

Category Theory

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1 Definitions and Examples

Definition 1.1. A *category* $\mathcal{C}$ consists of

1. A collection $\text{ob } \mathcal{C}$ of *objects*, $A, B, C, \dots$
2. A collection $\text{mor } \mathcal{C}$ of *morphisms*, $f, g, h, \dots$
3. Two operations dom, cod sending a morphism f to its *domain* and *codomain* respectively. We write $A \xrightarrow{f} B$ where f is a morphism, $A = \text{dom } f$ and $B = \text{cod } f$
4. An operation sending an object to its identity morphism $A \xrightarrow{1_A} A$
5. A partial binary operation $(f, g) \mapsto fg$ on pairs of morphisms which is defined if $\text{dom } f = \text{cod } g$, and in this case $\text{dom } fg = \text{dom } g$ and $\text{cod } fg = \text{cod } f$

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ & \searrow fg & \downarrow f \\ & & C \end{array}$$

subject to

1. $f1_A = f$ and $1_Bg = g$ whenever the composites are defined
2. $(fg)h = f(gh)$ whenever fg and gh are both defined

Remark. 1. $\text{ob } \mathcal{C}$ and $\text{mor } \mathcal{C}$ are not necessarily sets. If they are, we call $\mathcal{C}$ a *small* category

2. We could dispense with the objects, using the identity morphisms to do so, but we do not.
3. You could take the opposite (wrong) choice to denote fg for ‘first f , then g ’. We are not computer scientists (they use upside-down letters), so we will do things correctly.

Example 1.1.

1. Set, the category of sets, and functions between sets (strictly, the morphisms should be pairs of functions and assigned codomains - we will ignore similar complications in further examples)
2. Grp, the category of groups, and group homomorphisms.
Ring, the category of rings, and ring homomorphisms.
 Vect_k , the category of vector spaces over a field k and k -linear maps.
3. Top, the category of topological spaces and continuous functions.
Met, the category of metric spaces and non-expansive maps (maps f such that $d'(f(x), f(y)) \leq d(x, y)$).
Mfd, the category of smooth manifolds and smooth maps.
4. TopGrp, the category of topological groups and continuous group homomorphisms.
5. Htpy, the category of topological spaces and homotopy classes of continuous maps.
More generally, given an equivalence relation on morphisms such that equivalent morphisms have the same domain and codomain, and $f \simeq g$ implies $fh \simeq gh$ and $hf \simeq hg$ when defined, we can form the *quotient category* $\mathcal{C}/\simeq$, with the same objects as $\mathcal{C}$ and morphisms equivalence classes of $\text{mor } \mathcal{C}$.
6. Rel has the same objects as Set, but morphisms $A \rightarrow B$ are relations $R \subseteq A \times B$ with composition of $A \xrightarrow{R} B \xrightarrow{S} C$ by

$$SR = \{(a, c) \mid \exists b \in B. (a, b) \in R \text{ and } (b, c) \in S\}$$

Part, the category of sets and partial functions

7. For any category $\mathcal{C}$, the *opposite category* $\mathcal{C}^{op}$ has the same objects and morphisms as $\mathcal{C}$, but domain and codomain are interchanged, and (f, g) in $\mathcal{C}^{op}$ is (g, f) in $\mathcal{C}$. Hence we have a duality principle for categorical propositions, by interpreting theorems for $\mathcal{C}^{op}$.
8. A small category with one object $*$ is a monoid.
In particular, a group is a small category of one object whose morphisms are all isomorphisms
9. A *groupoid* is a category whose morphisms are all isomorphisms.
For example, the *fundamental groupoid* of a topological space, $\Pi(X)$ has points of X as objects, and morphisms $x \rightarrow y$ are homotopy classes of paths starting at x and ending at y
10. Suppose $\mathcal{C}$ is such that there is at most one morphism between any two objects. Then $\text{mor } \mathcal{C}$ becomes a binary relations on $\text{ob } \mathcal{C}$ via 'exists a morphism from x to y '. This must be reflexive and transitive, and we can reverse this construction to get a category for any preorder.
Therefore a partially ordered set is a small preorder whose only isomorphisms are identities.
11. Given a field k the category Mat_k has objects the natural numbers and morphisms $n \rightarrow p$ are $p \times n$ matrices in k , with composition by multiplication

Definition 1.2. A (covariant) functor $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of mappings $\text{ob } \mathcal{C} \xrightarrow{F} \text{ob } \mathcal{D}$ and $\text{mor } \mathcal{C} \xrightarrow{F} \text{mor } \mathcal{D}$ such that $F \text{ dom } f = \text{dom } Ff$, $F \text{ cod } f = \text{cod } Ff$, $F1_A = 1_{FA}$ and $F(fg) = F(f)F(g)$ where fg is defined

Example 1.2.

1. We have *forgetful functors* $\text{Grp} \rightarrow \text{Set}$, $\text{Ring} \rightarrow \text{Set}$, $\text{Top} \rightarrow \text{Set}$, $\text{Ring} \rightarrow \text{AbGrp}$ by ignoring the additional structure.
2. For any set A there is a group FA which contains a copy of A such that any set-map $A \rightarrow G$ for a group G can be extended to a unique group homomorphism $FA \rightarrow G$ (called the *free group* on A), and this extends to a functor $\text{Set} \rightarrow \text{Grp}$ via lifting $A \rightarrow B \rightarrow FB$ to $FA \rightarrow FB$. It is left to the reader to verify functoriality.
3. The powerset construction $A \rightarrow PA$ is a functor $\text{Set} \rightarrow \text{Set}$. Given $f: A \rightarrow B$, we define $Pf: PA \rightarrow PB$ by $Pf(A') = f(A')$.
4. There is also a powerset functor $P^*: \text{Set} \rightarrow \text{Set}^{op}$ defined by $P^*(A) = PA$ on objects and $P^*f(B') = f^{-1}(B')$ on morphisms $f: A \rightarrow B$
5. The dual space functor $\text{Vect}_k^{op} \rightarrow \text{Vect}_k$ sending a vector space to its dual, and the map $f: X \rightarrow Y$ is sent to $f^*: Y \rightarrow X$ which acts by precomposition

6. The mapping $\mathcal{C} \mapsto \mathcal{C}^{\text{op}}$ is a functor $\text{Cat} \rightarrow \text{Cat}$, where Cat is the category of small categories (note that this is well defined as we can compose functors)
7. A functor between monoids is a monoid homomorphism
8. A functor between posets is an order-preserving map
9. A functor from a group G (seen as a one object groupoid) to Set is a group action. A functor from a group G to Vect_k is a linear representation
10. The fundamental group construction $(X, x) \mapsto \pi_1(X, x)$ is a functor $\text{Top}_* \rightarrow \text{Grp}$ where Top_* is the space of topological spaces with a marked point with morphisms the continuous maps such that marked points are sent to marked points by

Definition 1.3. A *contravariant* functor $\mathcal{C} \rightarrow \mathcal{D}$ is a functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$

Definition 1.4. Suppose we have functors $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$. By a *natural transformation* $\alpha: F \rightarrow G$ we mean an operation assigning to each $A \in \text{ob } \mathcal{C}$ a morphism $FA \xrightarrow{\alpha_A} GA$ in $\mathcal{D}$ such that for each $f: A \rightarrow B$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes. We call this the *naturality square* for α and f

Remark. We can compose natural transformations $\alpha: F \rightarrow G$ and $\beta: G \rightarrow H$ pointwise to get a natural transformation $A \mapsto \beta_A \alpha_A$ from F to H . Therefore we can define the category of functors from $\mathcal{C}$ to $\mathcal{D}$ with morphisms natural transformations, denoted $[\mathcal{C}, \mathcal{D}]$

Example 1.3.

1. For any vector space V , we have a natural mapping $\alpha_V: V \rightarrow V^{**}$, where $\alpha_V(v)(f) = f(v)$. These define a natural transformation $\alpha: 1_{\text{Vect}_k} \rightarrow (\cdot)^{**}$
2. Let $U: \text{Grp} \rightarrow \text{Set}$ be the forgetful functor. The embeddings $A \hookrightarrow UFA$ for all sets A define a natural transformation from the identity functor to UF
3. For any set A , we have a mapping $\eta_A: A \rightarrow PA$ sending A to $\{a\}$. Then we have a natural transformation $1_{\text{Set}} \rightarrow P$.
4. For posets we have a natural transformations $f \rightarrow g$ if $f(x) \leq g(x)$ for all x in the domain

5. For group homomorphisms $G \xrightarrow[u]{u} H$, a natural transformation is an element h such that $hu(g) = v(g)h$, so that u and v are conjugate homomorphisms.
6. Given permutation representations of G on A, B , a natural transformation between them is a G -equivariant map of representations.
7. Given topological space X and a basepoint x , the n th Hurewicz homomorphism $h_n: \pi_n(X, x) \rightarrow H_n(X)$ induced by $\Delta^n \rightarrow S^n \rightarrow X$. It defines a natural transformation from $\pi_n \rightarrow H_n \circ U$ where U forgets the basepoint.

There is the obvious notion of *isomorphism* of categories, eg $\text{Rel} \cong \text{Rel}^{\text{op}}$ via $A \mapsto A, R \mapsto R^\circ = \{(a, b) \mid (b, a) \in R\}$. To define the weaker notion of equivalence, we need to talk about natural isomorphisms.

Lemma 1.1. *Let $\alpha: F \rightarrow G$ be a natural transformation between $F, G: \mathcal{C} \rightarrow \mathcal{D}$. Then α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$ if and only if each α_A is an isomorphism in $\mathcal{D}$.*

Proof. The forward direction holds as composition of natural transformations is pointwise. For the other direction suppose each α_D is an isomorphism with inverse β_D . We need to verify the naturality of β . Given $f: A \rightarrow B$ in $\mathcal{C}$, consider

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ \alpha_A \uparrow \downarrow \beta_A & & \alpha_B \uparrow \downarrow \beta_B \\ FA & \xrightarrow{Ff} & FB \end{array}$$

We have $\beta_B Gf = \beta_B Gf \alpha_A \beta_A = \beta_B \alpha_B Ff \beta_A = Ff \beta_A$ so we are done. $\square$

Definition 1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *equivalence* between $\mathcal{C}$ and $\mathcal{D}$ consists of functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms $\alpha: 1_{\mathcal{C}} \rightarrow GF, \beta: FG \rightarrow 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ if there exists such an equivalence.

Remark. Morally, anything we care about in category theory should behave well under equivalence.

Example 1.4.

1. The category Part of sets and partial functions is equivalent to the category Set_* of sets with basepoints, with $F: \text{Set}_* \rightarrow \text{Part}$ defined by $F((A, *)) = A \setminus \{*\}$ and for $f: A \rightarrow B$, Ff is defined by $Ff(x) = f(x)$ if $f(x) \neq *$, and undefined otherwise. In the other direction, $G: \text{Part} \rightarrow \text{Set}_*$ sends $A \mapsto (A \cup \{A\}, A)$ and $G(A \xrightarrow{f} B)(x) = f(x)$ if f is defined, and $f(x) = B$ otherwise. $FG = 1_{\text{Part}}$, and $GF \cong 1_{\text{Set}_*}$ by sending $A \in A \cup \{A\}$ to $*$, and acting as the identity otherwise. Note that these are not isomorphic, as Part has an isomorphism class consisting of one object $\emptyset$, but Set_* does not.

2. The category of finite dimensional vector spaces fd Vect_k is equivalent to $\text{fd Vect}_k^{\text{op}}$ via the dual-space functor in both directions, and the natural transformation $1_{\text{fd Vect}_k} \rightarrow (\cdot)^{**}$.
3. fd Vect_k is also equivalent to Mat_k . The functor $F: \text{Mat}_k \rightarrow \text{fd Vect}_k$ sends n to k^n , and a matrix B to the linear map it represents with respect to the standard basis. The functor $G: \text{fd Vect}_k \rightarrow \text{Mat}_k$ is defined by choosing a basis of each vector space, and define $G(V) = \dim V$ on objects, and $G(g)$ is the matrix of g with respect to the chosen bases. $GF = 1_{\text{Mat}_k}$ provided we choose the standard basis for k^n (otherwise it is essentially conjugation, which is naturally isomorphic to the identity). $FG \not\cong 1_{\text{fd Vect}_k}$, but the chosen basis for V yields an isomorphism $k^{\dim V} = FG(V) \cong V$ and these form a natural isomorphism as desired.

Definition 1.6. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. F is *faithful* if given $A \xrightarrow[g]{f} B$ in $\mathcal{C}$, $Ff = Fg$ implies $f = g$. It is *full* if given $g: FA \rightarrow FB$ in $\mathcal{D}$ there is some $f: A \rightarrow B$ in $\mathcal{C}$ with $Ff = g$. F is *essentially surjective* if for every $B \in \text{ob } \mathcal{D}$ there is $A \in \text{ob } \mathcal{C}$ such that B is isomorphic to FA .

Remark. Note that if F is full and faithful then it is essentially injective, given an isomorphism $g: FA \rightarrow FB$ the unique $f: A \rightarrow B$ with $Ff = g$ is an isomorphism with inverse the unique $h: B \rightarrow A$ with $Fh = g^{-1}$.

Lemma 1.2. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D}$ if and only if F is full, faithful and essentially surjective.

Proof. In the forwards direction, given G, α, β demonstrating F as an equivalence, and given $B \in \text{ob } \mathcal{D}$, β_B demonstrates an isomorphism $FGB \rightarrow B$ so F is essentially surjective. Given $f: A \rightarrow B$, $f = \alpha_B^{-1}(GFf)\alpha_A$, so f is recoverable from Ff , its domain and its codomain, so F is faithful. Suppose we have

$g: FA \rightarrow FB$. Then the composite $f: A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B$ satisfies $GFf = Gg$, so as G is faithful (symmetrically to F), $g = Ff$.

Suppose that F is full, faithful, and essentially surjective. For each $B \in \text{ob } \mathcal{D}$ choose an object $GB \in \text{ob } \mathcal{C}$ and an isomorphism $\beta_B: FGB \rightarrow B$ in $\mathcal{D}$. Given a morphism $g: B \rightarrow C$ in $\mathcal{D}$ define $Gg: GB \rightarrow GC$ to be the unique morphism

whose image under F is $FGB \xrightarrow{\beta_B} B \xrightarrow{g} C \xrightarrow{\beta_C^{-1}} FGC$. Functoriality follows from uniqueness, as $G(fg)$ and $G(f)G(g)$ are morphisms with the same image under F , so they're equal. By construction, β is a natural isomorphism from $FG \rightarrow \text{id}$. We define $\alpha_A: A \rightarrow GFA$ to be the unique morphism whose image

under F is $FA \xrightarrow{\beta_{FA}^{-1}} FGFA \xrightarrow{\beta_{FA}^{-1}} FGFB$. α_A is an isomorphism as F is faithful, and the image of a naturality square for α , under F , is a naturality square for β^{-1} , and therefore the square commutes. $\square$

Definition 1.7. We say a category $\mathcal{C}$ is *skeletal* if each isomorphism class in $\text{ob } \mathcal{C}$ has only one member. By a *skeleton* of $\mathcal{C}$, we mean a full subcategory (ie one where the inclusion functor is full) with exactly one object from each isomorphism class.

Example 1.5. Mat_k is skeletal, and the full subcategory of fdVect_k on the spaces k^n is a skeleton of fdVect_k .

Definition 1.8. Let $f: A \rightarrow B$ be a morphism in a category $\mathcal{C}$. We say f is a *monomorphism* if $fg = fh$ implies $g = h$ for $C \begin{smallmatrix} \xrightarrow{g} \\ \xrightarrow{h} \end{smallmatrix} A$, and f is an *epimorphism* if $hf = gf$ implies $h = g$. We denote monomorphisms with $A \xrightarrow{f} B$ and epimorphisms with $A \xrightarrow{f} B$. If every function which is monic and epic is an isomorphism, then $\mathcal{C}$ is *balanced*.

Example 1.6.

1. In Set , monic is the same as injective (using the correspondence between elements of A and morphisms $1 \rightarrow A$). Epic is the same as surjective (using the correspondence between subsets of A and morphisms $A \rightarrow \{0, 1\}$), so Set is balanced.
2. In Grp monic is equivalent to injective (using the argument of above, with $\mathbb{Z}$ instead of 1), and epic is equivalent to surjective (but proof is non-trivial, needs free products with amalgamation), so Grp is balanced.
3. In Ring , monic is equivalent to injective (using $\mathbb{Z}[X]$), but $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic, so Ring isn't balanced.
4. In Top , monic is equivalent to injective and epic is equivalent to surjective, with similar proof to Set (endowing the indiscrete topology on $\{0, 1\}$), but Top is not balanced.
5. In a preorder, every morphism is monic and epic, so it is balanced if and only if it is a groupoid.

We will see stronger notions of injectivity and surjectivity later.

2 The Yoneda Lemma

It is doubtful whether the Yoneda lemma is either a lemma, or due to Yoneda.

Definition 2.1. A category is *locally small* if for any two objects $A, B \in \text{ob } \mathcal{C}$, the morphisms from A to B form a set.

Example 2.1. Set , Grp , Top are all locally small, but functor categories $[\mathcal{C}, \mathcal{D}]$ need not be.

For any object A of a locally small category, we have a functor $\mathcal{C}(A, \cdot): \mathcal{C} \rightarrow \text{Set}$ which sends $B \in \text{ob } \mathcal{C}$ to $\mathcal{C}(A, B) \in \text{Set}$, and $f \in \mathcal{C}(B, C)$ is sent to $g \mapsto fg: \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$ (functoriality follows from the associative law). Dually, we have a functor $\mathcal{C}(\cdot, B): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ for each $B \in \text{ob } \mathcal{C}$.

Lemma 2.1. (*Yoneda lemma*) *Let $A \in \text{ob } \mathcal{C}$ where $\mathcal{C}$ is a locally small category, and let $F: \mathcal{C} \rightarrow \text{Set}$ be a functor. Then*

1. *There is a bijection between natural transformations $\mathcal{C}(A, \cdot) \rightarrow F$ and elements of FA .*
2. *Moreover, the bijection is natural in A and F .*

Proof. We prove 1 first, and defer the proof of 2 to later.

Given a natural transformation $\alpha: \mathcal{C}(A, \cdot) \rightarrow F$, consider $\Phi(\alpha) = \alpha_A(1_A) \in FA$. Given $x \in FA$ we define $\Psi(x): \mathcal{C}(A, \cdot) \rightarrow F$ by $\Psi(x)_B(A \xrightarrow{f} B) = (Ff)(x) \in FB$. Naturality of $\Psi(x)$ follows as F is a functor. We have $\Phi\Psi(x) = \Psi(x)_A(1_A) = F(1_A)(x) = x$ and $\Psi\Phi(\alpha)_B(A \xrightarrow{f} B) = (Ff)(\Phi(\alpha)) = (Ff)\alpha_A(1_A) = \alpha_B(\mathcal{C}(A, f)(1_A)) = \alpha_B(f)$, so Φ, Ψ are inverse bijections. $\square$

Corollary 2.1.1. *The mapping $A \rightarrow \mathcal{C}(A, \cdot)$ defines a full and faithful functor $\mathcal{C}^{\text{op}} \rightarrow [\mathcal{C}, \text{Set}]$*

Proof. Putting $F = \mathcal{C}(B, \cdot)$ in the Yoneda lemma, we get a bijection from $\mathcal{C}(B, A)$ to the collection of natural transformations $\mathcal{C}(A, \cdot) \rightarrow \mathcal{C}(B, \cdot)$ which sends f to precomposition by f , and hence is functorial. $\square$

Dually, the mapping $A \rightarrow \mathcal{C}(\cdot, A)$ is a full and faithful functor $\mathcal{C} \rightarrow [\mathcal{C}^{\text{op}}, \text{Set}]$. This functor is called the *Yoneda embedding*.

Definition 2.2. We say a functor $\mathcal{C} \rightarrow \text{Set}$ is *representable* if it is isomorphic to $\mathcal{C}(A, \cdot)$ for some A . By a representation of a functor $F: \mathcal{C} \rightarrow \text{Set}$ we mean a pair (A, x) where $A \in \text{ob } \mathcal{C}$ and $x \in FA$ such that $\Phi(x): \mathcal{C}(A, \cdot) \rightarrow F$ is an isomorphism. We call x a *universal element* of F .

Corollary 2.1.2. *If $(A, x), (B, y)$ are representations of $F: \mathcal{C} \rightarrow \text{Set}$, then there is a unique isomorphism $f: A \rightarrow B$ in $\mathcal{C}$ such that $Ff(x) = y$*

Proof. The equation $Ff(x) = y$ says

$$\begin{array}{ccc} \mathcal{C}(B, \cdot) & \xrightarrow{\Phi(f)} & \mathcal{C}(A, \cdot) \\ & \searrow \Phi(y) & \downarrow \Phi(x) \\ & & F \end{array}$$

commutes so f is the unique isomorphism in $\mathcal{C}$ whose image under the Yoneda embedding is $\Phi(x)^{-1}\Phi(y)$ $\square$

Proof. (of the second part of the Yoneda lemma) Suppose $\mathcal{C}$ is small so $[\mathcal{C}, \text{Set}]$ is locally small. We have two functors $\mathcal{C} \times [\mathcal{C}, \text{Set}] \rightarrow \text{Set}$. The first sends $(A, F) \rightarrow FA$ and $(A \xrightarrow{f} B, F \xrightarrow{\alpha} G)$ to the diagonal of

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

. The other is the composite $\mathcal{C} \times [\mathcal{C}, \text{Set}] \xrightarrow{y \times 1} [\mathcal{C}, \text{Set}]^{\text{op}} \times [\mathcal{C}, \text{Set}] \xrightarrow{[\mathcal{C}, \text{Set}](\cdot, \cdot)} \text{Set}$ where y is the Yoneda embedding. The naturality statement is that Φ, Ψ are natural transformations between these. What it says is that given $x \in FA$, if $y \in GB$ is its image under $\alpha_B(Ff)$, then $\Psi(y)$ is the composite $\mathcal{C}(B, \cdot) \xrightarrow{y(f)} \mathcal{C}(A, \cdot) \xrightarrow{\Psi(x)} F \xrightarrow{\alpha} G$. This statement makes sense in any not-necessarily-small category, and maps $1_B \mapsto f \mapsto (Ff)(x) \mapsto \alpha_B(Ff)(x) = y$ $\square$

Example 2.2.

1. The forgetful functor $\text{Grp} \rightarrow \text{Set}$ is representable by $(\mathbb{Z}, 1)$, $\text{Ring} \rightarrow \text{Set}$ by $(\mathbb{Z}[x], x)$ and $\text{Top} \rightarrow \text{Set}$ by 1
2. The functor $P^*: \text{Set}^{\text{op}} \rightarrow \text{Set}$ is representable by $\{0, 1\}$, but $P: \text{Set} \rightarrow \text{Set}$ is not representable since $P(\{*\})$ has two elements
3. The functor $\Omega: \text{Top} \rightarrow \text{Set}$ sending X to the set of its open subsets and $f: A \rightarrow B$ to $U \mapsto f^{-1}(U)$ is representable by the Sierpinski space $\{0, 1\}$ with open sets $\emptyset, \{1\}, \{0, 1\}$
4. The functor $\text{Vect}_k^{\text{op}} \xrightarrow{(\cdot)^*} \text{Vect}_k \xrightarrow{U} \text{Set}$ is representable by the 1 dimensional space k
5. For a group G considered as a category, the unique representable functor $G \rightarrow \text{Set}$ is the Cayley representation, where G acts on itself by left multiplication
6. Given A, B in a locally small category $\mathcal{C}$, we have a functor $\mathcal{C}^{\text{op}} \rightarrow \text{Set}$ sending $C \in \text{ob } \mathcal{C}$ to $\mathcal{C}(C, A) \times \mathcal{C}(C, B)$. We denote this representing object by $A \times B$ and call it a (categorical) *product* of A and B . The universal element of this is a pair $(A \times B \xrightarrow{\pi_1} A, A \times B \xrightarrow{\pi_2} B)$ with the property that for any $(C \xrightarrow{f} A, C \xrightarrow{g} B)$ there is a unique $C \xrightarrow{(f, g)} A \times B$ with $\pi_1(f, g) = f$ and $\pi_2(f, g) = g$. Dually we get the notion of a *coproduct* (or *sum*) with coprojections $(A \xrightarrow{\nu_1} A + B, B \xrightarrow{\nu_2} A + B)$
7. Given a parallel pair of morphisms $A \rightrightarrows B$ in a locally small $\mathcal{C}$, the assignment $C \mapsto \{h: C \rightarrow A \mid fh = gh\}$ defines a subfunctor of $\mathcal{C}(\cdot, A)$.

If this is representable, we call the representing object $(E, E \xrightarrow{e} A)$ an *equaliser* of (f, g) . If this does exist, then e must be a monomorphism since any $C \xrightarrow{h} A$ factors through e uniquely. We call monomorphisms occurring as equalisers *regular monomorphisms*. Dually we get the notions of *coequalisers* and *regular epimorphisms*.

Remark. In Set categorical products exist (and are just Cartesian products), coproducts exist (and are disjoint unions). Equalisers exist (and are the inclusion of the set on which $A \xrightarrow[f]{g} B$ agree, and the coequaliser is the quotient map $B \rightarrow B/R$ where R is the minimal equivalence relation such that $g(x) \sim f(x)$ for all $x \in A$.

In Set all monomorphisms and epimorphisms are regular, but in Top monomorphisms $X \xrightarrow{f} Y$ are regular if and only if X has the subspace topology, and epimorphisms are regular if and only if Y has the quotient topology (so not all monomorphisms and epimorphisms are regular)

Definition 2.3. Let $\mathcal{G}$ be a class of objects of a locally small category $\mathcal{C}$

1. We say $\mathcal{G}$ is a *separating family* if the functors $\mathcal{C}(G, \cdot)$ for $G \in \mathcal{G}$ are jointly faithful, ie given $A \xrightarrow[f]{g} B$ if $fh = gh$ for all $h: G \rightarrow A$ with $G \in \mathcal{G}$, then $f = g$
2. We say $\mathcal{G}$ is a *detecting family* if the functors $\mathcal{C}(G, \cdot)$ for $G \in \mathcal{G}$ jointly reflect isomorphisms, ie given $A \xrightarrow{f} B$, if every $G \xrightarrow{h} B$ with $G \in \mathcal{G}$ factors uniquely through f , then f is an isomorphism

If $\mathcal{G} = \{G\}$, we call G a *separator/detector* in the corresponding cases

Lemma 2.2. *If $\mathcal{C}$ has equalisers, then any detecting family in $\mathcal{C}$ is separating. If $\mathcal{C}$ is balanced, then any separating family is detecting.*

Proof. Let $\mathcal{G}$ be a detecting family and suppose $A \xrightarrow[f]{g} B$ satisfies the hypotheses of Definition 2.3(1). Then any $G \xrightarrow{h} A$ with $G \in \mathcal{G}$ factors uniquely through the equaliser of f, g , so the equaliser is an isomorphism, so $f = g$ and we are done with the first statement.

Let $\mathcal{G}$ be separating, and suppose $A \xrightarrow{f} B$ satisfies Definition 2.3(2). If $C \xrightarrow[g]{h} A$ satisfies $fg = fh$, then any $G \xrightarrow{k} C$ with $G \in \mathcal{G}$ satisfies $gk = hk$ since both are factorisations of $fgk = fhk$ through f , so $g = h$ and f is monic. Similarly if $B \xrightarrow[l]{m} D$ has $lf = mf$, then any $G \xrightarrow{n} B$ with $G \in \mathcal{G}$ satisfies $ln = mn$ since it factors through f . So $l = m$, and f is epic, so by balancedness, f is an isomorphism. $\square$

Example 2.3.

1. In $\mathbf{Set}$, $1 = \{x\}$ is a separator and a detector since represents $1_{\mathbf{Set}}: \{0, 1\}$ is a coseparator/codetector since it represents $P^*: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$ which is faithful and reflects isomorphisms.
2. In $\mathbf{Grp}$, $\mathbb{Z}$ is a separator/detector since it represents the forgetful functor. $\mathbf{Grp}$ has no codetecting set, as there are simple groups of arbitrary cardinality. If we take one whose cardinality is larger than that of any element of $\mathcal{G}$, then any map from $\mathcal{G}$ to that group is trivial, so $\mathcal{G}$ can't detect that $H \rightarrow 1$ isn't an isomorphism
3. For any small category $\mathcal{C}$, $\{\mathcal{C}(A, \cdot) \mid A \in \text{ob } \mathcal{C}\}$ is a separating/detecting set for $[\mathcal{C}, \mathbf{Set}^{\text{op}}]$ by Yoneda
4. In $\mathbf{Top}$, 1 is a separating set as it represents the forgetful functor to $\mathbf{Set}$, but $\mathbf{Top}$ has no detecting set as given a set $\mathcal{G}$ of spaces and κ is a larger cardinality than that of any set in $\mathcal{G}$, we can find two different topologies on a set of cardinality κ which agree on any subspace of smaller cardinality, so $\mathcal{G}$ can't detect the fact they're different.
5. Let $\mathcal{C}$ be the category with $\text{ob } \mathcal{C} = \mathbf{Ord}$, the class of ordinals, and $\mathcal{C}(\alpha, \beta) = \emptyset$ if $\alpha > \beta$, 1_α if $\alpha = \beta$ and $\{f, g\}$ if $\alpha < \beta$ with composition given by $f^2 = fg = gf = g^2 = f$. Then 0 is a detector for $\mathcal{C}$ as it can tell neither of $0 \xrightarrow[f]{g} \alpha$ are isomorphisms since neither factors through the other, and it can tell $\alpha \xrightarrow[f]{g} \beta$ aren't isomorphisms since $0 \xrightarrow{g} \beta$ doesn't factor through either, but $\mathcal{C}$ has no separating set, as if $\mathcal{G}$ is a set of ordinals and $\alpha > \gamma$ for all $\gamma \in \mathcal{G}$, then $\mathcal{G}$ can't separate $\alpha \xrightarrow[f]{g} \alpha + 1$

Any functor of the form $\mathcal{C}(A, \cdot)$ preserves monomorphisms, but they don't all preserve epimorphisms

Definition 2.4. In a locally small category $\mathcal{C}$, we say an object P is *projective* if, given the solid lines in the diagram

$$\begin{array}{ccc} & P & \\ & \downarrow f & \\ Q & \xrightarrow[g]{\quad} & R \end{array}$$

with g epic, then there is a $h: P \rightarrow Q$ with $gh = f$. If this holds for all g in some class of epimorphisms $\mathcal{E}$, we call P *$\mathcal{E}$ -projective*. Dually, P is *injective* in $\mathcal{C}$ if it is projective in $\mathcal{C}^{\text{op}}$.

Remark. We will consider the class of pointwise epimorphisms in $[\mathcal{C}, \mathbf{Set}]$, ie those $F \xrightarrow{\alpha} G$ such that α_A is surjective for all A (This will be proven to be the same as the class of epimorphisms in $[\mathcal{C}, \mathbf{Set}]$ in chapter 4)

Corollary 2.2.1. *For any small $\mathcal{C}$, the representable functors are pointwise projective in $[\mathcal{C}, \text{Set}]$*

Proof. Given

$$\begin{array}{ccc} & \mathcal{C}(A, \cdot) & \\ & \downarrow \alpha & \\ F & \xrightarrow[\beta]{\twoheadrightarrow} & G \end{array}$$

α corresponds to some $x \in GA$. If $y \in FA$ satisfies $\beta_A(y) = x$, then $\Psi(y): \mathcal{C}(A, \cdot) \rightarrow F$ completes the triangle. $\square$

Proposition 2.1. *For a small category $\mathcal{C}$, then for any $F: \mathcal{C} \rightarrow \text{Set}$, there is a pointwise epimorphism $G \rightarrow F$ where G is pointwise projective*

Proof. Given F , we define $G = \coprod_{(A,x)} \mathcal{C}(A, \cdot)$ ranging over $A \in \mathcal{C}$ and $x \in FA$. As G is a coproduct in $[\mathcal{C}, \text{Set}]$, ie morphisms $G \rightarrow H$ correspond to families of morphisms $\mathcal{C}(A, \cdot) \rightarrow H$, this implies G is pointwise projective. But the morphism $G \rightarrow F$ which is the natural transformation $\Psi(x)$ on the (A, x) -component is pointwise epic since $x = \Psi(x)_A(1_A)$ $\square$

3 Adjunctions

Definition 3.1. An *adjunction* consists of a pair of functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$, $\mathcal{D} \xrightarrow{G} \mathcal{C}$ together with, for each pair $A \in \text{ob } \mathcal{C}, B \in \text{ob } \mathcal{D}$ a bijection between morphisms $A \rightarrow GB$ and morphisms $FA \rightarrow B$ which are natural in A and B (ie $\mathcal{C}(\cdot, G\cdot)$ and $\mathcal{D}(F\cdot, \cdot)$ are naturally isomorphic functors $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \text{Set}$). We say F is *left adjoint* to G and G is *right adjoint* to F and we write $F \dashv G$

Example 3.1.

1. The free group functor $F: \text{Set} \rightarrow \text{Grp}$ is left adjoint to the forgetful functor $U: \text{Grp} \rightarrow \text{Set}$ since maps $A \rightarrow UG$ extend uniquely to homomorphisms $FA \rightarrow G$.
2. The forgetful functor $\text{Top} \xrightarrow{U} \text{Set}$ has a left adjoint D , which gives a set the discrete topology and a right adjoint I which gives a set the indiscrete topology
3. $\text{ob}: \text{Cat} \rightarrow \text{Set}$ has a left adjoint D sending A to the discrete category and a right adjoint sending A to the category with one morphism between any pair of objects. D has a further left adjoint sending $\mathcal{C}$ to its set of connected components, the quotient by smallest equivalence relation identifying $\text{dom } f, \text{cod } f$ for all f
4. For any set A , $(\cdot) \times A$ is a functor $\text{Set} \rightarrow \text{Set}$ which has a right adjoint $\text{Set}(A, \cdot)$. More generally if $\mathcal{C}$ is a category with binary products, we say $\mathcal{C}$ is *Cartesian closed* if $(\cdot) \times A$ has a right adjoint $(\cdot)^A$ for all $A \in \text{ob } \mathcal{C}$. For example Cat is Cartesian closed with $\mathcal{D}^{\mathcal{C}}$ taken to be $[\mathcal{C}, \mathcal{D}]$

5. Let $M = \{1, e\}$ with $e^2 = e$. An M -set is a pair (A, e) where $e: A \rightarrow A$ satisfies $e^2 = e$. We have $F: \text{Set} \rightarrow [M, \text{Set}]$ sending A to $(A, 1_A)$ and a functor $G: [M, \text{Set}] \rightarrow \text{Set}$ sending (A, e) to $\{a \in A \mid e(a) = a\}$. Then $(F \dashv G)$ and $(G \dashv F)$ but F, G don't form an equivalence
6. If 1 is the discrete category with one object, a left adjoint for $\mathcal{C} \rightarrow 1$ picks out an *initial object* (an object I such that for any $A \in \text{ob } \mathcal{C}$ there is a unique $I \rightarrow A$) and a right adjoint picks out a *terminal object*, which is the dual notion.
7. Let $A \xrightarrow{f} B$ be a function in Set . Then $Pf: PA \rightarrow PB$ is left adjoint to $P^*f: PB \rightarrow PA$ since $Pf(A') \subseteq B'$ is equivalent to $A' \subseteq P^*f(B')$
8. Given a relation $R \subseteq A \times B$ we have mappings $(\cdot)^r: PA \rightarrow PB$ and $(\cdot)^l: PB \rightarrow PA$ by $(S)^r = \{b \in B \mid sRb, \forall s \in S\}$ and $(S)^l = \{b \in B \mid bRs, \forall s \in S\}$. Then $S \subseteq T^l$ is equivalent to $T \subseteq S^r$ so the two functors are adjoint on the right
9. The functor $P: \text{Set}^{\text{op}} \rightarrow \text{Set}$ is self-adjoint on the right as functions $A \rightarrow PB$ and $B \rightarrow PA$ both correspond bijectively to relations $R \subseteq A \times B$
10. Similarly for the dual vector space functor

Theorem 3.1. Suppose we have $G: \mathcal{D} \rightarrow \mathcal{C}$. For each object A of $\mathcal{C}$, let $A \downarrow G$ denote the category whose objects are pairs $(B, f: A \xrightarrow{f} GB)$ and whose morphisms $(B, f) \rightarrow (B', f')$ makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & GB \\ & \searrow f' & \downarrow Gg \\ & & GB' \end{array}$$

commute. Then specifying a left adjoint for G is equivalent to defining an initial object of $A \downarrow G$ for each A

Proof. If G has a left adjoint F , for each A let $\eta_A: A \rightarrow GFA$ correspond to $FA \xrightarrow{1_{FA}} FA$. Then (FA, η_A) is initial in $A \downarrow G$ since

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ & \searrow f & \downarrow Gg \\ & & GB \end{array}$$

commutes if and only if (transposing over the adjunction) f corresponds to $g(1_{FA}) = g$ under the adjunction. For the converse, suppose we have an initial object (FA, η_A) of $A \downarrow G$ for each A , and on morphisms $A \xrightarrow{f} A'$ we define Ff

to be the unique morphism from (FA, η_A) to $(FA', \eta_{A'}f)$ in $A \downarrow G$. Then

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow \eta_A & & \downarrow \eta_{A'} \\ GFA & \xrightarrow{GFf} & GFA' \end{array}$$

commutes for all f , and functoriality of F follows from uniqueness. The adjunction bijection sends $A \xrightarrow{f} GB$ to the unique morphism from the initial object to (B, f) in $A \downarrow G$, with inverse sending $FA \xrightarrow{g} B$ to $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$. This is natural in A as η is natural, and in B as G is functorial. $\square$

Corollary 3.1.1. *If F, F' are both left adjoint to $G: \mathcal{D} \rightarrow \mathcal{C}$, then $F \cong G$ in $[\mathcal{C}, \mathcal{D}]$*

Proof. We know (FA, η_A) and $(F'A, \eta'_A)$ are both initial in $A \downarrow G$, so there is a unique isomorphism between them, called α_A . This is natural as given $f: A \rightarrow A'$, the composites $\alpha_{A'}(Ff)$ and $(F'f)\alpha_A$ are both morphisms $(FA, \eta_A) \rightarrow (F'A', \eta'_{A'}f)$ in $A \downarrow G$, so they're equal $\square$

Lemma 3.2. *Suppose we are given*

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{H} \\ \xleftarrow{K} \end{array} \mathcal{E}$$

with $F \dashv G$ and $H \dashv K$. Then $HF \dashv GK$

Proof. We have bijections $\mathcal{C}(A, GKC) \rightarrow \mathcal{D}(FA, KC) \rightarrow \mathcal{E}(HFA, C)$ which are natural in A, C . $\square$

Corollary 3.2.1. *Suppose we have a commutative diagram of categories and functors*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ G \downarrow & & \downarrow H \\ \mathcal{E} & \xrightarrow{K} & \mathcal{F} \end{array}$$

and all the functors have left adjoints. Then the square of left adjoints commutes up to natural isomorphism

Proof. The two ways around the left adjoint diagram are left adjoint to $KG = HF$ by Lemma 3.2, so are isomorphic by Corollary 3.1.1 $\square$

Given $F \dashv G$, the natural transformation $\eta: 1_{\mathcal{C}} \rightarrow GF$ in the proof of Theorem 3.1 is called the *unit*, and dually we have the *counit* $\varepsilon: FG \rightarrow 1_{\mathcal{D}}$ where ε_B corresponds to 1_{GB} under the adjunction

Theorem 3.3. Given functors $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$, specifying an adjunction $F \dashv G$ is equivalent to specifying natural transformations $\eta: 1_{\mathcal{C}} \rightarrow GF$ and $\varepsilon: FG \rightarrow 1_{\mathcal{D}}$ such that the two diagrams

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow 1_F & \downarrow \varepsilon_F \\ & & F \end{array} \quad \begin{array}{ccc} G & \xrightarrow{\eta_G} & GFG \\ & \searrow 1_G & \downarrow G\varepsilon \\ & & G \end{array}$$

commute. We call these the triangular identities.

Proof. Given $F \dashv G$, we have seen how to define η, ε . Now $\varepsilon_{FA}(F\eta_A)$ corresponds to $1_{GFA}\eta_A = \eta_A$ under the adjunction, so it is 1_{FA} . The second identity is dual.

Conversely given η, ε and $A \xrightarrow{f} GB$, we define $\Phi(f)$ to be $FA \xrightarrow{Ff} FGB \xrightarrow{\varepsilon_B} B$ and given $FA \xrightarrow{g} B$ we define $\Psi(g)$ to be $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$. Then $\Psi(\Phi(f))$ is $A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\varepsilon_B} GB$, which by naturality of η is the same as $A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\varepsilon_B} GB$ which is f , so $\Phi\Psi(f) = f$, and dually $\Psi\Phi(f) = f$, and Φ, Ψ are natural as η, ε are $\square$

When we defined equivalence of categories in section 1, we had natural isomorphisms $\alpha: 1_{\mathcal{C}} \rightarrow GF$ and $\beta: FG \rightarrow 1_{\mathcal{D}}$

Proposition 3.1. Suppose there is an equivalence $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ and natural isomorphisms α, β as above. Then there are natural isomorphisms α', β' as above which also satisfy the triangular identities. In particular, $F \dashv G$ and $G \dashv F$

Proof. We define $\alpha' = \alpha$ and take β' to be $FG \xrightarrow{\beta_{FG}^{-1}} FGFG \xrightarrow{(F\alpha_G)^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}}$. Note $FG\beta = \beta_{FG}$ as

$$\begin{array}{ccc} FGFG & \xrightarrow{FG\beta} & FG1_{\mathcal{D}} \\ \downarrow \beta_{FG} & & \downarrow \beta_{1_{\mathcal{D}}} \\ FG & \xrightarrow{\beta} & 1_{\mathcal{D}} \end{array}$$

by naturality, and β is monic. The triangular identities for α', β' are

$$\begin{array}{ccccc} F & \xrightarrow{F\alpha} & FGF & \xrightarrow{(\beta_{FGF})^{-1}} & FGFGF \\ & \searrow 1_F & \downarrow (F\alpha)^{-1} & & \downarrow (F\alpha_{GF})^{-1} \\ & & F & \xrightarrow{(\beta_F)^{-1}} & FGF \\ & & & \searrow 1_F & \downarrow \beta_F \\ & & & & F \end{array}$$

and

$$\begin{array}{ccccc}
G & \xrightarrow{\alpha_G} & GFG & \xrightarrow{(GFG\beta)^{-1}} & GFGFG \\
& \searrow 1_G & \downarrow (\alpha_G)^{-1} & & \downarrow (GF\alpha_G)^{-1} = (\alpha_{GFG})^{-1} \\
& & G & \xrightarrow{(G\beta)^{-1}} & GFG \\
& & & \searrow 1_G & \downarrow G\beta \\
& & & & G
\end{array}$$

so α', β' are the unit and counit of an adjunction $F \dashv G$, so done. $\square$

Lemma 3.4. *Let $F \dashv G$ be an adjunction with counit ε . Then*

1. ε is pointwise epic if and only if G is faithful
2. ε is an isomorphism if and only if G is full and faithful

Proof. (1) Given $B \xrightarrow{g} B'$ in $\mathcal{D}$, $Gg = (Gg)1_{GB}$ so corresponds under the adjunction to $g\varepsilon_B$, so ε_B is epic if and only if G acts injectively on morphisms with domain B and specified codomain, so the result follows

(2) Similarly, G is full and faithful if and only if for all B , the mapping $g \mapsto g\varepsilon_B$ is bijective. This implies there is a unique $g: B \rightarrow FGB$ so that $g\varepsilon_B = 1_{FGB}$ and $\varepsilon_B g = 1_B$ as $\varepsilon_B g\varepsilon_B = \varepsilon_B$, so G is full and faithful if and only if every ε_B is an isomorphism, so we are done. $\square$

Definition 3.2. A *reflection* is an adjunction satisfying the condition of 3.4(ii). A *reflective subcategory* is a full subcategory such that the inclusion has a left adjoint. The *reflector* is that left adjoint.

Example 3.2.

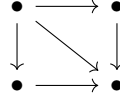
1. AbGrp is reflective in Grp, the reflector sends $G \mapsto G/[G, G]$. Any homomorphism from G into an abelian group factors uniquely through $G/[G, G]$
2. Recall that a group is called torsion if every element has finite order, and torsion free if the only element of finite order is the identity. In an abelian group the elements of finite order form a subgroup A_t and any homomorphism $B \rightarrow A$ with B torsion takes values in A_t , so $A \mapsto A_t$ defines a right adjoint to the inclusion of abelian torsion groups into the category of abelian groups, and hence tAbGrp is coreflective, and similarly the category of torsion free abelian groups is reflective.
3. Let CHaus $\subseteq$ Top be the full subcategory of compact Hausdorff spaces. The inclusion has a left adjoint, called the Stone-Ćech compactification.
4. We say a subset $A \subseteq X$ is *sequentially closed* if it is closed under taking limits of sequences. We say a space X is *sequential* if the set of sequentially closed sets and the set of closed sets are the same. The subcategory of such spaces is coreflective in Top: Given X define X_S to be the same set with topology given by having closed sets the sequentially closed subsets of X . The identity $X_S \rightarrow X$ is the counit of the adjunction.

5. In $\mathbf{Cat}$, the subcategory of preorders is reflective. The reflector sends $\mathcal{C}$ to $\mathcal{C}/\simeq$ where $f \simeq g$ if f, g are parallel.
6. For any space X , the poset $\Omega(X)$ of open subsets of X is coreflective in PX as if U is an open set and A is a subset, then $U \subseteq A$ if and only if $U \subseteq A^\circ$.

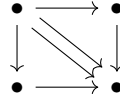
4 Limits and Colimits

Definition 4.1. Let J be a category (almost always small, often finite). By a *diagram of shape J* in $\mathcal{C}$ we mean a functor $D: J \rightarrow \mathcal{C}$. The objects $D(j)$ for $j \in \text{ob } J$ are called *vertices* and morphisms $D(f)$ for $f \in \text{mor } J$ are called *edges*.

Example 4.1. If J is the category

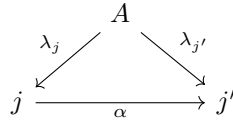


(plus identity morphisms, which will be omitted from here on), a diagram of shape J is a commutative square. If J is



where one diagonal arrow is the composite of the top and right, and the other the composite of the bottom and left, is a not-necessarily-commutative square

Definition 4.2. Let $D: J \rightarrow \mathcal{C}$ be a diagram. A *cone* over D consists of an object $A \in \text{ob } \mathcal{C}$ (its *apex*) together with morphisms $\lambda_j: A \rightarrow D_j$ for all $j \in \text{ob } J$, called the *legs* of the cone, satisfying $D(\alpha)\lambda_j = \lambda_{j'}$ for $\alpha: j \rightarrow j'$ in $\text{mor } J$, ie many copies of the following commute



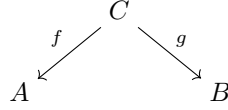
A *morphism* of cones $(A, \{\lambda_j \mid j \in \text{ob } J\}) \rightarrow (B, \{\mu_j \mid j \in \text{ob } J\})$ is a morphism $f: A \rightarrow B$ such that $\mu_j f = \lambda_j$ for all $j \in \text{ob } J$. We denote $\text{Cone}(D)$ for the category of cones over D . A *limit* for D is a terminal object in this category if it exists.

Dually we have *cones under D* and *colimits* of D (initial objects in the category of cones under D).

Remark. We have a functor $\Delta: \mathcal{C} \rightarrow [J, \mathcal{C}]$ sending A to the constant diagram with vertices A and edges 1_A . A cone over D with apex A is the same as a natural transformation $\Delta A \rightarrow D$. In fact, $\text{Cone}(D)$ is another name for the category $(\Delta \downarrow D)$. $\mathcal{C}$ has limits for all digrams of shape J if and only if $\Delta: \mathcal{C} \rightarrow [J, \mathcal{C}]$ has a right adjoint.

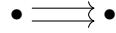
Example 4.2.

1. If $I = \emptyset$ then a cone over $D: \emptyset \rightarrow \mathcal{C}$ is just an object and a limit is a terminal object
2. If J is the discrete category on two objects, a diagram of shape J is a pair of objects A, B . A cone over it is a span

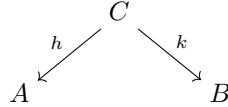


A limit is a product and a colimit is a coproduct.

3. If J is a small discrete category, a J -shaped diagram is $|\text{ob } J|$ many objects, and the limit (resp. colimit) is the product (resp. coproduct) of them
4. If J is

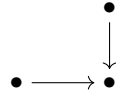


, a diagram of shape J is a parallel pair $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$, and a cone over it is a span

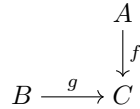


with $fh = gh = k$, or equivalently a morphism $C \xrightarrow{h} A$ with $fh = gh$. So a (co)limit is a (co)equaliser

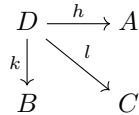
5. If J is



then a diagram of shape J is a cospan



then a cone over it is



with $l = fh = gk$, or equivalently a span

$$\begin{array}{ccc} & D & \\ h \swarrow & & \searrow k \\ A & & B \end{array}$$

completing the diagram to a commutative square, and a limit for it is called a *pullback*. If we have binary products and equalisers, we may construct this by forming $A \times B$ and taking the equaliser of the two maps formed into C and dually, colimits of shape J^{op} are *pushouts*

6. If J is the two element monoid $\{1, e\}$ with $e^2 = e$, then a J shaped diagram is an object A with an idempotent endomorphism $e: A \rightarrow A$. A cone over such a diagram is a morphism $B \xrightarrow{f} A$ satisfying $ef = f$, and a limit is the monic part of a splitting of e (resp. colimit is the epic part)
7. If $J = (\mathbb{N}, \leq)$, a diagram of shape J is a direct sequence $A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \dots$ and a colimit for this is a *direct limit* A_∞ with morphisms from each A_i which is universal such that the diagram commutes

$$\begin{array}{ccccccc} A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \dots \\ & & & & \searrow & \searrow & \searrow \\ & & & & & & A_\infty \end{array}$$

Dually a diagram of shape $\mathbb{N}^{\text{op}}$ is called an inverse sequence and a limit for it is an *inverse limit*. For example, an infinite-dimensional CW complex X is the direct limit in Top of its n -skeletons. The ring $\mathbb{Z}_p$ of p -adic integers is the inverse limit of $\mathbb{Z}/p^n\mathbb{Z}$ (with maps given by reduction mod p^k)

Proposition 4.1.

1. If $\mathcal{C}$ has equalisers and all small products, then $\mathcal{C}$ all small limits
2. If $\mathcal{C}$ has equalisers and all finite products, then $\mathcal{C}$ has all finite limits
3. If $\mathcal{C}$ has all pullbacks and a terminal object, then $\mathcal{C}$ has all finite limits

Proof. (1) and (2): Suppose we have a diagram $D: J \rightarrow \mathcal{C}$. Form the products $P = \prod_{j \in \text{ob } J} D(j)$ and $Q = \prod_{\alpha \in \text{mor } J} D(\text{cod } \alpha)$. We have morphisms

$$P \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Q \quad \text{defined by } \pi_\alpha f = \pi_{\text{cod } \alpha} \text{ and } \pi_\alpha g = D(\alpha)\pi_{\text{dom } \alpha}.$$

We form the equaliser of f, g , $E \xrightarrow{e} P$ and let $\lambda_j = \pi_j e: E \rightarrow D(j)$. These form a cone over D as for any $\alpha: j \rightarrow j'$ in D , $D(\alpha)\lambda_j = D(\alpha)\pi_j e = \pi_{\alpha \text{ cod } \alpha} e = \pi_{\alpha \text{ dom } \alpha} f e = \pi_{\alpha \text{ dom } \alpha} g e = \pi_{\alpha \text{ dom } \alpha} \lambda_{j'} = \lambda_{j'}$. This is a limit cone as given any cone (A, μ_j) over D , we get a unique factorisation $\mu: A \rightarrow P$ with $\pi_j \mu = \mu_j$, and μ has equal composite with f, g as the μ_j are compatible with the edges of the diagram, so μ factors uniquely through e .

(3): We show the hypotheses imply the existence of equalisers and finite products. If we denote the terminal object by 1 , a pullback of

$$\begin{array}{ccc} & A & \\ & \downarrow & \\ B & \longrightarrow & 1 \end{array}$$

is exactly a product of A and B as the compatibility criterion is trivial, hence it has the universal property of a binary product. We construct finite products inductively. The equaliser of $A \begin{smallmatrix} f \\ \longrightarrow \\ g \end{smallmatrix} B$ is exactly the limit of

$$\begin{array}{ccc} & A & \\ & \downarrow 1 \times f & \\ A & \xrightarrow{1 \times g} & A \times B \end{array}$$

so we are done. $\square$

Definition 4.3. Let $G: \mathcal{D} \rightarrow \mathcal{C}$ be a functor and J a small category.

1. We say G *preserves* limits of shape J if given $D: J \rightarrow \mathcal{D}$ and a limit $(L, (\lambda_j \mid j \in \text{ob } J))$, then $(GL, (G\lambda_j \mid j \in \text{ob } J))$ is a limit for GD
2. We say G *reflects* limits of shape J if J if given $D: J \rightarrow \mathcal{D}$ and a cone $(L, (\lambda_j \mid j \in \text{ob } J))$ over D which maps to a limit in $\mathcal{C}$, then the cone is in fact a limit.
3. We say G *creates* limits of shape J if given $D: J \rightarrow \mathcal{D}$ and a limit $(L, (\lambda_j \mid j \in \text{ob } J))$ of GD in $\mathcal{C}$, there is a cone $(L', (\lambda'_j \mid j \in \text{ob } J))$ in $\mathcal{D}$ whose image is isomorphic to $(L, (\lambda_j \mid j \in \text{ob } J))$ and any such cone is a limit in $\mathcal{D}$.

Corollary 4.0.1. *In each version of Proposition 4.1, we can replace ‘ $\mathcal{C}$ has’ by ‘ $\mathcal{D}$ has and $G: \mathcal{D} \rightarrow \mathcal{C}$ preserves’ or by ‘ $\mathcal{C}$ has and $G: \mathcal{D} \rightarrow \mathcal{C}$ creates’*

Definition 4.4. We call a category *(co)complete* if it has all small (co)limits

Example 4.3.

1. The forgetful functor $\text{Grp} \rightarrow \text{Set}$ creates all small limits. Given a family of groups $\{G_i \mid i \in I\}$ there is a unique group structure on their product as sets which makes that set the product in Grp , and the equalisers are inclusions of subgroups, so the same holds there. This does not preserve or reflect colimits, as $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is infinite, and their disjoint union has 4 elements, but it does create some colimits
2. The forgetful functor $\text{Top} \rightarrow \text{Set}$ preserves all small limits and colimits, but it doesn’t reflect them (as there are non-limit cones in Top that map to limit cones in Set , just topologise the limit set differently)

3. The inclusion $\text{AbGrp} \rightarrow \text{Grp}$ reflects coproducts, but doesn't preserve them as $A * B$ is always non-abelian if A, B are non-trivial, and $1 * B = B = B \oplus 1$
4. If $\mathcal{D} \subseteq \mathcal{C}$ is a reflective subcategory, the the inclusion $\mathcal{D} \rightarrow \mathcal{C}$ creates limits (and even in the strict sense if $\mathcal{D}$ is isomorphism-closed). Suppose we have $D: J \rightarrow \mathcal{D}$ and a limit cone $(L, \lambda_j \mid j \in \text{ob } J)$ in $\mathcal{C}$. Then the composite $FL \xrightarrow{F\lambda_j} FD(j) \xrightarrow{\eta_{D(j)}^{-1}} D(j)$ gives a cone over D (where F is the adjoint to the inclusion), and so this induces a unique $FL \xrightarrow{f} L$. Now $f\eta_L = 1_L$ as it's a factorisation of the limit cone through itself, and $\eta_L f = 1_{LF}$ for the same reasons. So we have given a limit cone for D in $\mathcal{C}$, and hence for D in $\mathcal{D}$
5. If $\mathcal{D}$ has limits of shape D , then so does any functor category $[\mathcal{C}, \mathcal{D}]$, and the forgetful functor $[\mathcal{C}, \mathcal{D}] \rightarrow \mathcal{D}^{\text{ob } \mathcal{C}}$ creates them (ie we can form pointwise limits). Given $D: J \rightarrow [\mathcal{C}, \mathcal{D}]$ we can regard it as a functor $J \times \mathcal{C} \rightarrow \mathcal{D}$, and for each $A \in \text{ob } \mathcal{C}$, $D(\cdot, A)$ is a J -shaped diagram in $\mathcal{D}$, which therefore has a limit $(LA, \lambda_{j,A} \mid j \in \text{ob } J)$. Given $A \xrightarrow{f} B$ in $\mathcal{C}$, the composites $LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$ are a cone over $D(\cdot, B)$, so we get a unique morphism $LA \xrightarrow{Lf} LB$, and functoriality holds by uniqueness. This is the unique way of making $A \mapsto LA$ a functor such that the $\lambda_{j,\cdot}$ are natural transformations. This is a limit cone as given a cone $(M, \mu_{j,\cdot} \mid j \in \text{ob } J)$. Fixing A , we get a cone of MA over $D(\cdot, A)$, so we get a map $MA \xrightarrow{\nu_A} LA$ for each $A \in \text{ob } J$ and the naturality squares commute by uniqueness of factorisations through limits.

Remark. In any category if $A \xrightarrow{f} B$ is monic if and only if

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ 1_A \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback, so we conclude from 4.7(e) that if $\mathcal{D}$ has pullbacks, then any monomorphism in $[\mathcal{C}, \mathcal{D}]$ is pointwise monic

Lemma 4.1. *If $G: \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint, then G preserves all limits which exist in $\mathcal{D}$*

Proof. (Proof 1) If $\mathcal{C}, \mathcal{D}$ both have limits of shape J . If $F \dashv G$, then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \Delta \downarrow & & \downarrow \Delta \\ [J, \mathcal{C}] & \xrightarrow{[J, F]} & [J, \mathcal{D}] \end{array}$$

So by Corollary 3.2.1,

$$\begin{array}{ccc} [J, \mathcal{D}] & \xrightarrow{[J, G]} & [J, \mathcal{C}] \\ \downarrow \lim & & \downarrow \lim \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

commutes up to isomorphism, so G preserves limits of shape J

(Proof 2) Suppose we have $D: J \rightarrow \mathcal{D}$ and a limit cone $(L, \lambda_j \mid j \in \text{ob } J)$ for it.

Given a cone $(A, A \xrightarrow{\alpha_j} GDj)$ over GD in $\mathcal{C}$, the transposes $FA \xrightarrow{\bar{\alpha}_j} D\lambda_j$ form a cone over D , so they induce a unique $f_i: FA \rightarrow L$ such that $\lambda_j \bar{f} = \bar{\alpha}_j$ for all j .

Then $A \xrightarrow{f} GL$ is the unique morphism satisfying $(G\lambda_j)f = \alpha_j$ by transposing again. $\square$

The ‘primitive’ adjoint functor theorem says that if $\mathcal{D}$ has and $G: \mathcal{D} \rightarrow \mathcal{C}$ preserves all limits, then G has a left adjoint (see later)

Lemma 4.2. *If $\mathcal{D}$ has and $G: \mathcal{D} \rightarrow \mathcal{C}$ preserves all limits of shape J , then for each $A \in \text{ob } \mathcal{C}$, the category $A \downarrow G$ has limits of shape J and the forgetful functor $A \downarrow G \rightarrow \mathcal{D}$ creates them*

Proof. Suppose we have $D: J \rightarrow A \downarrow G$, write $D(j) = (UD(j), \delta_j)$ and suppose we are given a limit $(L, \lambda_j: L \rightarrow UD(j))$ for UD . Then we claim $(GL, G\lambda_j)$ is a limit for GUD . The edges of D are morphisms of $A \downarrow G$ and $\delta_j: A \rightarrow GUD(j)$ form a cone over GUD with apex A , so induce a unique $A \xrightarrow{f} GL$. Then (L, f) is the apex of a cone over D which is the unique lifting of $(X_j \mid j \in \text{ob } J)$ to a cone in $A \downarrow G$. The argument that this is a limit is as in Example 4.3(5) $\square$

Lemma 4.3. *Let $\mathcal{C}$ be a category. Specifying an initial object of $\mathcal{C}$ is equivalent to specifying a limit cone for $\mathcal{C} \xrightarrow{1_{\mathcal{C}}} \mathcal{C}$*

Proof. Suppose $I \in \text{ob } \mathcal{C}$ is initial. Then we have a cone over $1_{\mathcal{C}}$ via the unique morphisms to every other object. This is a limit as given any other cone with apex B , B factors through I via the I component of the limit cone.

Conversely given a limit cone $(I, (f_A \mid A \in \text{ob } \mathcal{C}))$, I is weakly initial as we have a morphism to any other object. For any $g: I \rightarrow A$, we must have $gf_I = f_A$. In particular, $f_A f_I = f_A$ for all A , so f_I is a factorisation of the limit cone through itself and hence is the identity. Hence we deduce that $g = f_A$, and the morphism is unique. $\square$

Theorem 4.4. (General Adjoint Functor Theorem) *Suppose $\mathcal{D}$ is locally small and complete. Then a functor $f: \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint if and only if G preserves small limits and satisfies the solution set condition: for any $A \in \text{ob } \mathcal{C}$, there is a set $\{A \xrightarrow{f_i} GB_i \mid i \in I\}$ which is collectively weakly initial in $A \downarrow G$*

Proof. The forwards direction is Lemma 4.1 and Theorem 3.1, since if $F \dashv G$ then $\{A \xrightarrow{\eta_A} GFA\}$ is a singleton solution set for $A \downarrow G$.

For the backwards direction, Lemma 4.2 gives us that $A \downarrow G$ is complete and

it's locally small since morphisms $(B, f) \rightarrow (B', f')$ are a subset of morphisms $B \rightarrow B' \in \mathcal{D}$. We have therefore reduced the problem to proving that if $\mathcal{A}$ is locally small and complete with a weakly initial set of objects $\{S_i \mid i \in I\}$, then it has an initial object. We form $P = \prod_{i \in I} S_i$. This is weakly initial as every object admits a morphism from some S_i . Now form the limit $I \rightarrow P$ of the diagram

$$P \begin{array}{c} \xrightarrow{\quad} \\ \vdots \\ \xrightarrow{\quad} \end{array} P$$

where the edges are all morphisms $P \rightarrow P$. I is clearly weakly initial, and is initial as if we have $I \xrightarrow[g]{f} T$ we can form the equaliser $E \xrightarrow{e} I$. There is a morphism $h: P \rightarrow E$, then ieh and 1_P appear in the equaliser diagram, so $ieh = i$ and hence as i is monic, $ehi = 1_I$, so e is split epic, and $f = g$ $\square$

Example 4.4.

1. Consider $U: \text{Grp} \rightarrow \text{Set}$. Grp has and U preserves all small limits, by 4.3(1) and Grp is locally small, so we need to verify the solution set condition. Given a set A and any mapping $A \rightarrow UG$, this factors as $A \xrightarrow{f} UG' \rightarrow UG$ where G' is the subgroup generated by $f(A)$, and $\text{card } G' \leq \max\{\aleph_0, \text{card } A\}$. Take a fixed set B of this cardinality and consider all subsets $B' \subseteq B$ and all mappings $A \rightarrow B'$. These form a solution set at A .
2. Consider the forgetful functor $\text{CLat} \xrightarrow{U} \text{Set}$ where CLat is the category of complete lattices (posets with meets and joins) and maps preserving the meets and joins. This satisfies the hypotheses of Theorem 4.4 apart from the solution set criterion, but for $A = \{a, b, c\}$ there is no solution set at A (A. Hales 1965)

Remark. Knowing the solution set condition for an explicit functor G is usually enough to give an explicit construction of its left adjoint

Remark. If a locally small category $\mathcal{C}$ has all small coproducts, then a functor $G: \mathcal{C} \rightarrow \text{Set}$ has a left adjoint if and only if it is representable, as representability says $1 \downarrow G$ has an initial object and we get an initial object for $A \downarrow G$ by taking the coproduct of A copies of this object. Having small coproducts is necessary as $\text{CLat} \xrightarrow{U} \text{Set}$ is representable by the 3-element complete lattice, but the coproduct of 3 copies of this object doesn't exist.

Definition 4.5. A *subobject* of an object A , we mean a monomorphism with codomain A . We write $\text{Sub}_{\mathcal{C}}(A)$ for the collection of all subobjects of A , which has a natural preorder by $m \leq m'$ if m factors through m' . $\mathcal{C}$ is *well-powered* if every $\text{Sub}_{\mathcal{C}}(A)$ is equivalent to a small poset, equivalently there is a set $\{A_i \xrightarrow{m_i} A \mid i \in I\}$ meeting every isomorphism class in $\text{Sub}_{\mathcal{C}}(A)$

Example 4.5. Set is well powered as any subobject is isomorphic to the inclusion of a subset. It is also well copowered as epimorphisms $A \twoheadrightarrow B$ are represented by equivalence relations on A up to isomorphism.

Lemma 4.5. *Given a pullback square*

$$\begin{array}{ccc} D & \xrightarrow{h} & A \\ \downarrow k & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

with f monic, k is also monic

Proof. Suppose we have $E \xrightleftharpoons[m]{l} D$ with $kl = km$. Then $fhl = gkl = gkm = fhm$, so $hl = hm$, so l, m are both factorisations of (hl, kl) and hence are equal. $\square$

Theorem 4.6. (*Special Adjoint Functor Theorem*) *Suppose $\mathcal{C}, \mathcal{D}$ are both locally small, and $\mathcal{D}$ is complete and well-powered, and has a coseparating set of object. Then $G: \mathcal{D} \rightarrow \mathcal{C}$ has a left adjoint if and only if it preserves all small limits*

Proof. The forwards implication is Lemma 4.1. Fix $A \in \text{ob } \mathcal{C}$. Then $A \downarrow G$ inherits completeness and local smallness from $\mathcal{D}$, and is well-powered since subobjects of (B, f) are all subobjects of B (such that f factors through the image of the monomorphism under G). It also has a coseparating set as if $\mathcal{S}$ is coseparating in $\mathcal{D}$, then $\{(S, f) \mid S \in \mathcal{S}, f: A \rightarrow GS\}$ is coseparating in $A \downarrow G$. So we're reduced to the following claim: If $\mathcal{A}$ is locally small, complete, well powered and has a coseparating set $\mathcal{S}$ then $\mathcal{A}$ has an initial object. We first form $P = \prod_{S \in \mathcal{S}} S$. Now consider the diagram

$$\begin{array}{ccc} P'' & & P' \\ & \searrow & \downarrow \\ \vdots & & P \\ & \nearrow & \\ P^{(n)} & & \end{array}$$

where edges are all the members of a representative set of subobjects of P . If I is the apex of a limit cone for this, all the maps $I \rightarrow P^{(k)}$ of the limit cone must be monic by the argument of Lemma 4.5, and hence in particular $I \rightarrow P$ is monic, and it is a smallest subobject of P . If we had $I \xrightleftharpoons[g]{f} T$, then their equaliser would be a subobject of I , and hence of P , and hence isomorphic to I , so $f = g$. Given any $A \in \text{ob } \mathcal{A}$, consider $Q = \prod_{(S, f)} S$ indexed by pairs $S \in \mathcal{S}$ and $f: A \rightarrow S$, and the morphism $g: A \rightarrow Q$ defined by $\pi_{(S, f)} g = f$. That $\mathcal{S}$ is coseparating is equivalent to g being monic. We also have $h: P \rightarrow Q$ defined by $\pi_{(S, f)} h = \pi_S$, so we can form the pullback

$$\begin{array}{ccc} B & \xrightarrow{k} & A \\ p \downarrow & & \downarrow g \\ P & \xrightarrow{h} & Q \end{array}$$

and by Lemma 4.5, $B \rightarrow P$ is monic, hence $(I \rightarrow P) \leq p$, and so there is a morphism $I \rightarrow B \xrightarrow{k} A$ $\square$

Example 4.6. We show the inclusion $I: \mathbf{KHaus} \rightarrow \mathbf{Top}$ has a left adjoint. Tychonoff's theorem says $\mathbf{KHaus}$ has and I preserves all small products. The equaliser in $\mathbf{Top}$ of $X \xrightleftharpoons[g]{f} Y$, a diagram in $\mathbf{KHaus}$, is a closed subspace of X and hence is compact. So we have all small limits and I preserves them as $\mathbf{KHaus}$ is a full subcategory. $\mathbf{KHaus}$ is well-powered as subobjects are closed subspaces. Urysohn's Lemma says that $[0, 1]$ is a coseparator for $\mathbf{KHaus}$, and so the special adjoint functor theorem applies.

Remark.

1. This is essentially how Čech constructed β . Given X , Čech formed the product of a coseparating set for $X \downarrow I$ and took βX to be the closure of the image $X \rightarrow P$
2. We can form a solution set of β , and hence use the general adjoint functor theorem. Note that any $f: X \rightarrow IY$ factors as $X \rightarrow IY' \rightarrow IY$ where Y' is the closure of the image of X and if a Hausdorff space Y' has a dense subspace X' of cardinality at most κ , then the cardinality of Y' is at most 2^{2^κ} , so we can take a set of this size and coseparating set all compact Hausdorff topologies on sets at most this large.

5 Monads

Suppose we have an adjunction

$$\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$$

with $F \dashv G$, how much can we see from the point of view of $\mathcal{C}$? We can see the composite $GF = T: \mathcal{C} \rightarrow \mathcal{C}$, unit $\eta: 1_{\mathcal{C}} \rightarrow T$ and the natural transformation $\mu = G\varepsilon_F: TT \rightarrow T$. The triangular identities for η, ε give commutative diagrams

$$\begin{array}{ccc} T & \xrightarrow{T\eta} & TT \\ & \searrow 1_T & \downarrow \mu \\ & & T \end{array} \quad (1)$$

$$\begin{array}{ccc} T & \xrightarrow{\eta T} & TT \\ & \searrow 1_T & \downarrow \mu \\ & & T \end{array} \quad (2)$$

and naturality of $G\varepsilon$ yields

$$\begin{array}{ccc} TTT & \xrightarrow{T\mu} & TT \\ \downarrow \mu_T & & \downarrow \mu \\ TT & \xrightarrow{\mu} & T \end{array} \quad (3)$$

Definition 5.1. Let $\mathcal{C}$ be a category. A *monad* $\mathbb{T} = (T, \eta, \mu)$ on $\mathcal{C}$ consists of a functor $T: \mathcal{C} \rightarrow \mathcal{C}$ and natural transformations $\eta: 1_{\mathcal{C}} \rightarrow T, \mu: TT \rightarrow T$ satisfying 1-3

Example 5.1.

1. Let M be a monoid. The functor $M \times (\cdot): \text{Set} \rightarrow \text{Set}$ has a monad structure. The unit $\eta_A: A \rightarrow M \times A$ sends $a \mapsto (1, a)$ and $\mu_A: M \times M \times A \rightarrow M \times A$ sends $(m, m', a) \mapsto (mm', a)$. Diagrams 1-3 come from the unit and associative laws, and in fact monoids are monads on Set whose underlying functors have a right adjoint (Consider where $\{*\}$ is sent).
2. The covariant power set functor $P: \text{Set} \rightarrow \text{Set}$ has a monad structure. The unit $\eta_A: A \rightarrow PA$ sends $a \mapsto \{a\}$ and $\mu: PPA \rightarrow PA$ sends $\mathcal{A} \subseteq PA$ to $\bigcup \mathcal{A}$. The monad identities are easy to verify.

Does every monad come from an adjunction? Note that both of the above come from adjunctions. For (1) it's

$$\text{Set} \xrightleftharpoons[U]{F} [M, \text{Set}]$$

with M acting on FA by multiplication on the left factor. For (2) it's

$$\text{Set} \xrightleftharpoons[U]{F} \text{CSLat}$$

where CSLat is the category of complete (join-)semilattices since any $f: A \rightarrow UB$ extends uniquely to $\bar{f}: PA \rightarrow B$ sending A' to the join of all its elements

Definition 5.2. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. An (*Eilenberg-Moore*) *algebra* for $\mathbb{T}$ is a pair (A, α) where $A \in \text{ob } \mathcal{C}$ and $\alpha: TA \rightarrow A$ satisfies

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow 1_A & \downarrow \alpha \\ & & A \end{array}$$

$$\begin{array}{ccc} TTA & \xrightarrow{T\alpha} & TA \\ \downarrow \mu_A & & \downarrow \alpha \\ TA & \xrightarrow{\alpha} & A \end{array}$$

commuting. A homomorphism $f: (A, \alpha) \rightarrow (B, \beta)$ is a morphism $f: A \rightarrow B$ such that

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \downarrow \alpha & & \downarrow \beta \\ A & \xrightarrow{f} & B \end{array}$$

commutes. We denote the category of $\mathbb{T}$ -algebras and homomorphisms

Proposition 5.1. *The forgetful functor $\mathcal{C}^{\mathbb{T}} \xrightarrow{G^{\mathbb{T}}} \mathcal{C}$ has a left adjoint $G^{\mathbb{T}}$ and the adjunction induces $\mathbb{T}$*

Proof. We define $F^{\mathbb{T}}A = (TA, \mu_A)$ which is an algebra by 2, 3 and $F^{\mathbb{T}}(A \xrightarrow{f} B) = Tf$ which is a homomorphism by naturality of μ . This is clearly functorial and $G^{\mathbb{T}}F^{\mathbb{T}} = T$. To construct $(F^{\mathbb{T}} \dashv G^{\mathbb{T}})$ we construct a unit and counit. The unit is η and the counit is $F^{\mathbb{T}}A \xrightarrow{\varepsilon_{(A, \alpha)}} (A, \alpha)$ is α which is a homomorphism by the second requirement for algebras and is natural by the definition of a homomorphism. The triangular identities are 1 and the unit requirement for algebras, and multiplication of the induced monad is $G^{\mathbb{T}}\varepsilon_{F^{\mathbb{T}}A} = \mu_A$ by definition. $\square$

There is another adjunction inducing the monad of Example 5.1(1), namely

$$\text{Set} \xrightleftharpoons[P]{I} \text{Rel}$$

Note that I is bijective on objects. We aim to generalise this. Given an adjunction

$$\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$$

inducing $\mathbb{T}$, we can replace $\mathcal{D}$ by its full subcategory on the image of F , and hence we may assume F is bijective on objects. If so, morphisms $FA \rightarrow FB$ in $\mathcal{D}$ must correspond to morphisms $A \rightarrow GFB = TB$ in $\mathcal{C}$

Definition 5.3. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. The *Kleisli category* $\mathcal{C}_{\mathbb{T}}$ has objects $\text{ob } \mathcal{C}$ and morphisms $A \xrightarrow{f} B$ are morphisms $A \xrightarrow{f} TB$ in $\mathcal{C}$ (n.b purple arrows are always arrows in the Kleisli category). The identity morphism $A \xrightarrow{f} B$ is η_A and the composite $A \xrightarrow{f} B \xrightarrow{g} C$ is $A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC$

Remark. The unit and associativity laws follow from

$$\begin{array}{ccc} A & \xrightarrow{f} & TB \\ & \searrow 1_{TB} & \downarrow \mu_B \\ & & TB \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ f \downarrow & & \downarrow Tf \\ TB & \xrightarrow{\eta_{TB}} & TTB \\ & \searrow 1_{TB} & \downarrow \mu_B \\ & & TB \end{array}$$

and

$$\begin{array}{ccccccc} A & \xrightarrow{f} & TB & \xrightarrow{Tg} & TTC & \xrightarrow{TTh} & TTTD & \xrightarrow{T\mu_D} & TTD \\ & & & & \downarrow \mu_C & & \downarrow \mu_{TD} & & \downarrow \mu_D \\ & & & & TC & \xrightarrow{Th} & TTD & \xrightarrow{\mu_D} & TD \end{array}$$

Proposition 5.2. *There is an adjunction*

$$C \xrightleftharpoons[G_{\mathbb{T}}]{F_{\mathbb{T}}} C_{\mathbb{T}}$$

inducing the monad $\mathbb{T}$

Proof. We define $F_{\mathbb{T}}A = A$ and $A \xrightarrow{F_{\mathbb{T}}} B = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$. $F_{\mathbb{T}}$ preserves identities by definition and for composition we have

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{\eta_B} & TB \\ & & \downarrow g & & \downarrow Tg \\ & & C & \xrightarrow{\eta_C} & TC \\ & & & & \uparrow T\eta_C \\ & & & & TTC \\ & & & \nearrow 1_{TC} & \\ & & & & \downarrow \mu_C \\ & & & & TC \end{array}$$

We define $G_{\mathbb{T}} = TA$ and $G_{\mathbb{T}}(A \xrightarrow{f} B) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$. $G_{\mathbb{T}}$ preserves identities by 1 and for composition we have

$$\begin{array}{ccccccc} TA & \xrightarrow{Tf} & TTB & \xrightarrow{TTg} & TTTC & \xrightarrow{T\mu_C} & TTC \\ & & \downarrow \mu_B & & \downarrow \mu_{TC} & & \downarrow \mu_C \\ & & TB & \xrightarrow{Tg} & TTC & \xrightarrow{\mu_C} & TC \end{array}$$

We verify the adjunction by producing a unit and counit. The unit is η and the counit $TA \xrightarrow{\varepsilon_A} A$ is $TA \xrightarrow{1_A} TA$. Note that $F_{\mathbb{T}}G_{\mathbb{T}}(A \xrightarrow{f} B)$ is $TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{\eta_{TB}}$ which isn't Tf , but has the same composite with μ_B , so the following square commutes

$$\begin{array}{ccc} F_{\mathbb{T}}G_{\mathbb{T}}A & \xrightarrow{F_{\mathbb{T}}G_{\mathbb{T}}f} & F_{\mathbb{T}}G_{\mathbb{T}}B \\ \downarrow \varepsilon_A & & \downarrow \varepsilon_B \\ A & \xrightarrow{f} & B \end{array}$$

The triangular identities for η, ε both reduce to 2, and $G_{\mathbb{T}}\varepsilon_{F_{\mathbb{T}}A} = TTA \xrightarrow{T1_{TA}} TTA \xrightarrow{\mu_A} TA = \mu_A$, so the adjunction induces $\mathbb{T}$. $\square$

Given a monad $\mathbb{T}$ on $\mathcal{C}$ we write (non-standard) $\text{Adj}(\mathbb{T})$ for the category whose objects are adjunctions inducing $\mathbb{T}$ and whose morphisms are functors between the codomains of the left adjoint such that the obvious diagrams with the left/right adjoints commute.

Theorem 5.1. *Let $\mathbb{T}$ be a monad. Then the Eilenberg-Moore adjunction $\mathcal{C} \xrightleftharpoons[G^{\mathbb{T}}]{F^{\mathbb{T}}} \mathcal{C}^{\mathbb{T}}$*

is a terminal object in $\text{Adj}(\mathbb{T})$ and the Kleisli adjunction $\mathcal{C} \xrightleftharpoons[G_{\mathbb{T}}]{F_{\mathbb{T}}} \mathcal{C}_{\mathbb{T}}$ is initial

Proof. Given

$$\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$$

in $\text{Adj}(\mathbb{T})$ we define $K: \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ by $KB = (GB, G\varepsilon_B)$ where ε_B is the counit of $F \dashv G$. This is an algebra by a triangular identity for η, ε and naturality of ε . On morphisms we define $K(A \xrightarrow{g} B) = Gg$ (a homomorphism by naturality of ε) and clearly $G^{\mathbb{T}}K = G$, and $KFA = (GFA, G\varepsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A$, so $KF = F^{\mathbb{T}}$. Suppose we have a map K' from $F \dashv G$ to $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$. Since $G^{\mathbb{T}}K' = G$, we have $K'B = (GB, \beta_B)$ for some algebra structure on GB , and $K'g = Gg$, so this is an algebra homomorphism from $(GB, \beta_B) \rightarrow (GB', \beta_{B'})$, so β is a natural transformation $GFG \rightarrow G$. Also since $K'F = F^{\mathbb{T}}$, we have $\beta_{FA} = \mu_A = G\varepsilon_{FA}$ we have a naturality square

$$\begin{array}{ccc} GFGFGB & \xrightarrow{GF G \varepsilon_B} & GFGB \\ G\varepsilon_{FGB} \downarrow \beta_{FGB} & & G\varepsilon_B \downarrow \beta_B \\ GFGB & \xrightarrow{G\varepsilon_B} & GB \end{array}$$

whose left edges are equal, and $GFG\varepsilon_B$ is split epic by the triangular identity, so $G\varepsilon_B = \beta_B$ for all B , so $K' = K$

For the Kleisli category, we define $H: \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$ by $HA = FA$ and $H(A \xrightarrow{f} B) = FA \xrightarrow{Ff} FGF B \xrightarrow{\varepsilon_{FB}} FB$. Note that Hf corresponds to $A \xrightarrow{f} GFB$ under the adjunction, so H is full and faithful. H preserves identities as $\varepsilon_{FA}(F\eta_A) = 1_{FA}$ and $HF_{\mathbb{T}} = F$ for the same reason. H preserves the composite $A \xrightarrow{f} B \xrightarrow{g} C$ by

$$\begin{array}{ccccc} FA & \xrightarrow{Ff} & FGF B & \xrightarrow{FGF g} & FGFGFC & \xrightarrow{FG \varepsilon_{FC}} & FGFC \\ \downarrow \varepsilon_{FB} & & \downarrow \varepsilon_{FGB} & & \downarrow \varepsilon_{FGFC} & & \downarrow \varepsilon_{FC} \\ FB & \xrightarrow{Fg} & FGFC & \xrightarrow{\varepsilon_C} & FC & & \end{array}$$

and $GHA = GFA = G_{\mathbb{T}}A$, $GH(A \xrightarrow{f} B) = G_{\mathbb{T}}f$, so H is a morphism in $\text{Adj } \mathbb{T}$.

Suppose we have $H': \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{D}$. Since $H'F_{\mathbb{T}} = F$, H and H' agree on objects. Since $GH' = G_{\mathbb{T}}$, and $F \dashv G$ has the same unit as $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$, H' must send the transpose $A \xrightarrow{f} B$ of $A \xrightarrow{f} GFB$ to its transpose across $F \dashv G$, but that means $H = H'$ on morphisms and we are done. $\square$

$\mathcal{C}_{\mathbb{T}}$ has coproducts if $\mathcal{C}$ does, since $F_{\mathbb{T}}$ preserves them, but in general it has few other limits and colimits, eg for the power set monad, the Kleisli category is Rel which doesn't have splittings of all its idempotents.

Proposition 5.3. *Let $\mathbb{T}$ be a monad on $\mathcal{C}$*

1. *The forgetful functor $\mathcal{C}^{\mathbb{T}} \xrightarrow{G} \mathcal{C}$ creates all limits that exist in $\mathcal{C}$*
2. *If $\mathcal{C}$ has colimits of shape J , then G creates colimits of shape J if and only if T preserves them*

Proof. Suppose we have $D: J \rightarrow \mathcal{C}^{\mathbb{T}}$. We write $D(j) = (GD(j), \delta_j)$. Let $(L, (L \xrightarrow{\alpha_j} GD(j) \mid j \in \text{ob } J))$ be a limit cone for GD . Then the maps $TL \xrightarrow{T\alpha_j}$

$TGD(j) \xrightarrow{\delta_j} GD(j)$ form a cone over TGD , so they induce a unique map $TL \xrightarrow{\lambda} L$. λ is an algebra structure on L , as the requirements assert the equality of two morphisms with codomain L which commute with the limit cone, and it's the unique lifting of L which makes the α_j homomorphisms. These forming a limit cone over D is as usual.

If G creates limits of shape J , then it must preserve them, but so does F as F is a left adjoint, so $GF = T$ preserves them. In the other direction, if we have a diagram $D: J \rightarrow \mathcal{C}^{\mathbb{T}}$ and a colimit cone over L with morphisms α_j . Since $(TL, T\alpha_j)$ forms a colimit cone we get a unique $\lambda: TL \rightarrow L$ which is an algebra structure as TTL is also a colimit, and this makes (L, λ) a colimit in $\mathcal{C}^{\mathbb{T}}$ $\square$

Remark. There is a further result which says that $\mathcal{C}^{\mathbb{T}}$ has colimits of other shapes provided $\mathcal{C}$ has them and $\mathcal{C}^{\mathbb{T}}$ has certain coequalisers

It becomes of interest to ask that given $F \dashv G$ inducing $\mathbb{T}$, when is $K: \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ part of an equivalence.

Definition 5.4. If $K: \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ is part of an equivalence, we say $F \dashv G$ is *monadic*

The monadicity theorem gives criteria for this

Lemma 5.2. *Let $F \dashv G$ be an adjunction inducing $\mathbb{T}$ and suppose that for each $\mathbb{T}$ -algebra (A, α) the pair $FGFA \xrightleftharpoons[\varepsilon_{FA}]{F\alpha} FA$ has a coequaliser in $\mathcal{D}$. Then K has a left adjoint $L: \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{D}$*

Proof. Write $FA \xrightarrow{\lambda_{(A, \alpha)}} L(A, \alpha)$ for the coequaliser given in the statement. Given a homomorphism $f: (A, \alpha) \rightarrow (B, \beta)$ we get a diagram

$$\begin{array}{ccccc} FGFA & \rightrightarrows & FA & \longrightarrow & L(A, \alpha) \\ \downarrow FGf & & \downarrow Ff & & \downarrow Lf \\ FGFB & \rightrightarrows & FB & \longrightarrow & L(B, \beta) \end{array}$$

inducing the unique Lf shown. Functoriality of L follows from uniqueness. Morphisms $L(A, \alpha) \rightarrow B$ in $\mathcal{D}$ correspond to morphisms $FA \rightarrow B$ having $f(F\alpha) = f\varepsilon_{FA}$, or equivalently morphisms $\bar{f}: A \rightarrow GB$ with $\bar{f}\alpha = Gf$. But $f = \varepsilon_B(F\bar{f})$ so $Gf = (G\varepsilon_B)T\bar{f}$, so $\bar{f}$ is an algebra homomorphism $(A, \alpha) \rightarrow KB$. This establishes the adjunction $L \dashv K$ $\square$

Remark. Since $G^{\mathbb{T}}K = G$, if L exists we must have $LF^{\mathbb{T}} = F$. Since there's a coequaliser

$$F^{\mathbb{T}}TA \rightrightarrows F^{\mathbb{T}}A \longrightarrow (A, \alpha)$$

in $\mathcal{C}^{\mathbb{T}}$ for any (A, α) , we must have a coequaliser

$$FTA \rightrightarrows FTA \longrightarrow L(A, \alpha)$$

Definition 5.5.

1. We say a parallel pair $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$ is *reflexive* if there is $B \xrightarrow{r} A$ with $fr = gr = 1_B$
2. By a *split coequaliser diagram* we mean a diagram

$$\begin{array}{ccc} A & \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} & B & \begin{smallmatrix} \xrightarrow{h} \\ \xleftarrow{s} \end{smallmatrix} & C \\ & \searrow t & & \swarrow s & \\ & & & & \end{array}$$

with $hf = hg, hs = 1_C, gt = 1_B$ and $ft = sh$.

3. Given a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ we say a parallel pair $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$ in $\mathcal{D}$ is *G-split* if there is a split coequaliser diagram

$$\begin{array}{ccc} GA & \begin{smallmatrix} \xrightarrow{Gf} \\ \xrightarrow{Gg} \end{smallmatrix} & GB & \begin{smallmatrix} \xrightarrow{h} \\ \xleftarrow{s} \end{smallmatrix} & C \\ & \searrow t & & \swarrow s & \\ & & & & \end{array}$$

in $\mathcal{C}$.

Remark. The pair $FGFA \begin{smallmatrix} \xrightarrow{F\alpha} \\ \xrightarrow{\varepsilon_{FA}} \end{smallmatrix} FA$ with common right inverse $F\eta_A$ is reflexive.

If we have a split coequaliser, then h is a coequaliser of (f, g) as if $B \xrightarrow{k} D$ has $kf = kg$ then $k = kgt = kft = ksh$ so k factors through h and this is unique as h is split epic.

The pair $FGFA \begin{smallmatrix} \xrightarrow{F\alpha} \\ \xrightarrow{\varepsilon_{FA}} \end{smallmatrix} FA$ is *G-split* since

$$\begin{array}{ccc} GFGFA & \begin{smallmatrix} \xrightarrow{G\varepsilon_{FA}=\mu_A} \\ \xrightarrow{GF\alpha} \end{smallmatrix} & GFA & \begin{smallmatrix} \xrightarrow{\alpha} \\ \xleftarrow{\eta_A} \end{smallmatrix} & A \\ & \searrow \eta_{GFA} & & \swarrow \eta_A & \\ & & & & \end{array}$$

is a split coequaliser in $\mathcal{C}$

Theorem 5.3. (*Precise Monadicity Theorem*) $G: \mathcal{D} \rightarrow \mathcal{C}$ is monadic if and only if it has a left adjoint and creates coequalisers of *G-split* pairs

Theorem 5.4. (*Crude Monadicity Theorem*) If G has a left adjoint, G reflects isomorphisms, and $\mathcal{D}$ has and G preserves reflexive coequalisers, then G is monadic

Proof. We prove these together. We do the forwards direction of Theorem 5.3. Necessity of the left adjoint is obvious, so it suffices to show that $G^{\mathbb{T}}$ creates coequalisers of $G^{\mathbb{T}}$ -split pairs. The argument is the same as Proposition 5.3. If

$(A, \alpha) \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} (B, \beta)$ is a $G^{\mathbb{T}}$ -split pair on $\mathcal{C}^{\mathbb{T}}$, the coequaliser $B \xrightarrow{h} C$ of (f, g)

in $\mathcal{C}$ is preserved by T, TT so C acquires a unique $\mathbb{T}$ algebra structure making h a coequaliser in $\mathcal{C}^{\mathbb{T}}$.

For Theorem 5.4 and the backwards direction of Theorem 5.3, the coequalisers needed for Lemma 5.2 exist, so K has a left adjoint L . The unit of $L \dashv K$ of (A, α) is the factorisation of $GFA \xrightarrow{G\lambda_{(A, \alpha)}} GL(A, \alpha)$ through the (split) coequaliser

$$\begin{array}{ccc} GFGFA & \xrightleftharpoons[GFA]{G\varepsilon_{FA}=\mu_A} & GFA \\ & \nwarrow \eta_{GFA} & \nwarrow \eta_A \\ & & A \end{array}$$

in $\mathcal{C}$, which is an isomorphism since either set of hypotheses implies that G preserves this coequaliser. The counit at $B \in \text{ob } \mathcal{D}$ is the factorisation of $FGB \xrightarrow{\varepsilon_B} B$ through the coequaliser $FGB \xrightarrow{\lambda_{KB}} LKB$ of $FGFGB \xrightleftharpoons[F\varepsilon_{GB}]{\varepsilon_{FGB}} FGB$. Again either hypotheses implies that this factorisation is an isomorphism. $\square$

Remark.

1. Reflexive coequalisers are colimits of shape J where J is the category

$$\begin{array}{ccc} & \xrightarrow{s} & \\ & \curvearrowright & \\ A & \xrightleftharpoons[r]{f} & B \\ & \curvearrowleft & \\ & \xrightarrow{t} & \end{array}$$

with $fr = gr = 1_B, rf = s, rg = t$

2. All colimits can be constructed from coproducts and reflexive coequalisers, as in Proposition 4.1(1) we constructed limits from products, and equalisers that are actually coreflexive with common left inverse $r: Q \rightarrow P$ by projecting onto the identity morphisms
3. If $A \xrightleftharpoons[g]{f} B$ is reflexive, then $B \xrightarrow{h} C$ is a coequaliser for (f, g) if and only if

$$\begin{array}{ccc} & \xrightarrow{s} & \\ & \curvearrowright & \\ A & \xrightleftharpoons[r]{f} & B \\ & \curvearrowleft & \\ & \xrightarrow{t} & \end{array}$$

is a pushout, so reflexive coequalisers exist in any categories with pushouts

4. In a cartesian closed category, reflexive coequalisers commute with binary products. To prove this, consider

$$\begin{array}{ccccc} A_1 \times A_2 & \xrightleftharpoons{\quad} & A_1 \times B_2 & \longrightarrow & A_1 \times C_2 \\ \Downarrow & & \Downarrow & & \Downarrow \\ B_1 \times A_2 & \xrightleftharpoons{\quad} & B_1 \times B_2 & \longrightarrow & B_1 \times C_2 \\ \downarrow & & \downarrow & & \downarrow \\ C_1 \times A_2 & \xrightleftharpoons{\quad} & C_1 \times B_2 & \longrightarrow & C_1 \times C_2 \end{array}$$

in which all rows and columns are coequalisers by Cartesian closed. If $B_1 \times B_2 \xrightarrow{k} D$ coequalises $(f_1 \times f_2, g_1 \times g_2)$, then it also coequalises $(1 \times f_2, 1 \times g_2)$, and so factors through $1 \times h_2$. This factorisation coequalises $A_1 \times C_2 \rightrightarrows B_1 \times C_2$ and so factors uniquely through $B_1 \times C_2 \rightarrow C_1 \times C_2$

Example 5.2.

1. The forgetful functor from groups to sets satisfies Theorem 5.4. It has a left adjoint, it reflects isomorphisms, and actually creates reflexive coequalisers, as if $A \xrightarrow[f]{g} B$ is a reflexive pair in $\mathbf{Grp}$ with coequaliser

$$UA \xrightarrow[Ug]{Uf} UB \xrightarrow{h} C$$

, by the above remark we get a binary operation on C and this is a group structure, making the diagram a coequaliser of groups. Similarly for Abelian groups, rings, modules, lattices, etc.

2. Any reflection is monadic. To apply Theorem 5.3, let $I: \mathcal{D} \rightarrow \mathcal{C}$ be the inclusion of a reflective subcategory, and let $A \xrightarrow[f]{g} B$ be an I -split pair in $\mathcal{D}$. The morphism $B \xrightarrow{t} A$ in the split coequaliser diagram belongs to $\mathcal{D}$, and so does $ft = sh$, but this is a split idempotent and so $s, h \in \mathcal{D}$ up to isomorphism.

3. Consider the composite adjunction

$$\mathbf{Set} \xrightleftharpoons[U]{F} \mathbf{AbGrp} \xrightleftharpoons[I]{L} \mathbf{tfAbGrp}$$

The two factors are monadic by the above examples, but the composite is not, since free abelian groups are torsion free, so $(LF \dashv UI)$ and $(F \dashv U)$ induce the same monad on $\mathbf{Set}$, but the category of algebras is $\mathbf{AbGrp}$.

4. The functor $P^*: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$ is monadic and satisfies Theorem 5.3. It has a left adjoint P^* and it reflects isomorphisms as the 2-element set is a codetector, and represents the functor. Let

$$E \xrightarrow{e} A \xrightarrow[f]{g} B$$

be a coreflexive equaliser. Then

$$\begin{array}{ccc} E & \xrightarrow{e} & A \\ \downarrow e & & \downarrow f \\ A & \xrightarrow[g]{} & B \end{array}$$

is a pullback, so we deduce that

$$\begin{array}{ccc} PA & \xrightarrow{P^*e} & PE \\ Pf \downarrow & & \downarrow Pe \\ PB & \xrightarrow{P^*g} & PA \end{array}$$

commutes, but $P^*ePe = 1_{Pe}$ and $P^*gPg = 1_{PA}$ since e, g are monic, so

$$\begin{array}{ccccc} PB & \xrightleftharpoons[P^*g]{P^*f} & PA & \xrightarrow{P^*e} & PE \\ & \searrow \scriptstyle Pg & & \swarrow \scriptstyle Pe & \\ & & & & \end{array}$$

is a split coequaliser diagram in set.

5. The forgetful functor $\text{Top} \rightarrow \text{Set}$ has a left adjoint and preserves coequalisers, but isn't monadic. This shows we can't weaken reflecting of isomorphisms to faithfulness
6. The composite $\text{Set} \xrightarrow{D} \text{Top} \xrightarrow{\beta} \text{KHaus}$ is monadic. We will prove this with Theorem 5.3. Given $X \xrightleftharpoons[g]{f} Y$ in KHaus and a split coequaliser diagram

$$\begin{array}{ccccc} UX & \xrightleftharpoons[Ug]{Uf} & UY & \xrightarrow{h} & Z \\ & \searrow \scriptstyle t & & \swarrow \scriptstyle s & \\ & & & & \end{array}$$

There is a unique topology on Z making this a coequaliser in Top, and it's compact, so it's a coequaliser in KHaus if it is Hausdorff (if so, it is the only one that could work). It is a fact that if $S \subseteq Y \times Y$ is an equivalence relation, then Y/S is Hausdorff if and only if S is closed. If $h(y_1) = h(y_2)$ then there are x_1, x_2 with $x_i = t(y_i)$ satisfying $g(x_i) = y_i$ and $f(x_1) = f(x_2)$, and conversely if x_1, x_2 satisfy these, then $(y_1, y_2) \in S$, so $S = (g \times g)(R)$ where $R = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$, and R is closed as Y is Hausdorff, and hence R is compact, so S is compact and hence closed.

Definition 5.6. Suppose we are given an adjunction $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ where $\mathcal{D}$ has reflexive coequalisers. The *monadic tower* of the adjunction is the diagram

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ & & (\mathcal{C}^T)^S \\ & \nearrow & \uparrow \\ \mathcal{D} & \xrightarrow{K} & \mathcal{C}^T \\ & \nwarrow & \downarrow \\ & & \mathcal{C} \end{array}$$

$\begin{array}{ccc} & \xrightarrow{L} & \\ & \nwarrow & \downarrow \\ & & \mathcal{C} \end{array}$

$\begin{array}{ccc} & \xrightarrow{G} & \\ & \nwarrow & \downarrow \\ & & \mathcal{C} \end{array}$

$\begin{array}{ccc} & \xrightarrow{F} & \\ & \nwarrow & \downarrow \\ & & \mathcal{C} \end{array}$

where $\mathbb{T}$ is the monad induced by $F \dashv G$ and $\mathbb{S}$ is the monad induced by $L \dashv K$. We say $F \dashv G$ has *monadic length* n (or $\mathcal{D}$ has *monadic height* n over $\mathcal{C}$) if we reach an equivalence at level n

6 Filtered Colimits

Definition 6.1. A (small) category $\mathcal{C}$ is *filtered* if every finite diagram has a cone under it.

Remark. If $\mathcal{C}$ has finite colimits, it is filtered

Lemma 6.1. $\mathcal{C}$ is filtered if and only if

1. $\mathcal{C}$ is non-empty (there is a cone under the empty diagram)
2. Given $A, B \in \text{ob } \mathcal{C}$ there is a cospan

$$\begin{array}{ccc} A & & \\ & \searrow & \\ & & C \\ & \nearrow & \\ B & & \end{array}$$

3. Given a parallel pair $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$, there is $h: B \rightarrow C$ with $hf = hg$

Proof. These are all special cases of the existence of cones under a diagram. Given $D: J \rightarrow \mathcal{C}$ where J is finite and non-empty, we use condition 2 to get an object C with morphisms from every $D(j)$. We then use condition 3 to obtain a morphism $C \rightarrow D$ coequalising all pairs of the form

$$\begin{array}{ccc} D(j) & \xrightarrow{\quad} & C \\ & \searrow D(\alpha) & \nearrow \\ & D(j') & \end{array}$$

□

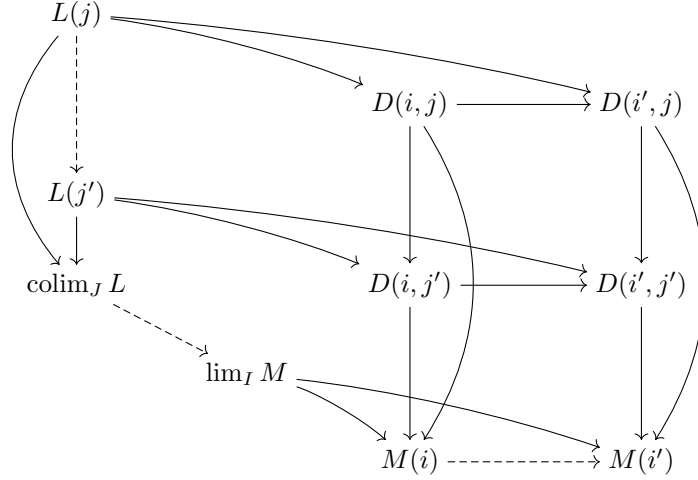
Remark. If a preorder is filtered, we may call it directed. Such preorders only need to satisfy 1 and 2.

Definition 6.2. A category has *filtered* (*directed*) colimits if it has colimits for all diagrams from small filtered categories (small directed preorders)

Lemma 6.2. If $\mathcal{C}$ has finite colimits and directed colimits, then it has all small colimits

Proof. We need only construct arbitrary coproducts, as we already have coequalisers. To do this, we form all finite products of subsets of $\{A_i \mid i \in I\}$, and form them into a directed poset by inclusion. The colimit of this diagram is their coproduct, so we are done. □

Suppose we have a diagram $D: I \times J \rightarrow \mathcal{C}$ and $\mathcal{C}$ has limits of shape I and colimits of shape J .



For fixed $j \in \text{ob } J$, $D(\cdot, j)$ has a limit $L(j)$ and similarly for fixed i we can form the colimit $M(i)$. For each j , the composites $L(j) \rightarrow D(i, j) \rightarrow M(i)$ form a cone over M , so induce a morphism $L(j) \rightarrow \lim_I M$, so this forms a cone under L and we get a map $\text{colim}_J L \rightarrow \lim_I M$. In this case we say limits of shape I *commute* with colimits of shape J if this map is always an isomorphism. Equivalently, $\text{colim}_J: [J, \mathcal{C}] \rightarrow \mathcal{C}$ preserves limits of shape I .

Theorem 6.3. *Let J be a small category. Then colimits of shape J commute with all finite limits in Set if and only if J is filtered.*

Proof. In the forwards direction, given $D: I \rightarrow J$ with I finite we define $E: I^{\text{op}} \times J \rightarrow \text{Set}$ via $(i, j) \mapsto J(D(i), j)$. For each i , $\text{colim}_J E(i, \cdot)$ is a singleton with every $D(i) \rightarrow j$ identified in the colimit with $D(i) \xrightarrow{1} D(i)$, so $\lim_I \text{colim}_J E$ is a singleton, but for any j , elements of $\lim_I E(\cdot, j)$ are in bijection with cones under I with apex j . So if $\text{colim}_J \lim_I E \neq \emptyset$ then there is a cone under D in J . In general, elements of $\text{colim}_J D$ are equivalence classes of elements modulo $x \sim D(\alpha)(x)$ for all $\alpha: j \rightarrow j'$ in J . If J is filtered, the equivalence relation identifies

$$x \in D(j) \text{ and } x' \in D(j') \text{ if there is a cospan with } D(\alpha)(x) = D(\alpha')(x') \quad \begin{array}{c} j \\ \searrow \alpha \\ j'' \\ \nearrow \alpha' \\ j' \end{array}$$

since filteredness implies this is an equivalence relation. Moreover if $j = j'$ we can require $\alpha = \alpha'$ by Lemma 6.1. Now given $D: I \times J \rightarrow \text{Set}$ and an element $x \in \lim_I \text{colim}_J D$, x corresponds to a family of elements $x_i \in \text{colim}_J D(i, \cdot)$ which can be represented by some $y_i \in D(i, j_i)$. By passing to a further j , we may assume j_i is independent of i . If $\alpha: i \rightarrow i'$ in I , the elements $D(\alpha, j)y_i$ and $y_{i'}$ both represent $x_{i'}$ in the colimit, so there is a $\beta: j \rightarrow j'$ such that they map to the same element of $D(i, j')$. Doing this for each edge, all the y_i represent

an element of $\lim_I D(\cdot, \bar{j})$, so the map is surjective. The other part (injectivity) is similar. $\square$

Corollary 6.3.1. *Let $\mathcal{C}$ be a finitary algebraic category as in Example 5.2(1). Then*

1. *The forgetful functor $\mathcal{C} \rightarrow \text{Set}$ creates filtered colimits*
2. *Finite limits and filtered colimits commute in $\mathcal{C}$*

Proof.

1. Given $D: J \rightarrow \mathcal{C}$ with J filtered, if L is the apex of a colimit cone under UD , then L^n is the colimit of UD^n for any n (ie D^n is the functor $A \rightarrow D(A) \rightarrow D(A)^n$), so we get finitary operations on $UD(j)$ making it an object of $\mathcal{C}$ and making the cone a colimit
2. This follows from part (1), reflection of isomorphisms, and Theorem 6.3

$\square$

Example 6.1. Consider the diagram of shape $\mathbb{N}^{\text{op}} \times 2$ in Set

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathbb{N} & \xrightarrow{5} & \mathbb{N} & \xrightarrow{5} & \mathbb{N} \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & 1 \end{array}$$

The limit of the top row is empty, and the limit of the bottom row is 1, so $\lim_{\mathbb{N}^{\text{op}}} : [\mathbb{N}^{\text{op}}, \text{Set}] \rightarrow \text{Set}$ doesn't preserve epimorphisms, and hence doesn't preserve pushouts, and so $\text{colim}_{\mathbb{N}} : [\mathbb{N}, \text{Set}^{\text{op}}] \rightarrow \text{Set}^{\text{op}}$ doesn't preserve pullbacks.

Proposition 6.1. *Let $\mathcal{C}$ be a small category with finite limits, and $F: \mathcal{C} \rightarrow \text{Set}$ a functor. The following are equivalent*

1. *F preserves finite limits*
2. *$(1 \downarrow F)$ is cofiltered*
3. *F is expressible as a filtered colimit of representable functors.*

Proof. (1) implies (2): By Lemma 4.2, if F preserves finite limits then $(1 \downarrow F)$ has them, so it's cofiltered.

(3) implies (1): Functors of the form $\mathcal{C}(A, \cdot)$ preserve any limits which exist, as colimits of functors can be done pointwise, this follows from Theorem 6.3.

(2) implies (3): Consider the diagram

$$(1 \downarrow F)^{\text{op}} \xrightarrow{U} \mathcal{C}^{\text{op}} \xrightarrow{Y} [\mathcal{C}, \text{Set}]$$

where Y is the Yoneda embedding. By Yoneda, a cone under this diagram with apex G assigns to each pair (A, x) with $x \in FA$, an element $\alpha_A(x) \in GA$ with compatibility conditions that say α is a natural transformation $F \rightarrow G$, so the cone corresponding to 1_F is a colimit, so given (2) this expresses F as a filtered colimit of representables. $\square$

Definition 6.3. We call a functor $\mathcal{C} \rightarrow \mathbf{Set}$ flat if it satisfies the conclusion (2) or (3) above.

If $\mathcal{A}$ is a finitary algebraic category in the sense of Example 5.2(1), we know $\mathcal{A} \rightarrow \mathbf{Set}$ is monadic and $T: \mathbf{Set} \rightarrow \mathbf{Set}$ underlying the monad preserves filtered colimits by Corollary 6.3.1

Definition 6.4. We call a functor T *finitary* if it preserves filtered colimits. Note that such a function is determined by its restriction to finite sets, since every set is a filtered colimit of finite sets.

For any $\mathcal{C}$ with filtered colimits, the restriction functor $[\mathbf{Set}, \mathcal{C}] \rightarrow [\mathbf{Set}_f, \mathcal{C}]$ has a left adjoint, and the image of the left adjoint is the finitary functors (and in that case the counit is an isomorphism)

Definition 6.5. A *Lawvere theory* is a small category $\mathcal{T}$ equipped with a functor $\mathbf{Set}_f \rightarrow \mathcal{T}$ which is bijective on objects and preserves finite coproducts. By a *model* of $\mathcal{T}$ in $\mathcal{C}$ which has finite products, we mean a functor $\mathcal{T}^{\mathrm{op}} \rightarrow \mathcal{C}$ preserving finite products. We write $\mathbf{Mod} \mathcal{T}$ for the category of models of $\mathcal{T}$ in $\mathbf{Set}$.

Remark. Given any monad $\mathbb{T}$ on $\mathbf{Set}$, the full subcategory $\mathbf{Set}_{\mathbb{T}}$ on the objects $[n]$ (or equivalently, the full subcategory of $\mathbf{Set}^{\mathbb{T}}$ on $F^{\mathbb{T}}[n]$) is a Lawvere theory

Proposition 6.2. *For any Lawvere Theory $\mathcal{T}$, the category $\mathbf{Mod} \mathcal{T}$ is equivalent to a finitary algebraic category in the sense of Example 5.2(1)*

Proof. Let M be a model of $\mathcal{T}$. We know $M[n] = M[1]^n$ and the effect of M on arbitrary morphisms is determined by its effect on morphisms $[1] \rightarrow [n]$, since $M([m] \rightarrow [n])$ is determined by its components, so we can think of M as a set A equipped with operations $\alpha_A: A^n \rightarrow A$ for each $\alpha: [1] \rightarrow [n]$ in $\mathcal{T}$ satisfying $(\nu_i)_A(a_1, \dots, a_n) = a_i$ and whenever we have

$$\begin{array}{ccc} [1] & \xrightarrow{\alpha} & [m] \\ & \searrow \gamma & \downarrow (\beta_1, \dots, \beta_m) \\ & & [n] \end{array}$$

the equation

$$\alpha_A((\beta_1)_A(a_1, \dots, a_n), \dots, (\beta_m)_A(a_1, \dots, a_n)) = \gamma_A(a_1, \dots, a_n)$$

holds. This defines an equivalence between $\mathbf{Mod} \mathcal{T}$ and a finitary algebraic category $\square$

Theorem 6.4. *For a category $\mathcal{A}$, the following are equivalent:*

1. $\mathcal{A}$ is equivalent to a finitary algebraic category in the sense of Example 5.2(1)
2. $\mathcal{A} \simeq \mathbf{Set}^{\mathbb{T}}$ for a finitary monad $\mathbb{T}$ on $\mathbf{Set}$

3. $\mathcal{A} \simeq \text{Mod } \mathcal{T}$ for a Lawvere theory $\mathcal{T}$

Proof. We proved (i) implies (ii) in Example 5.2(1) and Corollary 6.3.1. We proved (iii) implies (i) in Proposition 6.2.

For (ii) implies (iii), we need to show that given a finitary monad $\mathbb{T}$ on Set , the monad $\mathbb{T}'$ which we construct from the Lawvere theory $\mathcal{T}$ (namely, the finite part of the Kleisli category) is isomorphic to $\mathbb{T}$. For any $\mathbb{T}$ algebra A , the restriction of $\text{Set}^{\mathbb{T}}(\cdot, A)$ to finitely generated free algebras gives a functor $\mathcal{T}^{\text{op}} \rightarrow \text{Set}$ preserving finite products, ie a model of $\mathcal{T}$. This defines a functor $\text{Set}^{\mathbb{T}} \rightarrow \text{Mod } \mathcal{T} \cong \text{Set}^{\mathbb{T}'}$. Moreover, this functor can be taken to be the identity on underlying sets. It is a fact that such a functor is necessarily induced by a morphism of monads. In our case, $\theta_{[n]}$ is bijective for all n since the free $\mathbb{T}'$ algebra on $[n]$ since the free $\mathbb{T}'$ algebra on $[n]$ consists of all the morphisms $F^{\mathbb{T}}[1] \rightarrow F^{\mathbb{T}}[n]$ in $\text{Set}^{\mathbb{T}}$, but a finitary functor is determined by its action on finite sets, so we are done. $\square$

7 Additive and Abelian Categories

Definition 7.1. Let $\mathcal{A}$ be a category with a forgetful functor to set . By an *enrichment* of a locally small category $\mathcal{C}$ over $\mathcal{A}$, we mean a factorisation of $\mathcal{C}(\cdot, \cdot)$ through the forgetful functor.

Example 7.1. We consider categories enriched over Set_* (*pointed categories*), over CMon (*semi-additive categories*), and over AdGrp (*additive categories*).

Pointed categories have, for all A, B , a map $A \xrightarrow{0} B$ such that $f0 = f, 0g = 0$ for all composable f, g .

Semi-additive categories have addition of morphisms which is associative, commutative, unital and distributive

Additive categories are semi-additive categories with additive inverses

Lemma 7.1.

1. For an object A in a pointed category, the following are equivalent

- (a) A is terminal
- (b) A is initial
- (c) $1_A = 0_A$

2. For objects A, B, C in a semiadditive category, the following are equivalent

- (a) C is a product of A, B
- (b) C is a coproduct of A, B
- (c) There are morphisms

$$A \begin{array}{c} \xleftarrow{\pi_1} \\ \xrightarrow{\nu_1} \end{array} C \begin{array}{c} \xleftarrow{\pi_2} \\ \xrightarrow{\nu_2} \end{array} B$$

satisfying $\pi_i \nu_i = 1$, $\pi_2 \nu_1 = 0$, $\pi_1 \nu_2 = 0$ and $\nu_1 \pi_1 + \nu_2 \pi_2 = 1_C$

Proof. By duality we need only prove (a) is equivalent to (c) in both cases

1. (a) implies (c) as a terminal object only has one endomorphism.
(c) implies (a) as for any $B \xrightarrow{f} A$ we have $f = 1_A f = 0_A f = 0$.
2. (a) implies (c): We take the first 3 equations given in (c) as the definition of π_i, ν_i as C is a product. Now $\pi_1(\nu_1\pi_1 + \nu_2\pi_2) = 1_A\pi_1 = \pi_1$ and similar for π_2 , so $\nu_1\pi_1 + \nu_2\pi_2$ is the identity on both factors, so is the identity.
(c) implies (a): Given

$$\begin{array}{ccc} & D & \\ f \swarrow & & \searrow g \\ A & & B \end{array}$$

, we set $h = \nu_1 f + \nu_2 g: D \rightarrow C$. This satisfies $\pi_1 h = 1_A f + 0g = f$ and $\pi_2 h = g$. If h' has the same factorisation, then $h' = 1_C h' = (\nu_1\pi_1 h' + \nu_2\pi_2 h') = \nu_1 f + \nu_2 g = h$.

□

Definition 7.2. If an object is both initial and terminal, we call it a *zero object* and denote it 0. An object which is both a product and a coproduct is called a *biproduct*, denoted $A \oplus B$

Lemma 7.2. Let $\mathcal{C}$ be a locally small category.

1. If $\mathcal{C}$ has a zero object, it has a unique pointed structure.
2. If $\mathcal{C}$ has a zero object, finite products and finite coproducts, and for each (A, B) , the morphism $A + B \rightarrow A \times B$ defined by $\pi_i c \nu_i = 1$, and $\pi_i c \nu_j = 0$ if $i \neq j$ is an isomorphism. Then $\mathcal{C}$ has a unique semiadditive structure.

Proof.

1. The zero morphism of the zero object must be 1_0 , and hence for any two A, B , there is a unique morphism $A \rightarrow B$ factoring through 0, which must be a 0 morphism as it factors through 1_0 , and this gives a unique pointed structure.
2. From now on, denote morphisms $C \rightarrow A \times B$ by column vectors $\begin{pmatrix} f \\ g \end{pmatrix}$ and morphisms $A + B \rightarrow C$ by row vectors. Hence $c: A + B \rightarrow A \times B$ is denoted by $\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$. Given morphisms $A \xrightarrow[f]{g} B$ we define $f +_l g$ to be

$$A \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{(1_B, 1_B)} B$$

and $f +_r g$ to be

$$A \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{(f, g)} B$$

Note that $(f +_l g)h = fh +_l gh$ and $h(f +_r g) = hf +_r hg$. Both $+_l, +_r$ have 0 as a 2-sided identity as

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{1_B} & B \\
 \downarrow \begin{pmatrix} f \\ 0 \end{pmatrix} & & \searrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \swarrow \nu_1 & \nearrow (1,1) \\
 B \times B & \xrightarrow{c^{-1}} & B + B & &
 \end{array}$$

(and 3 other similar diagrams). Now given $f, g, h, k \in \mathcal{C}(A, B)$, consider the composite

$$A \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{\begin{pmatrix} f & g \\ h & k \end{pmatrix}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{(1,1)} B$$

The first three factors compose to $\begin{pmatrix} f +_r g \\ h +_r k \end{pmatrix}$ so the whole is $(f +_r g) +_l (h +_r k)$. Composing the last three, we get $(f +_l g) +_r (h +_l k)$. Putting $g = h = 0$ in this, we get $f +_l k = f +_r k$ so $+_l = +_r$. Putting $f = k = 0$ we get $g +_l h = g +_r h$ and putting $h = 0$ we get $(f + g) + k = f + (g + k)$ so we have a semi-additive structure. For uniqueness, suppose $\tilde{+}$ is any semiadditive structure. Then $(\nu_1 \pi_1 \tilde{+} \nu_2 \pi_2)$ must be the inverse of c by composing with projections, and so by the definition of $+_r, +_l$, $\tilde{+} = +$.

□

Remark. If $\mathcal{C}, \mathcal{D}$ are semiadditive categories with finite biproducts, then any functor between them is *semiadditive* if it preserves the additive structure, if and only if it preserves finite products or coproducts. In particular if F has either a left or right adjoint, then it is semiadditive. Moreover if $F \dashv G$, the bijection $\mathcal{C}(A, GB) \rightarrow \mathcal{D}(FA, B)$ is an isomorphism of commutative monoids as it sends f to $\varepsilon_B(Ff)$

Definition 7.3. In a pointed category $\mathcal{C}$, a *kernel* of $A \xrightarrow{f} B$ is an equaliser of f and 0. We say a monomorphism is *normal* if it is a kernel of some morphism. Dually we have *cokernels* and *normal* epimorphisms. f is a *pseudo-monomorphism* if its kernel is the 0 morphism, ie if $fg = 0$ then $g = 0$ (similarly we have *pseudo-epimorphisms*)

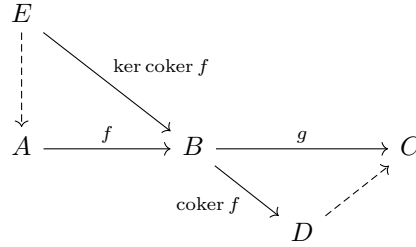
Remark. In additive categories, normal is equivalent to regular since an equaliser of (f, g) is a kernel of $f - g$, but in Grp any subgroup inclusion is regular monic, but normal monics are inclusions of normal subgroups. However, any surjection in Grp is normal epic.

In Set_* , any injection is normal monic, but a surjection is normal epic if it is injective on elements not mapped to the basepoint. Further f is pseudo-monic if the only preimage of the basepoint is the basepoint.

In Grp , $A \xrightarrow{f} B$ is pseudo-epic if the normal closure of the image is B

Lemma 7.3. *In a pointed category $\mathcal{C}$ with kernels and cokernels, a morphism $A \xrightarrow{f} B$ is normal monic if and only if $f \cong \ker \text{coker } f$*

Proof. The backwards direction is trivial.



Suppose $f = \ker g$. $gf = 0$ so g factors through $\text{coker } f$, so $g(\ker \text{coker } f) = 0$, hence $\ker \text{coker } f$ factors through $f = \ker g$, but also $(\text{coker } f)f = 0$ so f factors through $\ker \text{coker } f$, so the result follows $\square$

Corollary 7.3.1. *Under the hypotheses of the preceding lemma, $\ker, \text{coker}$ induce a bijection between isomorphism classes of normal subobjects and normal quotients of A*

Proof. Trivial $\square$

Remark. Note that for any f , $\ker \text{coker } f$ is the smallest subobject of $\text{cod } f$ through which f factors.

Definition 7.4. An *abelian category* is an additive category which has all finite limits and colimits in which all monomorphisms and epimorphisms are normal

Remark. It is equivalent to require finite biproducts, kernels, and cokernels instead of all finite limits and colimits.

Example 7.2.

1. AbGrp is abelian, as is $\text{Mod } R$ for any ring R .
2. If $\mathcal{A}$ is abelian, then so is $[\mathcal{C}, \mathcal{A}]$ for any small category $\mathcal{C}$ with pointwise constructed structure.
3. If $\mathcal{C}$ is additive in the above example, then so is the subcategory of $[\mathcal{C}, \mathcal{A}]$ of additive functors, denoted $\text{Add}(\mathcal{C}, \mathcal{A})$

Remark. Note that an additive category with one object is a ring, and an additive functor $R \rightarrow \text{AbGrp}$ is an R -module.

In an abelian category, $\ker \operatorname{coker} f = \operatorname{im} f$ is the smallest subobject of $\operatorname{cod} f$ through which f factors. Such a factorisation, $f = (\operatorname{im} f)g$ has g epic, as if $hg = kg$, then the equaliser of h, k yields a smaller subobject of B through which f factors, so $g = \operatorname{coim} f$, and this is the unique factorisation of f as an epi then a mono.

We need to show this factorisation is stable under pullback. We know monomorphisms are stable under pullback.

Lemma 7.4. *Suppose we have a (not necessarily commutative) diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{g} & D \end{array}$$

in an additive category $\mathcal{A}$. We define its flattening to be

$$A \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} B \oplus C \xrightarrow{(h, -k)} D$$

Then

1. The square commutes if and only if the composite of the flattening is 0
2. The square is a pullback if and only if $\begin{pmatrix} f \\ g \end{pmatrix} = \ker(h, -k)$
3. The square is a pushout if and only if $(h, -k) = \operatorname{coker} \begin{pmatrix} f \\ g \end{pmatrix}$

Proof.

1. Trivial as the composite is $hf - kg$
2. $\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & & \end{array}$ is universal among spans completing $\begin{array}{ccc} & B & \\ & \downarrow h & \\ C & \xrightarrow{k} & D \end{array}$ to a commutative square is equivalent to $\begin{pmatrix} f \\ g \end{pmatrix}$ being universal with respect to having 0 composite with $(h, -k)$
3. Nearly dual to (2), not quite because of signs, but same argument works

□

Corollary 7.4.1. *In an abelian categories epis are stable under pullback*

Proof. If we have a pullback square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

with h epic, then by (2) of the previous lemma, $\begin{pmatrix} f \\ g \end{pmatrix} = \ker(h, -k)$, but $(h, -k)$ is epic since h is, so $(h, -k) = \operatorname{coker} \begin{pmatrix} f \\ g \end{pmatrix}$ by lemma 7.3, so by (3) of above, the square is a pushout, so by the dual of “monomorphisms are stable under pullback”, we get that epimorphisms are stable under pushout, so we are done. $\square$

Definition 7.5. Given a sequence $\cdots \rightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \rightarrow \cdots$ in abelian category, we say it is *exact* at A_n if $\ker f_n = \operatorname{im} f_{n+1}$. We say the sequence is *exact* if it is exact at all vertices

Remark. By Corollary 7.3.1, this is equivalent to saying $\operatorname{coim} f_n = \operatorname{coker} f_{n+1}$, so both notions are self dual.

Example 7.3. A sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is exact at A if and only if f is monic, and is exact at B if $f = \ker g$

Definition 7.6. We say a function $F: \mathcal{A} \rightarrow \mathcal{B}$ is *exact* if it preserves exact sequences (equivalently, it preserves kernels and cokernels). We say F is *left exact* if it preserves kernels, and *right exact* if it preserves cokernels.

Proposition 7.1.

1. F is left exact if and only if it preserves all finite limits
2. F is left exact and preserves epimorphisms if and only if it is exact

Proof.

1. The backwards direction is trivial. For the forwards direction we need to show that F preserves 0 and binary biproducts. For 0, note that $0 \rightarrow A$ is the kernel of $A \xrightarrow{1_A} A$ for any A . For biproducts, note that

$$0 \rightarrow A \rightarrow A \oplus B \rightarrow B \rightarrow 0$$

is exact as it is the flattening of

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ 0 \downarrow & & \downarrow 0 \\ B & \xrightarrow{1} & B \end{array}$$

Conversely if

$$0 \rightarrow A \xrightarrow{f} C \xrightarrow{g} B \rightarrow 0$$

is exact and either f or g is split (WLOG g), then $gs = 1_B$, and $1 - sg: C \rightarrow C$ is idempotent. f is the kernel of sg as s is monic, so is an equaliser of $(1_C - sg, 1_c)$, and if r is the epic part of this splitting, then f, g, r, s satisfy the relations needed for biproducts. As any functor preserves split epimorphisms, F preserves the exactness of

$$0 \rightarrow A \rightarrow A \oplus B \rightarrow B \rightarrow 0$$

. Note this implies F is additive.

2. The backwards direction is trivial. For the forward direction, if F preserves epimorphisms, then $F(\text{coker } f)$ is epic, so is $\text{coker } \ker(F \text{ coker } f) = \text{coker}(F(\text{im } f))$, hence coincides with $\text{coker } Ff$ since F preserves images

□

Lemma 7.5. (Five lemma) Suppose we are given a commutative diagram

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ u_1 \downarrow & & u_2 \downarrow & & u_3 \downarrow & & u_4 \downarrow & & u_5 \downarrow \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

in an abelian category with exact rows.

1. If u_2, u_4 are monic and u_1 is epic. Then u_3 monic.
2. If u_2, u_4 are epic and u_1 is monic. Then u_3 epic.

Proof. We prove only (i) as (ii) is dual. We draw the following diagram, which is to be explained.

$$\begin{array}{ccccccccccc} E & \xrightarrow{r} & D & \xrightarrow{y} & C & & & & & & \\ \downarrow w & & \downarrow z & \nearrow & \downarrow x & & & & & & \\ A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 & & \\ u_1 \downarrow & & u_2 \downarrow & & u_3 \downarrow & & u_4 \downarrow & & u_5 \downarrow & & \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 & & \end{array}$$

The diagram includes additional arrows and points: a dot in the triangle $A_2 \rightarrow A_3 \rightarrow A_4$ with arrows from A_2 and A_3 ; a dot in the triangle $B_1 \rightarrow B_2 \rightarrow B_3$ with arrows from B_1 and B_2 ; an arrow from D to the dot in $A_2 \rightarrow A_3 \rightarrow A_4$; an arrow from E to the dot in $B_1 \rightarrow B_2 \rightarrow B_3$; and an arrow from C to the dot in $A_2 \rightarrow A_3 \rightarrow A_4$.

Suppose given $x: C \rightarrow A_3$ with $u_3x = 0$. Then $u_4f_3x = g_3u_3x = 0$, so as u_4 is monic, $f_3x = 0$. If we form the pullback D as indicated, y is epic. Now $g_2u_2z = u_3f_2z = u_3xy = 0$ so u_2z factors through $\text{im } g_1 = \ker g_2$. Forming the pullback E , r is epic as before, so now $u_2zr = g_1u_1r = u_2f_1w$ and u_2 is monic so $zr = f_1w$. So $xyr = f_2zr = f_2f_1w = 0$. As y, r are both epic, $x = 0$ as desired, so u_3 is monic. □

Lemma 7.6. (*Snake Lemma*) Suppose given a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 \\
 & & \downarrow u_1 & & \downarrow u_2 & & \downarrow u_3 \\
 & & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 \longrightarrow 0 \\
 & & \downarrow v_1 & & \downarrow v_2 & & \downarrow v_3 \\
 0 & \longrightarrow & C_1 & \xrightarrow{h_1} & C_2 & \xrightarrow{h_2} & C_3 \\
 & & \downarrow w_1 & & \downarrow w_2 & & \downarrow w_3 \\
 & & D_1 & \xrightarrow{k_1} & D_2 & \xrightarrow{k_2} & D_3 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

(Purple arrows indicate the sequence: $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{s} 0$, $0 \longrightarrow C_1 \xrightarrow{h_1} C_2 \xrightarrow{h_2} C_3 \xrightarrow{s} 0$, and $D_1 \xrightarrow{k_1} D_2 \xrightarrow{k_2} D_3 \xrightarrow{s} 0$.)

Then we get the purple sequence which is exact. If g_1, h_2 are monic and epic respectively, then so are f_1, k_2

Proof. Exercise. (Read: We can't be bothered to do it. It's long and boring) $\square$

Definition 7.7. A *complex* in an abelian category $\mathcal{A}$ is an infinite sequence

$$\dots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \dots$$

where every $f_n f_{n+1} = 0$. Equivalently it's an additive functor $\mathcal{Z} \rightarrow \mathcal{A}$ where $\mathcal{Z}$ has objects the integers, and morphisms are $\mathbb{Z}$ between m, n with $|m-n| \leq 1$, and trivial otherwise (composition must be multiplication)

Given a complex $A_\bullet$, we define $Z_n(A_\bullet) = \text{dom ker } f_n$ and $I_n(A_\bullet) = \text{dom im } f_{n+1}$. The codomain of the cokernel of $I_n \rightarrow Z_n$ is $H_n(A_\bullet)$. We call this the *nth homology object* of $A_\bullet$. Each of these are additive functors

Remark. The category of complexes in $\mathcal{A}$, denoted $c\mathcal{A}$, is abelian.

Lemma 7.7. The definition of homology is self-dual

Proof. Write $Q_n(A_\bullet)$ for $\text{cod coker } f_{n+1}$ so we have a diagram

$$\begin{array}{ccccccc}
 A_{n+1} & \xrightarrow{f_{n+1}} & A_n & \xrightarrow{f_n} & A_{n-1} \\
 \downarrow & & \nearrow & & \nwarrow \\
 I_n & \rightarrowtail & Z_n & \twoheadrightarrow & H_n & \twoheadrightarrow & Q_n & \twoheadrightarrow & I_{n-1}
 \end{array}$$

Since $A_{n+1} \rightarrow I_n$ is epic, $A_n \rightarrow Q_n = \text{coker}(I_n \rightarrow Z_n \rightarrow A_n)$ and $I_n \rightarrow A_n = \text{ker}(A_n \rightarrow Q_n)$. Since $Z_n \rightarrow A_n$ is monic, $I_n \rightarrow Z_n = \text{ker}(Z_n \rightarrow A_n \rightarrow Q_n)$. Therefore $Z_n \rightarrow H_n = \text{coim } Z_n \rightarrow Q_n$, hence $H_n \rightarrow Q_n$ is monic, so is the kernel of $Q_n \rightarrow I_n$ $\square$

Theorem 7.8. (Mayer-Vietoris) Suppose we have an exact sequence of complexes

$$0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0$$

where $\mathcal{A}$ is abelian. Then there is an infinite exact sequence

$$\dots \longrightarrow H_{n+1}(C_\bullet) \longrightarrow H_n(A_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(B_\bullet) \xrightarrow{H_n(g_\bullet)} H_n(C_\bullet) \longrightarrow \dots$$

Proof. First apply the snake lemma to

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_{n+1} & \longrightarrow & B_{n+1} & \longrightarrow & C_{n+1} \longrightarrow 0 \\ & & d_{n+1} \downarrow & & d_{n+1} \downarrow & & d_{n+1} \downarrow \\ 0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n \longrightarrow 0 \end{array}$$

to get exact sequences

$$0 \rightarrow Z_{n+1}(A_\bullet) \rightarrow Z_{n+1}(B_\bullet) \rightarrow Z_{n+1}(C_\bullet)$$

and

$$Q_n(A_\bullet) \rightarrow Q_n(B_\bullet) \rightarrow Q_n(C_\bullet) \rightarrow 0$$

Then apply the snake lemma to

$$\begin{array}{ccccccc} Q_{n+1}(A_\bullet) & \longrightarrow & Q_{n+1}(B_\bullet) & \longrightarrow & Q_{n+1}(C_\bullet) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ I_n(A_\bullet) & & I_n(B_\bullet) & & I_n(C_\bullet) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_n(A_\bullet) & \longrightarrow & Z_n(B_\bullet) & \longrightarrow & Z_n(C_\bullet) \end{array}$$

where $\text{coker}(Q_{n+1} \rightarrow Z_n) = \text{coker}(I_n \rightarrow Z_n) = Z_n \rightarrow H_n$ since $Q_{n+1} \rightarrow I_n$ is epic, and $\ker(Q_{n+1} \rightarrow Z_n) = \ker(Q_{n+1} \rightarrow I_n) = H_{n+1} \rightarrow Q_{n+1}$ by [7.15] $\square$

Remark. Z_n is left exact as a functor $c\mathcal{A} \rightarrow \mathcal{A}$ Q_n is right exact. In fact they are the right and left adjoints respectively to the functor $\mathcal{A} \rightarrow c\mathcal{A}$ sending $A \rightarrow A[n]$ with A in dimension n and 0 elsewhere. A morphism $A[n] \xrightarrow{f} C_\bullet$ is uniquely determined by f_n , which must factor through $Z_n(C_\bullet) \rightarrow C_n$

Corollary 7.8.1. Suppose given a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_\bullet & \longrightarrow & B_\bullet & \longrightarrow & C_\bullet \longrightarrow 0 \\ & & f_\bullet \downarrow & & g_\bullet \downarrow & & h_\bullet \downarrow \\ 0 & \longrightarrow & D_\bullet & \longrightarrow & E_\bullet & \longrightarrow & F_\bullet \longrightarrow 0 \end{array}$$

in $\mathcal{A}$ with exact rows, if any two of f, g, h induce isomorphisms on H_n , then the third one does

Proof. Apply Mayer-Vietoris, then the 5-lemma $\square$

Definition 7.8. Let $C_\bullet \xrightarrow[f_\bullet]{f_\bullet} D_\bullet$ be two morphisms of $\mathcal{A}$. By a *homotopy* from $f_\bullet$ to $g_\bullet$, we mean a sequence of morphisms $h_n: C_n \rightarrow D_{n+1}$ satisfying $g_n - f_n = d_{n+1}h_n + h_{n-1}d_n$. We write $f_\bullet \simeq g_\bullet$.

Remark. Homotopy is an equivalence relation, and in fact is a congruence on $c\mathcal{A}$. We write $H\mathcal{A}$ for the quotient category $\mathcal{A}/\simeq$. Note that $\simeq$ respects addition, and $\mathcal{A} \rightarrow H\mathcal{A}$ is an additive functor

Lemma 7.9. The functors $H_n: \mathcal{A} \rightarrow \mathcal{A}$ respect homotopy, so they induce functors $H\mathcal{A} \rightarrow \mathcal{A}$

Proof. Suppose we have $C_\bullet \xrightarrow[f_\bullet]{f_\bullet} D_\bullet$ and a homotopy $h_\bullet$ between them.

Then $Z_n(g_\bullet) - Z_n(f_\bullet): Z(C_\bullet) \rightarrow Z(D_\bullet)$ reduces to $Z_n(C_\bullet) \rightarrow C_n \xrightarrow{h_n} D_{n+1} \xrightarrow{d_{n+1}} Z_n(D_\bullet)$. This term disappears when we factor through $Z_n \rightarrow H_n$, so we are done. $\square$

Definition 7.9. A category $\mathcal{A}$ has *enough projectives* if for every $A \in \mathcal{A}$ there is an epimorphism $P \rightarrow A$ where P is projective

Remark. Mod_R has enough projectives for every ring R , as free modules are projective

Definition 7.10. By a *projective resolution* of $A \in \mathcal{A}$ we mean a complex $P_\bullet$ with P_n projective for all n , $P_n = 0$ for $n < 0$, and $H_0(P_\bullet) = A$, $H_n(P_\bullet) = 0$ for $n > 0$. Equivalently, it's an exact sequence

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$$

with all P_n projective

Lemma 7.10. If $\mathcal{A}$ has enough projectives, then any object $A \in \text{ob } \mathcal{A}$ has a projective resolution

Proof. First choose $P_0 \rightarrow A$ which is an epimorphism with P_0 projective. Inductively, we choose $P_n \rightarrow \ker(P_{n-1} \rightarrow P_{n-2})$ an epimorphism with P_n projective and we are done. $\square$

Lemma 7.11. Let $P_\bullet, Q_\bullet$ be projective resolutions of A, B , then for any $f: A \rightarrow B$, there is a map $f_\bullet: P_\bullet \rightarrow Q_\bullet$ with $H_0(f_\bullet) = f$, and any two such morphisms are homotopic

Proof.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & \ker(P_0 \rightarrow A) & \twoheadrightarrow & P_0 \longrightarrow \twoheadrightarrow A \\ & & \downarrow f_1 & & \downarrow k_0 & & \downarrow f_0 & & \downarrow f \\ \cdots & \longrightarrow & Q_1 & \longrightarrow & \ker(Q_0 \rightarrow B) & \twoheadrightarrow & Q_0 \longrightarrow \twoheadrightarrow B \end{array}$$

f_0 exists by projectivity, and K_0 is mapped to L_0 by f_0 . Call the resulting morphism k_0 . Repeat, and we get some $f_\bullet$.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & P_1 & \longrightarrow & \ker(P_0 \rightarrow A) & \twoheadrightarrow & P_0 \longrightarrow A \\
& & \downarrow f_1 & & \downarrow h_0 & \nearrow k_0 & \downarrow f_0 \\
\cdots & \longrightarrow & Q_1 & \longrightarrow & \ker(Q_0 \rightarrow B) & \twoheadrightarrow & Q_0 \longrightarrow B \\
& & & & \downarrow h'_0 & \nearrow h_0' & \downarrow f_0 \\
& & & & & & \downarrow f
\end{array}$$

Suppose we have $g_\bullet$ also inducing f . Then $g_0 - f_0$ factors through $L_0 \rightarrow Q_0$, so by projectivity of P_0 we get $h_0: P_0 \rightarrow Q_1$ with $d_1 h_0 = g_0 - f_0$. By induction we do the same with $g_n - f_n - h_{n-1} d_n$ to construct h_n inductively. $\square$

Remark. We did not need projectivity of Q , so the projective resolution functor $PR: \mathcal{A} \rightarrow \mathbf{H}\mathcal{A}$ is a left adjoint to H_0 on the subcategory of complexes $C_\bullet$ with $H_n(C_\bullet) = 0$ for all $n > 0$.

If $f: \mathcal{A} \rightarrow \mathcal{B}$ is an additive functor between abelian categories, it induces $cF: c\mathcal{A} \rightarrow c\mathcal{B}$ and $\mathbf{H}F: \mathbf{H}\mathcal{A} \rightarrow \mathbf{H}\mathcal{B}$ in the obvious way.

Definition 7.11. If $\mathcal{A}$ has enough projectives, we define the n th left derived functor $L^n F: \mathcal{A} \rightarrow \mathcal{B}$ to be the composite

$$\mathcal{A} \xrightarrow{PR} \mathbf{H}\mathcal{A} \xrightarrow{\mathbf{H}F} \mathbf{H}\mathcal{B} \xrightarrow{H_n} \mathcal{B}$$

Remark. If F is exact, it preserves projective resolutions, so $L^0 F = F$ and $L^n F = 0$ for $n > 0$. If F is right exact, it preserves the exactness of $P_2 \rightarrow P_1 \rightarrow A \rightarrow 0$ so we still have $L^0 F = F$.

Lemma 7.12. Suppose we have an exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ in an abelian category with enough projectives. Then we can choose projective resolutions $P_\bullet, Q_\bullet, R_\bullet$ of A, B, C and morphisms between them extending f, g so that they are exact, and the exactness is preserved by all additive functors.

Proof. Choose $P_\bullet, R_\bullet$ arbitrarily and define $Q_n = P_n \oplus R_n$. Q_n is projective as coproducts of projectives are projective. We need to define the morphisms $Q_n \rightarrow Q_{n-1}$ and $Q_0 \rightarrow B_\bullet$. Consider the diagram h exists by projectivity of R_0 and define $e_2 = (f e_1, h)$. Note that the two squares commute by definition. We need to know e_2 is epic. Suppose we have $k: B \rightarrow D$ with $k(f e_1, h) = 0$. Then $k f e_1 = 0$ so $k f = 0$ as e_1 is epic, hence k factors through g , say $k = l g$. Now $l e_3 = l g h = k h = 0$, and e_3 is epic, so $l = 0$ and hence $k = 0$. We form the kernels of e_1, e_2, e_3 called K_0, L_0, M_0 . By snake lemma, $0 \rightarrow K_0 \rightarrow L_0 \rightarrow M_0 \rightarrow 0$ is exact. Continuing inductively, we are done. $\square$

Remark. In general $Q_\bullet$ is not the biproduct of $P_\bullet, R_\bullet$ as the differentials are different.

Theorem 7.13. Given an additive functor $F: \mathcal{A} \rightarrow \mathcal{B}$ where $\mathcal{A}$ has enough projectives, then for any short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ in $\mathcal{A}$, there is an exact sequence

$$\cdots \rightarrow L^1 F A \rightarrow L^1 F B \rightarrow L^1 F C \rightarrow L^0 F A \rightarrow L^0 F B \rightarrow L^0 F C \rightarrow 0$$

Proof. The previous lemma and Mayer Vietoris. □

In particular, L^0F is always right exact. For a right exact functor, we can see the above sequence as repairing the left exactness of F . If F is merely additive, then $FP_0 \rightarrow FA$ factors through $FP_0 \rightarrow L^0FA$, so L^0F is the coreflection of F in the category of right exact functors