

Complex Manifolds

Amy Needham

January 2025

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Notes from Kovalev, Book: Huybrechts	

1 Linear Algebra over $\mathbb{C}$

Definition 1.1. A **Complex Number**, is a pair $(x, y) \in \mathbb{R}^2$, write $z = x + iy$. We write $\mathbb{C}$ for the set of all complex numbers

Remark. This is a real vector space with an injective map $\mathbb{R} \rightarrow \mathbb{C}$ where $x \mapsto x + 0i$

Proposition 1.1. *The complex numbers form a field under $(x, y)(u, v) = (xu - yv, xv + yu)$ with unit $1 = 1 + 0i$*

Proof. Trivial □

Definition 1.2. An abelian group, V , is a **Complex Vector Space** if it has an action $\mathbb{C} \times V \rightarrow V$ called scalar multiplication that is distributive and linear ie

1. $\lambda(u + v) = \lambda u + \lambda v$
2. $(\lambda + \mu)u = \lambda u + \mu u$

for $\lambda, \mu \in \mathbb{C}$ and $u, v \in V$

Example 1.1. $\mathbb{C}^n$ is a complex vector space

Definition 1.3. A **Complex Linear Map**, $T: V \rightarrow V'$ is a group homomorphism from V to V' such that $T(\lambda u) = \lambda T(u)$ for $\lambda \in \mathbb{C}, u \in V$

Definition 1.4. A **Complex Structure** on a real vector space is an endomorphism $J: V \rightarrow V$ such that $J^2 = -I$

Remark. 1. If V is finite dimensional and admits a complex structure then $\dim(V)$ is even

2. J is an automorphism of V

Proposition 1.2. 1. *If (V, J) is a complex structure on a real vector space V then it has a natural structure of a complex vector space*

2. *If V is a complex vector space the underlying real vector space has a natural complex structure called i*

Proof. 1. $z = x + iy \in \mathbb{C}$ acts on $v \in V$ by $xI + yJ$

2. Let $J = i$. It follows as $i(i(v)) = -v$ because $i^2 = -1$

□

Definition 1.5. If V is a real vector space its **Complexification** $V_{\mathbb{C}}$ is the real vector space $V \oplus V$ equipped with the complex structure $J(x, y) = (-y, x)$

Proposition 1.3. If $T: V \rightarrow V'$ is a linear map between real vector spaces, we get a natural induced complex-linear map $T_{\mathbb{C}}: V_{\mathbb{C}} \rightarrow V'_{\mathbb{C}}$

Definition 1.6. If $T: V \rightarrow V'$ is a real linear map and (V, J) and (V', J') are complex structures on V and V' respectively, we say T is **Complex Linear** if $J'T = TJ$. T is **Complex Anti-Linear** if $J'T = -TJ$

Proposition 1.4. 1. If (V, J) and (V', J') are real vector spaces with complex structures and $T: V \rightarrow V'$ is complex linear, there is a complex linear map T between the complex vector spaces naturally associated to V and V'

2. A map between complex vector spaces induces a map between their real vector spaces commuting with their complex structures

Proof. Trivial

□

Proposition 1.5. If $T: V \rightarrow V'$ is a real linear map and J, J' are complex structures on V, V' respectively, then we can write $T = T^{1,0} + T^{0,1}$ uniquely such that $T^{1,0}$ is complex linear and $T^{0,1}$ is complex anti-linear

Proof. Trivial

□

Proposition 1.6. If (V, J) is a finite dimensional real vector space with a complex structure then there is a complex linear isomorphism $T: (V, J) \rightarrow (\mathbb{R}^{2n}, J_0)$ where $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ (The **Standard Complex Structure** on $\mathbb{R}^{2n}$)

Proof. Trivial

□

Remark. Unfortunately, $(\mathbb{C}^n, i)$ has associated real vector space $\mathbb{R}^{2n}$ and complex structure $\text{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ which is not the same (but they are isomorphic)

Proposition 1.7. If (V, J) is a real vector space with a complex structure then there's a natural complex linear map $J_{\mathbb{C}}: V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$. $J_{\mathbb{C}}$ has two eigenvalues $\pm i$ and $V_{\mathbb{C}}$ can be written as $V^{1,0} \oplus V^{0,1}$ where $V^{1,0}$ is the i eigenspace of $J_{\mathbb{C}}$ and $V^{0,1}$ is the $-i$ eigenspace. They are both complex vector spaces

Proof. Trivial

□

(A useful note for the proof is that $v = \frac{1}{2}(v - iJv) = \frac{1}{2}(v + iJv)$)

Definition 1.7. If V^* is the real dual vector space of V and J is a complex structure on V then $J^*(\phi)(v) = \phi(J(v))$ gives a real linear map $J^*: V^* \rightarrow V^*$ such that $J^* = -I$ defining a **Dual Complex Structure** on V^*

Definition 1.8. If V is a complex vector space, V^* means the complex vector space of complex linear maps $V \rightarrow \mathbb{C}$

Proposition 1.8. If V is a real vector space with a complex structure J then $(V^{1,0})^* = (V^*)^{1,0} = \{\text{Complex Linear maps } (V, J) \rightarrow (\mathbb{C}, i)\}$

Proof. Trivial □

Proposition 1.9. If $x_1, \dots, x_n$ are a basis for a real vector space V , there is a basis for $V_{\mathbb{C}}$ given by $z_i = x_i + iy_i \in V^{1,0}$ and $\bar{z}_i = x_i - iy_i \in V^{0,1}$

Proof. Trivial □

2 Complex Analysis Review

Definition 2.1. Suppose $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is differentiable regarding $\mathbb{C}$ as $\mathbb{R}^2$. Write $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ for $u, v: \mathbb{R}^2 \rightarrow \mathbb{R}$. We say f is **Holomorphic** or **Complex analytic** if u, v satisfy the Cauchy Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proposition 2.1. Let $f: U \rightarrow \mathbb{C}$ be differentiable. Then f is holomorphic if and only if $d_p f$ is complex linear at every $p \in U$

Proof. Write it out in coordinates: $d_p f = \begin{pmatrix} \frac{\partial u}{\partial x}(p) & \frac{\partial u}{\partial y}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix}$ and check it commutes with J if and only if the Cauchy Riemann equations hold □

Theorem 2.1. If $f: U \rightarrow \mathbb{C}$ is holomorphic with $U \subset \mathbb{C}$ then

1. f has a convergent power series around every point $p \in U$ ie f is analytic
2. (Cauchy's formula) If $D_s(a) \subset U$ is a closed disk then $f(w) = \frac{1}{2\pi i} \int_{\partial D_s(a)} \frac{f(z)}{w-z} dz$ for $w \in \text{Int}(D_s(a))$

Proof. See IB Complex Analysis □

Proposition 2.2. If $f: U \rightarrow \mathbb{C}$ is holomorphic with $U \subset \mathbb{C}$ connected open

1. $|f|$ does not have a local maximum in U
2. If $f = 0$ on some connected open $V \subset U$ then $f = 0$ on all of U
3. (Cauchy's Theorem) If γ is a simple closed loop in simply connected U then $\int_{\gamma} f(z) dz = 0$

Proof. See IB Complex Analysis □

Remark. If $f: D_\epsilon(0) \setminus \{0\} \rightarrow \mathbb{C}$ is bounded holomorphic then f extends to a holomorphic function on $D_\epsilon(0)$

Theorem 2.2. (*Cauchy's residue theorem*) If $f: D_\epsilon(0) \setminus \{0\} \rightarrow \mathbb{C}$ is holomorphic and has a convergent laurent series expansion on $D_\epsilon(0)$ of the form $f(z) = \sum_{n=k}^{n=\infty} a_n z^n$ where k is finite negative then

$$\frac{1}{2\pi i} \int_{|z|=\epsilon/2} f(z) dz = a_{-1} = \text{Res}_{z=0} f(z)$$

Proof. See IB Complex Analysis □

Definition 2.2. Let $f: U \rightarrow \mathbb{C}^m$ be differentiable for $U \subset \mathbb{C}^n$. Then f is **holomorphic** if and only if $d_p f$ is complex linear at every $p \in U$

Proposition 2.3. $f: U \rightarrow \mathbb{C}$ with $U \subset \mathbb{C}^n$ is holomorphic if and only if it is separately holomorphic in each variable

Proof. Write it out in coordinates as in Proposition 2.1 □

Remark. We use the notation $\frac{\partial}{\partial z_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right)$, $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \right)$
A polydisk $\Delta_\epsilon(a) \subset \mathbb{C}^n$ is the product of disks $\{z \in \mathbb{C}^n \mid |z_i - a_i| < \epsilon_i\}$

Proposition 2.4. If $f: U \rightarrow \mathbb{C}$ is holomorphic with $U \subset \mathbb{C}^n$ then

1. f has a convergent power series around every point $p \in U$ ie f is analytic

2. (*Cauchy's formula*) If $\Delta_s(a) \subset U$ is a closed polydisk then

$$f(w) = \frac{1}{2\pi i} \int_{|z_i - a_i| = s_i} \frac{f(z)}{(w_1 - z_1) \dots (w_n - z_n)} dz_1 \dots dz_n \text{ for } w \in \text{Int}(\Delta_s(a))$$

Theorem 2.3. (*Hartogs's Theorem*) If $U \subset \mathbb{C}^n$ is open connected with $K \subset U$ compact and we have $f: U \setminus K \rightarrow \mathbb{C}$ holomorphic and $n > 1$ then there is a unique holomorphic extension of f to U

Proof. (Sketch) For simplicity, $n = 2$. Let U_i, K_i be the projections of U, K onto the i th factor. $f_2(u, z_2) = \frac{1}{2\pi i} \int_\gamma \frac{f(z_1, z_2)}{w - z_1} dz_1$ holomorphic in $z_2 \in U_2$. We can do the same in the other factor to get a function holomorphic in $z_1 \in U_1$. Let $\tilde{f} = \sum_{i_1, i_2=1}^{\infty} \frac{1}{i_1! i_2!} f_1^{(i_1)}(a) f_2^{(i_2)}(a) (z_1 - a_1)^{i_1} (z_2 - a_2)^{i_2}$. This works. We needed compactness to find such a γ □

Recall that if V is a real vector space its complexification $V_{\mathbb{C}}$ is the complex vector space with $V_{\mathbb{C}} = V \oplus V, J(x, y) = (-y, x)$

Remark. We will write an element (x, y) in the complexification as $x + iy$

Further recall that if $A: V \rightarrow V$ is a real linear map, it induces a complex linear map $A_{\mathbb{C}}: V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ by $A_{\mathbb{C}}(x + iy) = Ax + iAy$. If V itself has a complex structure J then $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ is a direct sum of eigenspaces of $J_{\mathbb{C}}$

Proposition 2.5. *If $A: V \rightarrow V$ is a real linear map, it is complex linear if and only if $A_{\mathbb{C}}$ preserves $V^{1,0}$ and $V^{0,1}$. ie $A_{\mathbb{C}} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$*

Proof. Trivial □

Remark. If $A: V \rightarrow V$ is a complex linear map with respect to J , it induces a complex linear map $A^{\mathbb{C}}$ from (V, J) to (V, J) considered as complex vector spaces. In this case $A_{\mathbb{C}} = \begin{pmatrix} A^{\mathbb{C}} & 0 \\ 0 & \overline{A^{\mathbb{C}}} \end{pmatrix}$

Proposition 2.6. $\det(A) = \det(A_{\mathbb{C}}) = |\det(A^{\mathbb{C}})|^2$. In particular A is invertible if and only if $A^{\mathbb{C}}$ is

Proof. Trivial □

Recall that if $f: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is smooth then $d_p f: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a real linear map. A smooth map $f: \mathbb{C}^n \rightarrow \mathbb{C}^n$ has a derivative $d_p f: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ which is real linear and by definition it is holomorphic if and only if $d_p f$ is complex linear, that is with respect to the standard complex structure on $\mathbb{R}^{2n}$ we have $J \circ d_p f = d_p f \circ J$. If we choose a basis for $\mathbb{C}^n$ given by $x_1, \dots, x_n, y_1, \dots, y_n$ then $d_p f = \begin{pmatrix} \mathcal{D}_x(u) & \mathcal{D}_y(u) \\ \mathcal{D}_x(v) & \mathcal{D}_y(v) \end{pmatrix}$.

The complex derivative of a complex function $f: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is $d_p^{\mathbb{C}} f = (d_p f)^{\mathbb{C}}: \mathbb{C}^n \rightarrow \mathbb{C}^n$. In $\mathbb{C}^n$ we have coordinates $\frac{1}{2}(x_i + iy_i)$ and in this basis, $d_p^{\mathbb{C}} f = (\frac{\partial}{\partial z_i} f_j)$. Lastly we can also consider $(d_p f)_{\mathbb{C}}: \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$.

By the proposition, $(d_p f)_{\mathbb{C}} = \begin{pmatrix} d_p^{\mathbb{C}} f & 0 \\ 0 & \overline{d_p^{\mathbb{C}} f} \end{pmatrix}$

Remark. Observe:

1. If $f: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is holomorphic, then it is orientation preserving as $\det(d_p f) = |\det(d_p^{\mathbb{C}} f)|^2 \geq 0$
2. $d_p f$ is invertible if and only if $d_p^{\mathbb{C}} f$ is invertible

Definition 2.3. A smooth map $f: U \rightarrow V$ with $U, V \subset \mathbb{C}^n$ both open is a **biholomorphism/biholomorphic** if f is holomorphic with holomorphic inverse

Observe that if $f: U \rightarrow V$ is holomorphic with smooth inverse $f^{-1}: V \rightarrow U$ then f^{-1} is automatically holomorphic

3 Complex Manifolds

Definition 3.1. Suppose X is a topological space

1. X is **Hausdorff** if for any two distinct $x, y \in X$ there are disjoint opens $x \in U, y \in V$

2. X is **second countable** if there's some countable τ_0 of open subsets of X such that any open subset of X is a union subsets in τ_0

Example 3.1. $\mathbb{C}^n$ in the Euclidean topology is Hausdorff and second countable

Definition 3.2. If X is a topological space and G is a group acting on X , then the **quotient space**, X/G has the structure of a topological space where, if $\pi: X \rightarrow X/G$ is the quotient map, then $U \subset X/G$ is open if and only if $\pi^{-1}(U)$ is open in X

Definition 3.3. A **complex manifold** X (of dimension n) is a hausdorff, second countable topological space equipped with an atlas of holomorphic charts. ie a collection $(U_\alpha, \phi_\alpha), \alpha \in I$ consisting of pairs where $U_\alpha \subset X$ is open and $\phi_\alpha: U_\alpha \rightarrow \mathbb{C}^n$ is a homeomorphism onto an open subset of $\mathbb{C}^n$ such that

1. the U_α cover X
2. If $U_\alpha \cap U_\beta \neq \emptyset$ then the transition function

$$\phi_\alpha \phi_\beta^{-1}: \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

is a biholomorphism

Remark. There are a few comments on this definition

1. If X is a complex manifold we can forget the fact that the transition maps are holomorphic to get a $2n$ -dimensional smooth manifold.
2. Warning: A complex manifold is not a smooth manifold with the property that the transition maps are holomorphic as being a complex manifold is a structure, not a property. A complex manifold is a smooth manifold with a choice of atlas for which all transition functions are biholomorphic
3. If X is a complex manifold, we can always add in new charts as long as they are compatible with the existing holomorphic charts and this won't change the complex manifold. We might as well assume we have a maximal atlas
4. We will abuse notation and if we have (U, ϕ) a holomorphic chart on the complex manifold X , we write $(z_1, \dots, z_n) = \phi(x)$ for $x \in U$ ie the z_i are local holomorphic coordinates and without loss of generality, $0 \in \text{Im}(\phi)$

4 Examples of Complex Manifolds

Example 4.1. We have some examples of complex manifolds

1. $U \subset \mathbb{C}^n$ is a complex manifold, with the single chart $\text{Id}: U \rightarrow U$

2. If X, Y are complex manifolds then $X \times Y$ (in the product topology) has a natural structure of a complex manifold by, for holomorphic charts (U_i, ϕ_i) of X for $i \in I$ and (V_j, φ_j) of Y for $j \in J$ we have an atlas of charts for $X \times Y$ given by $(U_i \times V_j, (\phi_i, \varphi_j))$ for all $i \in I, j \in J$ where $(\phi_i, \varphi_j): U_i \times V_j \rightarrow \mathbb{C}^n \times \mathbb{C}^m$ by $(x, y) \mapsto (\phi_i(x), \varphi_j(y))$ which is holomorphic as holomorphicity can be checked componentwise

3. (Riemann Sphere) Let $S^2 \subset \mathbb{R}^3$.

We may make this into a smooth manifold via charts on the sets

$U_1 = S^2 \setminus \{(0, 0, 1)\}$ and $U_2 = S^2 \setminus \{(0, 0, -1)\}$ given by $\phi_1: U_1 \rightarrow \mathbb{R}^2$ via $\phi_1(x, y, z) = \frac{1}{1-z}(x, y)$ and $\phi_2: U_2 \rightarrow \mathbb{R}^2$ via $\phi_2(x, y, z) = \frac{1}{1+z}(x, -y)$.

The transition function is $\phi_2 \circ \phi_1^{-1}: \phi_1(U_1 \cap U_2) \rightarrow \phi_2(U_1 \cap U_2)$ which you can check is smooth

We can consider these as complex charts, as $\mathbb{C} \cong \mathbb{R}^2$.

Suppose $w = \phi_1(x, y, z)$ then

$$\phi_2(x, y, z) = \bar{w} \frac{1-z}{1+z} = \bar{w} \frac{(1-z)^2}{1-z^2} = \bar{w} \frac{(1-z)^2}{x^2 + y^2} = \frac{\bar{w}}{|w|^2} = \frac{1}{w}$$

which is holomorphic on the punctured plane

4. (Complex Projective Space) Recall $\mathbb{CP}^n = \mathbb{P}^2$ is the set of lines passing through 0 in $\mathbb{C}^{n+1}$. As a topological space, this is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the action of $\mathbb{C}^\times$ by $t \cdot (z_1, \dots, z_n) = (tz_1, \dots, tz_n)$. We use the homogenous coordinates $[z_1: \dots: z_n]$ to denote the equivalence class of $(z_1, \dots, z_n)$

To give this the structure of a complex manifold, we have the complex charts on $U_i \subset \mathbb{CP}^n$ given by $U_i = \{[z_0: \dots: z_n] \in \mathbb{CP}^n \mid z_i \neq 0\}$ given by

$$\phi_i: U_i \rightarrow \mathbb{C}^n, \phi_i([z_0: \dots: z_n]) = \frac{1}{z_i}(z_0, \dots, \hat{z}_i, \dots, z_n)$$

This is well defined as $\phi_i([tz_0: \dots: tz_n]) = \frac{1}{tz_i}(tz_0, \dots, t\hat{z}_i, \dots, tz_n)$ and the t cancels out.

The transition functions for (without loss of generality) $0 < i < j < n$ are $\phi_j \phi_i^{-1}: \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ given by, for the choice of preimage $w = \phi_i^{-1}(z_0, \dots, z_n) = [z_0: \dots: z_{i-1}: 1: z_{i+1}: \dots: z_n]$

$$w \mapsto \frac{1}{z_j}(z_0, \dots, z_{i-1}, 1, z_{i+1}, \dots, \hat{z}_j, \dots, z_n)$$

which is holomorphic.

We will see $S^2 \cong \mathbb{CP}^1$ as complex manifolds

5. (Elliptic Curves) As topological spaces, elliptic curves are given by the quotient $E_\tau = \mathbb{C}/\mathbb{Z}^2$ where $\mathbb{Z}^2$ acts by $(a, b) \cdot z = z + a + b\tau$ for some $\tau \in \mathbb{C}$ with positive imaginary part. This is homeomorphic to the 2-torus T^2 . Let $D \subset \mathbb{C}$ be the half open parallelogram spanned by $1, \tau$ and

$\pi: \mathbb{C} \rightarrow E_\tau$ the quotient map. Then $\pi|_D$ is a bijection. We define charts by $U_0 = D^o, U_1 = D^o + \frac{1}{2}, U_2 = D^o + \frac{\tau}{2}, U_3 = D^o + \frac{\tau+1}{2}$. $\pi|_{U_i}$ are bijective when combined. Define $\phi_i: U_i \rightarrow \mathbb{C}$ by $[z] \mapsto z \in D^o + \text{offset}$. This is a complex manifold. $\phi_1\phi_0^{-1}: \phi_0(U_0 \cap U_1) \rightarrow \phi_1(U_0 \cap U_1)$ is either $z \mapsto z$ or $z \mapsto z + 1$ depending on which connected component of the domain, and both are holomorphic, and general i, j is similar

6. Complex tori are $\mathbb{C}^n/\mathbb{Z}^{2n}$ where $\mathbb{Z}^{2n}$ acts by translation, which is a complex manifold for the same reasons as 5, and as smooth manifolds, they are all isomorphic to $T^{2k} \times (\mathbb{C}^*)^m$
7. $GL_n(\mathbb{C})$, the group of $n \times n$ invertible complex matrices, which is the open subset of $\mathbb{C}^{n^2}$ where $\det(A) \neq 0$, so is a complex manifold and is in fact a complex lie group
8. $SL_n(\mathbb{C}), O(n, \mathbb{C}), \dots$ are also complex manifolds
9. Grassmannians, $Gr(k, n)$ is the set of all k dimensional complex subspaces of $\mathbb{C}^n$ (this generalises $\mathbb{CP}^{n-1}$). This is also a complex manifold
10. The Hopf Surface, $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}, 1 \circ z = q^{-1}z$ has the structure of a complex manifold, $q \in \mathbb{C} \setminus \{0\}$

Definition 4.1. If X, Y are complex manifolds and $f: X \rightarrow Y$ is a continuous function, we say f is **holomorphic** if for every holomorphic chart (U, ϕ) on X and (V, ψ) on Y , the composition $\psi \circ f \circ \phi^{-1}: \phi(U \cap f^{-1}(V)) \rightarrow \psi(V)$ is holomorphic.

f is a **biholomorphism** if it has a holomorphic inverse. Then X, Y are **isomorphic** complex manifolds

Remark. It suffices to check holomorphicity for a cover of holomorphic charts. It suffices for f to have a smooth inverse to be a biholomorphism

Remark. A **holomorphic function** on a complex manifold X will refer to a holomorphic map $f: X \rightarrow \mathbb{C}$

Proposition 4.1. If X is a compact connected complex manifold then every holomorphic function on X is constant

Proof. Suppose $f: X \rightarrow \mathbb{C}$ is holomorphic with $\dim(X) = n$. Since X is compact, there is some point $p \in X$ such that $|f|$ attains a maximum value. Take a holomorphic chart $(U, \phi), p \in U$ such that $\phi(p) = 0$. Then $f \circ \phi^{-1}: \phi(U) \rightarrow \mathbb{C}$ is holomorphic and $|f \circ \phi^{-1}(0)|$ is a maximum on $\phi(U)$. Fix $z_2, \dots, z_n$ and consider $f \circ \phi^{-1}$ as a function of only z_1 . Then this is a holomorphic function which attains a maximum value of $|f \circ \phi^{-1}|$ at $z = 0$. By the maximum principle, $f \circ \phi^{-1}$ doesn't depend on z_1 , and by doing similar for all z_i , we see that $f \circ \phi^{-1}$ and hence f is constant on $U \subset X$.

Let $g = f - f(p)$. If (V, ψ) is another holomorphic chart for X such that $U \cap V \neq \emptyset$ then g is similarly constant on V , and hence is 0 as $U \cap V \neq \emptyset$. If

(U_2, ϕ_2) intersects U_1 . As X is connected and, we have the open sets given by $U_0 = \{z \in X \mid g(z) = 0\}$, $U_1 = \{z \in X \mid g(z) \neq 0\}$ which are disjoint and cover X , and $p \in U_0$, we are done $\square$

5 Tangent Spaces

If X is a complex manifold, $P \in X$ and (U, ϕ) is a holomorphic chart such that $P \in U$ then we have an isomorphism $d_p\phi: T_pX \rightarrow T_{\phi(p)}\mathbb{C}^n \cong \mathbb{C}^n$ of real vector spaces

Proposition 5.1. *For every $p \in X$ the tangent space T_pX of a complex manifold has a complex structure given by $J(v) = (d_p\phi^{-1}) \circ i \circ d_p\phi(v)$, where i is the complex structure on $\mathbb{C}^n$, which is independent of choice of holomorphic chart*

Proof. If (V, ψ) is another holomorphic chart for X and $p \in V$ then

$$\begin{aligned} J_\psi(v) &= (d_p\psi)^{-1}i(d_p\psi)(v) \\ &= (d_p\phi)^{-1}(d_p\phi)(d_p\psi)^{-1}i(d_p\psi)(v) \\ &= (d_p\phi)^{-1}(d_p\phi \circ \psi^{-1})i(d_p\psi)(v) \end{aligned}$$

where the 2nd term in the last expression is complex linear so we get

$$(d_p\phi)^{-1}i(d_p\phi \circ \psi^{-1})(d_p\psi)(v) = (d_p\phi)^{-1}i(d_p\phi)(v)$$

and we are done $\square$

Definition 5.1. An **almost complex structure** J on a smooth manifold M is a smooth section of $T^*M \otimes TM \cong \text{End}(TM)$ such that $J^2 = -\text{Id}$

Proposition 5.2. *Every complex manifold X admits an almost complex structure. Such almost complex structures are **integrable***

Proof. Exactly the proposition above $\square$

Definition 5.2. If M is a smooth manifold with an almost complex structure J , the **Nijenhuis tensor**, $N_J: TM \otimes TM \rightarrow TM$ is defined by

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]$$

Theorem 5.1. (Wewlander-Nirenberg) *An almost complex structure J is integrable if and only if $N_J = 0$*

Proof. No Proof $\square$

The backwards direction will be on an example sheet

Example 5.1. $J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & x & 1+x^2 \end{pmatrix}$ on $\mathbb{R}^4$ with coordinates (x, y, z, w)

is an example of a non-integrable complex structure (or something like this)

In coordinates, if X is a complex manifold and $p \in X$ with (U, ϕ) a holomorphic chart where $\phi(p) = (x_1, \dots, x_n, y_1, \dots, y_n)$, we have a real basis of $T_p X$ given by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ and J acts on $T_p(X)$ by $J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}$ and $J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}$ so $J = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}$

But we have $\frac{\partial}{\partial z_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right)$ and $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \right)$. The complexified tangent space, $T^{\mathbb{C}}X$ is the complex vector bundle on X whose fibres are $T_p^{\mathbb{C}}X = (T_p X)_{\mathbb{C}}$ for every $p \in X$.

We have that for a holomorphic chart (U, ϕ) on X with $p \in U$, $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$ form a basis of $T_p^{\mathbb{C}}X$ as i ranges over $1, \dots, n$

Proposition 5.3. *For every $p \in X$ the complexified tangent space splits into eigenspaces of $J_{\mathbb{C}}$ as $T_p^{\mathbb{C}}X = T_p^{1,0}X \oplus T_p^{0,1}X$. In particular, the $\frac{\partial}{\partial z_i}$ form a basis of $T_p^{1,0}X$ and the $\frac{\partial}{\partial \bar{z}_i}$ forms a basis of $T_p^{0,1}X$.*

Proof. Trivial □

Remark. These assemble to form complex vector bundles $T^{1,0}X$ and $T^{0,1}X$ called the **holomorphic tangent bundle** and the **anti-holomorphic tangent bundle**

Warning: a section of the holomorphic tangent bundle need not be a holomorphic vector field. In fact,

Definition 5.3. A section v of $T^{1,0}X$ is a **holomorphic vector field** if it can be locally written as $v(z) = a_i(z) \frac{\partial}{\partial z_i}$ (using summation convention) such that a_i are holomorphic functions

Observe that sections of $T^{\mathbb{C}}X$ have a natural action on smooth functions $f: X \rightarrow \mathbb{C}$ by differentiation

Proposition 5.4. *A smooth functions $f: X \rightarrow \mathbb{C}$ is holomorphic if and only if $v(f) = 0$ for every section v of $T^{0,1}$*

Proof. (Since $\frac{\partial}{\partial \bar{z}_i} f = 0$ for $i = 1, \dots, n$ implies f is holomorphic) □

Remark. If $f: X \rightarrow Y$ is a holomorphic function then

1. $d_p f: d_p X \rightarrow d_p Y$ is complex linear
2. $(d_p f)_{\mathbb{C}}: T_p^{\mathbb{C}}X \rightarrow T_p^{\mathbb{C}}Y$ sends $T_p^{1,0}X$ to $T_p^{1,0}Y$ and $T_p^{0,1}X$ to $T_p^{0,1}Y$

and each of these is separately equivalent to holomorphicity

If f is holomorphic, we get a $\mathbb{C}$ linear map $d_p^{\mathbb{C}} f: T_p^{1,0}X \rightarrow T_{f(p)}^{1,0}Y$ given in components by $(d_p^{\mathbb{C}} f)_{ij} = \frac{\partial}{\partial z_i} f_j$. The real rank of $d_p f$ is twice the complex rank of $d_p^{\mathbb{C}} f$ and similarly with the nullity

6 Holomorphic Vector Bundles

Definition 6.1. If X is a complex manifold, a **holomorphic vector bundle of rank k** on X is a complex manifold E with $\pi: E \rightarrow X$ a holomorphic map such that

1. π is a submersion ($d_p\pi$ has full rank, which is equivalent to $d_p^{\mathbb{C}}\pi$ having full rank)
2. Every fibre $\pi^{-1}(p)$ has the structure of a k dimensional complex vector space for $p \in X$
3. There is a holomorphic local trivialisation: there is an open cover $\{U_i\}$ of X and biholomorphic maps $\Phi_i: \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}^k$ such that

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\Phi_i} & U_i \times \mathbb{C}^k \\ & \searrow \pi \quad \swarrow \pi_1 & \\ & U_i & \end{array}$$

commutes and Φ_i is an isomorphism of complex vector spaces on each fibre

Definition 6.2. If E is a holomorphic vector bundle with a holomorphic local trivialisation $\Phi_i: \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}^k$ for $\{U_i\}$ an open cover of X . On an overlap, $U_i \cap U_j$, we have

$$U_i \cap U_j \times \mathbb{C}^k \xleftarrow{\Phi_j} \pi^{-1}(U_i \cap U_j) \xrightarrow{\Phi_i} U_i \cap U_j \times \mathbb{C}^k$$

hence $\Phi_j \circ \Phi_i^{-1}(p, v) = (p, \psi_{ij}(p)v)$ where $\psi: U_i \cap U_j \rightarrow GL_k(\mathbb{C})$ is holomorphic. These are called the **transition functions**

Proposition 6.1. If $\pi: E \rightarrow X$ is a holomorphic vector bundle then the transition functions are holomorphic and satisfy the cocycle condition on $U_i \cap U_j \cap U_k$ given by $\psi_{ij}\psi_{jk}\psi_{ki} = \text{Id}$

Conversely given an open cover U_i of X and a collection of holomorphic maps $\psi_{ij}: U_i \cap U_j \rightarrow GL_k(\mathbb{C})$ satisfying the cocycle condition then there is a holomorphic vector bundle $\pi: E \rightarrow X$ of rank k whose transition functions are the ψ_{ij} and is uniquely determined up to isomorphism

Proof. Trivial □

Definition 6.3. If E, F are holomorphic vector bundles over X , a **morphism** from $E \rightarrow F$ is a holomorphic map $f: E \rightarrow F$ such that

1. The diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ & \searrow \pi_E \quad \swarrow \pi_F & \\ & X & \end{array}$$

commutes

2. f acts on fibres as a complex linear map of constant rank

An **isomorphism of holomorphic vector bundles** $f: E \rightarrow F$ is a morphism with an inverse morphism

Definition 6.4. A **holomorphic section** of a holomorphic vector bundle $\pi: E \rightarrow X$ is a holomorphic map $s: X \rightarrow E$ such that $\pi \circ s = \text{Id}_X$. Equivalently, there is an open cover $\{U_i\}$ of X and holomorphic functions $s_i: U_i \rightarrow \mathbb{C}^k$ such that $s_j = \psi_{ij}s_i$ on $U_i \cap U_j$

Proposition 6.2. If E, F are holomorphic vector bundles over X then so are $E \otimes F, E \oplus F, E^\vee, \text{End}(E), \bigwedge^p E, \dots$

Proof. Trivial □

Proposition 6.3. If $f: X \rightarrow Y$ is holomorphic and $\pi: E \rightarrow Y$ is a holomorphic vector bundle then so is $f^*E \rightarrow X$

Proof. Pull back transition functions □

Proposition 6.4. The holomorphic tangent bundle, $T^{1,0}X$, has a natural holomorphic vector bundle structure

Proof. Suppose (U_i, ϕ_i) is a cover of X by holomorphic charts. Then we have a holomorphic atlas on $T^{1,0}X$ given by $(\pi^{-1}(U_i), \Phi_i)$ where we have the map $\Phi_i: \pi^{-1}(U_i) \rightarrow \mathbb{C}^n \times \mathbb{C}^n$ by $(p, v) \mapsto (\phi_i(p), d_p^{\mathbb{C}}\phi_i(v))$. This is a holomorphic vector bundle since we have local trivialisations $\Psi: \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}^n$ by $(p, v) \mapsto (p, d_p^{\mathbb{C}}\phi_i(v))$. This is a biholomorphism which sends fibres to fibres, which is holomorphic by looking at coordinates. □

Remark. We use $\mathcal{T}_X$ to denote $T^{1,0}X$ as a holomorphic vector bundle

Example 6.1. Trivial bundle: If X is a complex manifold, Let $E = X \times \mathbb{C}^k$ and $\pi: E \rightarrow X$ be projection onto the first factor. This is a holomorphic vector bundle

Example 6.2. If X is a complex manifold, the holomorphic tangent bundle, $\mathcal{T}_X$ is a holomorphic vector bundle

Example 6.3. $\mathcal{T}_X^*$ is also a holomorphic vector bundle, called the **holomorphic cotangent bundle**, Ω_X . As a complex vector bundle, it is isomorphic to $(T^{1,0}X)^* = (T^*X)^{1,0} = (TX, J)^*$

Example 6.4. $\Omega_X^p = \bigwedge^p \Omega_X$ is called the **bundle of holomorphic p -forms**

Example 6.5. If $\dim(X) = n$, $K_X = \bigwedge^n \Omega_X$ is the **canonical line bundle**

Example 6.6. The **tautological line bundle** on $\mathbb{CP}^n$ is defined by total space, $E = \{(l, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid v \in l\}$. We have $\pi: E \rightarrow \mathbb{CP}^n$ by projection on the first factor. The fibre above l is a 1-dimensional complex vector space isomorphic to $l \subset \mathbb{C}^{n+1}$. Recall we have an open cover $\{U_i \mid i = 0, \dots, n\}$ where

$U_i = \{[z_0 : \dots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\}$. In U_i each line has a unique representative with $z_i = 1$ of the form $[z_0 : \dots : z_{i-1} : 1 : z_{i+1} : \dots : z_n]$ and so any $v \in l$ can be uniquely represented as $\lambda[z_0 : \dots : z_{i-1} : 1 : z_{i+1} : \dots : z_n]$ for some $\lambda \in \mathbb{C}$. Define $\Phi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}$ by $([z_0 : \dots : z_n], v) \mapsto ([z_0 : \dots : z_n], \lambda)$

Proposition 6.5. *This is a holomorphic vector bundle, called the tautological line bundle and denoted $\mathcal{O}(-1)$. We also have holomorphic vector bundles given by $\mathcal{O}(1) = \mathcal{O}(-1)^*$, $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$ and $\mathcal{O}(-k) = \mathcal{O}(1)^{\otimes k}$*

Proof. Trivial □

Proposition 6.6. $K_{\mathbb{CP}^n} \cong \mathcal{O}(-n-1)$

Proof. Deferred □

Proposition 6.7. *If $f \in \mathbb{C}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree $d \geq 1$ then it defines a holomorphic section of $\mathcal{O}(d)$ over $\mathbb{CP}^n$*

Proof. Take the cover $\{U_i\}$ of $\mathbb{CP}^n$ as before and define a holomorphic function $f_i : U_i \rightarrow \mathbb{C}$ by $f_i([z_0, \dots, z_n]) = \frac{1}{z_i^d} f(z_0, \dots, z_n)$. This is well defined as usual as $z_i \neq 0$. On an overlap $U_i \cap U_j \subset \mathbb{CP}^n$ we have

$$U_j \times \mathbb{C} \xleftarrow{\Phi_j} \pi^{-1}(U_i \cap U_j) \xrightarrow{\Phi_i} U_i \times \mathbb{C}$$

and $\Phi_i \circ \Phi_j([z_0 : \dots : z_n], \lambda) = \Phi_i([z_0 : \dots : z_n], \lambda(z_0/z_j, \dots, 1, \dots, z_n/z_j))$ and this is equal to $([z_0 : \dots : z_n], (z_i/z_j)\lambda)$ and this is a holomorphic function into $GL_1(\mathbb{C})$ as $z_j, z_i \neq 0$. So passing from $\mathcal{O}(-1)$ to $\mathcal{O}(d)$, we have that the transition functions are $\psi_{ij} = \left(\frac{z_j}{z_i}\right)^d$. The polynomial f gives a well defined section of $\mathcal{O}(d)$ since $\psi_{ij}f_i = f_j$. □

7 Complex Submanifolds

Definition 7.1. An closed subset Y of a complex manifold X is a **codimension- k complex submanifold** if every $p \in Y$ has a neighbourhood $U \subset X$ and a holomorphic function $f : U \rightarrow \mathbb{C}^k$ with $d_p f$ having full rank such that we have $f^{-1}(0) \cap U = Y \cap U$

Remark. If $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is a smooth function then $d_p f$ is complex linear if and only if $(d_p f)^{-1}$ is complex linear if it exists, so if f is a diffeomorphism, then $d_p f$ is complex linear if and only if $d_{f(p)} f^{-1}$ is complex linear

Theorem 7.1. *(Inverse function theorem for holomorphic functions)*

If $f : U \rightarrow \mathbb{C}^n$ with $U \subset \mathbb{C}^n$ is holomorphic and $d_p^{\mathbb{C}} f$ is invertible for some $p \in U$ then f maps some neighbourhood, V , of p biholomorphically onto $f(V) \subset \mathbb{C}^n$

Proof. Apply the real inverse function theorem for smooth maps, and use the remark □

Theorem 7.2. (*Holomorphic implicit function theorem*) Suppose $f: U \rightarrow \mathbb{C}^n$ where $U \subset \mathbb{C}^{k+n}$ is holomorphic and write $f(p) = f(w_1, \dots, w_k, z_1, \dots, z_n)$ for $p \in U$. If $(D_z^{\mathbb{C}} f)(p)$ is invertible for some $p = (w_0, z_0)$ with $f(p) = 0$ then there exist open sets $U_1 \subset \mathbb{C}^k, U_2 \subset \mathbb{C}^n$ such that $p \in U_1 \times U_2 \subset U$ and there is a unique holomorphic function $g: U_1 \rightarrow U_2$ with $g(w_0) = z_0$ such that

$$f^{-1}(0) \cap (U_1 \times U_2) = \{(w, z) \in U_1 \times U_2 \mid z = g(w)\}$$

Proof. As above □

Proposition 7.1. A closed subset $Y \subset X$ is a k -dimensional complex submanifold of X if and only if

1. Near every point $p \in Y$ there is a holomorphic chart (U, ϕ) for X where $p \in U, \phi(p) = 0$ such that $Y \cap U = \phi^{-1}(\{z_{k+1} = \dots = z_n = 0\})$
2. $Y \subset X$ is a $2k$ -dimensional smooth submanifold such that $T_y Y \subset T_y X$ is a J_y invariant subspace for every $y \in Y$
3. Y has the structure of a complex manifold and the inclusion map $\iota: Y \rightarrow X$ is a holomorphic embedding

Proof. Exercise □

Remark. If $Y \subset X$ is a complex submanifold and $\iota: Y \rightarrow X$ is the inclusion then we can pull back holomorphic vector bundles E on X to holomorphic vector bundles on Y

Example 7.1. We have a natural map $\mathcal{T}_Y \hookrightarrow \iota^* \mathcal{T}_X$ which gives a sequence of vector bundle morphisms

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \iota^* \mathcal{T}_X \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0$$

which is an exact sequence, where the last bundle is the holomorphic normal bundle

Definition 7.2. A closed subset $Y \subset X$, where X is a complex manifold of dimension n , is a **subvariety** of codimension k if at every point $p \in Y$ there is an open $U \subset X$ with $p \in U$ and a holomorphic function $f: U \rightarrow \mathbb{C}^k$ with $f(p) = 0$ such that $Y \cap U = f^{-1}(0) \cap U$.

Definition 7.3. We say the locus where $d_p^{\mathbb{C}} f$ is not of full rank is the **singular locus** of Y and denote it Y^s . $Y \setminus Y^s$ is a 'complex submanifold' called the **regular locus**. We say Y is **irreducible** if $Y \setminus Y^s$ is connected. $Y^s \subset X$ is itself a subvariety of codimension at least k as the vanishing of the derivative defines another equation. We say Y is a **hypersurface** if $\text{codim}(Y) = 1$

Example 7.2. Let $Y \subset \mathbb{C}^2$ where $Y = \{z_1 z_2 = 0\}$. Define $f(z_1, z_2) = z_1 z_2$ then we see that Y is a codimension 1 subvariety of X . This is also not a complex submanifold (or even a complex manifold). In this case $Y^s = \{(0, 0)\}$.

Example 7.3. We have a few examples.

1. $f \in \mathbb{C}[z_1, \dots, z_n]$ is a polynomial then $f^{-1}(0) \subset \mathbb{C}^n$ is an **affine subvariety** with regular locus being $\{p \in \mathbb{C}^n \mid f(p) = 0, d_p^{\mathbb{C}} f \neq 0\}$
2. If s is a holomorphic section of a holomorphic vector bundle $E \rightarrow X$ then $s^{-1}(0)$ gives a subvariety of codimension $\text{rank}(E)$ as each coordinate function of the section need be 0
3. If $f \in \mathbb{C}[z_0, \dots, z_n]$ is a homogeneous polynomial of degree d , it defines a holomorphic section of $\mathcal{O}(d)$ over $\mathbb{CP}^n$. We say that $f^{-1}(0)$ is a **projective hypersurface**. In general if $Y \subset \mathbb{CP}^n$ is a complex submanifold we say that Y is **projective**
4. $Y = \{0\} \subset \mathbb{C}$ is the zero locus of both $f(z) = z$ and $f(z) = z^2$ so we don't have uniqueness

8 Meromorphic Functions

Definition 8.1. A **meromorphic function** f on a complex manifold X is a function from an open subset of X to $\mathbb{C}$ such that in some open cover $\{U_i\}$ of X we can write $f|_{U_i} = \frac{g_i}{h_i}$ for holomorphic functions $g_i, h_i: U_i \rightarrow \mathbb{C}$ where h is not identically zero and $\frac{g_i}{h_i}$ doesn't extend to a holomorphic function on all of U_i

Proposition 8.1. (Weak Nullstellensatz) If $Y \subset X$ is a hypersurface where X is a complex manifold and $p \in Y$ then there is an open subset $U \subset Y$ with $p \in U$ and a holomorphic function $f: U \rightarrow \mathbb{C}$ such that $Y \cap U = f^{-1}(0)$ such that f divides any other function with the same property. We call f a **local defining function** for Y at p

Proof. No Proof □

Example 8.1. z is a local defining function for $\{0\} \subset \mathbb{C}$, and z^2 isn't as $z \mid z^2$ at 0

Example 8.2. $X = \{z_1 z_2 = 0\} \subset \mathbb{C}^2$. Then $Y = \{0\}$ is a hypersurface in X with defining functions z_1, z_2

Suppose $Y \subset X$ is an irreducible hypersurface and $p \in Y \setminus Y^s$ with local defining function $f: U \rightarrow \mathbb{C}$ such that $f(p) = 0$ and suppose g is a holomorphic function $g: V \rightarrow \mathbb{C}$ where $V \subset U$ is an open set with $p \in V, g \neq 0$. Then we can write $g = f^a h$ where h is non-vanishing on $V' \subset V$ and $a \in \mathbb{Z}_{\geq 0}$

Proposition 8.2. The integer a described above is independent of the choice of defining function and doesn't depend on p outside of a codimension 2 locus of X

Proof. No Proof □

Example 8.3. If $Y = \{z_1 z_2 = 0\} \subset \mathbb{C}^2$ and $p = 0$, $g = z_1^{b_1} z_2^{b_2}$ then we have points with $a = b_1$ and points with $a = b_2$ which each form a codimension 1 locus in $\mathbb{C}^2$, so this fails for singular points

The idea of the proof is that if f_1, f_2 are local defining functions for p at Y then $f_1 = k f_2$ for a nowhere zero holomorphic function $k: V'' \rightarrow \mathbb{C}$ so a is independent of choice of local defining function

Definition 8.2. The integer a is called the **order of vanishing** of g along Y denoted as $\text{Ord}_Y(g) = a$ for p in the complement of the codimension 2 locus of X from the above proposition. If h is a meromorphic function given by $\frac{h_1}{h_2}$ in U then we define $\text{Ord}_Y(h) = \text{Ord}_Y(h_1) - \text{Ord}_Y(h_2)$ and this is not dependent on the choice of h_1, h_2 as $\text{Ord}_Y(g_1 g_2) = \text{Ord}_Y(g_1) + \text{Ord}_Y(g_2)$. If $\text{Ord}_Y(h) = -k < 0$ we say that h has a **pole of order k** along Y

Remark. If $p \notin g^{-1}(0)$ we say $\text{Ord}_p(g) = 0$

Example 8.4. Let $Y = \{z_1 = 0\} \subset \mathbb{C}^2$ and $g(z_1, z_2)$. Then $\text{Ord}_Y(g) = 0$ and $\text{Ord}_Y(z_1^k) = k$

Proposition 8.3. If f is a meromorphic function on a complex manifold X and $\text{Ord}_Y(f) \geq 0$ for every irreducible hypersurface Y , then f extends to holomorphic function on all of X

Proof. This is proven via the following lemma

Lemma 8.1. Suppose $g: U \rightarrow \mathbb{C}$ is a holomorphic function with g not identically 0 on a subset $U \subset \mathbb{C}^n$ and $g^{-1}(0) \subset K$ for K a codimension 2 subvariety of $\mathbb{C}^n$, then $g^{-1}(0) = \emptyset$

Proof. Recall that for a subvariety K , the singular locus $K^s \subset K$ is a closed subset of codimension $\text{codim}(K^s) \geq \text{codim}(K)$ and around any point $p \in K \setminus K^s$ we can find an open subset $V \subset U$ and a holomorphic chart $\phi: V \rightarrow \mathbb{C}^n$ such that $(K \setminus K^s) \cap V = \phi^{-1}(\{z_1 = z_2 = 0\})$. Then for every $(z_3, \dots, z_n)$ there is a function $f_2(z, w) = g \circ \phi^{-1}(z, w, z_3, \dots, z_n)$ is a holomorphic function on some polydisk $\Delta_\delta(0) \subset \mathbb{C}^2$ that can only be zero at $(0, 0) \in \Delta_\delta(0)$. So by Hartogs's theorem on $\frac{1}{g_2}$, g_2 actually has no zero at $(0, 0)$. By the construction in Hartogs's theorem, the extension of $\frac{1}{g_2}$ depends holomorphically on $(z_3, \dots, z_n)$. Therefore g is non-zero over all of $K \setminus K^s$. So if $K^s = \emptyset$ we are done. Otherwise apply the same argument to K^s which is of codimension larger than the codimension of K . $\square$

$\square$

Corollary 8.1.1. If f is a meromorphic function on a complex manifold X , then if $\text{Ord}_Y(f) = 0$ for every irreducible hypersurface Y , then f extends to a nowhere vanishing holomorphic function

Proof. $\text{Ord}_Y(f) = -\text{Ord}_Y(\frac{1}{f})$ and apply the proposition to $1/f$ $\square$

Lemma 8.2. *If X is a compact complex manifold with f a meromorphic function on X , then $\text{Ord}_Y(f) \neq 0$ for finitely many irreducible hypersurfaces*

Proof. We take a covering of charts with representatives of f on them, and take a finite subcovering. Such irreducible hypersurfaces are then the union of finitely many zero loci of the numerators and denominators of the representatives $\square$

Until otherwise stated, X is a compact complex manifold

Definition 8.3. A **divisor**, D , on a complex manifold X is a formal $\mathbb{Z}$ -linear combination of irreducible hypersurfaces, $Y_i \subset X$, $D = \sum_{i=1}^k a_i Y_i$ with $a_i \in \mathbb{Z}$ for $k < \infty$. The set $\text{Div}(X)$ of all divisors forms an abelian group

Example 8.5. If X is 1 dimensional, then divisors are formal linear combinations of points

Definition 8.4. If f is a meromorphic function, the **principal divisor** of f is $(f) = \sum_{i=1}^k \text{Ord}_{Y_i}(f) \cdot Y_i$ for the irreducible hypersurfaces $Y_i \subset X$ where $\text{Ord}_{Y_i}(f) \neq 0$. We say a divisor D is **effective** if $a_i \geq 0$ for every coefficient a_i in D

Corollary 8.2.1. *If (f) is effective then f extends to a holomorphic function on all of X . if $(f) = 0$ then f extends to a nowhere-zero holomorphic function on all of X*

Definition 8.5. A meromorphic section s of a holomorphic line bundle E over X is a holomorphic section s of E defined over some open subset of X such that in a holomorphic trivialisation $\{U_i\}$ of E over X , we can write $s|_{U_i} = g_i/h_i$ with g_i, h_i both holomorphic such that h_i is not identically 0 and h_i doesn't divide g_i

If E is a holomorphic line bundle with a meromorphic section s , it can be written in some open cover $\{U_i\}$ of X as g_i/h_i with g_i, h_i holomorphic functions with h_i not dividing g_i , h_i not identically 0.

We can define the order identically to before, which is well defined on the intersections as the transition function is holomorphic.

We may also define the divisor of a meromorphic section in the same way.

Definition 8.6. We say two divisors D, D' on X are **linearly equivalent** if $D - D' = (f)$ for f a meromorphic function on X . We may think of D being the 'locus of zeros' of f , D' being the 'locus of poles' and f defining a 'homotopy' between them.

The quotient of $\text{Div}(X)/\sim$ is called the **divisor class group**, $cl(X)$ and forms an abelian group under addition (as $\text{Ord}_Y(fg) = \text{Ord}_Y(f) + \text{Ord}_Y(g)$ so we have $(fg) = (f) + (g)$)

Example 8.6. If $\dim(X) = 1$ then all divisors are sums of points (eg. $(0) - (\infty)$ is the divisor of the identity on $\mathbb{CP}^1$, and it is linearly equivalent to 0)

Recall that $(f) = 0$ if and only if f extends to a nowhere vanishing holomorphic function

Proposition 8.4. *If E is a holomorphic line bundle on X and s, s' are two meromorphic sections, then $(s) \sim (s')$. This means $[(s)] = [(s')]$ in $cl(X)$ and $[E] \in cl(X)$ is a well defined class*

Proof. We claim that $s = fs'$ for some meromorphic function f on X . The result easily follows from the claim. If we have a holomorphic trivialisation of E over an open cover $\{U_i\}$ of X we can write $s|_{U_i} = s_i, s'|_{U_i} = s'_i$ for some meromorphic functions on U_i such that $s_i = \psi_{ij}s_j$ and $s'_i = \psi_{ij}s'_j$ on $U_i \cap U_j$ where ψ_{ij} is a nowhere vanishing holomorphic function. So we have $\frac{s_i}{s'_i} = \frac{\psi_{ij}s_j}{\psi_{ij}s'_j} = \frac{s_j}{s'_j}$. Now $f|_{U_i} = \frac{s_i}{s'_i}$ is a global meromorphic function on X thus $s = fs'$ and we are done. $\square$

Example 8.7. Let $E = \mathcal{O}(1)$ on $\mathbb{CP}^n$. $s = z_0$ gives a holomorphic section of $\mathcal{O}(1)$ and $(s) = \{z_0 = 0\}$ is a hyperplane in $\mathbb{CP}^n$. We write $[\mathcal{O}(1)] = H$ is the hyperplane class in $cl(X)$

Suppose $D = \sum_i a_i Y_i$ is a divisor in X . Then D is **locally principal** as we can take some open cover $\{U_i\}$ of X such that each $Y_i \cap U_j$ has a defining function $f_{ij}: U_j \rightarrow \mathbb{C}$ so that $Y_i \cap U_j = (f_{ij})$. We can therefore write, on an open set, $D \cap U_j = \sum_i a_i (Y_i \cap U_j) = (\prod_i f_{ij}^{a_i})$. We say that $\{(U_j, g_j)\}$ is a **local defining function** of D (A **Cartier divisor** in algebraic geometry)

On an overlap $U_i \cap U_j$ we must have $g_j|_{U_i} = D \cap (U_i \cap U_j) = g_i|_{U_j}$ and so $(g_j|_{U_i}) - (g_i|_{U_j}) = 0$, and hence $\left(\frac{g_j}{g_i}|_{U_i \cap U_j}\right) = 0$, and so $\frac{g_j}{g_i}|_{U_i \cap U_j}$ is a nowhere vanishing holomorphic function. The functions $\psi_{ij} = \frac{g_j}{g_i}$ trivially satisfy the cocycle condition and so we get a holomorphic line bundle, $\mathcal{O}(D)$ associated to D

Lemma 8.3. *Suppose that E is a holomorphic line bundle on X with a holomorphic trivialisation over an open cover $\{U_i\}$ of X such that $\psi_{ij} = f_i/f_j$ for $f_i: U_i \rightarrow \mathbb{C}$ which are nowhere vanishing holomorphic functions. Then E is isomorphic to the trivial bundle $\mathcal{O}$*

Proof. We have a global nowhere vanishing holomorphic section s of E given by $s|_{U_i} = f_i$. Then $F: E \rightarrow X \times \mathbb{C}$ defined by $(p, \lambda) \mapsto (p, \lambda s(p))$ is an isomorphism of holomorphic line bundles. $\square$

Proposition 8.5. *$\mathcal{O}(D)$ is independent of choice of local defining function of D up to isomorphism*

Proof. After refining the cover, we have $\{(U_i, f_i)\}, \{(U_i, f'_i)\}$ two local defining functions. Now $D \cap U_i = (f_i) = (f'_i)$. Then $\lambda_i = f'_i/f_i$ is a nowhere vanishing holomorphic function. Then the transition functions are given by $f_i/f_j = (f_i/f'_i)(f'_j/f_j)(f'_i/f'_j)$ so the transition functions of the two locally defining functions differ by λ_j/λ_i . We therefore have $\mathcal{O}(\{(U_i, f_i)\}) \cong E \otimes \mathcal{O}(\{(U_i, f'_i)\})$ where E is a line bundle with transition functions λ_j/λ_i , hence is trivial by the lemma so we are done. $\square$

9 The Line Bundle - Divisor Correspondence

We have given a way to associate a divisor to any line bundle, by the divisor of any meromorphic section, and a way to associate a line bundle to a divisor, by ratios of local defining functions

Remark. We observe that

1. If $D = (s)$ for some meromorphic section s of $E \rightarrow X$ then over an open cover $\{U_i\}$ of X where we have holomorphic trivialisations of E , ie we can write $s|_{U_i}$ as a meromorphic function $s_i: U_i \rightarrow \mathbb{C}$ then $\{(U_i, s_i)\}$ give a local defining function for D
2. If $\{(U_i, g_i)\}$ is a local defining function for a divisor D then $g_i: U_i \rightarrow \mathbb{C}$ give a meromorphic section of $\mathcal{O}(D)$. We say this is the **defining section** of D

Theorem 9.1. (*Line Bundle-Divisor Correspondence*) For X a compact complex manifold, there is a bijection between $cl(X)$ and holomorphic line bundles admitting a meromorphic section up to holomorphic isomorphism given by $D \mapsto \mathcal{O}(D)$ and $E \mapsto (s)$ for D a divisor of E and s a meromorphic section of E

Proof. We prove that the given maps are inverses to each other

1. Claim: $[\mathcal{O}(D)] \sim D$ in $cl(X)$.
Choose the meromorphic section $\{(U_i, g_i)\}$ for a defining function of D . Then $(s) = D$ by definition and we are done.
2. Claim: $\mathcal{O}([E]) \cong E$ in $\text{Pic}(X)$.
This is because they have the same transition functions.

□

Proposition 9.1. The set $\text{Pic}(X)$ of holomorphic line bundles up to holomorphic isomorphism forms an abelian group under the tensor product and the map of theorem 9.1 is an isomorphism of groups

Proof. 1. The only non-trivial group axiom is the existence of inverses, but $E \otimes E^* \cong \mathcal{O}$

2. The isomorphism claim is equivalent to $\mathcal{O}(D + D') \cong \mathcal{O}(D) \otimes \mathcal{O}(D')$. If s is a meromorphic section of $\mathcal{O}(D)$ and s' is a meromorphic section of $\mathcal{O}(D')$, then $s \otimes s'$ is a meromorphic section of $\mathcal{O}(D) \otimes \mathcal{O}(D')$, and the divisor of $s \otimes s'$ will be $D + D'$ on any open cover $\{U_i\}$ trivialising both s_i, s'_i , and so the result holds globally. The isomorphism follows

□

Proposition 9.2. This isomorphism is natural in X

Proof. Example Sheet

□

Theorem 9.2. *If X is a compact complex manifold and E a holomorphic line bundle admitting a meromorphic section, there is a bijection between holomorphic sections of E , and meromorphic functions f on X such that $(f) + [E] \geq 0$*

Proof. Trivial $\square$

Remark. This is a form of Poincare duality: $c_1(\mathcal{O}(D))$ is Poincare dual to D

10 Differential Forms

Let V be a real vector space and J a complex structure on V . Recall that $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ as complex vector spaces

Proposition 10.1. *There is a direct sum decomposition*

$$\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \left(\bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1} \right)$$

as complex vector spaces. We define $\bigwedge^{p,q} V = \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1}$. In particular, $J_{\mathbb{C}}(\phi) = i^{p-q}\phi$ where $\phi \in \bigwedge^{p,q} V$ for $J_{\mathbb{C}}$ the extension of $J_{\mathbb{C}}$ to $\bigwedge^k V_{\mathbb{C}}$

Proof. Trivial $\square$

Recall that if X is a complex manifold, $T_p X$ is a real vector space with complex structure J given in holomorphic local coordinates by $J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ and $J \frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$ for $z_i = x_i + iy_i$. Then we have a direct sum decomposition given by $T_p^{\mathbb{C}} X = (T_p X)_{\mathbb{C}} = T_p^{1,0} X \oplus T_p^{0,1} X$ and there is a basis for $T_p^{\mathbb{C}} X$ given by $\frac{\partial}{\partial z_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right)$ which is a basis of $T_p^{1,0} X$, and $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \right)$ which is a basis of $T_p^{0,1} X$. The cotangent space $T_p^* X$ also has a complex structure given in local holomorphic coordinates by $J dx_i = -dy_i$ and $J dy_i = dx_i$. If we define $dz_i = dx_i + i dy_i$, a basis of $(T_p^* X)^{1,0}$ and $d\bar{z}_i = dx_i - i dy_i$ a basis of $(T_p^* X)^{0,1}$, we get a basis of $(T_p^* X)_{\mathbb{C}}$ which is dual to that of $T_p^{\mathbb{C}} X$

Corollary 10.0.1. *We have a complex basis for $\bigwedge^k (T_p^* X)_{\mathbb{C}}$ given by*

$$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \in \bigwedge^{p,q} T_p^* X$$

for $1 \leq i_1 < i_2 < \cdots < i_p \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_q \leq n$ such that $p + q = k$

Proof. Trivial $\square$

Observe that complex conjugation gives a map $(T_p^* X)^{1,0} \rightarrow (T_p^* X)^{0,1}$ sending $dz_i \mapsto d\bar{z}_i$ and $d\bar{z}_i \mapsto dz_i$ and so we also get a map $\bigwedge^{p,q} T_p^* X \rightarrow \bigwedge^{q,p} T_p^* X$. Recall that the holomorphic cotangent bundle is $\Omega_X = (T^* X)^{1,0}$ and we have $\Omega_X^k = \bigwedge^k \Omega_X$. Note that $\Omega_X^k = \bigwedge^{k,0} T^* X$. These are the forms that can be written in the basis $dz_{i_1} \wedge \cdots \wedge dz_{i_k}$

Proposition 10.2. *The de Rham differential d on differential forms extends to a complex linear map $d: C^\infty(X, \bigwedge^k(T^*X)_\mathbb{C}) \rightarrow C^\infty(X, \bigwedge^{k+1}(T^*X)_\mathbb{C})$ where $C^\infty(\cdot, \cdot)$ denotes smooth sections. We have the restriction of d ,*

$$d: C^\infty(X, \bigwedge^{p,q} T^*X) \rightarrow C^\infty(X, \bigwedge^{p+1,q} T^*X \oplus \bigwedge^{p,q+1} T^*X)$$

and we write $d = \partial \oplus \bar{\partial}$ to denote each of the two components of this, where ∂ is the $\bigwedge^{p+1,q} T^*X$ component and $\bar{\partial}$ is the $\bigwedge^{p,q+1} T^*X$ component

Proof. If $\alpha = \sum f dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$ we have

$$\begin{aligned} d\alpha &= \sum \frac{\partial f}{\partial z_l} dz_l \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \\ &\quad + \sum \frac{\partial f}{\partial \bar{z}_l} d\bar{z}_l \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \end{aligned}$$

The first component is $\partial\alpha$ and the latter $\bar{\partial}\alpha$ □

Definition 10.1. We say a smooth section α of $\bigwedge^{p,q} T^*X$ is **holomorphic** if $\bar{\partial}\alpha = 0$

Lemma 10.1. *If α is a smooth section of $\bigwedge^k \Omega_X$ then α is a holomorphic section if and only if $\bar{\partial}\alpha = 0$*

Proof. In local coordinates, $\alpha = \sum f(z, \bar{z}) dz_{i_1} \wedge \cdots \wedge dz_{i_k}$ and it is holomorphic if f is a function of z alone. The result follows □

Remark. If E is a holomorphic vector bundle, then $C^\infty(X, E)$ denotes smooth sections and $\Omega^0(X, E), H^0(X, E)$ denote holomorphic sections.

$\bar{\partial}^2 = \partial^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$ as the de Rham differential squares to 0, and considering which $\bigwedge^{p,q} T^*X$ component each term lands in

It is a fact that $\partial, \bar{\partial}$ satisfy a graded Leibniz rule with respect to $\wedge$

Definition 10.2. The **Dolbeault cohomology** $H^{p,\bullet}(X)$ is the cohomology of the complex $C^\infty(X, \bigwedge^{p,\bullet} T^*X)$ with differential $\bar{\partial}$. If $\alpha \in \ker \bar{\partial}$ then α is called $\bar{\partial}$ -closed and $\beta \in \text{im } \bar{\partial}$ is called $\bar{\partial}$ -exact

Remark. If $\bar{\partial}\alpha = 0$ then $\partial\bar{\alpha} = 0$ so we have a complex anti-linear isomorphism $H^{p,q}(X) \rightarrow H^{q,p}(X)$ (this is not immediately trivial as we get a complex with ∂ as its differential, but there is an isomorphism between the complexes)
It is a fact that $H^{p,q}(X) = H^q(X, \Omega_X^p)$ where the latter is sheaf cohomology

If $F: X \rightarrow Y$ is holomorphic, then we have a well defined map of the form $F^*: C^\infty(Y, \bigwedge^{p,q} T^*Y) \rightarrow C^\infty(X, \bigwedge^{p,q} T^*X)$

11 Chern Classes

Definition 11.1. Suppose E is a complex vector bundle on a smooth manifold X . A **connection** is a $\mathbb{C}$ -linear map $\nabla_A: C^\infty(X, E) \rightarrow C^\infty(X, E \otimes (T^*M)_\mathbb{C})$ satisfying the Leibnitz rule with respect to smooth complex valued functions $f: X \rightarrow \mathbb{C}$, ie

$$\nabla_A(fs) = f\nabla_A(s) + s \otimes df$$

From now on suppose that E is a line bundle and take an open cover $\{U_i\}$ of X and trivialisations $\Phi_i: E|_{U_i} \rightarrow U_i \times \mathbb{C}$ in which we can write $\nabla_A = d + A_i$ where A_i is a 1-form, called the **connection 1-form**. Over an overlap $U_i \cap U_j$ with transition function ψ_{ij} , we should have $A_i = A_j + \psi_{ij}^{-1}d\psi_{ij}$ in order for ∇_A to be well-defined on the overlap, ie $(d + A_i)(s_i) = (d + A_j)(\psi_{ij}s_j)$. Given a pair of connection 1-forms $\{A_i\}, \{A'_i\}$ over the same open cover $\{U_i\}$ of X then $A_i - A'_i = A_j - A'_j$ on $U_i \cap U_j$ so that $A_i - A'_i$ is a global 1-form on X . Recall that if ∇_A is a connection on E , we have an extension to E -valued k -forms, ie

$$\nabla_A: C^\infty(X, E \otimes \bigwedge^k T^*X) \rightarrow C^\infty(X, E \otimes \bigwedge^{k+1} T^*X)$$

defined by the Leibnitz rule $\nabla_A(s \otimes \alpha) = (-1)^k(\nabla_A s) \wedge \alpha + s \otimes d\alpha$

Proposition 11.1. *The composition*

$$\nabla_A \circ \nabla_A: C^\infty(X, E) \rightarrow C^\infty(X, E \otimes \bigwedge^2 T^*X)$$

is given by multiplication by a tensor $F_A \in C^\infty(X, \text{End}(E) \otimes \bigwedge^2 T^*X)$ which we call the **curvature form**

Proof. If $f: X \rightarrow \mathbb{C}$ is smooth then we expand $\nabla_A \circ \nabla_A(fs) = f\nabla_A \circ \nabla_A s$ for any section $s \in C^\infty(X, E)$ so $\nabla_A \circ \nabla_A$ is a tensor $\square$

Proposition 11.2. *If E is a line bundle then F_A gives a globally defined 2-form given in each local trivialisation by $F_{A_i} = dA_i$ on U_i and the de Rham cohomology class of F_A is independent of choice of connection*

Proof. Since $dF_A = 0$ and $A - A' \in C^\infty(X, T^*X)$, $F_A - F_{A'} = d\alpha$ so they give the same cohomology class $\square$

Definition 11.2. The **first Chern class** of a line bundle E is the de Rham cohomology class $c_1(E) = \frac{1}{2\pi i}[F_A]$ for any connection ∇_A on E

Remark. In fact $c_1(E) \in H^2(X, \mathbb{Z})$ classifies complex line bundles up to topological bundle isomorphism

Lemma 11.1. *The first Chern class gives a group homomorphism of the form $c_1: \text{Pic}(X) \rightarrow H^2(X, \mathbb{Z})$ with kernel those line bundles which are topologically trivial*

Proof. No Proof □

Definition 11.3. We say a complex vector bundle E has a **Hermitian inner product** if every fibre of E has a Hermitian inner product $\langle \cdot, \cdot \rangle_p$ which varies smoothly between fibres. That is, if s, s' are local smooth sections of E , then $\langle s, s' \rangle$ is a smooth function

Definition 11.4. If E is a holomorphic line bundle over a complex manifold X equipped with a specified Hermitian inner product $\langle \cdot, \cdot \rangle$ then a connection ∇_A is called a **Chern connection** if

1. (Compatibility with holomorphic structure) In a holomorphic trivialisation of E , the connection 1-forms are smooth sections of $(T^*X)^{1,0}$
2. (Compatibility with the Hermitian structure) $d\langle s, s' \rangle = \langle \nabla_A s, s' \rangle + \langle s, \nabla_A s' \rangle$ for any smooth sections s, s' of E

Proposition 11.3. *For a holomorphic line bundle E with a specified Hermitian inner product, a Chern connection exists and is unique*

Proof. Suppose we have a local holomorphic trivialisation of E over an open set $U \subset X$. Then the constant function $e(z) = 1$ gives a nowhere vanishing holomorphic section of E over U . Let $h(z) = \langle e(z), e(z) \rangle: U \rightarrow \mathbb{R}$, a smooth function on U . If s is another holomorphic section of E over U then $s = fe$ for some holomorphic function f and $\langle s, s \rangle = \langle fe, fe \rangle = f\bar{f}\langle e, e \rangle = f\bar{f}h$. In this local holomorphic trivialisation write $\nabla_A = d + A$. We want to express A in terms of h, e assuming ∇_A is a Chern connection.

We have $d\langle s, s' \rangle = \langle \nabla_A s, s' \rangle + \langle s, \nabla_A s' \rangle$, where $\nabla_A(s) \in C^\infty(X, (T^*X)_{\mathbb{C}} \otimes E)$ means we can apply the inner product between the E parts. In addition we can write $d(f\bar{f}h) = (df + Af)\bar{f}h + f(\bar{d}f' + Af')h$ so we get

$$\bar{f}hdf + fhd\bar{f} + f\bar{f}dh = \bar{f}hdf + \bar{f}hfA + fh\bar{d}f + f\bar{f}h\bar{A}$$

and hence

$$f\bar{f}h = f\bar{f}hA + f\bar{f}h\bar{A}$$

and so $dh = h(A + \bar{A})$, hence $\partial \log h + \bar{\partial} \log h = A + \bar{A}$. Since $A \in C^\infty(U, (T^*X)^{1,0})$, $A = \partial \log h$ and so it is unique. Conversely suppose we have a holomorphic trivialisation of E over an open cover $\{U_i\}$ of X , then in each trivialisation we have a nowhere vanishing holomorphic section e . Letting $A_i = \partial \log \langle e_i, e_i \rangle$, these patch together as $A_i = \partial \log |\psi_{ij}e_j|^2 = \partial \log |e_j|^2 + \partial \log \bar{\psi}_{ij}\psi_{ij} = A_j + \psi_{ij}^{-1}d\psi_{ij}$ since $\partial \log \bar{\psi}_{ij} = 0$ as the ψ_{ij} are holomorphic □

Corollary 11.1.1. *If E is a holomorphic line bundle with Hermitian inner product then we always have $F_A = \bar{\partial} \partial \log |s|^2$ for any local nowhere vanishing holomorphic section s , so $F_A \in C^\infty(X, \bigwedge^{1,1} T^*X)$*

Proof. Trivial □

Remark. We notate $d^c = i(\partial - \bar{\partial}) = d \circ J$ and so d^c acts on real differential forms and $-\bar{\partial}\partial = \frac{i}{2}dd^c$

Recall that the first Chern class is represented in $H_{dR}^2(X, \mathbb{C})$ by the 2-form $c_1(E) = \frac{1}{2\pi i}[F_A]$ when X is a compact complex manifold. Since we have that $F_A = \frac{-i}{2}dd^c \log |s|^2$, this means $c_1(E) \in H_{dR}^2(X, \mathbb{R})$

Remark. If there is a global nowhere vanishing holomorphic section of E then E is holomorphically trivial and $F_A = \bar{\partial}\partial \log |s|^2 = d\partial \log |s|^2$ is an exact form and so $c_1(E) = 0$ in $H_{dR}^2(X, \mathbb{C})$

11.1 Poincare Duality in de Rham Cohomology

Let X be a smooth compact n manifold. We have maps $H_{dR}^k(X, \mathbb{R}) \rightarrow H_{dR}^{n-k}(X, \mathbb{R})^*$ and $H_{n-k}(X, \mathbb{R}) \rightarrow H_{dR}^{n-k}(X, \mathbb{R})^*$ by $\alpha \mapsto (\beta \mapsto \int_X \alpha \wedge \beta)$ in the first case and by $D \mapsto (\beta \mapsto \int_D \beta)$ in the second.

Definition 11.5. Suppose X is a compact complex manifold and $D = \sum_i a_i Y_i$ is a divisor on X where $a_i \in \mathbb{Z}$ and $Y_i \subset X$ is an irreducible hypersurface. Then we define $[D] = \sum a_i [Y_i] \in H_{2n-2}(X, \mathbb{R})$

Lemma 11.2. Suppose that $Y = \{z_1 = 0\} \subset \mathbb{C}^n$ and define the ϵ neighbourhood of Y , $N_\epsilon = \{z \in \mathbb{C}^n \mid |z_1| < \epsilon\}$. Let $\Delta_\delta(0) \subset \mathbb{C}^n$ be a polydisk and α be a bounded real $2n-2$ form on $\Delta_\delta(0)$. Then

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\partial N_\epsilon \cap \Delta_\delta(0)} \frac{dz_1}{z_1} \wedge \alpha = \int_{Y \cap \Delta_\delta(0)} \alpha|_Y$$

Proof. Let $\alpha = \alpha_1 + \tilde{\alpha}$ where α_1 contains all terms involving $dz_1, d\bar{z}_1$, so that $\alpha|_Y = \tilde{\alpha}|_{z_1=0}$. Observe that $\lim_{\epsilon \rightarrow 0} \int_{|z_1|=\epsilon} \frac{dz_1}{z_1} \wedge \alpha_1 = 0$ since α is a real form so all terms must be of the form $\cdots \wedge dz_1 \wedge d\bar{z}_1$. In polar coordinates, we have that $idz_1 \wedge d\bar{z}_1 = 2rdr \wedge d\theta$. This means the term $\frac{dz_1}{z_1} \wedge \alpha_1$ is bounded on N_ϵ for all $\epsilon > 0$ as the r and the $\frac{1}{z_1}$ cancel. For the other terms,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\partial N_\epsilon \cap \Delta_\delta(0)} \frac{dz_1}{z_1} \wedge \tilde{\alpha} = \lim_{\epsilon \rightarrow 0} \int_{\{|z_1|=0\} \cap \Delta_\delta(0)} \frac{dz_1}{z_1} \wedge \tilde{\alpha}$$

which gives the result by the Cauchy integral formula $\square$

Theorem 11.3. $c_1(\mathcal{O}(D))$ is Poincare dual to $[D]$ in the sense that for every real closed $2n-2$ form β , we have

$$\frac{1}{2\pi i} \int_X F_A \wedge \beta = \int_D \beta$$

Proof. By linearity, we may assume D is a single irreducible hypersurface Y and choose some Hermitian inner product on $\mathcal{O}(D)$. Let ∇_A be its Chern connection.

Suppose s is some meromorphic section of $\mathcal{O}(D)$. Then $F_A = -\frac{i}{2}dd^c \log |s|^2$ where defined. Let N_ϵ be an epsilon neighbourhood of Y in X . Then

$$\int_X F_A \wedge \beta = -\lim_{\epsilon \rightarrow 0} \int_{X \setminus N_\epsilon} dd^c \log |s|^2 \wedge \beta$$

By Stokes, as β is closed,

$$\int_{X \setminus N_\epsilon} dd^c \log |s|^2 \wedge \beta = \int_{\partial N_\epsilon} d^c \log |s|^2 \wedge \beta$$

Away from some locus of measure 0, Y is smooth and we can find open sets $U_i \subset X$ such that $Y \cap U_i$ cover Y and there are local holomorphic coordinates on each U_i such that $Y = \{z_1 = 0\}$. Then $f_i = z_1$ is a local defining function for Y and so gives a holomorphic section of $\mathcal{O}(D)|_{U_i}$. Therefore in this holomorphic chart we have $\int_{\{|z_1|=\epsilon\} \cap U_i} d^c \log |z_1|^2 \wedge \beta$. Suppose $h_i: U_i \rightarrow \mathbb{R}_{>0}$ is the local coordinate expression for the Hermitian inner product on $\mathcal{O}(D)$. Expanding out, we get $d^c \log |z_1|^2 = -2i \operatorname{Im}(\partial \log z_1) + d^c \log h_i$ with the latter term bounded on U_i by compactness of X and so

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\{|z_1|=\epsilon\} \cap U_i} d^c \log |z_1|^2 \wedge \beta &= \lim_{\epsilon \rightarrow 0} 2i \operatorname{Im} \int_{\{|z_1|=\epsilon\} \cap U_i} \frac{dz_1}{z_1} \wedge \beta \\ &= 2i \operatorname{Im} \left(2\pi i \int_{Y \cap U_i} \beta|_Y \right) \end{aligned}$$

□

12 The Normal Bundle

Recall that if Y is a complex submanifold of a complex manifold X then we have $0 \rightarrow \mathcal{T}_Y \rightarrow \mathcal{T}_X|_Y \rightarrow \mathcal{N}_{Y/X} \rightarrow 0$ where the last term is the holomorphic normal bundle. If $\operatorname{codim}(Y) = 1$ then $\mathcal{N}_{Y/X}$ is a line bundle on Y and we have the following

Theorem 12.1. *Suppose Y is a codimension 1 complex submanifold of a compact complex manifold X . Then there is an isomorphism of holomorphic line bundles on Y $\mathcal{N}_{Y/X} \cong \mathcal{O}(Y)|_Y$*

Proof. Since Y is a smooth hypersurface, there is a collection of open subsets $\{U_i\}$ of X such that $\{Y \cap U_i\}$ cover Y and holomorphic functions $f_i: U_i \rightarrow \mathbb{C}$ such that $Y \cap U_i = f_i^{-1}(0)$ with f_i not identically zero and $f_i dy \neq 0$ for $y \in Y$. Recall that the line bundle $\mathcal{O}(Y)$ has transition functions $\psi_{ij}: U_i \cap U_j \rightarrow \mathbb{C}$ given by $\psi_{ij} = f_i/f_j$ and so $\mathcal{O}(Y)|_Y$ has transition functions $\psi_{ij} = f_i/f_j|_Y$. Suppose $p \in Y$. Then $(\mathcal{T}_Y)_p = T_p^{1,0}Y$, $(\mathcal{T}_X)_p = T_p^{1,0}X$ and $(\mathcal{N}_{Y/X})_p = T_p^{1,0}X/T_p^{1,0}Y$. We claim that if $p \in U_i \cap Y$ then $d_p f_i \in (\mathcal{N}_{Y/X})_p$ since $d_p f_i: T_p^{1,0}X \rightarrow \mathbb{C}$ vanishes along $T_p^{1,0}Y$. We therefore have that df_i gives a local holomorphic section of $(\mathcal{N}_{Y/X})^*$ on $Y \cap U_i$ which is nowhere vanishing on Y . So df_i on $Y \cap U_i$ give us a

collection of local holomorphic trivialisations of $(\mathcal{N}_{Y/X})^*$. The dual transition functions are given by

$$d_y f_i = d_y \left(\frac{f_i}{f_j} f_j \right) = dy(\psi_{ij} f_j) = (dy\psi_{ij})f_j|_Y + \psi_{ij}|_Y dyf_j = \psi_{ij}|_Y dyf_j$$

so the transition functions are given by $\psi_{ij}^{-1}|_Y$. Since $(\mathcal{N}_{Y/X})^*$ has the same transition functions as $(\mathcal{O}(Y)|_Y)^*$, they are isomorphic as holomorphic line bundles $\square$

Theorem 12.2. (*Adjunction formula*) Suppose Y is a codimension 1 complex submanifold of a compact complex manifold X . Then there is an isomorphism of holomorphic line bundles on Y , $K_Y \cong K_X|_Y \otimes \mathcal{O}(Y)|_Y$

Proof. This follows from the exact sequence

$$0 \rightarrow \mathcal{T}_Y \rightarrow \mathcal{T}_X|_Y \rightarrow \mathcal{N}_{Y/X} = \mathcal{O}(Y)|_Y \rightarrow 0$$

and the following lemma, after tensoring by $\mathcal{O}(Y)$ on both sides and using the fact $\mathcal{O}(Y) \otimes \mathcal{O}(Y)^*$ is the trivial bundle

Lemma 12.3. Suppose E_1, E_2, E_3 are holomorphic vector bundle on a complex manifold X fitting into an exact sequence $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ of maps of holomorphic vector bundles. Then there is an isomorphism of holomorphic line bundles $\det E_2^* \cong \det E_1^* \otimes \det E_3^*$ where $\det E = \bigwedge^k E$ for $k = \text{rank}(E)$

Proof. It is a linear algebra fact that there is a natural such isomorphism for complex vector spaces, and the lemma follows as an exercise $\square$

$\square$

Lemma 12.4. The canonical bundle $K_{\mathbb{CP}^n}$ is isomorphic to $\mathcal{O}(-n-1)$

Proof.

Proposition 12.1. (*Euler sequence*) There is an exact sequence of holomorphic vector bundles on $\mathbb{CP}^n$ of the form

$$0 \rightarrow \mathcal{O} \rightarrow \bigoplus_{i=0}^n \mathcal{O}(1) \rightarrow \mathcal{T}_{\mathbb{CP}^n} \rightarrow 0$$

Proof. No Proof $\square$

$\square$

By the lemma,

$$\mathcal{O}(-1) \otimes \cdots \otimes \mathcal{O}(-1) = \det\left(\left(\bigoplus_{i=0}^n \mathcal{O}(1)\right)^*\right) \cong \det(\mathcal{O}^*) \otimes \det \mathcal{T}_{\mathbb{CP}^n}^*$$

and so we are done.

Alternatively we have a global meromorphic section of $K_{\mathbb{CP}^n}$ given on $U_i \subset \mathbb{CP}^n$ by $\omega = \frac{dz_0}{z_0} \wedge \cdots \wedge \frac{\hat{dz_i}}{z_i} \wedge \cdots \wedge \frac{dz_n}{z_n}$ then observe that $(\omega) = (-n-1)[H]$ so the result follows $\square$

Corollary 12.4.1. *If Y is a smooth hypersurface in $\mathbb{CP}^n$ defined by the vanishing locus of a degree d homogeneous polynomial, then $K_Y \cong \mathcal{O}(d - n - 1)|_Y$ by the adjunction formula, and the fact $\mathcal{O}(Y) = \mathcal{O}(d)$*

Proof. Trivial □

13 Blowing Up

13.1 Local Picture

Recall the tautological bundle $\mathcal{O}(-1) \rightarrow \mathbb{CP}^{n-1}$ can be described as a set as $\mathcal{O}(-1) = \{(l, v) \in \mathbb{CP}^{n-1} \times \mathbb{C}^n \mid v \in l\} \subset \mathbb{CP}^{n-1} \times \mathbb{C}^n$. We have a holomorphic map $\pi: \mathcal{O}(1) \rightarrow \mathbb{CP}^{n-1}$ by projection on the first factor. This gives it the structure of a holomorphic line bundle.

Consider the second projection $\sigma: \mathcal{O}(-1) \rightarrow \mathbb{C}^n$

1. Away from $\sigma^{-1}(0)$, this is a biholomorphism as each $v \in \mathbb{C}^n$ is on a unique line through 0, with inverse $v \mapsto (\text{Span}(v), v)$ for $v \neq 0$
2. The fibre $\sigma^{-1}(0)$ is biholomorphic to $\mathbb{CP}^{n-1}$ (the zero section in $\mathcal{O}(-1)$)

Definition 13.1. If $\Delta \subset \mathbb{C}^n$ is a polydisk, define $\tilde{\Delta} = \sigma^{-1}(\Delta)$. We say the map $\sigma: \tilde{\Delta} \rightarrow \Delta$ is the **blowup** of Δ at 0

Recall that in the open cover $\{U_i\}$ on $\mathbb{CP}^{n-1}$ where $z_i \neq 0$, we have coordinate charts $\Phi_i: \mathcal{O}(-1)|_{U_i} \rightarrow \mathbb{C}^{n-1} \times \mathbb{C}$ by

$$\Phi_i([z_0: \dots z_{n-1}], (v_0, \dots, v_{n-1})) = \left(\frac{z_0}{z_i}, \dots, \frac{\hat{z}_i}{z_i}, \dots, \frac{z_{n-1}}{z_i}, v_i \right)$$

We call these coordinates $(w_1, \dots, w_n) \in \mathbb{C}^n$. In these local coordinates,

$$\sigma(w_1, \dots, w_n) = (w_1 w_n, \dots, w_{i-1} w_n, w_n, w_i w_n, \dots, w_{n-1} w_n)$$

If we fix $w_n \neq 0$, this gives a biholomorphism, $\mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$ but if $w_n = 0$, this maps the whole $\mathbb{C}^{n-1}$ to a point

Definition 13.2. Suppose X is a complex manifold, $p \in X$ and $\phi: U \rightarrow \Delta$ is a holomorphic chart near p sending p to 0. The **blowup** of X at p is the complex manifold $\tilde{X}$ obtained by gluing $X \setminus P$ to $\tilde{\Delta}$ along the map $\phi^{-1}\sigma$, together with the holomorphic map $\sigma: \tilde{X} \rightarrow X$

Remark. Since σ is a biholomorphism away from P this gives a well defined complex manifold. We define $\sigma: \tilde{X} \rightarrow X$ by extending with the identity outside Δ . Moreover, $\tilde{X}$ is compact if X

We claim this is independent of choice of chart $\phi: U \rightarrow \Delta$. Suppose we have another $\phi': U \rightarrow \Delta'$ holomorphic chart near p . Then $\Phi = \phi' \circ \phi^{-1}: \Delta \rightarrow \Delta'$ is a

biholomorphism, and so we have a biholomorphism $\tilde{\Phi}: \tilde{\Delta} \setminus \sigma^{-1}(0) \rightarrow \tilde{\Delta}' \setminus \sigma^{-1}(0)$. Let's consider

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \tilde{\Phi}([v], \epsilon v) &= \lim_{\epsilon \rightarrow 0} \sigma^{-1} \Phi(\epsilon v) \\ &= \lim_{\epsilon \rightarrow 0} ([\Phi(\epsilon v)], \Phi(\epsilon v)) \end{aligned}$$

Since $\Phi(0) = 0$, this tells us that $\tilde{\Phi}$ extends to a biholomorphism $\tilde{\Delta}' \rightarrow \tilde{\Delta}$ by defining $\tilde{\Phi}(l, 0) = ([d_0 \Phi(l)], 0)$ where d_0 is the derivative at 0. We can identify $\sigma^{-1}(p)$ with $\mathbb{P}(T_p^{1,0} X)$

Definition 13.3. We say $E = \sigma^{-1}(p)$ is the **exceptional divisor** of $\tilde{X}$

Corollary 13.0.1. For the exceptional divisor E of the blowup, we have

$$\mathcal{O}(E)|_E \cong \mathcal{O}(-1)$$

Proof. Observe that $\mathcal{N}_{E/\tilde{X}} \cong \mathcal{O}(-1)$ since the zero section in $\mathcal{O}(-1)$ has normal bundle $\mathcal{O}(-1)|_E$, then use the fact that $\mathcal{N}_{E/X} \cong \mathcal{O}(E)|_E$ $\square$

Proposition 13.1. If K_X admits a meromorphic section, the canonical bundle of the blowup $\tilde{X}$ is given by

$$K_{\tilde{X}} \cong \sigma^* K_X \otimes \mathcal{O}(E)^{\otimes (n-1)}$$

Proof. Suppose ω is a defining meromorphic section of K_X . Then $\sigma^* \omega$ is a defining meromorphic section of $\sigma^* K_X$ so $[\sigma^* K_X] = (\sigma^* \omega) = \sum_{Y \subseteq X \text{ irred}} \text{ord}_Y(\sigma^* \omega)$. We claim that for an irreducible hypersurface Y not contained in E in $\tilde{X}$, $\text{Ord}_Y(\sigma^* \omega) = \text{Ord}_{\sigma(Y)}(\omega)$ (as σ is a biholomorphism away from E). We therefore must have that $[\sigma^* K_X] = [K_{\tilde{X}}] + m[E]$ for some $m \in \mathbb{Z}$. In the local coordinates, $(w_1, \dots, w_n)$ we have

$$\omega = dz_1 \wedge \dots \wedge dz_n$$

and so

$$\begin{aligned} \sigma^* \omega &= d(w_n w_1) \wedge \dots \wedge d(w_n w_{i-1}) \wedge dw_n \wedge d(w_n w_i) \wedge d(w_{n-1} w_n) \\ &= \pm w_n^{n-1} dw_1 \wedge \dots \wedge dw_n \end{aligned}$$

. Since w_n is a local defining function for E , we see that $\text{Ord}_E(\sigma^* \omega) = n - 1$ so $m = n - 1$ and we are done. $\square$

14 Hermitian Inner Products

Definition 14.1. Suppose V is a real vector space. A **symplectic structure** on V is a bilinear form $\omega: V \times V \rightarrow \mathbb{R}$ such that

1. ω is antisymmetric ($\omega(v, w) = -\omega(w, v)$)

2. ω is non-degenerate (If $\omega(v, \cdot) = 0$ then $v = 0$)

If V has a complex structure J , we say ω is

1. **J -invariant** if $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in V$
2. **J -compatible** if in addition to being J -invariant, $\omega(v, Jv) > 0$ for every $0 \neq v \in V$

More generally if $g: V \times V \rightarrow \mathbb{R}$ is a bilinear form, we say g is **J -invariant** if $g(Jv, Jw) = g(v, w)$ for all $v, w \in V$

Proposition 14.1. *If V is a real vector space with complex structure J then the following are (naturally) in bijection*

1. J -invariant positive definite symmetric bilinear forms g on V
2. J -compatible symplectic structures
3. positive definite hermitian inner products on $V^{1,0}$

Proof. The maps are

- (2) to (1): Given ω , let $g(v, w) = \omega(v, Jw)$
- (1) to (2): Given g , let $\omega(v, w) = g(Jv, w)$
- (1) to (3): Given g on V , define a Hermitian inner product h on $V_{\mathbb{C}}$ by $h(\lambda v, \mu w) = \lambda \bar{\mu} g(v, w)$ for $\lambda, \mu \in \mathbb{C}$ and $v, w \in V$ and extend sesquilinearly. The decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ is h orthogonal, so we can restrict the inner product to $V^{1,0}$
- (3) to (1): Given h on $V^{1,0}$ then under the isomorphism $(V, J) \rightarrow (V^{1,0}, i)$, we have $h = \text{Re}(h) + i \text{Im}(h)$ where $g = \frac{1}{2} \text{Re}(h)$ is a positive definite symmetric bilinear form (and $\omega = -\frac{1}{2} \text{Im}(h)$ is a J compatible symplectic structure

□

Remark. Since ω is an antisymmetric bilinear form, $\omega \in \bigwedge^2 V^*$ and ω is J -invariant if and only if $\omega \in \bigwedge^{1,1} V^*$

14.1 Hodge Star Operator

If V is a real vector space with a symmetric non-degenerate bilinear form g , then g extends to give a symmetric non-degenerate bilinear form on the exterior powers $\bigwedge^k V$. In particular, $\bigwedge^n V$ is 1 dimensional so has two vectors $\text{vol } g$ of length 1. Choose one and call it the volume element. Recall that we have a non-degenerate bilinear pairing

$$\bigwedge^k V \times \bigwedge^{n-k} V \rightarrow \bigwedge^n V \cong \mathbb{R} \langle \text{vol } g \rangle$$

and so we get an isomorphism $*$: $\bigwedge^k V \rightarrow \bigwedge^{n-k} V$ given by β maps to the unique $*\beta$ such that for all $\alpha \in \bigwedge^k V$, $\alpha \wedge *\beta = g(\alpha, \beta) \text{vol } g$, which we call the **Hodge star**

Proposition 14.2. *If V is a real vector space with complex structure J and J -compatible symplectic form ω , then*

$$\text{vol } g = \omega^n / n!$$

where g is the bilinear form associated to ω

Proof. No Proof □

If V has a complex structure J and the inner product g is J -invariant then we can complexify $*$ taken with respect to $\text{vol } g \in \bigwedge^{2n}$ corresponding to the orientation of V induced by J to get

$$*_\mathbb{C}: \bigwedge^k V_\mathbb{C} \rightarrow \bigwedge^{n-k} V_\mathbb{C}$$

, which satisfies $\alpha \wedge *_\mathbb{C} \bar{\beta} = h(\alpha, \beta) \text{vol } g$ preserving the p, q decomposition, in that $*_\mathbb{C}: \bigwedge^{p,q} V \rightarrow \bigwedge^{n-p, n-q} V$. We have that $(*_\mathbb{C})^2 = (-1)^{p+q}$, and we also have that $\bigwedge^k V_\mathbb{C} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$ is a h orthogonal decomposition

15 Kähler Manifolds

Definition 15.1. A complex manifold X is called a **Hermitian manifold** if it has a **Hermitian metric** (a smoothly varying family of Hermitian inner products $h_p: T_p^{1,0} \times T_p^{1,0} \rightarrow \mathbb{C}$).

Remark. Any Hermitian manifold X also has

1. A J -invariant Riemannian metric $g_p: T_p X \times T_p X \rightarrow \mathbb{R}$
2. A J -compatible 2-form ω on X , since $\omega_p: T_p X \times T_p X \rightarrow \mathbb{R}$ is an alternating bilinear form, called the **fundamental/symplectic form** of h . We in fact have that $\omega_p \in \bigwedge^{1,1} T_p^* X$

Definition 15.2. A Hermitian manifold X is called a **Kähler manifold** if the fundamental form is closed (ie $d\omega = 0$). We say h is a **Kähler metric** and ω is a **Kähler form**.

Remark. In this case X has the structure of a symplectic manifold

In particular for any Kähler manifold, there is a well defined cohomology class $[\omega] \in H^2(X; \mathbb{R})$. We recall that $\omega^n = n! \text{vol } g^*$. Therefore

$$\int_X \omega^n / n! = \int_X d \text{vol } g = \text{vol}(X)$$

Thus if X is a compact Kähler manifold, $[\omega]$ is a non-zero cohomology class, as if it is zero then $\text{vol}(X) = 0$ by Stokes, contradiction.

Corollary 15.0.1. *The Hopf surface $\mathbb{C}^2 \setminus \{0\}/(z \sim 2z)$, which is diffeomorphic to $S^1 \times S^3$ has no Kähler metric*

Proof. As there are no non-zero classes in $H^2(S^1 \times S^3, \mathbb{R})$ □

Example 15.1.

1. $X = \mathbb{C}^n$ with the standard Riemannian metric on $\mathbb{R}^{2n}$, with symplectic form

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i = \frac{i}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$$

is a Kähler manifold, as clearly $d\omega = 0$. This is called the **standard symplectic form**

2. $X = \mathbb{C}^n / \mathbb{Z}^{2n}$ the complex torus. ω descends to X and $d\omega$ is still zero.
3. If $\dim_{\mathbb{C}} X = 1$ then every 2-form is closed and so every Hermitian metric on X is Kähler.
4. $X = \mathbb{CP}^n$ with the **Fubini-Study form** is Kähler. In the open cover $\{U_i\}$ of $\mathbb{CP}^n$ where $U_i = \{z_i \neq 0\}$, we write

$$\omega_{fs} = \frac{i}{2\pi} \partial \bar{\partial} \log \left(\sum_{k=0}^n \left| \frac{z_k}{z_i} \right|^2 \right)$$

and one can check these agree on overlaps. This is clearly a 1,1 form on T^*X and ω_{fs} is non-degenerate (as $\omega_{fs} = \frac{i}{2\pi} F_A$ for ∇_A the Chern connection on $\mathcal{O}(1)$), and hence ω_{fs} gives a Kähler metric on $\mathbb{CP}^n$

Corollary 15.0.2. *If X is a complex submanifold of $\mathbb{CP}^n$ then X admits a Kähler metric, given by $\omega_{fs}|_X$ (as d commutes with restriction)*

Proof. Trivial □

Definition 15.3. Suppose X is a Hermitian manifold with Hermitian metric h . Then we can define

- $d^* = * \circ d \circ * : C^\infty(X, \bigwedge^k T^*X) \rightarrow C^\infty(X, \bigwedge^{k-1} T^*X)$, the **adjoint of d**
- $\Delta = dd^* + d^*d : C^\infty(X, \bigwedge^k T^*X) \rightarrow C^\infty(X, \bigwedge^k T^*X)$, the **Laplacian**
- $\bar{\partial}^* = -* \circ \bar{\partial} \circ * : C^\infty(X, \bigwedge^{p,q} T^*X) \rightarrow C^\infty(X, \bigwedge^{p,q-1} T^*X)$, the **adjoint of $\bar{\partial}$**
- $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : C^\infty(X, \bigwedge^{p,q} T^*X) \rightarrow C^\infty(X, \bigwedge^{p,q} T^*X)$ the **holomorphic Laplacian**
- We similarly define $\partial^*, \Delta_{\partial}$

Definition 15.4. The **Lefschetz operator** is

$$L(\alpha) = \omega \wedge \alpha: C^\infty(X, \bigwedge^{p,q} T^*X), C^\infty(X, \bigwedge^{p+1,q+1} T^*X)$$

and the **dual Lefschetz operator**

$$\Lambda(\alpha) = (-1)^{p+q} * \circ L \circ *: C^\infty(X, \bigwedge^{p,q} T^*X), C^\infty(X, \bigwedge^{p-1,q-1} T^*X)$$

16 Hodge Theory

For now we assume X is a Hermitian manifold. Recall that for a Riemannian manifold we have the Laplacian $\Delta = dd^* + d^*d$ and in the complex case we have its complexification

$$\Delta: C^\infty\left(\bigwedge^k(T^*X)_\mathbb{C}\right) \rightarrow C^\infty\left(\bigwedge^k(T^*X)_\mathbb{C}\right)$$

We say that α is **harmonic** if $\Delta\alpha = 0$, **$\bar{\partial}$ -harmonic** if $\Delta_{\bar{\partial}}\alpha = 0$ and **∂ -harmonic** if $\Delta_{\partial}\alpha = 0$

We denote the space of complex harmonic k forms on X by $\mathcal{H}^k(X)$, of $\bar{\partial}$ harmonic (p, q) forms by $\mathcal{H}_{\bar{\partial}}^{p,q}(X)$ and of ∂ harmonic (p, q) forms by $\mathcal{H}_{\partial}^{p,q}(X)$

Definition 16.1. The L^2 inner product on $C^\infty(X, \bigwedge^{p,q} T^*X)$ for compact Hermitian X is

$$\int_X h(\alpha, \beta) d\text{vol } g = \int_X \alpha \wedge * \bar{\beta}$$

where h is the extension of the Hermitian form to $\bigwedge^{p,q} T^*X$ and $d\text{vol } g$ the volume form for the associated Riemannian metric

Lemma 16.1. $\bar{\partial}^*$ is the formal adjoint to $\bar{\partial}$ in the sense that for $\alpha \in C^\infty(X, \bigwedge^{p,q-1} T^*X)$ and $\beta \in C^\infty(X, \bigwedge^{p,q} T^*X)$ we have

$$\langle \bar{\partial}\alpha, \beta \rangle = \langle \alpha, \bar{\partial}^*\beta \rangle$$

Proof.

$$\begin{aligned} \int_X h(\bar{\partial}\alpha, \beta) &= \int_X \bar{\partial}\alpha \wedge \bar{\beta} \\ &= \int_X \bar{\partial}(\alpha \wedge * \bar{\beta}) \pm \int_X \alpha \wedge \bar{\partial}(* \bar{\beta}) \end{aligned}$$

The first integrand on the latter line is in fact $d(\alpha \wedge * \bar{\beta})$ by considering the type of form, so by Stokes theorem, the entire thing is equal to

$$\pm \int_X \alpha \wedge \bar{\partial}(* \bar{\beta}) = \int_X \alpha \wedge *(\bar{\partial}^* \bar{\beta}) = \langle \alpha, \bar{\partial}^* \beta \rangle$$

□

Lemma 16.2. *A smooth (p, q) -form α is $\bar{\partial}$ -harmonic if and only if we have that $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$*

Proof. Suppose $\Delta_{\bar{\partial}}\alpha = 0$. Then

$$\begin{aligned}\langle \Delta_{\bar{\partial}}\alpha, \alpha \rangle &= \langle \bar{\partial}\bar{\partial}^*\alpha, \alpha \rangle + \langle \bar{\partial}^*\bar{\partial}\alpha, \alpha \rangle \\ &= \langle \bar{\partial}^*\alpha, \bar{\partial}^*\alpha \rangle + \langle \bar{\partial}\alpha, \bar{\partial}\alpha \rangle \\ &= |\bar{\partial}^*\alpha|^2 + |\bar{\partial}\alpha|^2 = 0\end{aligned}$$

and so the forwards direction follows. The backwards direction is trivial $\square$

Lemma 16.3. *The image of $\bar{\partial}^*$ is orthogonal to the kernel of $\bar{\partial}$*

Proof. Trivial linear algebra $\square$

Theorem 16.4. *If (X, h) is a compact Hermitian manifold, then $\mathcal{H}_{\bar{\partial}}^{p,q}(X)$ is finite dimensional and there is an L^2 orthogonal decomposition*

$$\begin{aligned}C^\infty\left(X, \bigwedge^{p,q} T^*X\right) &= \Delta_{\bar{\partial}}\left(C^\infty\left(X, \bigwedge^{p,q} T^*X\right)\right) \oplus \mathcal{H}_{\bar{\partial}}^{p,q}(X) \\ &= \bar{\partial}\left(C^\infty\left(X, \bigwedge^{p,q} T^*X\right)\right) \oplus \bar{\partial}^*\left(C^\infty\left(X, \bigwedge^{p,q} T^*X\right)\right) \oplus \mathcal{H}_{\bar{\partial}}^{p,q}(X)\end{aligned}$$

Proof. No Proof $\square$

Corollary 16.4.1. *Every Dolbeault cohomology class in $H^{p,q}(X)$ can be represented by a unique $\bar{\partial}$ -harmonic (p, q) -form*

Proof. $H^{p,q}(X) = \ker \bar{\partial} / \text{Im } \bar{\partial}$. By the theorem any (p, q) -form α can be written uniquely as $\alpha = \bar{\partial}\beta + \bar{\partial}^*\gamma + \mu$ for some $\bar{\partial}$ -harmonic μ . Since the image of $\bar{\partial}^*$ is orthogonal to $\ker \bar{\partial}$, $\ker \bar{\partial} \cong \text{Im } \bar{\partial} \oplus \mathcal{H}_{\bar{\partial}}^{p,q}(X)$ and the result follows $\square$

In particular, $H^{p,0}(X) \cong \Omega^p(X)$ as if α is a $(p, 0)$ form then α is $\bar{\partial}$ harmonic if and only if $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$, but the latter is always 0 for degree reasons, and so it is if and only if $\bar{\partial}\alpha = 0$, if and only if α is holomorphic

Theorem 16.5. (*Hodge Decomposition*) *If X is a compact Kähler manifold then*

$$H^k(X; \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

and $H^{p,q}(X) \cong \overline{H^{q,p}(X)}$.

Proof. Deferred $\square$

Remark. The odd Betti numbers of a compact Kähler manifold are even (as p, q and q, p are distinct if $p + q = 2k + 1$ and $h^{p,q} = h^{q,p}$ so that $b_{2k+1} = \sum_{i=0}^k 2h^{2k+1-k,k}$). In particular the Hopf surface has $H^1(X; \mathbb{C}) = \mathbb{C}$ so X admits no Kähler metric

Theorem 16.6. *If X is a Kähler manifold then $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$, and so the Laplacian preserves (p, q) -forms. Furthermore the holomorphic p -forms on a Kähler manifold are harmonic*

Proof.

Theorem 16.7. *(Kähler identities) If X is a Kähler manifold with fundamental form ω then*

1. $[L, \partial] = [L, \bar{\partial}] = 0$
2. $[\Lambda, \partial^*] = [\Lambda, \bar{\partial}^*] = 0$
3. $[\Lambda, \bar{\partial}] = -i\partial^*, [\Lambda, \partial] = i\bar{\partial}^*$
4. $[\bar{\partial}^*, L] = i\partial, [\partial^*, L] = -i\bar{\partial}$

Proof. 1. Note that $[L, \partial] = 0$ implies $\overline{[L, \partial]} = 0$ and so $[L, \bar{\partial}] = 0$ so the second equality follows from the first.

$$\begin{aligned} [L, \partial]\alpha &= L\partial\alpha - \partial L\alpha \\ &= \omega \wedge \partial\alpha - \partial(\omega \wedge \alpha) \\ &= \omega \wedge \partial\alpha - \partial\omega \wedge \alpha - \omega \wedge \partial\alpha \\ &= 0 \end{aligned}$$

since $0 = d\omega = \partial\omega + \bar{\partial}\omega$ and one is a $(2, 1)$ form and the other is a $(1, 2)$ form

2.

$$\begin{aligned} [\Lambda, \partial^*] &= (-1)^{p+q} * L * (- * \bar{\partial}^*) - (- * \bar{\partial}^*)(-1)^{p+q} * L * \\ &= (-1)^{p+q} * L(-1)^{p+q+1} \bar{\partial}^* + * \bar{\partial}^*(-1)^{2p+2q} L * \\ &= - * L \bar{\partial}^* + * \bar{\partial}^* L * \\ &= - * [L, \bar{\partial}^*] * \\ &= 0 \end{aligned}$$

Similarly the only other one we need to prove is $[\Lambda, \bar{\partial}] = -i\bar{\partial}^*$ and the others will follow from applying $*$ and taking complex conjugates, but that is proven by a calculation in local coordinates

□

First,

$$i(\partial\bar{\partial}^* + \bar{\partial}^*\partial) = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = \partial\Lambda\partial - \partial\Lambda\partial = 0$$

and thus

$$\begin{aligned}
\Delta &= dd^* + d^*d \\
&= (\partial + \bar{\partial})(\bar{\partial}^* + \partial^*) + (\partial^* + \bar{\partial}^*)(\bar{\partial} + \partial) \\
&= \partial\partial^* + \partial\bar{\partial}^* + \bar{\partial}\partial^* + \bar{\partial}\bar{\partial}^* + \partial^*\partial + \partial^*\bar{\partial} + \bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} \\
&= \Delta_{\partial} + \Delta_{\bar{\partial}} + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) + (\bar{\partial}\partial^* + \partial^*\bar{\partial}) \\
&= \Delta_{\partial} + \Delta_{\bar{\partial}}
\end{aligned}$$

We now use $[\Lambda, \bar{\partial}] = -i\partial^*$ to get

$$\Delta_{\partial} = \partial\partial^* + \partial^*\partial = i\partial[\Lambda, \bar{\partial}] + i[\Lambda, \bar{\partial}]\partial = i\partial(\Lambda\bar{\partial} - \bar{\partial}\Lambda) + i(\Lambda\bar{\partial} - \bar{\partial}\Lambda)\partial$$

On the other hand

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = -i\bar{\partial}[\Lambda, \partial] - i[\Lambda, \partial]\bar{\partial} = -i\bar{\partial}(\Lambda\partial - \partial\Lambda) - i(\Lambda\partial - \partial\Lambda)\bar{\partial}$$

but these are exactly equal so $\Delta_{\partial} = \Delta_{\bar{\partial}}$ $\square$

Corollary 16.7.1. (*Hodge symmetries*) *We have*

$$\begin{aligned}
H^{p,q}(X) &\cong \overline{H^{p,q}}(X) \\
H^{p,q}(X) &\cong H^{n-q,n-p}(X)
\end{aligned}$$

The latter is called Kodaira-Serre duality

Proof. Every class in $H^{p,q}(X)$ has a unique $\bar{\partial}$ harmonic representative α . Then $\bar{\alpha}$ is a (q,p) -form and $\Delta_{\bar{\partial}}\bar{\alpha} = \overline{\Delta_{\partial}\alpha} = 0$ since $\Delta_{\partial} = \Delta_{\bar{\partial}}$. Therefore $\bar{\alpha}$ is $\bar{\partial}$ holomorphic, and we have a complex antilinear isomorphism

$$\mathcal{H}_{\bar{\partial}}^{p,q}(X) \rightarrow \mathcal{H}_{\bar{\partial}}^{q,p}(X)$$

For the other symmetry, use the Hodge star instead $\square$

Lemma 16.8.

1. $[\Delta_{\partial}, L] = [\Delta_{\bar{\partial}}, L] = 0$
2. $[\Delta_{\partial}, \Lambda] = [\Delta_{\bar{\partial}}, \Lambda] = 0$

and so L takes $\bar{\partial}$ harmonic (p,q) forms to $\bar{\partial}$ harmonic $(p+1, q+1)$ forms, and therefore L gives a map $H^{p,q}(X) \rightarrow H^{p+1,q+1}(X)$

Proof. Trivial $\square$

Theorem 16.9. *If X is a compact Riemannian manifold then there is an isomorphism*

$$H^k(X, \mathbb{R}) \cong \mathcal{H}^k(X)$$

where the latter is Δ harmonic k -forms on X

Proof. No Proof □

Proof. (Of 16.5) If X is a compact Kähler manifold we know any k -form is a sum of (p, q) forms with $p + q = k$ in a unique way and $\Delta = 2\Delta_{\bar{\partial}}$ which takes (p, q) forms to (p, q) -forms, and so any harmonic k -form on X can be uniquely written as a sum of $\bar{\partial}$ harmonic (p, q) forms with $p + q = k$, recalling corollary 16.4.1, there is therefore an isomorphism

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} \mathcal{H}_{\bar{\partial}}^{p,q}(X) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

□

16.1 The Hodge Diamond

For any complex manifold, let $h^{p,q} = \dim H^{p,q}(X)$ be the Hodge numbers. We have the **Hodge Diamond**

$$\begin{array}{ccccc}
 & & h^{n,n} & & \\
 & & & & \\
 & h^{n,n-1} & & h^{n-1,n} & \\
 & & & & \\
 h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} \\
 & & & & \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & \\
 h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & & & & \\
 & h^{1,0} & & h^{0,1} & \\
 & & & & \\
 & & h^{0,0} & &
 \end{array}$$

For a Kähler manifold we have

1. The rows add up to the Betti numbers, b_k
2. The diamond is symmetric around the vertical axis
3. $h^{p,q} = h^{n-q,n-p}$

Example 16.1.

1. For $\mathbb{CP}^1$ we get

$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ 0 & & 0 \\ & & \downarrow \\ & & 1 \end{array}$$

2. If $X = \mathbb{C}/\Lambda$ is an elliptic curve, we get

$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ 1 & & 1 \\ & & \downarrow \\ & & 1 \end{array}$$

3. If X is a compact connected genus g surface then every complex structure is integrable and every Hermitian metric is Kähler. We also get the Hodge theorem

$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ g & & g \\ & & \downarrow \\ & & 1 \end{array}$$

Corollary 16.9.1. *The space of holomorphic 1-forms on a compact connected genus g surface has dimension g*

Proof. Trivial □

Lemma 16.10. (*$\partial\bar{\partial}$ lemma*) *If X is a compact Kähler manifold and if α is a closed (p, q) form, the following are equivalent*

1. α is exact ($\alpha = d\beta$)
2. α is $\bar{\partial}$ -exact ($\alpha = \bar{\partial}\mu$)
3. α is ∂ -exact ($\alpha = \partial\eta$)
4. α is $\partial\bar{\partial}$ -exact (ie $\alpha = \partial\bar{\partial}\rho$ for some ρ)

Proof. We recall that for X compact and Kähler, $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$ and $\Delta_{\bar{\partial}}\alpha = 0$ if and only if $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$. We also have the L^2 orthogonal decompositions

$$C^\infty(X, \bigwedge^{p,q} T^*X) = \ker \Delta_{\bar{\partial}} \oplus \text{im } \bar{\partial} \oplus \text{im } \bar{\partial}^*$$

and similar for ∂ .

Note that clearly (4) implies all of the others as $\partial\bar{\partial} = -\bar{\partial}\partial$ and $d = \partial + \bar{\partial}$ with $\partial^2 = \bar{\partial}^2 = 0$ so $\alpha = \partial\bar{\partial}\rho = d\bar{\partial}\rho$. Conversely assume that α is a closed (p, q) -form that is L^2 orthogonal to the harmonic (p, q) -forms (which are also the ∂ or $\bar{\partial}$ harmonic (p, q) forms). This is implied by any of 1-3. We want to show $\alpha = \partial\bar{\partial}\rho$ for some ρ . Since α is of type (p, q) and $d\alpha = 0$, $\partial\alpha = \bar{\partial}\alpha = 0$. We write $\alpha = \partial\beta + \partial^*\mu$ by the decomposition. Since $\partial\alpha = 0$ we have $\partial\partial^*\mu = 0$, and also $\partial^*\partial^*\mu = 0$ so $\partial^*\mu$ is ∂ harmonic, contradiction so $\alpha = \partial\beta$. By the same result applied to β , we get $\beta = \bar{\partial}\gamma + \bar{\partial}^*\epsilon + \eta$ where η is $\bar{\partial}$ -harmonic, and thus ∂ -harmonic. Thus $\alpha = \partial\beta = \partial\bar{\partial}\gamma + \partial\bar{\partial}^*\epsilon = \partial\bar{\partial}\gamma - \bar{\partial}^*\partial\epsilon$. Then

$$\langle \bar{\partial}\bar{\partial}^*\partial\epsilon, \partial\epsilon \rangle = \langle \bar{\partial}^*\partial\epsilon, \bar{\partial}^*\partial\epsilon \rangle = |\bar{\partial}^*\partial\epsilon|^2$$

but since $\bar{\partial}\alpha = 0$, $\bar{\partial}\bar{\partial}^*\partial\epsilon = 0$ and thus $\bar{\partial}^*\partial\epsilon = 0$ and $\alpha = \partial\bar{\partial}\gamma$ □

Lemma 16.11. (*Poincaré lemma*) *If $\Delta \subseteq \mathbb{C}^n$ is a polydisk then $H^{p,q}(\Delta) = 0$ for $q \geq 1$. Thus every $\bar{\partial}$ -closed form is locally $\bar{\partial}$ -exact*

Proof. No Proof □

Remark. $H^{p,0}(\Delta)$ is not in general 0, for example $H^{0,0}(\Delta)$ is just holomorphic functions on Δ

16.2 Lefschetz Theorems

Recall that we have the operators $L(\alpha) = \alpha \wedge \omega$ and $\Lambda(\alpha) = *^{-1} \circ L \circ *$. We claim $[L, \Delta] = [\Lambda, \Delta] = 0$ (exercise with the Kähler identities) and thus we get maps

$$L^k: H^{p,q}(X) \rightarrow H^{p+k, q+k}(X)$$

$$\Lambda^k: H^{p,q}(X) \rightarrow H^{p-k, q-k}(X)$$

Define the counting operator H by $H|_{\wedge^k V} = (k - n) \text{Id}$

Theorem 16.12. (*Hard Lefschetz Theorem*) *The map*

$$L^{n-k}: H^k(X) \rightarrow H^{2n-k}(X)$$

is an isomorphism for X compact and Kähler, and moreover $[l, H] = -2L$, $[\Lambda, H] = 2\Lambda$, $[L, \Lambda] = H$ (In other words $H^(X)$ has the structure of an $sl_2(\mathbb{C})$ representation)*

Proof. No Proof □

16.3 Complex Submanifolds

Lemma 16.13. *If Y is a complex submanifold of a compact Kähler manifold X then $[Y] \neq 0$ in $H_*(X; \mathbb{Z})$*

Proof. If $\dim Y = k$ then

$$[\omega]^k \cdot [Y] = \int_Y \omega^k = \text{vol}_Y$$

so $[Y] \neq 0$

□

Proposition 16.1. *If Y is a complex submanifold of a compact Kähler manifold X then $[Y] \in \text{Im}(H^{n-p, n-p}(X) \rightarrow H^{2n-2p}(X, \mathbb{Z})_{\mathbb{C}})$ for $p = \dim_{\mathbb{C}} Y$*

Proof. If $\eta = [Y]$ is the Poincaré dual of Y in homology then

$$\int_X Y \wedge \alpha = \int_Y \alpha$$

so $\eta \wedge \alpha$ is an (n, n) and α is a (p, p) form, so η is a $(n - p, n - p)$ form

□

Proposition 16.2. *(Lefschetz $(1, 1)$ theorem) For X compact and Kähler, the map $c_1: \text{Pic}(X) \rightarrow H^{1,1}(X) \cap H^2(X; \mathbb{Z})$ is surjective*

Proof. No Proof

□