Costs of Group Actions

Amy Needham

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1 Introduction

We want to consider $\Gamma \curvearrowright (X, \mu)$ where Γ is a group and (X, μ) is a measure space. We have an equivalence relation given by orbits, and want to consider how "much" of $X \times X$ is needed to generate it. In this introduction will discuss when actions are "orbit equivalent" and look at some examples and invariants.

We assume X is a standard Borel space and has no atoms (an *atom* is a set A with $\mu(A) > 0$ such that for all $B \subseteq A$, $\mu(B) = 0, \mu(A)$). Under this assumption, X is isomorphic to $([0, 1], \lambda)$. We assume Γ is countable. In the standard setup we work with, actions will be (almost everywhere) free and ergodic.

Example 1.1.

1. $\mathbb{Z}^n \curvearrowright S^1$ by rationally independent rotations.
2. $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{T}^n$ by its action on the universal cover.
3. Free ergodic volume preserving actions of groups on finite volume manifolds.
4. If Λ, Γ are lattices in the Lie group G, then $\Gamma \curvearrowright G/\Lambda$ and $\Lambda \curvearrowright G/\Gamma$ under the Haar measures.
5. The Bernoulli shift on X^Γ where Γ acts by permuting coordinates. In particular, in this way (for $X = \{0, 1\}$) every countable Γ acts in a probability measure preserving (pmp) way. The action is free and ergodic if and only if Γ is infinite.

Proposition 1.1. *Consider the orbit equivalence relation which we write $R_\alpha = R$. Then the following hold:*

1. *The equivalence classes of R are countable.*

2. *R is borel in $X \times X$.*

3. ***Partial isomorphisms*** *(Borel isomorphisms between measurable subsets) that restrict to act on the orbits also preserve the measure.*

Proof. Trivial □

Definition 1.1. An R satisfying the conditions above are called *countable standard measure preserving equivalence relations.*

Theorem 1.1. *(Feldmann-Moore, 1977) Every countable standard measure preserving equivalence relation R comes from a pmp action, but they are not necessarily free.*

Proof. No Proof □

Definition 1.2. Let $\Gamma_i \curvearrowright (X_i, \mu_i)$ with corresponding orbit equivalence relations R_i. We say R_1 is *orbit equivalent* to R_2 if there is a Borel isomorphism $f\colon X_1 \to X_2$ which preserves the measure and $f(\Gamma_1 x) = \Gamma_2 f(x)$ for almost all x.

From this we can define a cocycle, stable orbit equivalence, measure equivalence, etc, but we will not go into this here.

We can now ask some questions:

- Is there a group with all action OE? Yes, see below
- Can different groups have orbit equivalent actions? Yes, see below.
- Are there actions of a single group which are not orbit equivalent? Not in general, but see below. How many? See below.

Theorem 1.2. *(Dye, 1959) Any two pmp $\mathbb{Z}$ actions are orbit equivalent.*

Proof. No Proof □

Theorem 1.3. *(Dye, 1963) Let Γ have polynomial growth or be infinite abelian. Then any any free ergodic Γ-action is orbit equivalent to a $\mathbb{Z}$ action.*

Proof. No Proof □

Theorem 1.4. *(Zimme, 1978) If Γ has a free pmp action orbit equivalent to a $\mathbb{Z}$ action then Γ is amenable.*

Proof. No Proof □

Theorem 1.5. *(Ornstein-Weiss, 1980) Any pmp ergodic action of an amenable group is orbit equivalent to a $\mathbb{Z}$ action.*

Proof. No Proof □

Theorem 1.6. *(Connes-Weiss, 1980) Let Γ be neither amenable nor Kazhdan. Then Γ has two non-orbit equivalent free ergodic actions.*

Proof. No Proof □

Theorem 1.7. *(Guften-Gulodith, 1988) Non-uniform lattices in higher rank simple Lie groups with finite centre has uncountably many non-orbit equivalent free ergodic actions.*

Proof. No Proof □

Theorem 1.8. *(Hjanth, 2002) Infinite Kazhdan groups have uncountably many non-orbit equivalent free ergodic actions.*

Proof. No Proof □

So amenability is an orbit equivalence invariant. What are some others?

- Kazhdan's property (T).
- Cost.
- l^2-Betti numbers (and therefore Euler characteristic)
- a-T-menability

We will define cost now. One motivation for it is that it can distinguish between the ranks of free group actions

Definition 1.3. A *graphing* is a countable family of μ-preserving partial isomorphisms, $\Phi = \{\phi_i \colon A_i \to B_i\}$.

Remark 1.1. A graphing Φ generates a μ-preserving standard countable equivalence relation by the generating set $\{(a, \phi_i(a)) \mid a \in A_i\}$. We denote this relation by R_Φ.
Equivalently, $(x, y) \in R_\Phi$ if there is a word w in $\Phi \cup \Phi^{-1}$ such that $y = w(x)$ and $w(x)$ makes sense.

Definition 1.4. The *cost* of a graphing Φ is $\mathcal{C}(\Phi) = \sum_{i \in I} \mu(A_i)$.
The *cost* of an equivalence relation R is the infimum of $\mathcal{C}(\Phi)$ over all Φ such that $R = R_\Phi$. The *cost* of a group G is $\mathcal{C}(G) = \inf\{\mathcal{C}(R_\alpha) \mid \alpha \text{ is a free pmp action of } G\}$

2 Treeings

Definition 2.1. We write $\Gamma(\Phi)$ for the graph with vertices X and an edge from x to $\phi_i(x)$ for each x. We write $\Gamma(\Phi, x)$ for the component containing x. We call this the *graph associated to* Φ. Ψ is a *treeing* if every $\Gamma(\Psi, x)$ is a tree for almost every x.

Definition 2.2. A group is *treeable* if it has an action with a treeing. It is *strongly treeable* if every action has a treeing.

Remark 2.1. We use letters without loops for treeings and letters with loops for not-necessarily treeings.

Theorem 2.1. *$\mathcal{C}(\Phi) = \mathcal{C}(R_\Phi)$ if and only if Φ is a treeing.*

Proof. Forward direction: Suppose Φ is not a treeing. Then we have a loop of positive measure in our graph. We can make the domains of the loop disjoint in X, then remove an edge (see Gaborieau's notes for the formal proof) and we have reduced the cost.
For the backwards direction we need some lemmas.

Lemma 2.2. *Let Ψ be a graphing. Then there is a graphing Θ of R_Ψ such that*

1. *$\mathcal{C}(\Theta) \leqslant \mathcal{C}(R_\Phi) + \varepsilon$*
2. *For all $\theta \in \Theta$ there is a word in Ψ called m_θ such that $\theta(x) = m_\theta(x)$ for all $x \in \operatorname{dom}\theta$ and $|m_\theta| \leqslant L$*
3. *Each $\operatorname{dom}\Psi_i$ decomposes into finitely many pieces where Φ_i coincides with a θ-word of length at most L*

Proof. Let Φ be a graphing of R with $\mathcal{C}(\Phi) \leqslant \mathcal{C}(R) + \varepsilon$. Enumerate the Φ words as $w_1, \dots, w_n, \dots$. Let $w_n^i = \{x \in \operatorname{dom}\Psi_i \mid \Psi_i(x) \neq w_k(x), \forall k \leqslant n\}$. We have $w_n = \bigcup w_n^i$, and $\bigcap_{i=1}^\infty w_n = \varnothing$.
Let n_0 be such that $\sum_{i\in I} \mu(w_{n_0}^i) \leqslant \varepsilon$, and set $\Theta = \Phi_0 \cup \{\Psi_i|_{w_{n_0}^i}\}$. Then we have $\times \leqslant \mathcal{C}(R) + 2\varepsilon$, and all the desired properties hold. □

Let $\theta = m_\theta = \psi_{l(\theta),\theta} \dots \psi_{1,\theta}$. We take spaces $A_\theta \times \{j\}$ where $A_\theta \cong \operatorname{dom}\theta$ for $j = 0, \dots, l(\theta)$, and set $\bar{\theta}_j \colon A_\theta \times \{j\} \to A_\theta \times \{j+1\}$ the identity on the first component. We then consider $\bar{X} = \left(X \sqcup \bigsqcup_{j=1}^{l(\theta)} A_\theta \times \{0\}\right) / (A_\theta \times \{0\} \sim \operatorname{dom}\theta, A_\theta \times \{l(\theta)\} \sim \operatorname{im}\theta)$ with the obvious measure. The cost of $\bar{\theta}$ on $\bar{X}$ satisfies $\mathcal{C}(\bar{\theta}) - \mu(\bar{X}) = \mathcal{C}(\theta) - \mu(X)$. We define a map $p \colon \bar{X} \to X$ by $p(x, j) = \Psi_{\theta,j} \dots \Psi_{\theta,1}(x)$, and the identity on X. We want to show p is a quasi isometry. More specifically we want to show

Lemma 2.3.

$$d_\Psi(p(x), p(y)) \leqslant d_{\bar{\theta}}(x, y) \leqslant C_1 d_\Psi(x, y) + C_2$$

and x, y are in the same $\bar{\theta}$ orbit if and only if their projections are in the same Ψ orbit.

Proof. The first inequality is trivial. If we have $x, y \in \bar{X}$, and we have a Ψ word connecting $p(x), p(y)$, and this turns into a θ word of length at most L times longer, which turns into a $\bar{\theta}$ word of length at most L times longer, so we get $d_{\bar{\theta}}(p(x), p(y)) \leqslant L^2 d_\Psi(p(x), p(y))$. We note that the distance between x and the copy of x in the 0 component is at most L in the $\bar{\theta}$ metric, and the distance from that copy of x to $p(x)$ is bounded by L in the Ψ metric, so L^3 in the $\bar{\theta}$ metric.
The second comment is clear. □

We now wish to reduce the size of fibres. We write $\bar{\theta} = \{\bar{\theta}_i \mid i \in J\}$. Consider a pair $(\bar{\theta}_i^{\varepsilon_i}, \bar{\theta}_j^{\varepsilon_j})$ which are not equal but have the same projection. If a fibre is not of size one, by the fact Ψ is a treeing, we have a pair of morphisms as descibed (by considering a path between two elements of a fibre). Let $U = \{x \in \mathrm{dom}(\bar{\theta}_i^{\varepsilon_i}) \cap \mathrm{dom}(\bar{\theta}_j^{\varepsilon_j}) \mid \bar{\theta}_i^{\varepsilon_i}(x) \neq \bar{\theta}_j^{\varepsilon_j}(x)\}$. We quotient $\bar{X}$ to $\bar{X}'$ by identifying the two different images of U under $\bar{\theta}_i^{\varepsilon_i}, \bar{\theta}_j^{\varepsilon_j}$, and we have $\bar{\theta}'$ given by removing U from one of the two domains of $\bar{\theta}_i^{\varepsilon_i}, \bar{\theta}_j^{\varepsilon_j}$. We have $\mathcal{C}(\bar{\theta}') - \mu(\bar{X}') = \mathcal{C}(\bar{\theta}) - \mu(\bar{X})$. We are decreasing the distance between fibres, so it will eventually reach 0, so we may assume p is injective.

Suppose we have x and $\psi_i(x)$, and a $\bar{\Theta}'$ word between them. Using our folded projection, we map $\bar{\Theta}'$ to the single edge ψ_i, so the $\bar{\Theta}'$ word has length one, and so $\mathcal{C}(\Psi) \leqslant \mathcal{C}(\bar{\Theta}')$. Chaining the two inequalities we have, $\mathcal{C}(\Psi) - \mu(\bar{X}') \leqslant \mathcal{C}(\theta) - \mu(X)$, so $\mathcal{C}(\Psi) \leqslant \mathcal{C}(\theta) \leqslant \mathcal{C}(R) + 2\varepsilon$. □