $k_m = k_m(\epsilon)$ such that $|E \setminus E_{k_m}^m| \leq \epsilon 2^{-m}$. Then on $A_\epsilon = \bigcap_{m=1}^\infty E_{k_m}^m$ we have $|f_j(x) - f(x)| < \frac{1}{m}$ for all $j > k_m$ and hence $f_j \rightarrow f$ uniformly on A_ϵ . Computing, we have $|E \setminus A_\epsilon| = |\bigcup_{m=1}^\infty E \setminus E_{k_m}^m| \leq \epsilon \sum_{m=1}^\infty 2^{-m} = \epsilon$ $\square$

Theorem 1.13. (*Lusin*) Let $|E| < \infty$, $f: E \rightarrow \mathbb{R}$ (or $\mathbb{C}$) a borel function. Then for all $\epsilon > 0$, there is F_ϵ such that $|E \setminus F_\epsilon| < \epsilon$ and $f: F_\epsilon \rightarrow \mathbb{R}$ (or $\mathbb{C}$) is continuous

Remark. Note however that f is not necessarily continuous on F_ϵ regarded as a map defined on E

Proof. First let $f = \sum_{i=1}^m a_i 1_{A_i}$ be simple and without loss of generality, assume the A_i are disjoint and $E = \bigcup_{i=1}^m A_i$. Since μ is inner regular we can find compact $K_k \subset A_k$ such that $|A_k \setminus K_k| < \frac{\epsilon}{m}$ so that on the compact set $\bigcup_{k=1}^m K_k$, the function f is locally constant, and as the distance between K_i, K_j is strictly positive for $i \neq j$ (as they are compact), f is continuous.

Moreover, $|E \setminus \bigcup_{k=1}^m K_k| \leq |\bigcup_{k=1}^m A_k \setminus K_k| \leq m \frac{\epsilon}{m} = \epsilon$.

For general f , take f_m simple such that $f_m \rightarrow f$ on E and take A_ϵ such that $|E \setminus A_\epsilon| < \frac{\epsilon}{2}$ and $f_m \rightarrow f$ uniformly on A_ϵ by Egorov. Moreover take C_m compact such that $|E \setminus C_m| < \frac{\epsilon}{2} s^{-m}$ such that f_m is continuous on C_m .

Then on $F_\epsilon = A_\epsilon \cap (\bigcap_{m=1}^\infty C_m)$, f is the uniform limit of the continuous f_m and hence is continuous.

A brief calculation shows $|E \setminus F_\epsilon| < \epsilon$ and we are done $\square$

1.3 L^2 as a Hilbert space

Recall that $L^2(E, \mathcal{E}, \mu)$ is a Hilbert space for the inner product

$$(f, g)_{L^2} = \int_E f(x) \overline{g(x)} d\mu$$

and norm $(f, f) = \|f\|_{L^2}^2$. A collection $(u_j \mid j \in J) \subset L^2$ is **orthogonal** if $(u_i, u_j)_{L^2} = 0$ for $i \neq j$, is **orthonormal** if further $(u_i, u_i) = 1$ for all $i \in J$. It is called an **orthonormal basis** if $\text{Span}(u_j \mid j \in J) = L^2$, in which case any $f \in L^2$ equals $f = \sum_{j \in J} (f, u_j)_{L^2} u_j$ with convergence in L^2

Example 1.1. We have examples of orthonormal bases

1. For $E = [0, 1]$ with $\mu = dx$ we can take $u_n = e^{2\pi i n(\cdot)}$
2. $E = \mathbb{R}, \mu = dx$ we can take $\{\psi_{nk} = 2^{n/2} \psi(2^n(\cdot) - k) \mid k, n \in \mathbb{Z}\}$ where $\psi = 1_{[0, 1/2)} - 1_{[1/2, 1)}$ called the **Haar basis**

Recall the following result from Linear Analysis

Theorem 1.14. (*Riesz representation theorem in Hilbert spaces*) Let $\Lambda: H \rightarrow \mathbb{R}$ (or $\mathbb{C}$) be a bounded linear functional on a real (complex) Hilbert space H then there is $g \in H$ such that $\Lambda(h) = \langle h, g \rangle$ for $h \in H$

Proof. See II Linear Analysis $\square$

Definition 1.2. Consider two measures μ, ν on a measurable space $(E, \mathcal{E})$. We say that ν is **absolutely continuous** with respect to μ if $\mu(A) = 0$ implies $\nu(A) = 0$ for any $A \in \mathcal{E}$. We denote this $\nu \ll \mu$ and if also $\mu \ll \nu$ we

say μ, ν are **mutually absolutely continuous**. If there is $B \in \mathcal{E}$ such that $0 = \mu(B) = \nu(B^c)$ then we say μ, ν are **mutually singular**. We denote this $\mu \perp \nu$

Theorem 1.15. (*Radon-Nikodym*) Let μ, ν be finite measures on $(E, \mathcal{E})$ such that $\nu \ll \mu$. Then there is a function $w \in L^1(\mu)$ with $w \geq 0$ such that for all $A \in \mathcal{E}$, $\nu(A) = \int_A d\nu = \int_A w d\mu$

Remark. w is unique.

We will in fact show $\int_E h d\nu = \int_E h w d\mu$ for all non-negative measurable h . In particular $w = \frac{d\nu}{d\mu}$ (in Leibniz notation) is called the **Radon-Nikodym derivative**, or **density** of ν with respect to μ

The result extends to the case where μ, ν are σ -finite

Proof. (von Neumann) Define new measure $\alpha = \mu + 2\nu$ and $\beta = 2\mu + \nu$ and on $L^2(\alpha)$ consider the linear map $\Lambda(f) = \int_E f d\beta$ which is bounded as we have $|\Lambda(f)| \leq \int_E |f| d\beta = 2 \int_E |f| d\mu + \int_E |f| d\nu \leq 2 \int_E |f| d\alpha$. By applying Cauchy Schwartz with $f, 1$ we get $2 \int_E |f| d\alpha \leq 2 \sqrt{\alpha(E)} \|f\|_{L^2(\alpha)}$. Therefore by the Riesz representation theorem, we can write $\Lambda(f) = \int_E g f d\alpha$ hence $\int_E f(2d\mu + d\nu) = \int_E g f(d\mu + 2d\nu)$ so

$$\int_E f(2 - g) d\mu = \int_E f(2g - 1) d\nu$$

for all $f \in L^2(\alpha)$. We can extract information by testing $f = 1_{A_j}$ for the measurable sets $A_j = \{x \mid g(x) \leq \frac{1}{2} - \frac{1}{j}\}$ for $j \in \mathbb{N}$. Thus on the right hand side of the displayed equation, $\int_E f(2g - 1) d\nu \leq \frac{-2}{j} \int_{A_j} d\nu = \frac{-2}{j} \nu(A_j)$ and on the left hand side, $\int_E f(2 - g) d\mu \geq \frac{3}{2} \mu(A_j)$ so we have, as we have that a negative quantity is larger than a positive one, $\mu(A_j) = \nu(A_j) = 0$ and $g \geq \frac{1}{2}$ almost everywhere for μ, ν . Similarly, considering $A'_j = \{x \in E \mid g(x) \geq 2 + \frac{1}{j}\}$ we see that $g \leq 2$, μ and ν almost everywhere

Since $L^2(\alpha)$, for α a finite measure, contains all simple functions and since for $f \geq 0$ the integrands of the displayed equation are non-negative, we can use the monotone convergence theorem to extend it to

$$\int_E f(2 - g) d\mu = \int_E f(2g - 1) d\nu$$

for all $f \geq 0$ measurable. Now consider $f = 1_{\{x \mid g(x) = \frac{1}{2}\}}$. Then the left-hand side is $\frac{3}{2} \mu(\{x \mid g(x) = \frac{1}{2}\})$ and the right hand side is 0, so $\mu(\{x \mid g(x) = \frac{1}{2}\}) = 0$ and hence $\nu(\{x \mid g(x) = \frac{1}{2}\}) = 0$ (This will give the singular part on example sheet 1). Now let $h \geq 0$ be measurable and define $f = \frac{h}{2g-1}$ and $w = \frac{2-g}{2g-1}$ and 0 where they aren't defined (which is a null set under μ and ν) and furthermore, the set where $f \leq 0$ is a null set (as proven before). We now have

$$\int_E h d\nu = \int_E h w d\mu$$

for all measurable $h \geq 0$ and the result is done, since $w \geq 0$ and w is integrable as $\|w\|_{L^1(\mu)} = \int_E w d\mu = \nu(E) < \infty$ by considering the indicator of an arbitrary measurable set $\square$

Remark. If $\mathbb{P}$ is a probability measure on $\mathcal{B}$ such that $\mathbb{P} \ll dx$ then $\frac{d\mathbb{P}}{dx} = p(x)$ is the (Lebesgue) **probability density** of $\mathbb{P}$. Moreover in general, there is a unique decomposition $\mathbb{P} = \mathbb{P}_{ac} + \mathbb{P}_\perp$ such that $\mathbb{P}_{ac} \ll dx$, $\mathbb{P}_\perp \perp dx$. Note that $\mathbb{P}_\perp$ is not necessarily discrete

2 Dual Spaces

Definition 2.1. Let X be a topological vector space. Then the **topological dual space** is $X' = \{\Lambda: X \rightarrow \mathbb{R} \text{ (or } \mathbb{C}) \mid \Lambda \text{ is linear and continuous}\}$. If X is normal with norm $\|\cdot\|_X$, then continuous is equivalent to bounded, and X' has the norm $\|\Lambda\|_{X'} = \sup_{X'} |\Lambda(x)|$

Definition 2.2. For any dual space X' we have a further dual space $X'' = (X')'$ called the **bidual space**. For $\Lambda \in X'$ the **point evaluation** $\Lambda \mapsto \Lambda(x)$ at $x \in X$ is a linear functional on X' that is continuous since $|\Lambda(x)| \leq \|\Lambda\|_{X'} \|x\|_X \leq c \|\Lambda\|_{X'}$. Identifying x with this map, we can regard $X \subset X''$ (we will address injectivity later via Hahn Banach). If $X'' = X$, we say X is **reflexive**

Remark. By the Riesz Representation Theorem, for a Hilbert space H , $H' = H$ and hence $H'' = H$ so H is reflexive

Let us study now the dual space of $L^p = L^p(\mathbb{R}^n, dx)$ for $1 \leq p \leq \infty$. The remark above shows that $(L^2)' = L^2$. For general p we consider the **conjugate index**, q such that $\frac{1}{p} + \frac{1}{q} = 1$. Note that any $g \in L^q$ defines a linear functional on L^p via $\Lambda_g(f) = \int_{\mathbb{R}^n} f(x)g(x)dx$ for $f \in L^p$ which is clearly linear, and is continuous by Holder, and $\|\Lambda_g\|_{(L^p)'} = \|g\|_{L^q}$ (on Example Sheet 1). Identifying g with Λ_g , we say $L^q \subset (L^p)'$

Theorem 2.1. For $1 \leq p < \infty$, we have $(L^p)' = L^q$ isometrically, and for $1 < p < \infty$ L^p is therefore reflexive

Remark. This is false for $p = \infty$. In particular, L^1 is not reflexive

Proof. Note first that (Example sheet 2) any $\Lambda \in (L^p)'$ can be uniquely decomposed as $\Lambda = \Lambda^+ - \Lambda^-$ where $\Lambda^\pm$ are positive linear functionals (ie $\Lambda^\pm(h) \geq 0$ for all $h \geq 0$ almost everywhere). Since we already proved $L^q \subset (L^p)'$, we only need to prove the following lemma

Lemma 2.2. Let $U \in (L^p)'$ be positive. Then there is $g \in L^q$ which is positive almost everywhere such that $U(f) = \int_{\mathbb{R}^n} f(x)g(x)dx$ for all $f \in L^p$ and $\|U\|_{(L^p)'} = \|g\|_{L^q}$

Proof. On $\mathbb{R}^n$ consider the finite measure μ with density $d\mu(x) = e^{-|x|^2} dx$. Further define, for $A \in \mathcal{B}(\mathbb{R}^n)$ the set function $\nu(A) = U(e^{-\frac{|x|^2}{p}} 1_A) \geq 0$. We

have $\nu(\mathbb{R}^n) \leq \|U\|_{(L^p)'} \|e^{-\frac{|x|^2}{p}}\|_{L^p} < \infty$. To show that ν is countably additive, consider $A_m \in \mathcal{B}$ decreasing monotonically to $\emptyset$. To prove their measure goes to 0, note that $\nu(A_m) = U(e^{-\frac{|x|^2}{p}} 1_{A_m}) \leq \|U\|_{(L^p)'} \|e^{-\frac{|x|^2}{p}} 1_{A_m}\|_{L^p} \rightarrow 0$ by Dominated Convergence. Hence ν is a finite measure on $\mathcal{B}$ that is also absolutely continuous with respect to μ since for A such that $\mu(A) = 0$, we have that $\nu(A) \leq \|U\|_{(L^p)'} \|e^{-\frac{|x|^2}{p}} 1_A\|_{L^p} = \|U\|_{(L^p)'} \mu(A)^{1/p} = 0$ as $p < \infty$. Hence the Radon-Nikodym theorem implies there is some $G \geq 0$ such that $\nu(A) = \int_A G d\mu$ where $G \in L^1(\mu)$. Then for $F = \sum_{i=1}^n a_i 1_{A_i}$ set $f = e^{-\frac{|x|^2}{p}} F$, we can write the functional's value,

$$\begin{aligned} U(f) &= \sum_{i=1}^n a_i U(e^{-\frac{|x|^2}{p}} 1_{A_i}) \\ &= \int_{\mathbb{R}^n} \sum_{i=1}^n a_i 1_{A_i} G e^{-\frac{|x|^2}{p}} dx = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{p}} \sum_{i=1}^n (a_i 1_{A_i}) G e^{-\frac{|x|^2}{q}} dx \\ &= \int_{\mathbb{R}^n} f(x) g(x) dx \end{aligned}$$

where $g = G e^{\frac{|x|^2}{q}} \geq 0$.

Since the set of all such f is dense in $L^p \cap \{f \geq 0\}$ and since $fg \in L^1$ as $\int_{\mathbb{R}^n} |f(x)g(x)| dx = \int_{\mathbb{R}^n} |f(x)|g(x) dx = U(|f|) \leq \|U\|_{(L^p)'} \|f\|_{L^p} < \infty$. Decomposing $fg = f^+g - f^-g$ and taking limits, the identity extends to show that $U(f) = \int_{\mathbb{R}^n} f(x)g(x) dx$ for all $f \in L^p(\mathbb{R}^n)$. From the example sheet, for all g

$$\|g\|_{L^q} = \sup \left\{ \int_{\mathbb{R}^n} |fg| : \|f\|_{L^p} \leq 1 \right\} = U(|f|) \leq \|U\|_{(L^p)'} < \infty$$

. Also $\|U\|_{(L^p)'} = \sup_{\|f\|_{L^p} \leq 1} \left| \int f g \right| \leq \|g\|_{L^q}$ by Holder so the result follows $\square$

$\square$

We can characterise duals of subspaces of L^∞ . For instance any finite measure μ on $\mathcal{B}$ defines a linear functional on $C_c(\mathbb{R})'$ via $f \mapsto \int_{\mathbb{R}^n} f d\mu$. Call a measure **regular** on $\mathbb{R}^n$ if for all $\epsilon > 0$ and $A \in \mathcal{B}$ there are closed C , open O such that $C \subset A \subset O$ and $\mu(O \setminus C) < \epsilon$

Theorem 2.3. (Riesz) Let $\Lambda \in (C_c(\mathbb{R}^n))'$ be positive. Then there is a σ -field $\mathcal{M} \supset \mathcal{B}$ and a regular measure μ on $\mathcal{M}$ such that $\Lambda(f) = \int_{\mathbb{R}^n} f d\mu$

Proof. No Proof $\square$

3 Weak and Weak-* Topologies

On a Banach space X and its dual space X' , we have 'strong' topologies induced by the norms $\|\cdot\|_X, \|\cdot\|_{X'}$. But the unit balls of these spaces are only compact

when they are finite dimensional. To consider 'weaker' topologies, we make the following definition

Definition 3.1. A **semi-norm**, p , on a vector space X is a functional called $p: X \rightarrow \mathbb{R}_{\geq 0}$ such that

1. $p(x + y) \leq p(x) + p(y)$ for $x, y \in X$
2. $p(\lambda x) = |\lambda|p(x)$ for all $x \in X, \lambda \in \mathbb{R}$ (or $\mathbb{C}$)

Remark. Note that we do not require the third condition to be a norm, that $x = 0$ if and only if $p(x) = 0$

A collection $\mathcal{P}$ of seminorms p induces a 'locally convex' topology $\tau_{\mathcal{P}}$ by declaring the sets $V_x(p, n) = \{y \in X \mid p(y - x) < \frac{1}{n}\}$ to be open for all $x \in X, p \in \mathcal{P}, n \in \mathbb{N}$.

The family $\mathcal{P}$ is said to **separate points** if for any $0 \neq x \in X$ there is $p \in \mathcal{P}$ such that $p(x) \neq 0$. Therefore (example sheet 2) a sequence $(x_n)_{n \in \mathbb{N}} \rightarrow x$ for $\tau_{\mathcal{P}}$ if and only if $p(x - x_n) \rightarrow 0$ for all $p \in \mathcal{P}$.

This topology is not generally metrisable unless $\mathcal{P}$ is countable in which case a metric is given by $d_{\mathcal{P}}(x, y) = \sum_{i=1}^{\infty} 2^{-i} \frac{p_i(x-y)}{1+p_i(x-y)}$ where $(p_i \mid i \in I)$ is an enumeration of $\mathcal{P}$ (use dominated convergence for the counting measure to prove this).

Definition 3.2. We call $(X, \tau_{\mathcal{P}})$ a **locally convex** topological vector space, and if it is complete we call it a **Frechet** space

Definition 3.3. Returning to the normed space X , we can consider semi-norms given by $p_{\Lambda}(x) = |\Lambda(x)|$ and the topology $\tau_{\mathcal{P}}$ induced by the family of semi-norms called $\mathcal{P} = \{p_{\Lambda} \mid \Lambda \in X'\}$ is called the **weak topology**, τ_w such that $x_n \rightarrow x$ weakly in X , or $x_n \rightharpoonup x$ if $\Lambda(x_n) \rightarrow \Lambda(x)$ for all $\Lambda \in X'$. It is not yet clear that τ_w separates points, but will be proven later.

Definition 3.4. When X' is a dual space, we can consider the **weak-* topology**, τ_{w*} induced by the family of seminorms $\mathcal{P} = \{p_x(\Lambda) = |\Lambda(x)| \mid x \in X\}$ so $\Lambda_n \rightarrow \Lambda$ weak-*, or $\Lambda_n \rightharpoonup^* \Lambda$ if $\Lambda_n(x) \rightarrow \Lambda(x)$ for all $x \in X$ (pointwise convergence). In particular, τ_w, τ_{w*} always separate points, so are hausdorff

Example 3.1. We have a few examples

1. For $L^p = L^p(\mathbb{R}^n, dx), 1 \leq p < \infty$, we have dual space L^q for q conjugate to p , so $f_m \rightarrow f$ weakly in L_p if

$$\int_{\mathbb{R}^n} f_m(x)g(x)dx \rightarrow \int_{\mathbb{R}^n} f(x)g(x)dx$$

for all $g \in L^q$. Similarly as $L^p = (L^q)'$ for $1 < p \leq \infty, \Lambda_{f_n} \rightarrow \Lambda_f$ weak-* (or $f_n \rightarrow f$ weak-*) if and only if that same limit holds. In particular for $1 < p < \infty$ these notions of convergence coincide and this holds more generally for any reflexive space

2. Consider a Borel probability measure on a metric space D and let $C_b(D)$ denote the Banach space of bounded continuous functions on D . Then $\mu(f) = \int_D f d\mu$ defines an element of $C_b(D)'$ since $|\mu(f)| \leq \|f\|_\infty$ as $\mu(D) = 1$ and a sequence of probability measures μ_n converges to μ for τ_{w^*} on $C_b(D)'$ if

$$\int_D f d\mu_n \rightarrow \int_D f d\mu$$

for all $f \in C_b(D)$ (This is weak convergence of laws $\mu_n \rightarrow \mu$ from probability theory)

3.1 Compact Sets in Banach Spaces

Recall from Linear Analysis

Theorem 3.1. (*Ascoli-Arzelà*) Let $I = [0, 1]$ (or any compact metric space) and let f_k be a sequence in $C(I)$ such that $\|f_k\|_\infty \leq M$ for all $k \in \mathbb{N}$ and for all $\epsilon > 0$ there is a δ such that $|x - y| < \delta$ implies $|f_k(x) - f_k(y)| < \epsilon$ for all $k \in \mathbb{N}$. Then there is some $f \in C(I)$ such that for some sequence $n_i \in \mathbb{N}$, $f_{n_i} \rightarrow f$ uniformly on I

Proof. See Linear Analysis □

A sufficient criterion for such equicontinuity arises from Hölder continuity defined by norms of the form $\|f\|_{C^{0,\gamma}} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}$ for $0 < \gamma < 1$ and we can also define

$$\|f\|_{C^{m,\gamma}} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_\infty + \max_{|\alpha|=m} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^\gamma}$$

for all $0 < \gamma < 1, m \in \mathbb{N}$. Then $\{f \mid \|f\|_{C^{0,\gamma}} \leq 1\}$ is compact in $C(I)$ by Ascoli-Arzelà if $\gamma > 0$

The other approach to compactness is via the following theorem

Theorem 3.2. (*Banach-Alaoglu*) If X is a normed space. Then the unit ball, $B'_1 = \{\Lambda \in X' \mid \|\Lambda\|_{X'} \leq 1\}$ of X' is compact for the weak-* topology

Remark. In $C_b(D)'$, any sequence of probability measures has a weak-* convergent subsequence

Proof. (Separable case). Let us first show the following lemma, of independent interest

Lemma 3.3. For a countable dense subset $D = \{x_1, x_2, \dots\}$ of X , consider the seminorms $\tilde{\mathcal{P}} = \{p_{x_k}(\Lambda) = |\Lambda(x_k)| \mid k \in \mathbb{N}\}$ with topology $\tau_{\tilde{\mathcal{P}}}$. Then $\tau_{\tilde{\mathcal{P}}} = \tau_{w^*}$ as (trace-)topologies on $B'_1 = \{\Lambda \in X' \mid \|\Lambda\|_{X'} \leq 1\}$ and are metrised by $d_{\tilde{\mathcal{P}}}(\Lambda, \Lambda') = \sum_{k=1}^{\infty} 2^{-k} \frac{|\Lambda(x_k) - \Lambda'(x_k)|}{1 + |\Lambda(x_k) - \Lambda'(x_k)|}$

Proof. The open sets for the $\tau_{\mathcal{P}}$ topology are generated by the following open sets, $V(x_k, m) = \{\Lambda \mid |\Lambda(x_k)| < \frac{1}{m}\}$, indexed by $k, m \in \mathbb{N}$ and these are elements of τ_{w^*} so to prove $\tau_{\mathcal{P}} = \tau_{w^*}$ it suffices to show that $V(x, n)$ contain some $V(x_k, m)$ for any $x \in X, n \in \mathbb{N}$. To show this, let $x \in X \setminus D$ and take $x_k \in D$ such that $\|x - x_k\|_X < \epsilon$ and for $\Lambda \in V(x_k, m)$, $|\Lambda(x)| \leq |\Lambda(x_k - x)| + |\Lambda(x_k)| \leq \epsilon + \frac{1}{m}$ as $\Lambda \in B'_1$, hence we have shown what we wanted. That the metric is as written was previously proven $\square$

Since the topology τ_{w^*} on D'_1 is metric, it suffices to show

Theorem 3.4. *Let $\Lambda_j \in B'_1$ for $j \in \mathbb{N}$ then there is $\Lambda \in B'_1$ such that $\Lambda_{j_k} \rightarrow \Lambda$ weak-* along a subsequence*

Proof. Again, let $D = \{x_1, x_2, \dots\}$ be a countable dense subset of X . Then $|\Lambda_j(x_1)| \leq \|x_1\|_X < \infty$ so we may apply Bolzano-Weirstrass theorem, we may extract a subsequence Λ_{j_k} such that $\Lambda_{j_k}(x_1) \rightarrow \Lambda(x_1) \in \mathbb{R}$ so that we have $\Lambda(x_1) \leq \|x_1\|_X$. Call the sequence $\Lambda_{1,j}$. Repeating this argument for further coordinates, we define $\Lambda_{l,j}$ for all $l, j \in \mathbb{N}$ with $\Lambda_{l,j}(x_k)$ converging to the number $\Lambda(x_k)$ with $|\Lambda(x_k)| \leq \|x_k\|_X$ for all $k \leq l$ as j goes to infinity. Taking the diagonal sequence $\Lambda_{j,j}$, we have $\Lambda_{j,j}(x_k) \rightarrow \Lambda(x_k)$ and $|\Lambda(k)| \leq \|x_k\|_X$ for all $k \in \mathbb{N}$. To show that Λ in fact defines an element of B'_1 , first note that Λ is uniformly continuous on D since for $x, y \in D$ with $\|x - y\| < \frac{\epsilon}{2}$ and j is large enough that $|\Lambda_{j,j}(x) - \Lambda(x)| + |\Lambda_{j,j}(y) - \Lambda(y)| < \frac{\epsilon}{2}$ then $|\Lambda(x) - \Lambda(y)| < \epsilon$ by a usual triangle inequality argument. We may now extend Λ as a (not necessarily linear) function to all of $\overline{D} = X$ by a usual topological argument. To check that Λ is linear, let $x, y \in X$ and set $z = x + ay$ for $a \in \mathbb{R}$ (or $\mathbb{C}$) and pick $x', y', z' \in D$ such that $\|x - x'\|_X + |a| \|y - y'\|_X + \|z - z'\|_X < \frac{\epsilon}{4}$ and decompose, for sufficiently large j ,

$$\begin{aligned} |\Lambda(z) - \Lambda(x) - a\Lambda(y)| &\leq |\Lambda(z) - \Lambda(z')| + |\Lambda(x) - \Lambda(x')| + |a| |\Lambda(y) - \Lambda(y')| \\ &\quad + |\Lambda(z') - \Lambda_{j,j}(z')| + |\Lambda(x') - \Lambda_{j,j}(x')| + |a| |\Lambda(y') - \Lambda_{j,j}(y')| \\ &\quad + |\Lambda_{j,j}(z') - \Lambda_{j,j}(x') - a\Lambda_{j,j}(y')| \\ &\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + c < \epsilon \end{aligned}$$

since

$$\begin{aligned} c &\leq |\Lambda_{j,j}(z) - a\Lambda_{j,j}(y) - \Lambda_{j,j}(x)| \\ &\quad + |\Lambda_{j,j}(x - x')| + |a| |\Lambda_{j,j}(y - y')| + |\Lambda_{j,j}(z - z')| \rightarrow 0 \end{aligned}$$

Next $|\Lambda(x)| \leq |\Lambda(x' - x)| + |\Lambda(x')| < \epsilon + \|x'\|_X \leq 2\epsilon + \|x\|_X$ so $|\Lambda(x)| \leq \|x\|_X$, where we can bound $\Lambda(x' - x)$ by uniform continuity, so Λ is bounded with operator norm at most 1.

Finally $|\Lambda_{j,j}(x) - \Lambda(x)| \leq |\Lambda_{j,j}(x') - \Lambda(x')| + |\Lambda_{j,j}(x - x')| + |\Lambda(x - x')| \rightarrow 0$ so we are done $\square$

$\square$

4 The Hahn Banach Theorem and its Consequences

Definition 4.1. A functional $p: X \rightarrow \mathbb{R}$ on a real vector space is called **sub-linear** if

1. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$
2. $p(tx) = tp(x)$ for $t \geq 0, x \in X$

Example 4.1. Clearly any linear functional $l: X \rightarrow \mathbb{R}$ is sublinear and so is any semi-norm, but the converse isn't true

If p is a seminorm and $l: X \rightarrow \mathbb{R}$ is linear such that $l(x) \leq p(x)$ for all $x \in X$, then in fact we have $|l(x)| \leq p(x)$ since $-l(x) = l(-x) \leq p(-x) = p(x)$, and so 'dominated by p ' is equivalent to p -continuity

Lemma 4.1. (*Bounded Extension Lemma*) Let X be a real vector space and $p: X \rightarrow \mathbb{R}$ be sub-linear. Let $M \subset X$ be a vector subspace and $x \notin M$ and define $\widetilde{M} = \text{Span}(M, x)$. Then let $l: M \rightarrow \mathbb{R}$ be linear such that $l(x) \leq p(x)$ for all $x \in M$. Then there is some $\tilde{l}: \widetilde{M} \rightarrow \mathbb{R}$ which is linear such that $\tilde{l}|_M = l$ and $\tilde{l}(x) \leq p(x)$ for all $x \in \widetilde{M}$

Proof. Let $y_1, y_2 \in M$ be arbitrary. Then

$$l(y_1) + l(y_2) = l(y_1 + y_2) \leq p(y_1 + y_2) \leq p(y_1 - x) + p(y_2 + x), \forall x \in X \setminus M$$

Now $l(y_1) - p(y_1 - x) \leq p(y_2 + x) - l(y_2)$. Taking the supremum over y_1 and infimum over y_2 , we get

$$\sup\{l(y) - p(y - x) \mid y \in M\} \leq a \leq \inf\{p(y + x) - l(y) \mid y \in M\}$$

for some $a \in \mathbb{R}$. If $z \in \widetilde{M}$, it has a unique decomposition as $z = y + \lambda x$ for some $\lambda \in \mathbb{R}, y \in M$. Define $\tilde{l}(z) = \tilde{l}(y + \lambda x) = l(y) + \lambda a$ which is linear on $\widetilde{M}$ by definition, and restricts to l on M . To see that $\tilde{l} \leq p$ on $\widetilde{M}$, take first $\lambda > 0$ and note that $\tilde{l}(y + \lambda x) = \lambda(l(\frac{y}{\lambda}) + a) \leq \lambda(l(\frac{y}{\lambda}) + p(\frac{y}{\lambda} + x) - l(\frac{y}{\lambda})) = p(y + \lambda x)$. In the other direction, for $\lambda < 0$, set $\mu = -\lambda$ and we similarly get the inequality $\tilde{l}(y + \lambda x) = \tilde{l}(y - \mu x) = \mu(l(\frac{y}{\mu}) - a) \leq \mu(l(\frac{y}{\mu}) - l(\frac{y}{\mu}) + p(\frac{y}{\mu} - x)) = p(y + \lambda x)$ $\square$

To extend l to all of X , for X separable we could apply this inductively to $M_n = \text{Span}(M, x_1, \dots, x_n)$ where $x_1, \dots, x_n, \dots$ is a countable dense subset and take limits (messy). Instead we use transfinite induction via Zorn's lemma.

Theorem 4.2. (*Hahn Banach*) Let X be a real vector space and $p: X \rightarrow \mathbb{R}$ a sublinear functional. For $M \subset X$ a vector subspace, let $l: M \rightarrow \mathbb{R}$ be a linear functional such that $l(x) \leq p(x)$ for all $x \in M$. Then there is a linear map $\tilde{l}: X \rightarrow \mathbb{R}$ such that $\tilde{l}(x) \leq p(x)$ for all $x \in X$

Proof. Consider

$$S = \{(N, \tilde{l}) \mid M < N \leq X, \tilde{l}: N \rightarrow \mathbb{R} \text{ is linear}, \tilde{l}|_M = l, \tilde{l} \leq p \text{ on } N\}$$

. We declare $(N_1, \tilde{l}_1) \leq (N_2, \tilde{l}_2)$ if $N_1 \subset N_2$ and $\tilde{l}_2|_{N_1} = \tilde{l}_1$. Let $T \subset S$ be totally ordered and define $N_T = \bigcup_{(N, \tilde{l}) \in T} N$ with $l_T: N_T \rightarrow \mathbb{R}$ equal to $\tilde{l}$ on N which is an upper bound for T . Thus by Zorn's lemma there is a maximal $(\mathcal{N}, L)$ which is a maximal element of S . Now $\mathcal{N} = X$ as otherwise we could apply bounded extension again and so we are done. $\square$

Remark. Note the following

1. $\tilde{l}$ is not necessarily unique
2. If X is non-separable then this depends on the axiom of choice

Corollary 4.2.1. (*Norming functionals*) For X a normed linear space and $x \in X$ there is a continuous $\Lambda = \Lambda_x \in X'$ such that $\|\Lambda\|_{X'} = 1$ such that $|\Lambda(x)| = \|x\|_X$. In particular, if $\Lambda(x - y) = 0$ for all $\Lambda \in X'$ then $x = y$

Proof. Fix $x \in X$. Define $M = \{cx \mid c \in \mathbb{R}\} \leq X$ and the linear functional $l(cx) = c\|x\|_X$, which is dominated by the sublinear functional $p = \|\cdot\|_X$. Thus by Hahn Banach, there is a $\Lambda = \Lambda_x: X \rightarrow \mathbb{R}$ such that $|\Lambda_x(y)| \leq \|y\|_X$ so $\Lambda \in X'$ and $\|\Lambda\|_{X'} \leq 1$. Note that $\|\Lambda\|_{X'} \geq \sup_{x \in M \cap \{\|y\|_X \leq 1\}} |\Lambda(x)| = \|x\|_X$ and so $|\Lambda(x)| = |l(x)| = \|x\|_X$ $\square$

So the weak topology induced by all $\Lambda: X \rightarrow \mathbb{R}$ which are bounded, linear does separate points

Corollary 4.2.2. The canonical injection $i: X \hookrightarrow X'$ given by $x \mapsto (\Lambda \mapsto \Lambda(x))$ is an isometry

Proof. First $\|i(x)\|_{X''} = \sup_{\|\Lambda\|_{X'} \leq 1} |\Lambda(x)| \leq 1 \cdot \|x\|_X$, and taking a norming functional, the last supremum is larger than $|\Lambda_x(x)| = \|x\|_X$ so we are done $\square$

We further deduce that if $X = X''$, then X is isometric to X'' and hence is complete. Hence reflexive normed linear spaces are Banach spaces. If X is not reflexive, then X'' provides (up to isomorphism) the completion of X for $\|\cdot\|_X$. In particular if X is reflexive, then the weak topology coincides with the weak-* topology on X'' , and so by Banach Alagolu, the unit ball B_1 is compact in the weak topology

Theorem 4.3. (*Hyperplane Separation Theorem*) Let A, B be disjoint non-empty convex sets in a Banach space X

1. If A is open, there is $\Lambda \in X'$ and some real γ such that $\Lambda(a) < \gamma \leq \Lambda(b)$ for $a \in A, b \in B$
2. If instead A is compact and B is closed then there is $\Lambda \in X'$ and reals γ_1, γ_2 such that $\Lambda(a) < \gamma_1 < \gamma_2 < \Lambda(b)$

Proof. Let $a_0 \in A, b_0 \in B$ and $x_0 = b_0 - a_0 \in X$. Define

$$C = A - B + x_0 = \{a - b + x_0 \mid a \in A, b \in B\}$$

which is again convex containing 0 but not x_0 , as if $x_0 \in C$ then there is $a \in A \cap B$. C is also open since A is open, so we can consider its **Minkowski functional** $P_C(x) = \inf\{t \mid x \in tC\} < \infty$. We can show (example sheet) that P_C is sublinear on X and there is a constant $k > 0$ such that $P_C(x) \leq k\|x\|_X$ and $P_C(x) < 1$ for $x \in C$ and $P_C(x_0) \geq 1$. Now take the vector subspace $M = \{tx_0 \mid t \in \mathbb{R}\}$ and consider the linear functional $l: M \rightarrow \mathbb{R}$ by $l(tx_0) = t$. Then l is dominated by P_C so we can apply Hahn Banach. There is therefore some $\Lambda: X \rightarrow \mathbb{R}$ such that $-k\|x\|_X \leq -P_C(-x) \leq \Lambda(x) \leq P_C(x) \leq k\|x\|_X$ where the outer inequalities come from linearity and so $|\Lambda(x)| \leq k\|x\|_X$ hence $\Lambda \in X'$. Now pick $a \in A, b \in B$ and note

$$\Lambda(a) - \Lambda(b) + 1 = \Lambda(a - b + x_0) \leq P_C(a - b + x_0) < 1$$

. Then $\Lambda(a) < \Lambda(b)$ for such a, b . Now if A is open, so is $\Lambda(A)$ by open mapping theorem and so $\Lambda(a) < \sup(\Lambda(A)) \leq \Lambda(b)$ with strict inequality if B is open, proving 1. For 2, note that $\Lambda(A)$ is now compact in $\mathbb{R}$ and if $d = \|A - B\|_X > 0$ then $\tilde{A} = A + B_{d/2}$ where $B_{d/2}$ is the open ball of radius $d/2$, then applying 1 to this, we are done $\square$

5 Generalised Functions and Distributions

Consider a topological vector space $X \subset \bigcap_{q \geq 1} L^q$ where $L^q = L^q(\Omega, dx)$ for Ω an open subset of $\mathbb{R}^n$. Suppose X contains $C_c^\infty(\Omega)$ and let $f \in L^p$ then we obtain a linear functional $\Lambda_f(g) = \int_\Omega fg dx, g \in X$. If the embedding $X \hookrightarrow L^q$ for q conjugate to p is continuous, then $\Lambda_f \in X'$. Since X contains $g = \phi_\epsilon(\cdot - y)$ for all sufficiently small $y \in \Omega$, we can use the mollification theorem to see that if $\Lambda_f = 0$ in X' then $f = 0$ almost everywhere. We can therefore identify Λ_f with f and study the weak-* topology of X' on L^p . Choosing X as 'small as possible' obtains the 'weakest possible topologies' on functions and the completion of (L^p, τ_{w*}) contains objects that aren't functions anymore.

First define $C^\infty(\Omega) = \{\phi: \Omega \rightarrow \mathbb{R}(\text{or } \mathbb{C}) \mid \phi \text{ is infinitely differentiable}\}$ and consider the seminorms $p_N(\phi) = \max_{0 \leq |\alpha| \leq N} \sup_{x \in K_N} |D^\alpha \phi(x)|$ where we have $K_i \subset K_{i+1}$ a sequence of compact sets such that $\bigcup_{i=1}^\infty K_i = \Omega$. We define the Frechet space $\mathcal{E}(\Omega) = (C^\infty(\Omega), \tau_{\mathcal{P}})$ where $\mathcal{P} = \{p_N \mid N \in \mathbb{N}\}$. Note that $\mathcal{E}(\Omega)$ may contain non-integrable functions, such as $\phi(x) = x^{2025}$

To obtain a similar topology on $C_c^\infty(\Omega)$ is more delicate, but we can prove

Theorem 5.1. *There is a topology on $C_c^\infty(\Omega)$ such that*

1. *The vector space operations are continuous*
2. *$\phi_j \rightarrow 0$ for τ as $j \rightarrow \infty$ if and only if there is compact $K \subset \Omega$ such that $\text{supp}(\phi_j) \subset K$ for all $j \in \mathbb{N}$ and $D^\alpha \phi_j \rightarrow 0$ uniformly on K for all $0 \leq |\alpha| < \infty$*

3. If $T: C_c^\infty(\Omega) \rightarrow \mathbb{R}$ (or $\mathbb{C}$) is linear, then it is continuous if and only if $T(\phi_j) \rightarrow 0$ for all $\phi_j \rightarrow 0$

We define $D = D(\Omega) = (C_c^\infty(\Omega), \tau)$ is called the **space of test functions**

Example 5.1. Take $e^{-j}\phi(\frac{\cdot}{j})$ for $\phi \in C_c^\infty$. This converges to 0 in τ , but $j^{-2025}\phi(\frac{\cdot}{j})$ doesn't converge to 0 for τ

Also on $\Omega = \mathbb{R}$, $e^{-j}\phi(\cdot - j) \not\rightarrow 0$ in τ since there is no common K on which they are all supported

For $\Omega = \mathbb{R}^n$, there is a third space

Definition 5.1. Call $\phi \in C^\infty(\mathbb{R}^n)$ **rapidly decreasing** if

$$\sup_{x \in \mathbb{R}^n} (1 + |x|)^n |D^\alpha \phi(x)| < \infty$$

for all $0 \leq |\alpha| < \infty$, $N \in \mathbb{N}$

Example 5.2. $e^{-|x|^2}$ is rapidly decreasing, but $(1 + |x|)^{-2025}$ is not

Define the seminorms $\tilde{\mathcal{P}} = \{\tilde{p}_N \mid N \in \mathbb{N}\}$ with

$$\tilde{p}_N = \max_{0 \leq |\alpha| < N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |D^\alpha \phi(x)|$$

and define the Frechet space $S(\mathbb{R}^n) = (\{\phi \text{ rapidly decreasing}\}, \tau_{\tilde{\mathcal{P}}})$ which is metrisable as $\tilde{\mathcal{P}}$ is countable, and this is often called the Schwartz class.

Clearly $D(\Omega) \subset \mathcal{E}(\Omega)$ and $D(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$ are continuous embeddings and strict inclusions (example sheet).

We can now define

$$D'(\Omega) = \{T: D(\Omega) \rightarrow \mathbb{R} \mid T \text{ is linear and continuous}\}$$

, called the space of **Schwartz distributions**. We also define the space of **'compactly supported' Schwartz distributions**,

$$\mathcal{E}'(\Omega) = \{T: \mathcal{E}(\Omega) \rightarrow \mathbb{R} \mid T \text{ is linear and continuous}\}$$

. For $\Omega = \mathbb{R}^n$, we define the space of **tempered distributions**

$$S'(\mathbb{R}^n) = \{T: S(\mathbb{R}^n) \rightarrow \mathbb{R} \mid T \text{ is linear and continuous}\}$$

. We can use $\mathbb{C}$ instead of $\mathbb{R}$ throughout. These are equipped with their weak-* topologies. We have the reverse continuous embeddings $\mathcal{E}'(\Omega) \subset D'(\Omega)$ and $\mathcal{E}'(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$

Example 5.3. Consider $\delta_x(\phi) = \phi(x)$, $x \in \Omega$. If $\phi_j \rightarrow 0$ in $D, \mathcal{E}, S$ then $\delta_x(\phi_j) = \phi_j(x) \rightarrow 0$ as $j \rightarrow \infty$ so $\delta_x \in \mathcal{E}', D', S'$

Let $f \in L^1_{loc}(\Omega)$. Then $T_f(\phi) = \int_{\Omega} f\phi dx$ for $\phi \in D(\Omega)$ and $T_f \in D'(\Omega)$ since for $\phi_j \rightarrow 0$ in $D(\Omega)$, we have $T_f(\phi) \rightarrow 0$ by Dominated Convergence Theorem with dominating function $\sup_{j \in \mathbb{N}} \|\phi_j\|_{\infty} 1_K |f| \in L^1$ for some compact K on which all ϕ_j are . Also if $T_f = 0$ on $D(\Omega)$ we still get $f = 0$ almost everywhere by applying mollification on $f1_B$ where B is any ball in Ω containing $x \in \Omega$. Therefore $L^1_{loc} \subset D'(\Omega)$. Also if $\phi_{\epsilon} = \epsilon^{-1}\phi(\frac{\cdot}{\epsilon})$ for $\phi \in C_c^{\infty}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \phi dx = 1$ where ϕ is even, then for $g \in D(\mathbb{R}^n)$,

$$\begin{aligned} T_{\phi_{\epsilon}}(g) &= \int_{\mathbb{R}^n} g(x)\phi_{\epsilon}(x)dx \\ &= \int_{\mathbb{R}^n} g(0-x)\phi_{\epsilon}(x)dx \\ &= g * \phi_{\epsilon}(0) \\ &\rightarrow g(0) = \delta_0(g) \end{aligned}$$

So $T_{\phi_{\epsilon}} \rightarrow \delta_0$ in $D'(\mathbb{R}^n)$

5.1 Generalised (distributional) Derivatives in $D'(\Omega)$

Let $f \in C^1(\Omega)$, $D_i f = \frac{\partial}{\partial x_i} f \in L^1_{loc}$. Then

$$T_{D_i f}(\phi) = \int_{\Omega} \left(\frac{\partial}{\partial x_i} f \right) \phi dx = - \int_{\Omega} f \frac{\partial}{\partial x_i} \phi dx = -T_f(D_i \phi)$$

This motivates us to define, for any multi-index α ,

Definition 5.2. The **distributional derivative** of any $T \in D'(\Omega)$ is

$$(D^{\alpha}T)(\phi) = (-1)^{|\alpha|} T(D^{\alpha}\phi)$$

for $\phi \in D(\Omega)$. So $D^{\alpha}T \in D'(\Omega)$ as well. If $T = T_f$ and $D_{\alpha}T = T_g$ for some $f, g \in L^1_{loc}$ then we call $g = D_w^{\alpha}f$ the **weak partial derivative of f**

Example 5.4. Consider $T_f \in D'(\mathbb{R})$ where $f(x) = 0$ for $x \leq 0$ and $f(x) = x$ otherwise. Then

$$DT_f(\phi) = -T_f(\phi') = - \int_0^{\infty} x\phi'(x)dx = \int_0^{\infty} \phi(x)dx = \int_{\mathbb{R}} H\phi dx = T_H(\phi)$$

where H is the Heaviside step function. So $D_w f = H$

Computing the second derivative, we see

$$D^2 T_f(\phi) = DT_H(\phi) = -T_H(\phi') = - \int_0^{\infty} \phi'(x)dx = \phi(0) = \delta_0(\phi)$$

This is not a weak derivative, but is a distributional derivative. So in some sense we can write $\delta_0 = H'$

Finally, similarly, $D^3 T_f = -\phi'(0)$ which is a Schwartz distribution but not a measure

If $f \in L^1_{loc}(\Omega)$ and $a \in C^\infty(\Omega)$ then

$$T_{af}(\phi) = \int_{\Omega} af\phi = \int_{\Omega} f(a\phi) = T_f(a\phi)$$

which motivates the following definition

Definition 5.3. If $T \in D^1(\Omega)$, $a \in C^\infty(\Omega)$, we define $(aT)(\phi) = T(a\phi)$

5.2 Compactly Supported Distributions

Proposition 5.1. A linear map $T: \mathcal{E}(\Omega) \rightarrow \mathbb{R}$ (or $\mathbb{C}$) is continuous if and only if there are $K \subset \Omega$ compact and $N \in \mathbb{N}, c > 0$ such that

$$|T(\phi)| \leq c \max_{0 \leq |\alpha| \leq N} \sup_{x \in K} |D^\alpha \phi(x)|$$

for all $\phi \in \mathcal{E}(\Omega)$

Proof. Suppose the inequality holds and let $\phi_j \rightarrow 0$ in $\mathcal{E}(\Omega)$. Then by definition of the topology on $\mathcal{E}(\Omega)$, for some j large enough in the definition of the topology, $K \subset K_j$ and the right hand side of the inequality for $\phi = \phi_j$ converges to 0 and so $T(\phi_j) \rightarrow 0$ so $T \in \mathcal{E}'(\Omega)$

Conversely assume T is continuous but that the inequality doesn't hold. Hence if $K_j \subset K_{j+1}$ is any exhaustion of compact sets of Ω we get a sequence $\phi_j \in \mathcal{E}(\Omega)$ such that

$$|T(\phi_j)| \geq j \max_{1 \leq |\alpha| \leq j} \sup_{x \in K} |D^\alpha(\phi_j)|$$

for $j \in \mathbb{N}$. Define $\psi_j = \frac{\phi(j)}{|T(\phi_j)|}$ where the denominator is not zero as it is lower bounded as above and note that for $x \in K_j$ and $|\beta| \leq j$,

$$|D^\beta \psi_j(x)| \leq \frac{1}{j} \frac{|D^\beta \phi_j(x)|}{\max_{0 \leq |\alpha| \leq j} \sup_{x \in K_j} |D^\alpha \phi_j(x)|} \leq \frac{1}{j} \rightarrow 0$$

so $\psi_j \rightarrow 0$ in $\mathcal{E}(\Omega)$ but $T(\psi_j) = 1 \not\rightarrow 0$ contradiction. $\square$

Definition 5.4. $T \in D'(\Omega)$ has support in a closed set $K \subset \Omega$ if $T(\phi) = 0$ for $\phi \in C_c^\infty(\Omega \setminus K)$. The last proposition implies that $T \in \mathcal{E}'(\Omega)$ is supported in some compact subset of Ω

Example 5.5. If $f \in L^1_{loc}$ is such that $f = 0$ outside a compact set K then T_f is compactly supported

δ_x for $x \in \Omega$ is also compactly supported, in $K = \{x\}$

If $T \in D^1(\Omega)$ is compactly supported then so is $D^\alpha T$ for any multi-index α

Proposition 5.2. Any $T \in \mathcal{E}'(\Omega)$ restricts to a $T \in D'(\Omega)$ of compact support, and any $T \in D'(\Omega)$ which is compactly supported extends to $\tilde{T} \in \mathcal{E}'(\Omega)$

Proof. The first claim follows from the preceding proposition. Conversely, let T be supported in the compact set K . Then we take $\xi \in C_c^\infty$ such that $\xi = 1$ on K and define $\tilde{T}(\phi) = T(\xi\phi)$ for $\phi \in \mathcal{E}(\Omega)$, which defines an element of $\mathcal{E}'(\Omega)$ $\square$

5.3 Convolutions of Distributions

Recall that for $f, g \in C_c(\mathbb{R}^n)$, their convolution is $f * g(x) = \int_{\mathbb{R}^n} g(x-y)f(y)dy$ which is x almost everywhere defined whenever $f \in L^p, g \in \tilde{L}^1$ for $1 \leq p < \infty$ in view of Minkowski's inequality for integrals. To extend this definition to distributions, we need some notation

Remark. Recall that $\tau_x g = g(\cdot - x)$, set $\check{g} = g(-\cdot)$ and $\widetilde{\tau_x g} = g(x - \cdot)$. In this notation, $f * g(x) = T_f[\widetilde{\tau_x g}]$. Since $g \in C_c^\infty(\mathbb{R}^n)$, we have $\widetilde{\tau_x g} \in C_c^\infty(\mathbb{R}^n)$, we can define for $T \in D'(\mathbb{R}^n)$ and $\phi \in D(\mathbb{R}^n)$ the convolution $x \mapsto T * \phi(x) = T(\widetilde{\tau_x \phi})$

Theorem 5.2. *Let $T \in D'(\mathbb{R}^n)$, $\phi \in D(\mathbb{R}^n)$ and α be a multi-index. Then $T * \phi \in C^\infty(\mathbb{R}^n)$ and $D^\alpha(T * \phi) = (D^\alpha T) * \phi = T * (D^\alpha \phi)$*

Proof. Take $e_i \in \mathbb{R}^n$ a basis vector and $h \rightarrow 0 \in \mathbb{R}$. Write

$$\begin{aligned} \frac{1}{h}[T * \phi(x + he_i) - T * \phi(x)] &= T \left[\frac{\phi(x + he_i - \cdot) - \phi(x - \cdot)}{h} \right] \\ &\rightarrow T(D_i \phi(x - \cdot)) \\ &= T[\widetilde{\tau_x D_i \phi}] \\ &= T * (D_i \phi) \end{aligned}$$

where the convergence (where the internal term converges in $D(\mathbb{R}^n)$) is from the example sheet. By induction, we get that $T * \phi$ is C^∞ and the two outside terms of the theorem are equal.

To prove the first equality, note

$$D^\alpha(\widetilde{\tau_x \phi}) = D^\alpha \phi(x - \cdot) = (-1)^{|\alpha|} (D^\alpha \phi)(x - \cdot) = (-1)^{|\alpha|} \widetilde{\tau_x D^\alpha \phi}$$

Thus

$$(D^\alpha T) * \phi(x) = D^\alpha T[\widetilde{\tau_x \phi}] = (-1)^{|\alpha|} T[D^\alpha(\widetilde{\tau_x \phi})] = T[\widetilde{\tau_x D^\alpha \phi}] = T * (D^\alpha \phi)$$

so we are done $\square$

We notice that if $T \in \mathcal{E}'(\mathbb{R}^n)$, supported in K and if K_x is the shifted compact support of $\widetilde{\tau_x \phi} = \phi(x - \cdot)$, $\phi \in C_c^\infty(\mathbb{R}^n)$, then for $|x|$ sufficiently large $K \cap K_x = \emptyset$ and $T * \phi \in D(\mathbb{R}^n)$

Definition 5.5. Let $T_1 \in D'(\mathbb{R}^n)$ and $T_2 \in \mathcal{E}'(\mathbb{R}^n)$ then we define their convolution by the action $(T_1 * T_2) * \phi = T_1 * (T_2 * \phi)$ for $\phi \in D(\mathbb{R}^n)$

Remark. This defines a distribution as $((T_1 * T_2) * \phi)(0) = (T_1 * T_2)\check{\phi}$ for $\phi \in D(\mathbb{R}^n)$

Remark. Note that δ_0 has compact support and $\delta_0 * \phi = \delta_0[\phi(x - \cdot)] = \phi$ and so for any $T_1 \in D'(\mathbb{R}^n)$, we see that $(T_1 * \delta_0) * \phi = T_1 * \phi$, so δ_0 is a right identity for convolution on $D'(\mathbb{R}^n)$

Theorem 5.3. Let $T_1 \in D'(\mathbb{R}^n)$, $T_2 \in \mathcal{E}'(\mathbb{R}^n)$ and α any multi-index. Then $D^\alpha(T_1 * T_2) = (D^\alpha T_1) * T_2 = T_1 * (D^\alpha T_2)$

Proof. Using the previous theorem and the definitions, for any $\phi \in D(\mathbb{R}^n)$,

$$\begin{aligned} D^\alpha(T_1 * T_2) * \phi &= (T_1 * T_2) * D^\alpha \phi \\ &= T_1 * (T_2 * D^\alpha \phi) \\ &= T_1 * (D^\alpha T_2 * \phi) \\ &= (T_1 * D^\alpha T_2) * \phi \end{aligned}$$

and so $D^\alpha(T_1 * T_2) = T_1 * (D^\alpha T_2)$ and the other identity is similar $\square$

5.4 Fundamental Solutions of Linear PDEs

Consider the partial differential operator $L = \sum_{|\alpha| \leq k} a_\alpha D^\alpha$ with the coefficients $a_\alpha \in C^\infty(\mathbb{R}^n)$, $k \in \mathbb{N}$. The PDE $Lu = u_0$ for $u, u_0 \in D'(\mathbb{R}^n)$ can be interpreted in $D'(\mathbb{R}^n)$ and a 'weak solution' u is one such that $(Lu)(\phi) = u_0(\phi)$ for all $\phi \in D(\mathbb{R}^n)$. An element $G \in D'(\mathbb{R}^n)$ is called a **fundamental solution** for L if $LG = \delta_0$ in $D'(\mathbb{R}^n)$ and if $G = T_g$ for $g \in L^1_{loc}(\mathbb{R}^n)$ then we call g the **Green kernel** of G

Theorem 5.4. Suppose that L has constant coefficients, $a_\alpha \in \mathbb{R}$ (or $\mathbb{C}$) and that $G \in D'(\mathbb{R}^n)$ is its fundamental solution. If $u_0 \in \mathcal{E}'(\mathbb{R}^n)$, a solution to $Lu = u_0$ is given by $u = G * u_0$

Proof. Because L is linear and using the previous theorem,

$$Lu = \sum_{|\alpha| \leq k} a_\alpha D^\alpha (G * u_0) = \sum_{|\alpha| \leq k} (D^\alpha G * u_0) = (LG) * u_0 = \delta_0 * u_0 = u_0$$

since

$$(\delta_0 * u) * \phi = \delta_0 * (u_0 * \phi) = u_0 * \phi(x) = u_0 * \phi$$

$\square$

Remark. If $u_0 \in D$ then $G * u_0 \in C^\infty(\mathbb{R}^n)$ and the equation $Lu = u_0$ holds pointwise on $\mathbb{R}^n$ by testing $\phi_\epsilon(\cdot - x)$ for $\epsilon > 0$ and an appropriate ϕ_ϵ

For example if $L = -\Delta = -\sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ we can show that the fundamental solution is $G = T_g$ for the Green kernel $g(x) = \frac{1}{4\pi|x|}$. Thus for $f \in D(\mathbb{R}^n)$, the solutions to the Laplace equation $-\Delta u = f$ on $\mathbb{R}^3$ are given by

$$T_g * f = \frac{1}{4\pi} \int_{\mathbb{R}^3} f(y) \frac{1}{|x - y|} dy$$

5.5 Fourier Transforms of Distributions

Recall that for $f \in L^1(\mathbb{R}^n)$, the Fourier transform is the map $\hat{f}: \mathbb{R}^n \rightarrow \mathbb{C}$ given by

$$\hat{f}(u) = \int_{\mathbb{R}^n} e^{-ix \cdot u} f(x) dx, u \in \mathbb{R}^n$$

where $x \cdot u = \langle x, u \rangle_{\mathbb{R}^n}$. We sometimes write $Ff = F[f] = \hat{f}$. Since $\hat{f} \in L^1_{loc}(\mathbb{R}^n)$, we can consider

$$T_f(\phi) = \int_{\mathbb{R}^n} \hat{f}(u) \phi(u) du = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot u} \phi(u) du dx = T_f(\hat{\phi})$$

for $\phi \in D$, but we have a problem as $\hat{\phi}$ is not necessarily in D , so this definition does not necessarily extend to D' . We therefore work in $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ instead

Lemma 5.5. (*Riemann Lebesgue Lemma*) *Let $f \in L^1(\mathbb{R}^n)$. Then*

$$\hat{f} \in C_0(\mathbb{R}^n) = \{h: \mathbb{R}^n \rightarrow \mathbb{C} \mid h \text{ continuous, } \lim_{|u| \rightarrow \infty} |h(u)| = 0\}$$

and $\|\hat{f}\|_{\infty} \leq \|f\|_{L^1}$

Proof. Clearly for any $u_j \rightarrow u$ in $\mathbb{R}^n$ we have $e^{-ix \cdot u_j} f(x) \rightarrow e^{-ix \cdot u} f(x)$ and $|e^{-ix \cdot u_j} f(x)| \leq |f(x)| \in L^1$ and so by DCT, $\hat{f}(u_j) \rightarrow \hat{f}(u)$ for $\hat{f}$ is continuous and $\|\hat{f}\|_{\infty} \leq \int_{\mathbb{R}^n} |f(x)| dx = \|f\|_{L^1}$.

Finally for $f \in L^1$ take a sequence $f_k \in C_c^\infty(\mathbb{R}^n)$ such that $f_k \rightarrow f$ in L^1 so that also $\|\hat{f}_k - \hat{f}\|_{\infty} \leq \|f_k - f\|_{L^1} \rightarrow 0$ so $\hat{f}_k \rightarrow \hat{f}$ uniformly on $\mathbb{R}^n$ and $\hat{f}_k \in C_0(\mathbb{R}^n)$ since $|u_j| |\hat{f}_k(u)| = |\widehat{D_j f_k}(u)| \leq \|D_j f_k\|_{L^1} < \infty$ therefore since $C_0(\mathbb{R}^n)$ is complete, $\hat{f} \in C_0(\mathbb{R}^n)$ $\square$

Remark. Note that F does not map L^1 onto $C_0(\mathbb{R}^n)$

Lemma 5.6. *Let $f \in L^1(\mathbb{R}^n)$*

1. *If $f_\lambda = \lambda^{-n} f(\frac{\cdot}{\lambda})$ for $\lambda > 0$ then $\hat{f}_\lambda(u) = \hat{f}(\lambda u)$ for all $u \in \mathbb{R}^n$*
2. *$\widehat{\tau_x f}(u) = e^{-ix \cdot u} \hat{f}(u)$ and $\widehat{e^{i\langle y, \cdot \rangle} f} = \tau_y \hat{f}$*
3. *If $g \in L^1$ then $f * g \in L^1$ and $\widehat{f * g} = \hat{f} \cdot \hat{g}$*

Proof. Fubini and substitution $\square$

Theorem 5.7. 1. *Let $f \in C^1(\mathbb{R}^n)$ be such that $f, D_j f \in L^1$ for $j = 1, \dots, n$.*

Then $\widehat{D_j f}(u) = iu_j \hat{f}(u)$ for all $u \in \mathbb{R}^n$

2. *If $f \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} (1+|x|)|f(x)| dx < \infty$ then for any $j = 1, \dots, n$ and $u \in \mathbb{R}^n$ then $D_j \hat{f}(u) = -ix_j \widehat{f(x)}$. In particular, $\hat{f} \in C^1(\mathbb{R}^n)$*

Proof. 1. For any $\epsilon > 0$ we can pick a function $f_\epsilon \in C_c^\infty(\mathbb{R}^n)$ such that $\|f_\epsilon - f\|_{L^1} + \|D_j f_\epsilon - D_j f\|_{L^1}$ (we can restrict to compact support by multiplying by a smooth cut off function, then approximate by $\phi_\epsilon * f$ as the derivative also converges to $D_j f$). For such f_ϵ we see that

$$\begin{aligned}\widehat{D_j f_\epsilon}(u) &= \int_{\mathbb{R}^n} e^{-ix \cdot u} D_j f_\epsilon(x) dx \\ &= - \int_{\mathbb{R}^n} (D_i e^{-x \cdot u}) f_\epsilon(x) dx \\ &= \int_{\mathbb{R}^n} i u_j e^{-ix \cdot u} f_\epsilon(x) dx \\ &= i u_j \hat{f}_\epsilon(u)\end{aligned}$$

Now

$$\begin{aligned}|\widehat{D_j f}(u) - i u_j \hat{f}(u)| &\leq |\widehat{D_j f}(u) - \widehat{D_j f_\epsilon}(u)| + |i u_j (\hat{f}_\epsilon - \hat{f})(u)| \\ &\leq \|D_j f - D_j f_\epsilon\|_{L^1} + |u_j| \|f_\epsilon - f\|_{L^1} \\ &\leq (1 + |u_j|) \epsilon\end{aligned}$$

so we are done

2. By the Riemann Lebesgue lemma, it suffices to prove the identity. Take e_j the j th basis vector in $\mathbb{R}^n$ and $h \rightarrow 0$. Then

$$\begin{aligned}\frac{1}{h}(\hat{f}(u + h e_j) - \hat{f}(u)) &= \int_{\mathbb{R}^n} \frac{1}{h} (e^{i \cdot (u + h e_j)} - e^{ix \cdot u}) f(x) dx \\ &= \int_{\mathbb{R}^n} e^{ix \cdot u} \left(\frac{e^{ix \cdot h e_j} - 1}{h} \right) f(x) dx \\ &\rightarrow -i \int_{\mathbb{R}^n} e^{ix \cdot u} x_j f(x) dx \\ &= -i \widehat{x_j f}(u)\end{aligned}$$

We can apply DCT as the middle term is bounded by $|x_j|$ by MVT, which is $|f| dx$ integrable by hypothesis

□

Recall from Probability and Measure

Theorem 5.8. (*Fourier Inversion*) Let $f \in L^1$ and $\hat{f} \in L^1$. Then for almost all $x \in \mathbb{R}^n$,

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot u} \hat{f}(u) du = F^{-1}[\hat{f}](x)$$

and for the unique continuous representative $f \in [f] = \{g \mid g = f \text{ a.e}\}$ the formula holds everywhere in $\mathbb{R}^n$

Proof. See Probability and Measure

□

Remark. $F^{-1}[F\phi] = \frac{1}{(2\pi)^n} F[F\phi](\cdot)$ so F^{-1} is itself a Fourier transform itself

Recall that $S(\mathbb{R}^n) = \{\phi \mid \sup_{x \in \mathbb{R}^n} (1 + |x|^n) |D^\alpha(x)| < \infty, \forall \alpha, n\}$ with seminorms p_N

Theorem 5.9. F is a continuous homeomorphism of the $S(\mathbb{R}^n)$ into itself

Proof. Clearly $f \in L^1$ since

$$\int_{\mathbb{R}^n} |f(x)| dx \leq \sup_{x \in \mathbb{R}^n} (1 + |x|)^{n+1} |f(x)| \int_{\mathbb{R}^n} \frac{1}{|x+1|^{n+1}} dx < \infty$$

To see that $\hat{f} \in S(\mathbb{R}^n)$ as well, let α, β be multiindices. Then

$$\begin{aligned} |u^\alpha| |D^\beta \hat{f}(u)| &= |D^\alpha \widehat{(x^\beta f)}|(u) \\ &\leq \|D^\alpha (x^\beta f)\|_{L^1} \\ &\leq p_N(f) \end{aligned}$$

for sufficiently large N , with the last inequality as before. Similarly if $\phi_j \rightarrow 0$ in $S(\mathbb{R}^n)$ the previous inequality implies $p_N(\hat{\phi}_j) \rightarrow 0$ so F is continuous. Moreover if $F\phi = 0$ for $\phi \in S(\mathbb{R}^n) \subset L^1$, then by Fourier inversion, $\phi = F^{-1}(0) = 0$ so F is injective. Moreover any $\phi \in S(\mathbb{R}^n)$ is $F^{-1}F(\phi) = \frac{1}{(2\pi)^n} F\check{F}\phi$, which is the Fourier transform of some ϕ , hence $F: S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$ is surjective and we are done $\square$

We can therefore make the following definition

Definition 5.6. For $T \in S'(\mathbb{R}^n)$, we define its **distributional Fourier transform** $\hat{T}(\phi) = T(\hat{\phi})$ for all $\phi \in S(\mathbb{R}^n)$

Remark. Clearly by the previous theorem, $\hat{T} \in S'(\mathbb{R}^n)$, using sequential continuity. Also if $f \in L^1$ then $T_f(\phi) = T_f(\hat{\phi})$ as at the start of this section.

Definition 5.7. Call ϕ **slowly increasing** if $\sup_{x \in \mathbb{R}^n} (1 + |x|)^{-N} |\phi(x)| < \infty$ for some $N \in \mathbb{N}$.

Remark. If ϕ is slowly increasing, $T_\phi \in S'(\mathbb{R}^n)$. In this case, even if $\hat{T}_\phi$ is given by some $T_g, g \in L^1_{loc}$, we can't conclude that $\hat{\phi}$ makes sense pointwise

Note further that for $T_j, T \in S'(\mathbb{R}^n)$, if $T_j \rightarrow T$ in $S'(\mathbb{R}^n)$ weak-*, then we have $\hat{T}_j(\phi) = T_j(\hat{\phi}) \rightarrow T(\hat{\phi}) = \hat{T}(\phi)$ so $F: S'(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ is sequentially continuous (and turns out to be continuous). One can further show that if $\hat{T} = 0$ in $S'(\mathbb{R}^n)$ then $T = 0$ in $S'(\mathbb{R}^n)$ so $F: S'(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ is injective. If we define $F^{-1}T$ via $F^{-1}T(\phi) = T(F^{-1}\phi)$ for all $\phi \in S(\mathbb{R}^n)$, then $F^{-1}T \in S'(\mathbb{R}^n)$ as $F^{-1} = \frac{1}{(2\pi)^n} \check{F}$ and moreover

$$\begin{aligned} F^{-1}[FT](\phi) &= FT[F^{-1}\phi] = T[FF^{-1}\phi] \\ &= T[F \frac{1}{(2\pi)^n} \check{F}^{-1}\phi] = T[F^{-1}F\phi] \\ &= T\phi \end{aligned}$$

for all $\phi \in S(\mathbb{R}^n)$ and so we have proven

Theorem 5.10. *The Fourier transform F defines a continuous homeomorphism of $S'(\mathbb{R}^n)$ with inverse F^{-1}*

Example 5.6. Take δ_z the Dirac measure in $\mathbb{R}^n$ which is in $\mathcal{E}'(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$. Then

$$\begin{aligned}\hat{\delta}_z(\phi) &= \delta_z(\hat{\phi}) = \hat{\phi}(z) \\ &= \int_{\mathbb{R}^n} e^{-ix \cdot z} \phi(x) dx \\ &= T_{e^{-i \langle \cdot, z \rangle}}(\phi) \in S'(\mathbb{R}^n)\end{aligned}$$

for all $\phi \in S(\mathbb{R}^n)$ as $e^{-i \langle \cdot, z \rangle}$ is slowly increasing. Similarly

$$\begin{aligned}\widehat{e^{i \langle \cdot, z \rangle}}(\phi) &= \widehat{T_{e^{i \langle \cdot, z \rangle}}}(\phi) = T_{e^{i \langle \cdot, z \rangle}}(\hat{\phi}) \\ &= \int_{\mathbb{R}^n} e^{iz \cdot u} \hat{\phi}(u) du \\ &= (2\pi)^n \phi(z) \\ &= (2\pi)^n \delta_z(\phi)\end{aligned}$$

for all $\phi \in S(\mathbb{R}^n)$ (But note that when $z = 0$, $\hat{T}_1 = (2\pi)^n \delta_0$, but $\int_{\mathbb{R}^n} e^{ix \cdot u} du \neq \delta_0$ as a Lebesgue integral)

Remark. (Relationship to other Fourier transforms)

1. Recall from P&M that for $f, g \in L^1 \cap L^2$, $\hat{f}, \hat{g} \in L^2$ and Plancherel's theorem gives that $(f, g)_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} (\hat{f}, \hat{g})_{L^2(\mathbb{R}^n)}$. Thus F extends by uniform continuity to an isometry $\frac{1}{(2\pi)^{n/2}} \bar{F}$ of $L^2 = \overline{L^1 \cap L^2}^{L^2}$. Thus if we define $\bar{F}T(\phi) = T(\bar{F}\phi) = T(\hat{\phi}) = \hat{T}(\phi)$ for all $\phi \in S(\mathbb{R}^n) \subset L^1 \cap L^2$ so $\bar{F} = F$ on $S'(\mathbb{R}^n)$
2. For any finite measure μ on $\mathcal{B}(\mathbb{R}^n)$ we can define $\hat{\mu}(u) = \int_{\mathbb{R}^n} e^{-ix \cdot u} d\mu(x)$ for $u \in \mathbb{R}^n$. Then

$$\begin{aligned}\hat{T}_\mu(\phi) &= T_\mu u(\hat{\phi}) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix \cdot u} \phi(u) du d\mu(x) \\ &= T_{\hat{\mu}}(\phi) \quad \text{by Fubini}\end{aligned}$$

so $FT_\mu = T_{\hat{\mu}}$ in $S'(\mathbb{R}^n)$

3. For $T \in \mathcal{E}'(\mathbb{R}^n)$ we can define $\bar{T}(u) = T(e^{i \langle \cdot, u \rangle})$ for $u \in \mathbb{R}^n$ and one may show that $\bar{T} = T_{\bar{T}}$ (with $\bar{T}$ slowly increasing) in $S'(\mathbb{R}^n)$

To derive further results, note that if $a \in C^\infty(\mathbb{R}^n)$ is slowly increasing and $\phi \in S(\mathbb{R}^n)$, we have $a\phi \in S(\mathbb{R}^n)$ and therefore we can define, for $T \in S'(\mathbb{R}^n)$, the product $aT \in S'(\mathbb{R}^n)$ by $(aT)(\phi) = T(a\phi)$ where $\phi \in S(\mathbb{R}^n)$. Also for any $T \in D'(\mathbb{R}^n)$ we can define $\tau_u T$ by $(\tau_u T)(\phi) = T(\tau_{-u}\phi)$ for $u \in \mathbb{R}^n$

Lemma 5.11. Let $T \in S'(\mathbb{R}^n)$, $y \in \mathbb{R}^n$ and α a multi-index

1. $\widehat{\tau_y T} = e^{-i\langle y, \cdot \rangle} \hat{T}$ and $\widehat{e^{i\langle y, \cdot \rangle} T} = \tau_y T$
2. $\widehat{D^\alpha T} = i^{|\alpha|} u^\alpha \hat{T}$ and $D^\alpha \hat{T} = (-i)^{|\alpha|} \widehat{x^\alpha T}$

Proof. For instance for all $\phi \in S$,

$$\widehat{e^{i\langle y, \cdot \rangle} T} = e^{i\langle y, \cdot \rangle} T(\hat{\phi}) = T(e^{i\langle y, \cdot \rangle} \hat{\phi}) = T(\widehat{\tau_{-y} \phi}) = \tau_y \hat{T}(\phi)$$

and the others are similar □

Remark. Since $D^\alpha \delta_0 \in \mathcal{E}'(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ so $\widehat{D^\alpha \delta_0} = i^{|\alpha|} u^\alpha$ so the Fourier transforms of the partial derivatives of the Dirac measure δ_0 span the space of polynomials

5.6 Periodic Distributions

Definition 5.8. Call $T \in D'(\mathbb{R}^n)$ **periodic** if $\tau_k T = T$ for all $k \in \mathbb{Z}^n$

Example 5.7. If $f \in L^1_{loc}$ is such that $f(\cdot - k) = f$ for all $k \in \mathbb{Z}^n$ then T_f is periodic as $\tau_k T_f = \int_{\mathbb{R}^n} f(x) \phi(x+k) dx = \int_{\mathbb{R}^n} f(x-k) \phi(x) dx = T_f$. For $T \in \mathcal{E}'(\mathbb{R}^n)$, its **periodisation**, $T_{per} = \sum_{k \in \mathbb{Z}^n} \tau_k T$ which is a finite sum at each point is trivially periodic as $\tau_n T_{per} = \sum_{k \in \mathbb{Z}^n} \tau_{n+k} T = \sum_{k \in \mathbb{Z}^n} \tau_k T$

The indicator 1_q for $q = \{x \in \mathbb{R}^n \mid -\frac{1}{2} \leq x_i < \frac{1}{2}, 1 \leq i \leq n\}$, the **fundamental cell of the lattice** (since $\sum_{k \in \mathbb{Z}^n} 1_q(x+k) = 1$) is not in $C^\infty(\mathbb{R}^n)$. Instead we require the following

Lemma 5.12. There is some $\psi \in C_c^\infty(\mathbb{R}^n)$ such that

1. $\psi \geq 0$
2. $\text{supp } \psi \in Q^o$ for $Q = \{x \in \mathbb{R}^n \mid -1 \leq x_i \leq 1, 1 \leq i \leq n\}$
3. $\sum_{k \in \mathbb{Z}^n} \psi(x-k) = 1$ for all $x \in \mathbb{R}^n$

We call this a **periodic partition of unity** and if ψ' is another periodic partition of unity and T is a periodic distribution, then $T(\psi) = T(\psi')$

Proof. Start with $\psi_0 \in C_c^\infty(\mathbb{R}^n)$ such that $\psi_0 \geq 0$ is supported in Q^o and $\psi_0 = 1$ on q . Define $S(x) = \sum_{k \in \mathbb{Z}^n} \psi_0(x-k)$ which is a finite sum for fixed x as ψ_0 has compact support. This is non-negative and in fact at least 1 since $\psi_0 \geq 1_q$. Therefore we can normalise $\psi(x) = \frac{\psi_0(x)}{S(x)}$ which clearly satisfies 1 and 2, and 3

is easy by the definition of $S(x)$. Finally for T periodic,

$$\begin{aligned}
T(\psi) &= T(1\psi) \\
&= T\left(\psi \sum_{k \in \mathbb{Z}^n} (\tau_k \psi')\right) \\
&= \sum_{k \in \mathbb{Z}^n} \tau_k T((\tau_k \psi')\psi) \quad \text{by periodicity} \\
&= \sum_{k \in \mathbb{Z}^n} T(\psi'(\tau_k \psi)) \\
&= T\left(\psi' \sum_{k \in \mathbb{Z}^n} \tau_k \psi\right) \\
&= T(\psi')
\end{aligned}$$

□

In the previous proof we can take $\psi_{0,j} \rightarrow 1_q$ pointwise but such that we have $\sup_j \|\psi_{0,j}\|_\infty < \infty$. In particular since $\sum_n \tau_n 1_q = 1$ we obtain a sequence of periodic partition of unitys $\psi_j \rightarrow 1_q$ pointwise such that $\sup_j \|\psi_j\|_\infty < \infty$. Now for $f \in L^1_{loc}$ periodic, $T_f(\psi_j) = T_f(\psi_{j+i}) = \int_{\mathbb{R}^n} f(x) \psi_j(x) dx \xrightarrow{DCT} \int_q f(x) dx$ which is the mean of f . We hence can define, for general $T \in D'(\mathbb{R}^n)$ which are periodic, the **mean of T** , $M(T) = T(\psi)$ where ψ is any periodic partition of unity.

Theorem 5.13. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$. Then $T_{per} = \sum_{k \in \mathbb{Z}^n} \tau_k(T)$ converges in $S'(\mathbb{R}^n)$. Also if $T \in D'(\mathbb{R}^n)$ is periodic, there is some $V \in \mathcal{E}'(\mathbb{R}^n)$ such that $T = \sum_{k \in \mathbb{Z}^n} \tau_k V$ in $D'(\mathbb{R}^n)$*

Proof. Recall from earlier that for $T \in \mathcal{E}'(\mathbb{R}^n)$ we have a compact set $K \subset B_R$ (some ball of radius R) and $N, C > 0$ with $N \in \mathbb{Z}$ such that we have that $|T(\phi)| \leq C \sup_{x \in K, |\alpha| \leq N} |D^\alpha \phi(x)|$ for all $\phi \in \mathcal{E}'(\mathbb{R}^n)$. Now for $x \in K$, we have

$$\begin{aligned}
1 + |g| &\leq 1 + |g + x| + |x| \\
&\leq 1 + |g + x| + R \\
&\leq (1 + R)(1 + |g + x|)
\end{aligned}$$

which gives

$$1 \leq \frac{(1 + R)^M (1 + |g + x|)^M}{(1 + |g|)^M}$$

Therefore for all $\phi \in S(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$,

$$|T(\phi)| \leq C \frac{(1 + R)^M}{(1 + |g|)^M} \sup_{x \in K, |\alpha| \leq N} (1 + |g + x|)^M |D^\alpha \phi(x)|$$

This applies as well to $\tau_g \phi \in S$ so

$$\begin{aligned} |\tau_g T(\phi)| &\leq C \frac{(1+R)^M}{(1+|g|)^M} \sup_{x \in K, |\alpha| \leq N} (1+|g+x|)^M |D^\alpha \phi(x+g)| \\ &\leq C \frac{(1+R)^M}{(1+|g|)^M} \sup_{y \in \mathbb{R}^n, |\alpha| \leq N} (1+|y|)^M |D^\alpha \phi(y)| \end{aligned}$$

So since if $M = n+1$, we have $\sum_{g \in \mathbb{Z}^n} \frac{1}{(1+|g|)^{n+1}} < \infty$, we deduce

$$\left| \sum_{k \in \mathbb{Z}^n} \tau_k T(\phi) \right| \leq C' \sup_{y \in \mathbb{R}^n, |\alpha| \leq N} (1+|y|)^{n+1} |D^\alpha \phi(y)|$$

which shows that $\sum_{k \in \mathbb{Z}^n} \tau_k T(\phi) \in S'(\mathbb{R}^n)$ by example sheet 3. For the converse let T be periodic and $\phi \in D(\mathbb{R}^n)$, then if ψ is a periodic partition of unity we have

$$\begin{aligned} T(\phi) &= T \left(\left(\sum_{g \in \mathbb{Z}^n} \tau_g \psi \right) \phi \right) \\ &= \sum_{g \in \mathbb{Z}^n} \tau_g T((\tau_g \psi) \phi) \\ &= \sum_{g \in \mathbb{Z}^n} T(\psi(\tau_{-g} \phi)) \\ &= \sum_{g \in \mathbb{Z}^n} (\psi T)(\tau_{-g} \phi) \\ &= \sum_{g \in \mathbb{Z}^n} \tau_g (\psi T)(\phi) \end{aligned}$$

so we define $V = \psi T \in \mathcal{E}'(\mathbb{R}^n)$ as $\psi \in C_c^\infty(\mathbb{R}^n)$ □

Theorem 5.14. (Convergence of Fourier Series in S^1) Let $U \in D'(\mathbb{R}^n)$ be periodic. Then $U = \sum_{g \in \mathbb{Z}^n} u_g T_{e_{2\pi g}}$ in $S'(\mathbb{R}^n)$ where $e_{2\pi k} = e^{i\langle 2\pi k, \cdot \rangle}$ with Fourier coefficients $u_g = M(e_{-2\pi k} u)$

Proof. The idea is to use the Fourier transform on $S'(\mathbb{R}^n)$. We get the result $\widehat{\tau_k(U)} - U = (e_{-k} - 1)\hat{U} = 0$ for all k

Lemma 5.15. If $T \in S'(\mathbb{R}^n)$ such that $(e_{-k} - 1)T = 0$ for all $k \in \mathbb{Z}^n$. Then

$$T = \sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g}$$

in $S'(\mathbb{R}^n)$. In particular, $c_g \in \mathbb{C}$ and $|c_g| \leq c(1+|g|)^N$ for some $N, c > 0$

Proof. Denote by $\Lambda^* = \{2\pi g \mid g \in \mathbb{Z}^n\}$ the set of zeros of $e_{-k} - 1$ and take $\phi \in D(\mathbb{R}^n)$ such that $\text{supp } \phi \cap \Lambda^* = \emptyset$. Therefore $(e_{-k} - 1)^{-1}\phi \in D$ and so $T(\phi) = (e_{-k} - 1)T((e_{-k} - 1)^{-1}\phi) = 0$ and so T is also supported in Λ^* . Now take a periodic partition of unity ψ and rescale it to $\tilde{\psi} = \psi(\frac{\cdot}{2\pi})$ so that $\text{supp } \tilde{\psi} \subset \{x \in \mathbb{R}^n \mid -2\pi < x_i < 2\pi\}$ and $\sum_{g \in \mathbb{Z}^n} \tau_{2\pi g} \tilde{\psi} = 1$ on $\mathbb{R}^n$. Now define $T_g = (\tau_{2\pi g} \tilde{\psi})T$, which is supported in $\{2\pi g\}$ and we have $\sum_{g \in \mathbb{Z}^n} T_g = 1$ in $D'(\mathbb{R}^n)$ as $\tilde{\psi}$ is a periodic partition of unity and $(e_{-k} - 1)T_g = (\tau_{2\pi g} \tilde{\psi})(e_{-k} - 1)T = 0$. Choose $k = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the j th position, so

$$\begin{aligned} 0 &= (e_{-k} - 1)T_g \\ &= (e^{-ix_j} - 1)T_g \\ &= (e^{i(x_j - 2\pi g)} - 1)T_g \\ &= (x_j - 2\pi g)K(x_j)T_g \end{aligned}$$

Where the Taylor polynomial K doesn't vanish near $\text{supp } T_g = \{2\pi g\}$, implying $(x_j - 2\pi g)T_g = 0$. Next take $\phi \in S(\mathbb{R}^n)$ and apply Taylor's theorem again to see

$$\phi(x) = \phi(2\pi g) + \sum_{j=1}^n (x_j - 2\pi g)\phi_j(x)$$

for some $\phi_j \in S(\mathbb{R}^n)$. Applying T_g we get

$$\begin{aligned} T_g(\phi) &= T_g(\phi(2\pi g)) + \sum_{j=1}^n (x_j - 2\pi g)T(\phi_j(x)) \\ &= \delta_{2\pi g}(\phi)T_g(1) \end{aligned}$$

Next $|c_g| = \left| T_g \left(\sum_{g' \in \mathbb{Z}^n} \tau_{2\pi g'} \tilde{\psi} \right) \right| = |T_g(\tau_{2\pi g} \tilde{\psi})|$, which, since we know that $T_g \in \mathcal{E}'(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$, by the characterisation of $S'(\mathbb{R}^n)$ from example sheet 3, is at most

$$c_0 \sup_{x \in \mathbb{R}^n, |\alpha| \leq N} (1 + |x|)^N |D^\alpha \tilde{\psi}(x - 2\pi g)|$$

for some $N \in \mathbb{N}, c_0 > 0$. This is at most

$$c_0(1 + |g|)^N \sup_{y \in \mathbb{R}^n, |\alpha| \leq N} (1 + |y|)^N |D^\alpha \tilde{\psi}(y)| \leq c(1 + |g|)^N$$

where the supremum is finite as $\tilde{\psi}$ is compactly supported. Therefore $T = \sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g}$ in $S'(\mathbb{R}^n)$ $\square$

Now we apply the lemma to U (which is in $S'(\mathbb{R}^n)$ by theorem 5.13) to see $\hat{U} = (2\pi)^n \sum_{g \in \mathbb{Z}^n} u_g \delta_{2\pi g}$ where $u_g \in \frac{C_g}{(2\pi)^g}$. Taking the inverse Fourier transform, $U = \sum_{g \in \mathbb{Z}^n} u_g T_{e_{2\pi k}}$ in $S'(\mathbb{R}^n)$. To find u_g notice that $T \mapsto M(T)$ is continuous on $S'(\mathbb{R}^n)$ and so $M(e_{-2\pi k} U) = \sum_{g \in \mathbb{Z}^n} u_g M(e_{-2\pi k} T_{e_{2\pi k}})$. Each term is equal to $\int_{\mathbb{R}^n} e^{2\pi i \langle g-k, x \rangle} dx$ which is 1 if $g = k$ and 0 otherwise and so the result follows. $\square$

If we apply this to $U = \sum_{k \in \mathbb{Z}^n} \delta_k \in S'(\mathbb{R}^n)$ as it is the periodisation of δ_0 , we get Fourier coefficients, with periodic partition of unity ψ such that $\psi(0) = 1$, are

$$M(e_{-2\pi g}U) = U(e_{-2\pi g}\psi) = \sum_{k \in \mathbb{Z}^n} \delta_k(e_{-2\pi g}\psi) = 1$$

So by the theorem, $\sum_{k \in \mathbb{Z}^n} \delta_k = \sum_{k \in \mathbb{Z}^n} T_{e_{2\pi k}}$ in $S'(\mathbb{R}^n)$ (Poisson summation formula). Testing $\phi(x - \cdot)$ for $\phi \in S(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, we get

$$\sum_{k \in \mathbb{Z}^n} \phi(x - k) = \sum_{k \in \mathbb{Z}^n} T_{e_{2\pi k}} \phi(x - \cdot) = \sum_{k \in \mathbb{Z}^n} e^{i2\pi k \cdot x} \hat{\phi}(2\pi k)$$

For $x = 0$ we get

$$\sum_{k \in \mathbb{Z}^n} \phi(k) = \sum_{k \in \mathbb{Z}^n} \hat{\phi}(2\pi k)$$

6 Sobolev Spaces and Introduction to Elliptic PDEs

We now will construct solutions to PDEs such as Laplace's equation,

$$\Delta u = f \text{ on } \Omega \subset \mathbb{R}^n$$

subject to boundary conditions on $\partial\Omega$ and for f a source function. For this we require certain function spaces over open Ω . From now on we denote by $C^k(\Omega)$ the normed space $\{f: \Omega \rightarrow \mathbb{R} \mid D^\alpha f \text{ exists for all } 0 \leq |\alpha| \leq k, \|f\|_{C^k} < \infty\}$ with $\|f\|_{C^k} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_\infty$. Similarly, for $k \in \mathbb{N}, 0 < \eta < 1$ we define **Holder spaces** $C^{k,\eta}(\Omega) = \{f \in C^k(\Omega) \mid \|f\|_{C^{k,\eta}} < \infty\}$ where

$$\|f\|_{C^{k,\eta}} = \|f\|_{C^k} + \sum_{|\alpha|=k} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^\eta}$$

. The spaces $C^k, C^{k,\eta}$ are Banach spaces for these norms. We can replace $\|\cdot\|_\infty, D^\alpha$ by $\|\cdot\|_{L^p}, 1 \leq p \leq \infty$ and the weak derivative D^α (denoted the same as the normal derivative). We get

Definition 6.1. Let $k \in \mathbb{N} \cup \{0\}, 1 \leq p \leq \infty$. Then $f \in W^{k,p}(\Omega)$ if $D^\alpha f \in L^p$ for all $0 \leq |\alpha| \leq k$. The norm on $W^{k,p}(\Omega)$ is given by

$$\|f\|_{W^{k,p}} = \left(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_{L^p}^p \right)^{1/p}$$

for $p < \infty$ and

$$\|f\|_{W^{k,p}} = \max_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_{L^\infty}$$

for $p = \infty$. We call $W^{k,p}$ a **Sobolev space**

Remark. One shows that $W^{k,p}(\Omega)$ is a Banach space and considering the weak derivative instead of the normal derivative is essential for this

When $\Omega = \mathbb{R}^n, p = 2$ we have

Definition 6.2. Let $s \in \mathbb{R}$. Then $H^s(\mathbb{R}^n)$ consists of $f \in S'(\mathbb{R}^n)$ such that $\hat{f} \in L^2_{loc}(\mathbb{R}^n)$ and

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}^n} |\hat{f}(u)|^2 (1 + |u|^2)^s du < \infty$$

Remark. Note that $H^s(\mathbb{R}^n)$ is a Hilbert space for the inner product

$$(f, g)_{H^s} = \int_{\mathbb{R}^n} \hat{f}(u) \overline{\hat{g}(u)} (1 + |u|^2)^s du$$

which is an L^2 space for an appropriate measure.

By Plancherel's theorem, for $s > 0$, $H^s(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n, dx)$

In fact we have

Proposition 6.1. For $s \geq 0$ we have $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ with equivalent norms

Proof. We have Plancherel's theorem,

$$\int_{\mathbb{R}^n} |f(x)|^2 dx + \int_{\mathbb{R}^n} |D^\alpha f(x)|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{f}(u)|^2 (1 + |u|^{|\alpha|})^2 du$$

from which the result follows by standard inequalities for $|u|^{|\alpha|}$ □

Remark. (Examp sheet) One shows

1. $H^s \subseteq H^t$ for $s \geq t$
2. $S(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ (but this is not true for $D(\Omega) \subseteq W^{s,2}(\Omega)$)
3. $\delta_x \in H^{-s}$ for $s > n/2$
4. $(H^s)' = H^{-s}$

Theorem 6.1. (Sobolev Embedding Theorem) Let $s > \frac{n}{2} + k$ for $k \in \mathbb{N}$ and suppose $f \in H^s(\mathbb{R}^n)$. Then there is $f^* \in C^k(\mathbb{R}^n \setminus \{0\})$ such that $f^* = f$ almost everywhere and $\|f^*\|_{C^k} \leq C_{s,n,k} \|f\|_{H^s}$. In particular, identifying f and f^* , we have the embedding $H^s(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n)$

Proof. First take $f \in S(\mathbb{R}^n)$ and note

$$\begin{aligned} |D^\alpha f(x)| &= |F^{-1}[u^\alpha \hat{f}](x)| \\ &\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |u|^{|\alpha|} |\hat{f}(u)| \frac{(1 + |u|^2)^{s/2}}{(1 + |u|^2)^{s/2}} du \quad \text{By Riemann-Lebesgue} \\ &\stackrel{C^{-s}}{\leq} \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} \frac{|u|^{2|\alpha|}}{(1 + |u|^2)^s} du \right)^{1/2} \left(\int_{\mathbb{R}^n} |\hat{f}(u)|^2 (1 + |u|^2)^s du \right)^{1/2} \end{aligned}$$

which proves the Sobolev inequality for $f \in S(\mathbb{R}^n)$. For $f \in H^s(\mathbb{R}^n)$, take $f_k \in S(\mathbb{R}^n)$ such that $f_k \rightarrow f$ in $H^s(\mathbb{R}^n)$. Take a subsequence converging almost everywhere, as $H^s(\mathbb{R}^n)$ convergence implies $L^2(\mathbb{R}^n)$ convergence, implies an almost everywhere converging subsequence. This sequence is Cauchy in $H^s(\mathbb{R}^n)$ and by the inequality, also in C^k , hence it has a limit in C^k and so $f_n \rightarrow f^*$ in C^k . Hence by uniqueness of almost everywhere limits, $f = f^*$ almost everywhere, and taking limits in the inequality gives the result. $\square$

Remark. 1. This upgrades to $H^s \subseteq C^{k,\eta}$ if $s > \frac{n}{2} + k + \eta$ (example sheet)

Consider the PDE

$$-\Delta v + v = f \quad (\dagger)$$

where $f \in H^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$ is given, and $v \in S'(\mathbb{R}^n)$ and $\Delta = \sum_{i=1}^n \frac{\partial^2}{(\partial x_i)^2}$. Δ has Fourier transform

$$\hat{\Delta} = i^2 \sum_{j=1}^n u_j^2 = -|u|^2$$

Theorem 6.2. *There is a unique solution $v \in H^{s+2}(\mathbb{R}^n)$ to $(\dagger)$ and we have the elliptic regularity estimate $\|v\|_{H^{s+2}(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)}$*

Proof. Fourier transform $(\dagger)$ to obtain $(1 + |u|^2)\hat{v} = \hat{f}$ in $S'(\mathbb{R}^n)$ which for $\hat{f} \in L^1_{loc}$ has the unique solution

$$\hat{v}(u) = \frac{\hat{f}(u)}{1 + |u|^2}, u \in \mathbb{R}^n$$

in $S'(\mathbb{R}^n)$. So $v = F^{-1}\hat{v}$ solves the equation $(\dagger)$ and

$$\begin{aligned} \|v\|_{H^{s+2}(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} (1 + |u|^2)^{s+2} \frac{|\hat{f}(u)|^2}{(1 + |u|^2)^2} du \\ &= \|f\|_{H^s}^2 \end{aligned}$$

so we are done (note that the elliptic regularity estimate is stated as an inequality as in the more general situation, it is an inequality, and not an equality as in this case) $\square$

To study equations such as $(\dagger)$ restricted to open sets $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$ we need to define the restriction of $f \in H^s(\mathbb{R}^n)$ to $\partial\Omega$. If $f \in H^s(\mathbb{R}^n)$ for $s > n/2$ then by Sobolev embedding $f \in C^\epsilon$ for $\epsilon > 0$ and the **boundary trace** $f|_{\partial\Omega}$ exists by uniform continuity. For general $s > 1/2$ we have

Theorem 6.3. (*Trace Theorem*) *There exists a bounded linear operator of the form $T: H^s(\mathbb{R}^n) \rightarrow H^{s-1/2}(\mathbb{R}^{n-1})$ such that for all $f \in S(\mathbb{R}^n)$, we have $Tf = f|_{\mathbb{R}^{n-1} \times \{0\}}$*

Proof. Example sheet $\square$

We call $T = T_\Sigma$ with $\Sigma = \mathbb{R}^{n-1} \times \{0\}$, the boundary trace of $f \in H^s(\mathbb{R}^n)$. By a change of coordinates, the operator extends to $T_{\partial\Omega}$ for sufficiently regular Ω (eg any ball in $\mathbb{R}^n$), and in particular we have $T_{\partial\Omega}: H^1\mathbb{R}^n(\hookrightarrow H^{1/2}(\partial\Omega)) \rightarrow L^2(\partial\Omega)$ (where the middle term won't be defined in this course) which is bounded and linear

6.1 The $H_0^1(\Omega)$ Spaces

Any $f \in C_c^\infty(\Omega)$ for $\Omega \subseteq \mathbb{R}^n$ open with sufficiently regular boundary extends by 0 to an element in $H^1\mathbb{R}^n$ (and in fact in $H^s(\mathbb{R}^n)$ for any s). The Hilbert norm for such f is

$$\begin{aligned} \|f\|_{H^1(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} (1 + |u|^2) |\hat{f}(u)|^2 du \\ &= (2\pi)^n \int_{\Omega} (|f(x)|^2 + |Df(x)|^2) dx \end{aligned}$$

where $Df = \nabla f$ is the gradient vector. We define $H_0^1 = \overline{C_c^\infty(\Omega)}^{||\cdot||_{H^1}}$ in $H^1(\mathbb{R}^n)$. This space is not equal to $W^{1,2}(\Omega)$, as $1 \in W^{1,2}(\Omega)$, in view of

Proposition 6.2. *Let $f \in H_0^1(\Omega)$. Then $f(x) = 0$ for almost every $x \in \Omega^c$ and if $\partial\Omega$ is sufficiently regular then $T_{\partial\Omega}(f) = 0$*

Proof. Take $\phi \in C_c^\infty(\text{int } \Omega^c)$ and take $f_n \in C_c^\infty(\Omega)$ such that $f_n \rightarrow f$ in $H^1(\mathbb{R}^n)$. Then, since $\Lambda_\phi(h) = \int_{\mathbb{R}^n} \phi h$, we have $\Lambda_\phi \in (L^2)^\circ \subseteq (H^1)^\circ$. Then

$$0 = \int_{\mathbb{R}^n} \phi f_n dx = \Lambda_\phi(f_n) \rightarrow \Lambda(f) = \int_{\mathbb{R}^n} \phi f dx$$

and so $\int_{\mathbb{R}^n} \phi f dx = 0$ hence f is supported in Ω . Similarly

$$0 = T_{\partial\Omega}(f_n) \rightarrow T_{\partial\Omega}(f)$$

and so $f = 0$ on $\partial\Omega$ □

Now consider the boundary value problem

$$\begin{aligned} -\Delta + v &= f && \text{on } \Omega \\ v &= 0 && \text{on } \partial\Omega \end{aligned} \quad (*)$$

We therefore have, for $\phi \in C_c^\infty(\Omega)$,

$$\begin{aligned} \int_{\Omega} (-\Delta v + v) \phi dx &= \int_{\Omega} \nabla v \cdot \nabla \phi dx + \int_{\Omega} v \phi \\ &= \int_{\Omega} f \phi dx \end{aligned}$$

where the first equality is from integration by parts. We interpret this with $f \in L^2(\Omega)$, $v \in H_0^1(\Omega)$.

The last equality is just $\langle v, \phi \rangle_{H^1} = \langle f, \phi \rangle_{L^2}$. We call this the **weak formulation** of (*). Since $C_c^\infty(\Omega)$ is dense in H_0^1 and L^2 , this is equivalent to solving

$$\langle v, \phi \rangle_{H^1} = \langle f, \phi \rangle_{L^2}, \forall \phi \in H_0^1(\Omega) \quad (\dagger')$$

Theorem 6.4. *For every $f \in L^2(\Omega)$ there is a unique $v \in H_0^1(\Omega)$ such that $(\dagger')$ holds and $\|v\|_{H^1} \leq \|f\|_{L^2}$. Moreover the solution map $S: f \mapsto v = v_f$ is a bounded linear map from $L^2(\Omega) \rightarrow H_0^1(\Omega)$ which is self adjoint for $L^2(\Omega)$*

Proof. Define $\Lambda_f(\phi) = \int_\Omega f \phi dx$ so that $\Lambda_f \in H_0^1(\Omega)'$ as

$$|\Lambda_f(\phi)| \leq \|f\|_{L^2} \|\phi\|_{L^2} \leq \|f\|_{L^2} \|\phi\|_{H^1} \quad (\circ)$$

Hence by the Riesz representation theorem on $H_0^1(\Omega)$, there is some unique $v \in H_0^1(\Omega)$ such that

$$\langle v, \phi \rangle_{H^1} = \langle f, \phi \rangle_{L^2}$$

for all $\phi \in H_0^1(\Omega)$ as required. Now take $f_1, f_2 \in L^2(\Omega)$ and $a \in \mathbb{R}$, and take $v_i = S(f_i)$ for $i = 1, 2$. Define $v = v_1 + \alpha v_2$. Then

$$\begin{aligned} \langle v, \phi \rangle_{H^1} &= \langle v_1 + \alpha v_2, \phi \rangle_{H^1} \\ &= \langle v_1, \phi \rangle_{H^1} + \alpha \langle v_2, \phi \rangle_{H^1} \\ &= \langle f_1, \phi \rangle_{L^2} + \langle \alpha f_2, \phi \rangle_{L^2} \\ &= \langle f_1 + \alpha f_2, \phi \rangle_{L^2} \end{aligned}$$

and so $S(f_1) + \alpha S(f_2) = v = S(f_1 + \alpha f_2)$, hence S is linear. It is also bounded as

$$\begin{aligned} \|S(f)\|_{H^1} &= \|v_f\|_{H^1} \\ &\stackrel{\text{Riesz}}{=} \|\Lambda_f\|_{H^1 \rightarrow L^2} \\ &\stackrel{(\circ)}{\leq} \|f\|_{L^2} \end{aligned}$$

Finally for $g \in L^2(\Omega)$ we have

$$\begin{aligned} \langle S(f), g \rangle_{L^2} &= \langle g, S(f) \rangle_{L^2} \\ &\stackrel{(\dagger')}{=} \langle S(g), S(f) \rangle_{H^1} \\ &= \langle S(f), S(g) \rangle_{H^1} \\ &= \langle f, S(g) \rangle_{L^2} \end{aligned}$$

and so S is self adjoint □

To study further regularity properties of v , we introduce

Definition 6.3. We define

$$H_{loc}^s(\Omega) = \{f \in L_{loc}^2 \mid f \cdot \zeta \in H^s(\mathbb{R}^n), \forall \zeta \in C_c^\infty(\Omega)\}$$

Proposition 6.3. *If $f \in H_{loc}^s(\Omega)$ for $s > k + n/2$, $k, n \in \mathbb{N}$ then $f \in C^k(U)$ for any U open such that $\bar{U} \subseteq \Omega$*

Proof. Given U choose $\zeta \in C_c^\infty(\Omega)$ such that $\zeta = 1$ on $\bar{U}$ and note that $f \cdot \zeta \in H^s(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$ so as $f = f \cdot \zeta$ on U , the result follows $\square$

Corollary 6.4.1. $\bigcap_{s>0} H_{loc}^s(\Omega) \subseteq C^\infty(\Omega)$

Proof. Trivial $\square$

Note that $f \in C^\infty(\Omega)$ may still be unbounded at $\partial\Omega$, but we have

Theorem 6.5. *(Interior regularity) Let $f \in L^2(\Omega)$ and suppose $v \in H_0^1(\Omega)$ solves $(\dagger')$. Then $v \in H_{loc}^2(\Omega)$ and if $f \in L^2(\Omega) \cap H_{loc}^k(\Omega)$, then $v \in H_{loc}^{k+2}(\Omega)$*

Proof. Let $K \subseteq \Omega$ be any compact set and take $\chi \in C_c^\infty(\Omega)$ such that $\chi = 1$ on K . Take $\varphi \in S(\mathbb{R}^n)$ and set $\phi = \chi\varphi \in H_0^1(\Omega)$. Then $(\dagger')$ implies

$$\int_{\Omega} (Dv \cdot D(\varphi\chi) + v\varphi\chi) dx = \int_{\Omega} f\varphi\chi dx, \forall \varphi \in S(\mathbb{R}^n)$$

Using the chain rule and integration by parts, this rearranges to

$$\int_{\Omega} (D(v\chi) \cdot D\varphi + v\chi\varphi) dx = \int_{\Omega} g\varphi dx, \forall \varphi \in S(\mathbb{R}^n)$$

where $g = -Dv \cdot D\chi - v\Delta\chi + f\chi \in L^2(\mathbb{R}^n)$ by extending by 0, as $\chi \in C_c^\infty(\Omega)$. By another application of integration by parts, we see that $v\chi$ solves the equation

$$-\Delta(v\chi) + v\chi = g$$

in $S'(\mathbb{R}^n)$. Hence by the elliptic regularity estimate,

$$\|v\chi\|_{H^2} \leq \|g\|_{L^2} < \infty$$

Now to prove that $v \in H_{loc}^2(\Omega)$, take $\zeta \in C_c^\infty$ and $K = \text{supp } \zeta$ such that $v\zeta = v\chi\zeta$. Then

$$\|v\zeta\|_{H^2} = \|v\chi\zeta\|_{H^2} \leq C_n \|v\chi\|_{H^2} \|\zeta\|_{C^2}$$

(proved just as $\|fg\|_{L^2} \leq \|f\|_{L^2} \|g\|_{L^\infty}$ plus chain rule). We recognise that in fact $g \in H_{loc}^1(\Omega)$ whenever $f \in H_{loc}^1(\Omega)$ and so repeating the preceding argument gives $v \in H_{loc}^3(\Omega)$. Bootstrapping this and using the multiplier inequality

$$\|f \cdot g\|_{H^s} \leq C_{n,s} \|f\|_{H^s} \|g\|_{C^s}$$

(which is easy to prove), we are finished $\square$

Corollary 6.5.1. *If $f \in C^\infty(\Omega) \cap L^2(\Omega)$ then $v \in H_0^1(\Omega) \cap C^\infty(\Omega)$ solves*

$$-\Delta v + v = f$$

pointwise on Ω

Proof. Trivial □

Theorem 6.6. (*Rellich Kondrashov*) Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Let $u_j \in H_0^1(\Omega)$ such that $\|u_j\|_{H^1} \leq K$ for all $j \in \mathbb{N}$ and some $K > 0$. Then there is $u \in H_0^1(\Omega)$ such that $u_{j_k} \rightarrow u$ in $L^2(\Omega)$ along a subsequence

Proof. By Banach-Alaoglu, in the (reflexive) Hilbert space $H_0^1(\Omega)$ we obtain some subsequence such that $u_{j_k} \rightharpoonup u$ in $H_0^1(\Omega)$, and so it converges weakly in L^2 . Also u_{j_k}, u vanish almost everywhere on Ω^c so

$$\begin{aligned} \|u_{j_k} - u\|_{L^2(\Omega)}^2 &= \|u_{j_k} - u\|_{L^2(\mathbb{R}^n)}^2 \\ &= \frac{1}{(2\pi)^n} \|\hat{u}_{j_k} - \hat{u}\|_{L^2(\mathbb{R}^n)}^2 \\ &= \frac{1}{(2\pi)^n} \left(\int_{|z|>R} |\hat{u}_{j_k} - \hat{u}|^2 dz + \int_{|z|\leq R} |\hat{u}_{j_k} - \hat{u}|^2 dz \right) \end{aligned}$$

Given $\epsilon > 0$ arbitrary,

$$\begin{aligned} \int_{|z|>R} |\hat{u}_{j_k} - \hat{u}|^2 dz &\leq 2 \int_{|z|>R} \frac{1 + |z|^2}{1 + |z|^2} (|\hat{u}_{j_k}|^2 + |\hat{u}|^2) dz \\ &\leq \frac{2}{1 + R^2} (\|u_{j_k}\|_{H^1}^2 + \|u\|_{H^1}^2) \\ &\leq \frac{4K^2}{1 + R^2} \\ &< \epsilon \end{aligned}$$

and also, for $z \in \mathbb{R}^n$ fixed, we compute

$$\hat{u}_{j_k}(z) = \int_{\mathbb{R}^n} e^{-ix \cdot z} u_{j_k}(x) dx = \int_{\Omega} e^{-ix \cdot z} u_{j_k}(x) dx = \langle e^{-i\langle \cdot, z \rangle}, u_{j_k} \rangle_{L^2(\Omega)}$$

By weak convergence this converges to $\langle e^{-i\langle \cdot, z \rangle}, u \rangle_{L^2(\Omega)} = \hat{u}(z)$. Also

$$\begin{aligned} |\hat{u}_{j_k}(z)| + |\hat{u}(z)| &\leq \|u_{j_k}\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)} \\ &\stackrel{C-S}{\leq} C_{\Omega} (\|u_{j_k}\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \\ &\leq 2C_{\Omega} K \end{aligned}$$

, which is integrable on $\{z \mid |z| \leq R\}$ as the domain is bounded, so the result follows by dominated convergence □

Corollary 6.6.1. *The solution operator S from $(\dagger)$ is a compact linear self adjoint operator on $L^2(\Omega)$*

Proof. Use that S maps $L^2(\Omega)$ into $H_0^1(\Omega)$ in a bounded way, and the previous theorem □

Thus the spectral theorem from linear analysis implies there is an orthonormal basis of $L^2(\Omega)$, $\{\omega_k \mid k \in \mathbb{N}\}$ with real eigenvalues $\mu_k \downarrow 0$ approaching 0 such that

$$S\omega_k = \mu_k \omega_k \quad (S)$$

in $L^2(\Omega)$. Therefore $\omega_k \in H_0^1(\Omega)$ and for all $\phi \in H_0^1(\Omega)$,

$$\langle \omega_k, \phi \rangle_{L^2(\Omega)} \stackrel{(\dagger)}{=} \langle S\omega_k, \phi \rangle_{H^1(\Omega)} \stackrel{(S)}{=} \mu_k \langle \omega_k, \phi \rangle_{H^1(\Omega)}$$

Taking $\phi = \omega_k$ we see that $1 = \mu_k \|\omega_k\|_{H^1}^2$ so $\mu_k > 0$ for all $k \in \mathbb{N}$. Thus in $D'(\Omega)$,

$$(-\Delta + \text{Id})\omega_k = (-\Delta + \text{Id})\frac{\mu_k}{\mu_k}\omega_k = \frac{1}{\mu_k}(-\Delta + \text{Id})S\omega_k = \frac{\omega_k}{\mu_k}$$

and so

$$-\Delta\omega_k = \left(\frac{1}{\mu_k} - 1\right)\omega_k = \lambda_k \omega_k$$

with $\lambda_k = \left(\frac{1}{\mu_k} - 1\right) \uparrow \infty$ are the eigenvalues of $-\Delta$. Note that the weak form is

$$\langle D\omega_k, D\phi \rangle_{L^2} = \lambda_k \langle \omega_k, \phi \rangle_{L^2} \quad (\square)$$

for all $k \in \mathbb{N}$, $\phi \in D(\Omega)$, and in fact for $\phi \in H_0^1(\Omega)$.

Note that $\omega_k = f$ in $(\dagger)$ and in $H_0^1(\Omega)$ and so iterating interior regularity, we get $\omega_k \in C^\infty(\Omega)$. Thus $-\Delta\omega_k = \lambda_k \omega_k$ holds pointwise on Ω pointwise

Theorem 6.7. (*Poincare Inequality*) For all $u \in H_0^1(\Omega)$,

$$\frac{\langle Du, Du \rangle_{L^2}}{\langle u, u \rangle_{L^2}} \geq \lambda_1 > 0$$

Proof. Example sheet □

We can now solve the Dirichlet problem for the Laplace equation, which is to find v such that

$$\begin{aligned} -\Delta v &= f & \text{on } \Omega \\ v &= 0 & \text{on } \partial\Omega \end{aligned}$$

for given $f \in L^2(\Omega)$. In the weak form this is to find some $v \in H_0^1(\Omega)$ such that

$$\langle Dv, D\phi \rangle_{L^2} = \langle f, \phi \rangle_{L^2} \quad (\star)$$

Theorem 6.8. There is a unique solution $v \in H_0^1(\Omega)$ to $(\star)$, for any $f \in L^2(\Omega)$ and in fact

$$v = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle \omega_k, f \rangle_{L^2} \omega_k$$

Proof. Take partial sums $v_j = \sum_{k=1}^j \lambda_k^{-1} m\langle \omega_k, f \rangle_{L^2} \omega_k$ for $j \in \mathbb{N}$. Then

$$\begin{aligned}
\|v_j - v_{j'}\|_{H^1}^2 &= \langle v_j - v_{j'}, v_j - v_{j'} \rangle_{L^2} + \langle D(v_j - v_{j'}), D(v_j - v_{j'}) \rangle_{L^2} \\
&= \sum_{k=j'+1}^j \lambda_k^{-2} \langle \omega_k, f \rangle_{L^2}^2 \\
&\quad + \sum_{k, k'=j'+1}^j \lambda_k^{-1} \lambda_{k'}^{-1} \langle f, \omega_k \rangle_{L^2} \langle f, \omega_{k'} \rangle_{L^2} \langle D\omega_k, D\omega_{k'} \rangle_{L^2} \\
&\stackrel{(\square)}{=} \sum_{k=j'+1}^j \lambda_k^{-2} \langle \omega_k, f \rangle_{L^2}^2 + \sum_{k=j'+1}^j \lambda_k^{-1} \langle f, \omega_k \rangle_{L^2}^2 \\
&\leq \sum_{k=j'+1}^{\infty} (\lambda_k^{-2} + \lambda_k^{-1}) \langle \omega_k, f \rangle_{L^2}^2 \\
&\leq C(\lambda_1) \sum_{k=j'+1}^{\infty} \langle \omega_k, f \rangle_{L^2}^2 \\
&\rightarrow 0
\end{aligned}$$

So v_j is Cauchy in $H_0^1(\Omega)$, hence also $v \in H_0^1(\Omega)$. v solves $(\star)$ since

$$\begin{aligned}
\langle Dv, D\phi \rangle_{L^2} &= \langle v, -\Delta\phi \rangle_{L^2} \\
&= \sum_{k=1}^{\infty} \lambda_k^{-1} \langle v, \omega_k \rangle_{L^2} \langle \omega_k, -\Delta\phi \rangle_{L^2} \\
&\stackrel{(\square)}{=} \sum_{k=1}^{\infty} \langle v, \omega_k \rangle_{L^2} \langle \omega_k, \phi \rangle_{L^2} \\
&= \langle v, \phi \rangle_{L^2}
\end{aligned}$$

so we have proven existence. To prove uniqueness, suppose $v' \in H_0^1(\Omega)$ also solves $(\star)$ and let $w = v - v' \in H_0^1(\Omega)$ where

$$\langle Dw, D\phi \rangle_{L^2} = \langle f - f, \phi \rangle_{L^2} = 0$$

for all $\phi \in H_0^1(\Omega)$. Now

$$\begin{aligned}
\|w\|_{H^1}^2 &= \langle w, w \rangle_{L^2} + \langle Dw, Dw \rangle_{L^2} \\
&\leq \left(\frac{1}{\lambda_1} + 1 \right) \langle Dw, Dw \rangle_{L^2} \quad \text{By Poincaré} \\
&= 0
\end{aligned}$$

So $w = v - v' = 0$ almost everywhere, hence $v = v'$ □

Remark. Similarly to before, we can show interior regularity estimates to deduce that if $f \in C^\infty(\Omega) \cap L^2(\Omega)$ then $v \in C^\infty(\Omega)$

6.2 Variational Problems

Consider minimising a functional

$$F(u) = \|u\|_{H^1}^2 - 2\langle f, u \rangle_{L^2}$$

where $f \in L^2(\Omega)$ is fixed, over $H_0^1(\Omega)$.

Theorem 6.9. *Let $f \in L^2(\Omega)$ be fixed. Then*

$$\inf\{F(u) \mid u \in H_0^1(\Omega)\} \geq \sigma > -\infty$$

and there is a unique $w \in H_0^1(\Omega)$ such that $F(w) = \sigma$, and w solves (†')

Proof. We use the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ for $a, b \geq 0$. By Cauchy Schwartz,

$$\begin{aligned} F(u) &= \|u\|_{H^1}^2 - 2\langle f, u \rangle_{L^2} \\ &\geq \|u\|_{H^1}^2 - 2\|f\|_{L^2}\|u\|_{L^2} \\ &\geq \|u\|_{H^1}^2 - 2\|f\|_{L^2}^2 - \frac{1}{2}\|u\|_{L^2}^2 \\ &\geq \frac{1}{2}\|u\|_{H^1}^2 - 2\|f\|_{L^2}^2 \\ &\geq -2\|f\|_{L^2}^2 \\ &> -\infty \end{aligned}$$

Then taking $u_k \in H_0^1(\Omega)$ such that $F(u_k) \rightarrow \sigma$. Without loss of generality we may assume $|F(u_k)| \leq \bar{F} < \infty$. Thus

$$\begin{aligned} \|u_k\|_{H^1}^2 &= F(u_k) + 2\langle f, u_k \rangle_{L^2} \\ &\leq \bar{F} + 2\|f\|_{L^2}\|u_k\|_{L^2} \\ &\leq \bar{F} + 2\|f\|_{L^2}^2 + \frac{1}{2}\|u_k\|_{H^1}^2 \end{aligned}$$

and so for all k ,

$$\|u_k\|_{H^1}^2 \leq 2\bar{F} + 4\|f\|_{L^2}^2$$

so u_k is bounded in $H_0^1(\Omega)$. By Banach Alaoglu theorem, there is some subsequence $u_{k_j} \rightharpoonup w$ in $H_0^1(\Omega)$ and $L^2(\Omega)$ to some $w \in H_0^1(\Omega)$. By example sheet 2, we know that

$$\|w\|_{H^1}^2 \leq \liminf_{j \rightarrow \infty} \|u_{k_j}\|_{H^1}^2$$

and

$$\lim_{j \rightarrow \infty} \langle f, u_{k_j} \rangle_{L^2} = \langle f, w \rangle_{L^2}$$

Thus

$$\begin{aligned} F(w) &= \|w\|_{H^1}^2 - 2\langle f, w \rangle_{L^2} \\ &\leq \liminf_{j \rightarrow \infty} (\|u_{k_j}\|_{H^1}^2 - 2\langle f, u_{k_j} \rangle_{L^2}) \\ &= \liminf_{j \rightarrow \infty} F(u_{k_j}) = \sigma \end{aligned}$$

By definition of σ we have $\sigma \leq F(w)$, so $F(w) = \sigma$. To prove uniqueness, it suffices to show that w solves $(\dagger')$. Then for all $v \in H_0^1(\Omega)$ and $t \in \mathbb{R}$ we have $F(w) \leq F(w + tv)$ and $\frac{d}{dt}F(w + tv)|_{t=0} = 0$. Then

$$\begin{aligned} F(w + tv) &= \langle w + tv, w + tv \rangle_{H^1} - 2\langle f, w + tv \rangle_{L^2} \\ &= \|w\|_{H^1}^2 + t^2\|v\|_{H^1}^2 + 2t\langle w, v \rangle_{H^1} - 2t\langle f, v \rangle_{L^2} - 2\langle f, w \rangle_{L^2} \end{aligned}$$

Differentiating this we get

$$\frac{d}{dt}F(w + tv) = 2t\|v\|_{H^1}^2 + 2(\langle w, v \rangle_{H^1} - \langle f, v \rangle_{L^2})$$

so at $t = 0$ we have

$$\langle w, v \rangle_{H^1} = \langle f, v \rangle_{L^2}$$

for all $v \in H_0^1(\Omega)$, so w solves $(\dagger')$

□