

Coxeter Groups

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1 Definition and Examples

1.1 Abstract Definition

Definition 1.1. We recall a couple definitions

1. Let X be a set. Define the *Free Group* on X to be the set $\text{Free}(X)$ of words in elements of X and their inverses. The group operation is concatenation and then reduction of words
2. Let G be a group. A *Presentation* for G is a set X of *Generators* and a set R which is a subset of $\text{Free}(X)$ of *Relations* and an isomorphism $\text{Free}(X)/\langle\langle R \rangle\rangle \cong G$. We write $G = \langle x \in X \mid r = 1, \forall r \in R \rangle = \langle X \mid R \rangle$

Example 1.1. We have the following example of presentations

1. $\mathbb{Z}/n\mathbb{Z} = \langle x \mid x^n = 1 \rangle$
2. $G = \langle S \mid R \rangle, H = \langle Y \mid S \rangle$ then
$$G \times H = \langle X \sqcup Y \mid R, S, xy = yx \forall x \in X, y \in Y \rangle$$
3. $\text{Free}(X) = \langle X \mid \emptyset \rangle$
4. G is any group, and $g_1, \dots, g_\alpha$ be generators (not necessarily finitely many). Let X be in bijection with the g_i . There is a surjection $\text{Free}(X) \rightarrow G$. If R is the kernel, $G = \langle X \mid R \rangle$

Definition 1.2. Let I be a set and $X = \{x_i \mid i \in I\}$.

1. A *Cartan Matrix* is a symmetric $|I| \times |I|$ matrix $M = (m_{ij})_{i,j \in I}$ such that $m_{ij} \in \mathbb{N} \cup \infty$ with $m_{ii} = 1$ and $m_{ij} > 1$ for $i \neq j$.
2. A *Coxeter System* is a triple (W, I, M) where M is a Cartan matrix over I and W has presentation

$$W = \langle x_i, i \in I \mid (x_i x_j)^{m_{ij}} = 1 \rangle$$

$|I|$ is the *Rank* of the system and W is a *Coxeter Group*

Remark. We have two immediate consequences

1. For $i \in I$, we have $m_{ii} = 1$ so $x_i^2 = 1$

2. x_i, x_j commute if and only if $m_{ij} = 2$ (if is trivial, only if is not immediate, but will come later)

Example 1.2. We have a couple examples

1. Let $I = \{1\}$. Then $M = (1)$ and $W = \langle x_1 \mid x_1^2 = 1 \rangle = \mathbb{Z}/2\mathbb{Z}$
2. Let I be any set and $M = (m_{ij})$ where $m_{ij} = 1$ if $i = j$ and 2 otherwise. Then $W = \mathbb{Z}/2\mathbb{Z}^{\oplus |I|}$. These are the only abelian Coxeter groups

1.2 The Coxeter Graph

Fix a Coxeter System (W, I, M) .

Definition 1.3. The *Coxeter Graph*, $\Gamma(W, I, M)$ or $\Gamma(W)$ is the following labelled graph. The vertices are the elements of the generating set. Two vertices are joined by an edge, labelled n , if $m_{ij} = n > 3$, and unlabelled if $m_{ij} = 2$.

1.3 First Examples

Example 1.3. We have the following examples

1. $I = \{1\}$ then the graph is

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2. I is any set, $m_{ij} = 2$, $W = \mathbb{Z}/2\mathbb{Z}^{\oplus |I|}$ is the graph without edges on $|I|$ vertices

3. Let $I = \{1, 2\}$ and $3 \leq n \leq \infty$ and $M = \begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$. All 2×2 Cartan matrices have this form, and $W = \langle x_1, x_2 \mid (x_1 x_2)^n \rangle \cong D_{2n}$. For ∞ we get $\langle x_1, x_2 \mid x_1^2 = x_2^2 = 1 \rangle \cong D_\infty$. The Coxeter Graph is

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for $n > 3$, and the same with no label for $n = 3$

4. Let $X = \{1, \dots, n\}$ and $S_n = \text{Sym}(X)$. This is generated by all transpositions. For all $i < j - 1$, $(i, j) = (j, j - 1)(i, j - 1)(j, j - 1)$ so by induction, S_n is generated by $s_i = (i, i + 1)$ for $i < n$. $s_i s_j = (i, i + 2, i + 1)$ if $i = j - 1$, $(j, j + 2, j + 1)$ if $i = j + 1$, 1 if $i = j$ and $s_j s_i$ otherwise. So $m_{ij} = 1$ if $i = j$, $m_{ij} = 3$ if $|i - j| = 1$ and $m_{ij} = 2$ otherwise. The Coxeter Graph is

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called A_{n-1} and there is a surjection $W \rightarrow S_n$ by $s_i \rightarrow (i, i + 1)$

Proposition 2.1. *An isometry $\alpha: V \rightarrow V$ is linear if and only if it fixes 0*

Proof. Given $u, v \in V$, apply α to the parallelogram $0, u, v, u + v$. These map to the parallelogram $0, \alpha(u), \alpha(v), \alpha(u + v)$. But the fourth vertex of the parallelogram with $0, \alpha(u), \alpha(v)$ is $\alpha(u) + \alpha(v)$ so done.

If $0 \neq \lambda \in \mathbb{R}$ then as $0, u, \lambda u$ are colinear, $0, \alpha(u), \alpha(\lambda u)$ are colinear. We therefore have $\alpha(\lambda u) = \mu u$. Orthogonality gives (by taking lengths) $|\lambda| = |\mu|$ so $\mu = \pm \lambda$. By considering the distance from u to λu , if we assume $\mu = -\lambda$ we similarly get $|\lambda - 1| = |\lambda + 1|$ so $\lambda = 0$ and we are done. $\square$

Let $\mathcal{I}(V)$ be the set of isometries of V , which is a group under composition. For each $v \in V$ define the translation $T_v: V \rightarrow V$ by $u \mapsto u + v$. This is clearly an isometry, so if $T(V)$ is the set of translations, $V \cong T(V) \leq \mathcal{I}(V)$

Corollary 2.0.1. *$\mathcal{I}$ is generated by $O(V)$ and $T(V)$ and these have trivial intersection. An isometry α has a fixed point if and only if $\alpha = T\beta T^{-1}$ for $T \in T(V), \beta \in O(V)$*

Proof. Example sheet 1 $\square$

Definition 2.2. A *hyperplane* in V is a linear subspace of V of codimension 1 and an *affine hyperplane* in V is a subset of the form $T(H)$ for some translation T and hyperplane H

Definition 2.3. An isometry α is a *reflection* if $\mathcal{H}_\alpha = \{v \in V \mid \alpha(v) = v\}$ is an affine hyperplane

2.2 Standard Reflections

Let V be any real vector space and let $B: V \times V \rightarrow \mathbb{R}$ be any symmetric bilinear form

Definition 2.4. For each subspace $U \leq V$ define

$$U^\perp = \{v \in V \mid B(u, v) = 0 \ \forall u \in U\}$$

We say B is

1. *Non-degenerate* if $V^\perp = \{0\}$
2. *Positive semi-definite* if $B(u, u) \geq 0$ for all $u \in V$. It is *positive definite* if it is positive semi-definite and $B(u, u) = 0$ if and only if $u = 0$

Definition 2.5. Given $v \in V$ such that $B(v, v) \neq 0$ then we define the *standard reflection* at v to be the linear map $s_v: V \rightarrow V$ by $s_v(u) = u - \frac{2B(u, v)}{B(v, v)}v$

Given $v \in V$, a linear map $T: V \rightarrow V$ is *orthogonal with respect to B* if $B(Tu, Tv) = B(u, v)$

Lemma 2.1. *Given $v \in V$ with $B(v, v) \neq 0$*

1. $s_v^2 = \text{Id}$
2. $s_v(v) = -v$
3. $s_{\lambda v} = s_v$ for all $\lambda \neq 0$
4. $\mathcal{H}_v = \{u \in V \mid s_v(u) = u\} = \{u \in V \mid B(v, u) = 0\} = \langle v \rangle^\perp$ is a hyperplane in V
5. s_v is orthogonal with respect to B
6. If $T \in GL(V)$ is orthogonal with respect to B , $Ts_vT^{-1} = s_{Tv}$

Proof. Exercise □

From now on, V is finite dimensional and B is an inner product

Proposition 2.2. *An isometry α is a reflection if and only if $\alpha = Ts_vT^{-1}$ for some $T \in T(V)$, $v \in V$*

Proof. It suffices to show that if $s \in O(V)$ is a linear reflection then $s = s_v$ for some $v \in V$ by the following argument:

If $\alpha = Ts_vT^{-1}$, we want to prove

$$\mathcal{H}_\alpha = \{v \in V \mid \alpha(v) = v\} = \{u \in V \mid s_v(Tu) = Tu\} = \{u \in V \mid Tu \in \mathcal{H}_v\} = T^{-1}(\mathcal{H}_v)$$

If $\mathcal{H}_\alpha = T(H)$, let $\beta = T^{-1}\alpha T$, then $0 \in \mathcal{H}_\beta = \{u \in V \mid \beta(u) = u\} = T^{-1}(\mathcal{H}_\alpha)$. β is linear by 2.1, so we have shown it suffices.

s is linear so $\mathcal{H}_s$ is a hyperplane. Let $e_1, \dots, e_{n-1}$ be an orthogonal basis for $\mathcal{H}_s$ and extend it to an orthogonal basis of V . By considering the action, we only need to prove $s(e_n) = -e_n$. Write $s(e_n) = \lambda_1 e_1 + \dots + \lambda_n e_n$. Now $0 = (e_i, e_n) = (s(e_i), s(e_n)) = \lambda_i$, so $\lambda_i = 0$ for $i < n$ and $\lambda_n = \pm 1$. If $\lambda_n = 1$ then $s = \text{Id}$, contradiction so we are done. □

2.3 Reflection Groups and Root Systems

Let $(V, (\cdot, \cdot))$ be a finite dimensional inner product space

Definition 2.6. A *reflection group* in V is a subgroup of $\mathcal{I}(V)$ generated by reflections

A *linear reflection group* is a subgroup of $O(V)$ generated by linear reflections (ie those of form s_v for $v \in V$)

A *finite reflection group* is a linear reflection group that is finite

Example 2.1. We have the following examples of reflection groups

1. $O(V)$ itself is a linear reflection group and $\mathcal{I}(V)$ is a reflection group but $T(V)$ is not a reflection group
2. Let $V = \mathbb{R}$ and $O(\mathbb{R}) = \{\pm \text{Id}\} \cong \mathbb{Z}/2\mathbb{Z}$ is a finite reflection group. If $S = -\text{Id}$ and $t = T_{1/2} \circ S \circ T_{-1/2}$, then we get $D_\infty \cong \langle s, t \rangle \leq \mathcal{I}(\mathbb{R})$

3. Let $V = \mathbb{R}^2$ and Γ a regular n -gon. Consider the linear L_1 passing through a vertex and L_2 passing through the centre of an adjacent side. Let s_i be reflection in L_i .
We have $D_{2n} = \langle s_1, s_2 \rangle \leq O(V)$ is a finite reflection group, and is $\text{Sym}(\Gamma) = \{\alpha \in \mathcal{I}(V) \mid \alpha(\Gamma) = \Gamma\}$
4. Let $V = \mathbb{R}^n$. We define a *convex n -polyhedron*, Γ , to be the boundary of the convex bounded intersection of a set of half spaces (the region of V on one side of an affine hyperplane). Γ is the union of its faces (ie the intersection of the defining affine hyperplanes with Γ) and the symmetry group $\text{Sym}(\Gamma) = \{\alpha \in \mathcal{I} \mid \alpha(\Gamma) = \Gamma\}$ acts on the set of faces, each of which is an $n - 1$ convex polyhedron. We say a convex n polyhedron is regular if $\text{Sym}(\Gamma)$ acts transitively on the faces
5. If $V = \mathbb{R}^3$ then Γ is a regular polyhedron if and only if it satisfies the following 3 properties
 - (a) All faces are similar
 - (b) Each face is a regular polygon
 - (c) Each vertex meets the same number of faces

There are 5 of these, called the *platonic solids* These are

- (a) The tetrahedron, Γ_4 with 4 triangle faces, 6 edges and 4 vertices
- (b) The cube, Γ_6 , with 6 square faces, 12 edges and 8 vertices
- (c) The octahedron, Γ_8 , with 8 triangle faces, 12 edges and 6 vertices
- (d) The dodecahedron, Γ_{12} with 12 pentagon faces, 30 edges and 20 vertices
- (e) The icosahedron, Γ_{20} with 20 triangle faces, 30 edges and 12 vertices

These have the following symmetry group

- (a) $\Gamma_4 = S_4$ is generated by linear reflections, so is a finite reflection group. For a general n -simplex, we get $\text{Sym}(\Delta_n) = S_n$ as a finite reflection group
- (b) $\Gamma_6 \cong \Gamma_8 = \{p \in \text{Sym}\{\pm 1, \pm 2, \pm 3\} \mid p(-i) = -p(i)\} = SS_3$ is generated by linear reflections, so is a finite reflection group. Taking O_n to be the n hypercube, we get $\text{Sym}(O_n) = SS_n$
- (c) $\Gamma_{12} \cong \Gamma_{20}$ called the *isocahedronal isometry group* is a finite reflection group

Definition 2.7. A non-empty subset $\Phi \subset V$ with $|\Phi| < \infty$ with $0 \notin \Phi$ is a *root system* if

1. If $\alpha \in \Phi$ then $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$
2. If $\alpha, \beta \in \Phi$ then $s_\alpha(\beta) \in \Phi$, that is $s_\alpha(\Phi) = \Phi$

The elements of Φ are called the *roots*.

1. Φ is *crystallographic* if further we have $\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$ for $\alpha, \beta \in \Phi$

Definition 2.8. If $\Phi \subset V$ is a root system, the *reflection group* of Φ is

$$W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle \leq O(V)$$

which is a linear reflection group. This is the *Weyl group* if Φ is crystallographic

Remark. $W(\Phi)$ acts on Φ via $w \cdot \alpha = w(\alpha)$

Theorem 2.2. 1. For a root system Φ , $W(\Phi)$ is a finite reflection groups

2. If $W \leq O(V)$ is a finite reflection group, there is a root system Φ such that $W = W(\Phi)$

Proof. (1): $W(\Phi)$ acts on Φ so there is a group homomorphism $\phi: W(\Phi) \rightarrow \text{Sym}(\Phi)$, and the latter is finite. Suppose $w_0 \in W(\Phi)$ and that $\phi(w_0) = 1$, that is $w_0(\alpha) = \alpha$ for all α . Let $U = \text{Span}(\Phi)$ then $w_0(u) = u$ for all $u \in U$.

Decompose $V = U \oplus U^\perp$ and if $v \in U^\perp$ then for all $\alpha \in \Phi$, $s_\alpha(v) = v - \langle \alpha, v \rangle \alpha$ so $w_0 = \text{Id}$ and ϕ is injective so $W(\Phi)$ is a finite reflection group

(2): Let $\Phi = \{v \in V \mid \|v\| = 1, s_v \in W\}$. If $0 \neq v \in V$ is such that $s_v \in W$ then $s_v = s_{v/\|v\|}$, so $\frac{v}{\|v\|} \in \Phi$ so Φ is non-empty finite as W is finite. Now $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$ and $0 \notin \Phi$ by definition. Suppose $\alpha \in \Phi$ and $\beta \in \Phi \cap \mathbb{R}\alpha$. Then $\|\beta\| = 1$ and $\beta = \lambda\alpha$ so $|\lambda| = 1$ so we have (1) in the definition of a root system.

Suppose $\alpha, \beta \in \Phi$. Then $s_\alpha, s_\beta \in W$ so $s_\alpha s_\beta s_\alpha \in W$ so $s_{s_\alpha(\beta)} \in W$ and $\|s_\alpha(\beta)\| = \|\beta\| = 1$, so $s_\alpha(\beta) \in \Phi$ so we have (2) from the definition and we are done $\square$

Example 2.2. We give a couple examples

1. Let $V = \mathbb{R}, \Phi = \{\pm\alpha\}$ for any $\alpha \in \mathbb{R}$ then $W(\Phi) = \mathbb{Z}/2\mathbb{Z}$
2. $V = \mathbb{R}^2, \alpha \in V, \theta = \frac{2\pi}{n}$ and r a reflection by θ . For $\Phi = \{r^m(\alpha) \mid m \in \mathbb{Z}\}$ is a root system with $W(\Phi) = D_{2n}$

2.4 Fundamental Systems

Let $(V, (\cdot, \cdot))$ be an inner product space, $\Phi \leq V$ a root system and $U = \text{Span}(\Phi)$

Definition 2.9. A subset of the root system Φ that we will call Δ is a *fundamental system* if

1. Δ is a basis of U
2. Every $\alpha \in \Phi$ can be written as a linear combination, $\alpha = \sum_{\gamma \in \Delta} c_\gamma \gamma$ where $c_\gamma \geq 0$ for all $\gamma \in \Delta$ or $c_\gamma \leq 0$ for all $\gamma \in \Delta$.

The elements of Δ are called *simple roots*

Fix a basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\} \subset \Phi$ of U .
Let $\Pi = \{\alpha \in \Phi \mid \alpha = \sum_{i=N}^n b_i \alpha_i, b_N > 0\}$

Remark. For all $\alpha \in \Phi$, $\alpha \in \Pi$ or $-\alpha \in \Phi$. That is, $\Phi = \Pi \sqcup -\Pi$. Such a Π is called a *positive system*

Let $\Delta \subset \Pi$ minimal such that for all $\beta \in \Pi$, $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$

Lemma 2.3. For all $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$, $(\alpha, \beta) \leq 0$

Proof. Suppose $(\alpha, \beta) > 0$. We know that $s_\alpha(\beta) \in \Phi$, that is we know that $s_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \alpha$. Let $c = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} > 0$. Either $\beta - c\alpha \in \Pi$ or $c\alpha - \beta \in \Pi$. In either case we have $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$ such that $a\alpha - b\beta \in \Pi$ for some $a, b > 0$. That is, $a\alpha - b\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$ with $c_\gamma \geq 0$. We get two cases

1. $c_\alpha < a$. We get $(a - c_\alpha)\alpha = b\beta + \sum_{\gamma \neq \alpha} c_\gamma \gamma$. As all coefficients are positive, we get $\alpha = \sum_{\gamma \neq \alpha} d_\gamma \gamma$ and hence we get a smaller set, $\Delta \setminus \{\alpha\}$ with the required properties, contradiction
2. $c_\alpha \geq a$. We get $0 = (c_\alpha - a)\alpha + b\beta + \sum_{\gamma \neq \alpha} c_\gamma \gamma = \sum_{\gamma \in \Delta} d_\gamma \gamma$ with $d_\gamma \geq 0$ for all $\gamma \in \Delta$ and $d_\beta > 0$. For each $\gamma \in \Delta$, $\gamma = \sum_{i=N_\gamma}^n a_{i,\alpha} \alpha_i, a_{N_\gamma, \gamma} > 0$. Let $N = \min\{N_\gamma \mid \gamma \in \Delta, d_\gamma > 0\}$. Then $0 = \sum_{\gamma \in \Delta, d_\gamma > 0} d_\gamma \gamma = \sum_{\gamma \in \Delta, d_\gamma > 0} (\sum_{i=N}^n d_\gamma a_{i,\gamma}) \alpha_i$ so by linear independence $\sum_{i=N}^n d_\gamma a_{i,\gamma} = 0$ for all i . But we know this is positive, contradiction

□

Theorem 2.4. Δ is a fundamental root system in Φ

Proof. Since $\Phi = \Pi \sqcup -\Pi$, clearly every $\alpha \in \Phi$ can be written as $\sum_{\gamma \in \Delta} c_\gamma \gamma$ with c_γ all having the same sign. Clearly Δ spans U , It remains to prove that Δ is linearly independent.

Suppose that $0 = \sum_{\gamma \in \Delta} c_\gamma \gamma$ where $c_\gamma \neq 0$ for some γ . Assume the number of the γ such that $c_\gamma \neq 0$ is minimal. Let $\Delta_1 = \{\gamma \in \Delta \mid c_\gamma > 0\}$ and $\Delta_2 = \{\gamma \in \Delta \mid c_\gamma < 0\}$ and for each $\beta \in \Delta_2$ let $d_\beta = -c_\beta > 0$. Then $\sum_{\alpha \in \Delta_1} c_\alpha \alpha = \sum_{\beta \in \Delta_2} d_\beta \beta = \sigma$. Then $(\sigma, \sigma) = (\sum_{\alpha \in \Delta_1} c_\alpha \alpha, \sum_{\beta \in \Delta_2} d_\beta \beta) = \sum_{\alpha \in \Delta_1, \beta \in \Delta_2} c_\alpha d_\beta (\alpha, \beta) \leq 0$. Hence $\sigma = 0$ so $\sum_{\alpha \in \Delta_1} c_\alpha \alpha = 0$ and so $\Delta_1 = \Delta$, ie $c_\gamma \geq 0$ for all $\gamma \in \Delta$. For each $\gamma \in \Delta$ write $\gamma = \sum_{i=N_\gamma}^n a_{i,\gamma} \alpha_i$ with $a_{N_\gamma, \gamma} > 0$. Let $N = \min\{N_\gamma \mid c_\gamma > 0\}$. Write

$$0 = \sum_{\gamma \in \Delta, c_\gamma > 0} c_\gamma \gamma = \sum_{\gamma \in \Delta, c_\gamma > 0} c_\gamma \sum_{i=N}^n a_{i,\gamma} \alpha_i = \sum_{i=N}^n \left(\sum_{\gamma \in \Delta, c_\gamma > 0} c_\gamma a_{i,\gamma} \right) \alpha_i$$

. Hence $\sum_{\gamma \in \Delta, c_\gamma > 0} c_\gamma a_{i,\gamma} = 0$. But this is a sum with positive terms, with one term strictly positive, contradiction, so Δ is linearly independent □

We have proven

1. Fundamental systems exist for any Φ
2. For any fundamental system $\Delta \subset \Phi$, $\alpha \neq \beta \in \Phi$, $(\alpha, \beta) \leq 0$

Definition 2.10. For a fundamental system $\Delta \subset \Phi$ the *Dynkin diagram* of Δ is the graph $\Gamma(\Delta)$ with multiple edges defined by

1. Vertices: the elements of Δ
2. Edges: If $\alpha \neq \beta \in \Delta$ and their angle is of the form $\pi - \frac{\pi}{n}$, then we have $n - 2$ edges between α, β

Example 2.3. Let $\alpha \in \mathbb{R}^2$ and r a rotation by π/n and $\Phi = \{r^m(\alpha) \mid m \in \mathbb{Z}\}$. Then $\Delta = \{\alpha, r^{n-1}\alpha\}$ has $\Gamma(\Delta) =$

$$\alpha \xrightarrow{n-2} \beta$$

. For $n = 3$, $\Phi = \{\alpha, \alpha + \beta, \beta, -\alpha, -\alpha - \beta, -\beta\}$ and $\Delta = \{\alpha, \beta\}$ and $\Gamma(\Delta) =$

$$\alpha \text{ --- } \beta$$

2.5 The Reflection Group

Let Φ be a root system, $\Delta \subset \Phi$ a fundamental system and the positive system of Δ is $\Pi = \{\alpha \in \Phi \mid \alpha = \sum_{\gamma \in \Delta} c_\gamma \gamma, c_\gamma \geq 0\}$ so we have $\Phi = \Pi \sqcup -\Pi$

Lemma 2.5. For $\alpha \in \Delta$, $s_\alpha(\Pi \setminus \{\alpha\}) = \Pi \setminus \{\alpha\}$

Proof. We want to show that for all $\alpha \neq \beta \in \Pi$, $s_\alpha(\beta) \in \Pi$ and $s_\alpha(\beta) \neq \alpha$. If $s_\alpha(\beta) = \alpha$ then $\beta = s_\alpha(\alpha) = -\alpha \notin \Pi$ so it remains to prove that $s_\alpha(\beta) \in \Pi$. So $s_\alpha(\beta) \in \Phi$. Assume $s_\alpha(\beta) \in -\Pi$ that is $s_\alpha(\beta) = \sum_{\gamma \in \Delta} c_\gamma \gamma$ with $c_\gamma \leq 0$. But $\beta = \sum_{\gamma \in \Delta} d_\gamma \gamma$ with $d_\gamma \geq 0$ and so $s_\alpha(\beta) = \sum_{\gamma \in \Delta} d_\gamma (\gamma - \frac{2(\alpha, \gamma)}{(\alpha, \alpha)} \alpha) = \beta + c_\alpha \alpha$ so $c_\gamma = 0$ for $\gamma \neq \alpha$, ie $\beta = c_\alpha \alpha = \pm \alpha$ and hence as $\beta \in \Pi$ we have $\beta = \alpha$, contradiction $\square$

Theorem 2.6. $W(\Phi) = \langle s_\delta \mid \delta \in \Delta \rangle$

Proof. Let $W' = \langle s_\delta \mid \delta \in \Delta \rangle \leq W(\Phi)$. We will prove that for all $\beta \in \Phi$ there is some $\alpha \in \Delta$ and $w \in W'$ such that $w(\alpha) = \beta$. Thus $s_\beta = s_{w(\alpha)} = w s_\alpha w^{-1} \in W'$ then we will be done.

We may assume that $\beta \in \Pi$, since $s_\beta = s_{-\beta}$. It remains to prove $W'\beta \cap \Delta \neq \emptyset$. Clearly $W'\beta \cap \Pi \neq \emptyset$ as it contains β .

Let $m = \min\{\sum_{\gamma \in \Delta} c_\gamma \mid \sum_{\gamma \in \Delta} c_\gamma \gamma \in W'\beta \cap \Pi\} > 0$, Choose $\alpha \in W'\beta \cap \Pi$ such that $\alpha = \sum_{\gamma \in \Delta} c_\gamma \gamma$ with $c_\gamma \geq 0$ and $\sum_{\gamma \in \Delta} c_\gamma = m$. Now $\alpha \neq 0$ so $0 < (\alpha, \alpha) = \sum_{\gamma \in \Delta} c_\gamma (\gamma, \alpha)$ and therefore $(\delta, \alpha) > 0$ for some $\delta \in \Delta$. If $\delta = \alpha$ we are done. Assume for contradiction that $\alpha \neq \delta$. Therefore $\alpha \in \Pi \setminus \{\delta\}$ and so $s_\delta(\alpha) \in \Pi \setminus \{\delta\}$ by the preceeding lemma. But also $s_\delta \in W'$ so $s_\delta(\alpha) \in W'\beta \cap \Pi$. $s_\delta(\alpha) = \alpha - \frac{2(\alpha, \delta)}{(\delta, \delta)} \delta = \alpha - c\delta$ for some $c > 0$, and hence $s_\delta(\alpha)$ has a smaller sum of coefficients than α . Contradiction, so we are done $\square$

From now on, let $m_{\alpha,\beta}$ be the order of the isometry $s_\alpha s_\beta \in W(\Phi)$ for $\alpha, \beta \in \Delta$ and $M_\Delta = (m_{\alpha,\beta})$ which is a Coxeter matrix. Let $(\widetilde{W(\Delta)}, \Delta, M_\Delta)$ be the associated Coxeter system. There is clearly a surjection $\widetilde{W(\Delta)} \rightarrow W(\Phi)$ by $x_\alpha \mapsto s_\alpha$.

Proposition 2.3. *The Coxeter graph of $(\widetilde{W(\Delta)}, \Delta, M_\Delta)$ is equivalent to the Dynkin diagram of Δ*

Proof. (Sketch - exercise to fill in) $s_\alpha s_\beta$ 'is' rotation through $2\theta_{\alpha,\beta}$ and we have $\theta_{\alpha,\beta} = \pi - \frac{\pi}{n}$ for some $n \in \mathbb{N}$ and so the equivalence follows. $\square$

3 Coxeter Groups and Representations

Fix I an index set, M a Coxeter matrix and W the associated Coxeter group, so (W, I, M) is a Coxeter system

3.1 Geometric Representation

Definition 3.1. The *associated space* of M is the $\mathbb{R}$ -vector space denoted $V_M = \text{span}\{e_i \mid i \in I\}$. Define the *associated bilinear form* to be the bilinear form $(\cdot, \cdot)_M: V_M \times V_M \rightarrow \mathbb{R}$ given on the basis by $(e_i, e_j)_M = -\cos(\frac{\pi}{m_{i,j}})$ if $m_{i,j} < \infty$ and -1 otherwise. This is a symmetric bilinear form on V_M

Remark. The associated bilinear form need not be an inner product, even if I is finite, it need not be positive definite (or even positive semi-definite).

For $\alpha \in V_m$ such that $(\alpha, \alpha) \neq 0$, we can still define s_α . We know that

1. $(e_i, e_i)_M = 1$
2. $(e_i, e_j) = 0$ if and only if $m_{i,j} = 2$
3. $(e_i, e_j) = -1$ if and only if $m_{i,j} = \infty$

Definition 3.2. The *geometric representation* of W is the group homomorphism $\sigma_W: W \rightarrow GL(V_M)$ by $x_i \mapsto s_{e_i}$

Define $\Phi_M = \{\sigma_W(w)(e_i) \mid i \in I, w \in W\}$

Proposition 3.1. *If I is finite, W is finite and $(\cdot, \cdot)$ is an inner product on V_M . Then Φ_M is a root system in V_M*

Proof. $\emptyset \neq \Phi_M \leq V_M$ and $|\Phi_M| < \infty$ as $|I|, |W| < \infty$ and $0 \notin \Phi_M$. Given $\alpha, \beta \in \Phi_M$, write $\alpha = \sigma(w)(e_i), \beta = \sigma(w')(e_j)$.

1. $\Phi_M \cap \mathbb{R}\alpha = \{\pm\alpha\}$: If $\beta \in \Phi_M \cap \mathbb{R}\alpha, \beta = \lambda\alpha$ then $\sigma(w')(\lambda e_j) = \sigma(w)(e_i)$ so $\lambda e_j = \sigma(w'^{-1}w)(e_i)$ so by orthogonality, $|\lambda| = 1$. $-\alpha = \sigma(w)s_{e_i}(e_i) \in \Phi_M$ so we are done

2. $s_\alpha(\beta) \in \Phi_M$: $s_\alpha(\beta) = s_{\sigma(w)(e_i)}(\beta) = \sigma(w)s_{e_i}\sigma(w^{-1})(\beta) = \sigma(wx_iw^{-1})(\beta)$
and so $s_\alpha(\beta) = \sigma(wx_iw^{-1}w')(e_j) \in \Phi_M$ (the second equality needs it to be an inner product)

□

Choose a fundamental system $\Delta \subset \Phi_M$. We will prove $\widetilde{W(\Delta)} \cong W \cong W(\Phi)$ and so all finite Coxeter groups are finite reflection groups

3.2 The Length Function

Let (W, I, M) be a Coxeter system with generators $x_i, i \in I$

Definition 3.3. If $w \in W$, define the *length* of w , denoted $l(w)$, to be the smallest integer $m \geq 0$ such that there are $i_1, \dots, i_m \in I$ with $w = x_{i_1} \dots x_{i_m}$

Lemma 3.1. For all $w, w' \in W$ and $i \in I$

1. $l(w) = 0$ if and only if $w = 1$
2. $l(w) = l(w^{-1})$
3. $l(ww') \leq l(w) + l(w')$
4. $l(w) - 1 \leq l(wx_i) \leq l(w) + 1$

Proof. Exercise

□

It follows from this that we can define a metric on W via $d(w, w') = l(w^{-1}w')$ which gives W the discrete topology as it is integer valued

Proposition 3.2. There is a unique group homomorphism $\epsilon: W \rightarrow \mathbb{Z}/2\mathbb{Z}$ which sends each generator x_i to 1 mod 2. In fact $\epsilon(w) = l(w) \pmod{2}$

Proof. Uniqueness is trivial, by definition of what a generator is $W \cong \text{Free}\{y_i \mid i \in I\}/N(R)$ where $N(R) = \langle \langle (y_i y_j)^{m_{ij}} \mid i, j \in I \rangle \rangle$ and under this isomorphism, $x_i \mapsto y_i$. Define $\epsilon_0: \text{Free}\{y_i \mid i \in I\} \rightarrow \mathbb{Z}/2\mathbb{Z}$ by $y_i \mapsto 1$. This is a group homomorphism, and $\epsilon_0((y_i y_j)^{m_{ij}}) = m_{ij}(\epsilon_0(y_i) + \epsilon_0(y_j)) = 2m_{ij}$ and so the map from W to $\mathbb{Z}/2\mathbb{Z}$ is well defined and we are done. □

Example 3.1. If $W = D_{2n}$ then $\ker(\epsilon) = \{r^k \mid 0 \leq k < n\}$, and this extends to ∞ . If $W = S_n$ then $\ker(\epsilon) = A_n$

Corollary 3.1.1. If $w \in W, i \in I$ then $l(wx_i) \in \{l(w) \pm 1\}$

Proof. We know that $l(w) - 1 \leq l(wx_i) \leq l(w) + 1$ so we only need prove that $l(wx_i) \neq l(w)$. Now $\epsilon(wx_i) = \epsilon(w) + 1 \pmod{2}$ and so $l(wx_i) = l(w) + 1 \pmod{2}$ hence we are done. □

3.3 Faithful Representations

Let V_M be the associated space of the Coxeter matrix M and let $(\cdot, \cdot)_M$ be the associated bilinear form. Let $\sigma_W = \sigma: W \rightarrow GL(V_M)$ by $x_i \mapsto s_{e_i}$

Example 3.2. If $|I| = 2$ then $V_M = \mathbb{R}^2$, $M = \begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$ for $n \leq 2 \leq \infty$ and $(e_1, e_2) = -\cos(\frac{\pi}{n})$. If $n < \infty$, $\sigma(x_1 x_2) = s_{e_1} s_{e_2}$ is rotation by $\frac{2\pi}{n}$

Remark. If $v \in V_M$ we say $v > 0$ (resp $v < 0$) if, when $v = \sum_{i \in I} a_i v_i$ then $a_i > 0$ (resp $a_i < 0$) for all $i \in I$
For $w \in W, v \in V_M$ we write $w \cdot v = \sigma(w)(v)$

Theorem 3.2.

1. For all $w \in W, i \in I$, if $l(wx_i) > l(w)$ then $w \cdot e_i > 0$
2. For all $w \in W, i \in I$, if $l(wx_i) < l(w)$ then $w \cdot e_i < 0$
3. The map $\sigma: W \rightarrow GL(V_M)$ is a faithful representation

Proof. If $|I| \leq 2$, this is an exercise

If $|I| \geq 3$. Given $i, j \in I$ let $W_{ij} = \langle x_i, x_j \rangle \leq W$ and let

$$\widetilde{W}_{ij} = \langle y_1, y_2 \mid y_i^2, y_j^2, (y_i y_j)^{m_{ij}} \rangle$$

$\cdot (\widetilde{W}_{ij}, \{i, j\}, \begin{pmatrix} 1 & m_{ij} \\ m_{ij} & 1 \end{pmatrix})$ is a Coxeter system and there is a surjective map

$\pi: \widetilde{W}_{ij} \rightarrow W$ by $y_k \mapsto x_k$ for $k = i, j$ and $\sigma_{\widetilde{W}_{ij}} = \sigma_W|_{\widetilde{W}_{ij}} \circ \pi$ and by statement 3, $\sigma_{\widetilde{W}_{ij}}$ is faithful, so π is injective hence $\widetilde{W}_{ij} \cong W_{ij}$ as a Coxeter group with length function l_{ij} and $l_{ij}(w) \geq l(w)$ for all $w \in W_{ij}$.

To prove 3 from 1, 2, suppose there is $1 \neq w \in W$ and $\sigma(w) = \text{Id}$ so that $w \cdot v = v$ for all $v \in V$. But there is $i \in I$ such that $l(wx_i) < l(w)$ (like the last element of a minimal representation of w). Hence by 2, $w \cdot e_i < 0$ so $e_i = w \cdot e_i < 0$ contradiction.

To prove 2 from 1, suppose $l(wx_i) < l(w)$, so $l(wx_i) = l(w) - 1$. Let $w' = wx_i$ so $w = w'x_i$ and $l(w'x_i) = l(w) = l(w') + 1$ hence $w' \cdot e_i > 0$ so we have that $w \cdot e_i = w'x_i \cdot e_i = w' \cdot (x_i \cdot e_i) = -w'(e_i) < 0$

To prove statement 1, we use induction on $n = l(w)$.

If $w = 1$, then $l(wx_i) > l(w)$ for all $i \in I$ and $w \cdot e_i = e_i > 0$

Suppose $l(w) = n > 0$ and $l(wx_i) > l(w)$. $w \neq 1$ so there is $j \in I$ such that $l(wx_j) < l(w)$. Consider again W_{ij} with length function l_{ij} .

Define $A = \{w' \subset W \mid w'^{-1}w \in W_{ij}, l(w) = l(w') + l_{ij}(w'^{-1}w)\}$. This is non-empty as it contains $wx_j \in A$. Choose an element $w_0 \in A$ such that $l(w_0)$ is minimal and thus $l(w_0) \leq l(wx_j) = l(w) - 1 < l(w) = n$. We may therefore apply the inductive hypothesis. Now $v = w_0^{-1}w \in W_{ij}$ and $w = w_0 v$, $l(w) = l(w_0) + l_{ij}(v) \geq l(w_0) + l(v)$. We need to prove that all $l_{ij}(vx_i) > l_{ij}(v)$, $l(w_0 x_i) > l(w_0)$ and $l(w_0 x_j) > l(w_0)$ are true as it then follows that $ve_i > 0$

by induction and $w_0e_i, w_0e_j > 0$ so $ve_i = \alpha e_i + \beta e_j$ with $\alpha, \beta \geq 0$ and hence we know $we_i = w_0ve_i \geq 0$. First we aim to prove $l_{ij}(vx_i) > l_{ij}(v)$. Assume for contradiction that $l_{ij}(vx_i) < l_{ij}(v)$. Then

$$\begin{aligned} l(wx_i) &= l(w_0vx_i) \leq l(w_0) + l(vx_i) \\ &\leq l(w_0) + l_{ij}(vx_i) \\ &< l(w_0) + l_{ij}(v) = l(w) \end{aligned}$$

contradiction by definition of w .

For $k = i, j$ we want to prove $l(w_0x_k) > l(w_0)$. Assume the opposite. Then

$$\begin{aligned} l(w) &= l(w_0v) = l(w_0x_kx_kv) \leq l(w_0x_k) + l(x_kv) \leq l(w_0x_k) + l_{ij}(x_kv) \\ &= l(w_0) - 1 + l_{ij}(x_kv) \\ &\leq l(w_0) - 1 + l_{ij}(v) + 1 = l(w) \end{aligned}$$

so we have equality throughout, and hence $l(w_0x_k) + l(x_kv) = l(w)$ but by definition, $x_kv = x_kw_0^{-1}w = (w_0x_k)^{-1}w$ so $w_0x_k \in A$ with lower length than w_0 contradiction. $\square$

Corollary 3.2.1. *Let V be a finite dimensional inner product space, $\Phi \subset V$ be a root system and Δ a fundamental system in Φ , so that $W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$. Let $m_{\alpha\beta}$ be the order of the isometry $s_\alpha s_\beta$ where $\alpha, \beta \in \Phi$ and $M_\Delta = (m_{\alpha\beta})_{\alpha, \beta \in \Delta}$. Then $(W(\Phi), \Delta, M_\Delta)$ is a Coxeter system and so all finite reflection groups are Coxeter groups*

Proof. We know that $W(\Phi) = \langle s_\delta \mid \delta \in \Delta \rangle$ and $s_\delta = s_{\delta/\|\delta\|}$. For each $\delta \in \Delta$, let $e_\delta = \frac{\delta}{\|\delta\|}$ then $\{e_\delta \mid \delta \in \Delta\}$ is a basis for V and the inner product is given by $(e_\delta, e_\delta) = 1, (e_\delta, e_\gamma) = -\cos(\frac{\pi}{m_{\delta, \gamma}})$. Let $\widetilde{W(\Delta)}$ be the Coxeter group of M_Δ . We have that, by definition, $V_M = \text{Span}(e_\delta \mid \delta \in \Delta) \cong V$ and $(e_\delta, e_\gamma)_{M_\Delta} = (e_\delta, e_\gamma)$. We know that the geometric representation $\sigma_{\widetilde{W(\Delta)}}: \widetilde{W(\Delta)} \rightarrow GL(V)$ is faithful with image $\langle s_\delta \mid \delta \in V \rangle = W(\Phi)$ so $\widetilde{W(\Delta)} \cong W(\Phi)$. We know any finite reflection group has the form $W(\Phi)$ for some root system Φ and so we are done. $\square$

Corollary 3.2.2. *If $|W| < \infty$ and $(\cdot, \cdot)_M$ is positive definite then W is isomorphic to a finite reflection group*

Proof. $\sigma_W: W \rightarrow GL(V_M)$ is faithful with image $\langle s_\alpha \mid \alpha \in \Phi_M \rangle$. $\Phi_M \subset V_M$ is a root system and so $\sigma(W) \cong W(\Phi_M)$ $\square$

3.4 The Exchange Condition and the Deletion Condition

Given $w \in W$ we can write $w = x_{i_1} \dots x_{i_r}$ for $i_1, \dots, i_r \in I$. The expression is *reduced* if $r = l(w)$

Theorem 3.3. *Let $w = x_{i_1} \dots x_{i_r}$ and suppose that $k \in I$ and $l(wx_k) < l(w)$. Then there is $j \in \{1, \dots, r\}$ such that $wx_k = x_{i_1} \dots \widehat{x_{i_j}} \dots x_{i_r}$. If the expression for w is reduced, then this j is unique*

Proof. Set $x_j = x_{i_j}$ so that $w = w_1 \dots w_r$. Since $l(wx_k) < l(w)$ we know $w e_k < 0$. But $e_k > 0$ so there is $j \in \{1, \dots, r\}$ such that $x_{j+1} \dots x_r e_k > 0$ but $x_j \dots x_r e_k < 0$. Let $w' = x_{j+1} \dots x_r$, $\beta = w' e_k > 0$ and $x_j \beta < 0$. Now $\beta = \sum_{i \in I} \alpha_i e_i$ for $\alpha_i \geq 0$ and

$$\begin{aligned} x_j \beta &= \sigma(x_j)(\beta) = s_{e_j}(\beta) = \beta - 2(\beta, e_j)e_j \\ &= \sum_{i \neq j} \alpha_i e_i + (\alpha_j - 2(e_j, \beta))e_j \\ &< 0 \end{aligned}$$

so $\alpha_j \leq 0$ for all $j \neq i$ and hence $\beta = \alpha_j e_j$ and

$$(\beta, \beta)_M = (w' e_k, w' e_k) = (e_k, e_k)_M = 1$$

and also $(\beta, \beta)_M = \alpha_j^2$ so $\alpha_j = 1$ as it is positive by assumption so $\beta = e_j$ so that $w' e_k = e_j$. So

$$\begin{aligned} \sigma(w' x_k) &= \sigma(w') \sigma(e_k) \sigma(w'^{-1}) \sigma(w') \\ &= s_{\sigma(w') e_k} \sigma(w') = s_{e_j} \sigma(w') \\ &= \sigma(x_j w') \end{aligned}$$

but σ is faithful so $w' x_k = x_j w'$ so $x_j w' = x_j \dots x_r$ and

$$\begin{aligned} wx_k &= x_1 \dots x_r x_k \\ &= x_1 \dots x_j (x_{j+1} \dots x_r x_k) \\ &= x_1 \dots x_{j-1} x_{j+1} \dots x_r \end{aligned}$$

If $l(w) = r$, suppose $wx_k = x_1 \dots \widehat{x_j} \dots x_r = x_1 \dots \widehat{x_m} \dots x_r$ with $j < m$. We get that $x_{j+1} \dots x_m = x_j \dots x_{m-1}$. Now

$$\begin{aligned} w &= x_1 \dots x_r \\ &= x_1 \dots x_j x_{j+1} \dots x_m x_{m+1} \dots x_r \\ &= x_1 \dots x_j x_j \dots x_{m-1} x_{m+1} \dots x_r \end{aligned}$$

has length at most $r - 2 = l(w) - 2$, contradiction $\square$

Corollary 3.3.1. *(Deletion condition)*

1. *If $w = x_{i_1} \dots x_{i_r}$ and $l(w) < r$ then there are $k < j$ such that we have $w = x_{i_1} \dots \widehat{x_{i_k}} \dots \widehat{x_{i_j}} \dots x_{i_r}$.*
2. *A reduced expression for w can be obtained by removing successive pairs of generators and so $l(w) = r \pmod 2$*

Proof. (2) follows from (1)

To prove (1), for $j = 0, \dots, r$ define $w_j = x_{i_1} \dots x_{i_j}$ so $w_0 = 1$ and $w_r = w$. Let $j \geq 0$ be maximal such that $l(w_j) = j$, so that $l(w_{j+1}) < j+1$ and so we know $l(w_{j+1}) = j-1$ as $w_{j+1} = w_j x_{j+1}$. Therefore $w_{j+1} = w_j x_{j+1} = x_1 \dots \widehat{x_m} \dots x_j$ for some $m \leq j$ so

$$\begin{aligned} w &= x_1 \dots x_r \\ &= w_{j+1} x_{j+2} \dots x_r \\ &= x_1 \dots \widehat{x_m} \dots \widehat{w_{j+1}} \dots x_r \end{aligned}$$

and we are done $\square$

3.5 Braid Relations

Given $i, j \in I$, $x_i x_j$ has order m_{ij} so $x_i x_j \dots = x_j x_i \dots$ each side with m_{ij} terms. We call such a relation a *braid relation*

Example 3.3. If $m_{ij} = 2$ we get $x_i x_j = x_j x_i$

If $m_{ij} = 3$ we get $x_i x_j x_i = x_j x_i x_j$.

For general even m , we get $(x_i x_j)^{m_{ij}/2} = (x_j x_i)^{m_{ij}/2}$

We say that $(i_1, \dots, i_k) \sim_{be} (j_1, \dots, j_r)$ if $x_{i_1} \dots x_{i_k} = x_{j_1} \dots x_{j_r}$, and there is a sequence of braid relations connecting the

Example 3.4. Suppose $m_{ij} = 3, m_{jk} = 2$. Then $(i, j, i, k) \sim_{be} (j, i, k, j)$

Theorem 3.4. (*Matsumoto's Theorem*) If $l(w) = r, w = x_{i_1} \dots x_{i_r} = x_{j_1} \dots x_{j_r}$ then we have that $(i_1, \dots, i_r) \sim_{be} (j_1, \dots, j_r)$

Proof. If $r = 0, 1$ then $w = 1, x_i$ and these are the only reduced expressions, so we are done.

We induct on r . If $i_r = j_r$ then $w' = w x_{i_r} = x_{i_1} \dots x_{i_{r-1}} = x_{j_1} \dots x_{j_{r-1}}$ so $(i_1, \dots, i_{r-1}) \sim_{be} (j_1, \dots, j_{r-1})$ and we are done.

So assume $i_r \neq j_r$ and $l(w x_{i_r}), l(w x_{j_r}) < l(w) = r$. Choose $s \geq 1$ maximal such that $w = x_{m_1} \dots x_{m_{r-s}} \dots x_i x_j x_i$ where $i \neq j \in \{i_r, j_r\}$.

$l(w x_j) < l(w)$ so using the exchange condition, we either have

1. $w x_j = x_1 \dots \widehat{x_k} \dots x_{m-s} \dots x_i x_j x_i$
2. $w x_j = x_1 \dots x_{m-s} \dots x_j x_i x_j$
3. $w x_j = x_1 \dots x_{m-s} \dots x_i x_j x_i$ (where the one at the beginning has cancelled)

One in the middle of the last s can't cancel, as then the length would go down by at least 3 after canceling.

1. In the first case, we get $w = x_1 \dots \widehat{x_k} \dots x_{m-s} \dots x_i x_j x_i x_j$, contradicting the maximality of s

2. In the second case, $wx_j = x_1 \dots x_{m-s} \dots x_i x_j$ so we get an expression for w of length $r - 2$, namely $w = x_1 \dots x_{m-s} \dots x_i x_j x_i$, contradiction
3. In the third case, $wx_j = x_1 \dots x_{m-s} \dots x_i x_j x_i x_j$ which differs by a braid relation, as we may divide out by the first $m - s$ terms. We are therefore done as we can remove the x_i after applying a braid relation

□

Corollary 3.4.1. *If $\Gamma(W, I, M)$ contains a cycle, then W is infinite. So for finite W , $\Gamma(W)$ is a forest*

Proof. All subscripts in this proof are taken mod n . Suppose we have a cycle of minimal length in the Coxeter graph, with vertices $i_1, \dots, i_n$. Let $x_j = x_{i_j}$. If $i - j \not\equiv \pm 1 \pmod 2$ then $m_{ij} = 2$ so $x_i x_j = x_j x_i$. For each $1 \leq k \leq n$ set

$$w_0^{(k)} = 1, w_1^{(k)} = x_k, w_2^{(k)} = x_k x_{k+1}, \dots, w_j^{(k)} = w_{j-1}^{(k)} x_{k+j-1}$$

. Assume for contradiction that W is finite. Then there is $j \in \mathbb{N}$ which is minimal such that $l(w_j^{(k)}) < j$ for some k . Without loss of generality, $k = 1$ and denote $w_j = w_j^{(1)}$.

We have that $l(w_{j-1}) = j - 1$, $l(w_j) = l(w_{j-1} x_j) = l(w_{j-1}) - 1 = j - 2$ and we know that $l(w_{j-1} x_j) = l(w_{j-1})$. $w_{j-1} = x_1 \dots x_{j-1}$ so by the exchange condition, $w_{j-1} x_j = x_1 \dots \hat{x}_k \dots x_{j-1}$. If $k > 1$, we therefore have

$$w_{j-k}^{(k+1)} = x_{k+1} \dots x_{j-1} = x_{k-1} \dots x_1 w_{j-1} x_j$$

so

$$w_{j-k}^{(k+1)} x_j = x_{k+1} \dots x_{j-1} = x_{k+1} \dots x_{j-1}$$

so $l(w_{j-k}^{(k+1)}) < j - r + 1$, contradiction.

So $k = 1$ and $w_{j-1} x_j = x_2 \dots x_{j-1} = x_1 w_{j-1}$ and $w_{j-1} = x_1 \dots x_{j-1} = x_2 \dots x_j$ are reduced expressions for w_{j-1} . So by Matsumoto's theorem,

$$(1, 2, \dots, j-1) \sim_{be} (2, 3, \dots, j)$$

. But these don't contain any braid relations, contradiction so we are done. □

4 Inner Products and Finite Coxeter Groups

Let (W, I, M) be a Coxeter system.

Definition 4.1. The system is *irreducible* if $\Gamma(W)$ is connected.

From now on, we assume I is finite, so that $I = I_1 \sqcup \dots \sqcup I_r$ where each I_j is a connected component of the Coxeter graph. If we set $W_j = \langle x_i \mid i \in I_j \rangle$ then W_j is an irreducible Coxeter and $W = W_1 \times \dots \times W_r$. Throughout, let $\sigma: W \rightarrow GL(V_M)$ be the geometric representation

4.1 Subrepresentations of (V_M, σ)

Definition 4.2. Let G be a group and $\rho: G \rightarrow GL(V)$ be a representation.

1. We say the representation (V, ρ) is *irreducible* if it contains no proper non-zero subrepresentations
2. The representation (V, ρ) is *completely reducible* if there are irreducible subrepresentations $U_1, \dots, U_m$ such that $V = \bigoplus_{i=1}^m U_i$
3. The dual representation of (V, ρ) , denoted (V^*, ρ^*) , has the underlying space $V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ and representation $\rho^*(g)(f)(u) = f(\rho(g^{-1})u)$

Theorem 4.1. (*Maschke's Theorem*) If G is a finite group and (V, ρ) is a finite dimensional representation of G then V is completely reducible

Proof. See II Representation Theory □

Let $V = V_M, \sigma = \sigma_w$ be the geometric representation, $I = I_1 \sqcup \dots \sqcup I_r$ and $V_j = \text{span}\{e_i \mid i \in I_j\}$ and $V = \bigoplus_{i=1}^r V_r$ as vector spaces

Lemma 4.2. Each V_j is a subrepresentation of V , and W_k acts trivially on V_j for $k \neq j$, and moreover V_j is the geometric representation of W_j

Proof. If $i \in I$ and $u \in V_j$ then $x_i u = u - 2(e_i, u)e_i$ so if $i \notin I_j$ then $i \in I_k$ for some $k \neq j$ and $(e_i, e_m) = 0$ for all $m \in I_j$, and so $(e_i, u) = 0$ by linearity, so $x_i u = u$ so w_i acts trivially on V_j . If $i \in I_j$ then $x_i u = u - w(e_i, u)e_i \in V_j$ so $V_j \leq V$ is a subrepresentation. Clearly if V_{m_j} is the geometric representation of W_j then $V_{m_j} \cong V_j$ as vector spaces, and by the formula before, the actions coincide □

Lemma 4.3. (V, σ) is completely reducible if and only if each V_j is completely reducible

Proof. Trivial □

Recall that $V^\perp = \{v \in V \mid (v, u)_M = 0, \forall u \in V\}$ and $(\cdot, \cdot)_M$ is non-degenerate if $V^\perp = 0$

Proposition 4.1. We have the following facts

1. $V^\perp$ is a W -subrepresentation of V and $V^\perp = V_1^\perp \oplus \dots \oplus V_r^\perp$
2. If W is already irreducible, then any proper subrepresentation of V is contained in $V^\perp$
3. V is completely reducible if and only if $(\cdot, \cdot)_M$ is non-degenerate
4. If W is finite then $(\cdot, \cdot)_M$ is non-degenerate

Proof. 1. If $v \in V^\perp$ then $x_i v = v \in V^\perp$ so $V^\perp$ is a subrepresentation. If $v \in V^\perp$, $v = v_1 + \dots + v_r$ with $v_j \in V_j$ and $i \in I_j$ then $0 = (e_i, v) = (e_i, v_j)$ so $v_j \in V_j^\perp$

2. As (W, I, M) is irreducible, $\Gamma(W)$ is connected. Suppose $u \in U \leq V$ and $i \in I$. Then $x_i u \in U$ so $-2(e_i, u)e_i \in U$ and so for all $i \in I$, $e_i \in U$ or $(e_i, u) = 0$ for all $u \in U$. If for all $i \in I$, we have $e_i \notin U$ then $(e_i, u) = 0$ for all $i \in I, u \in U$ so $U \subset V^\perp$. Assume that $e_i \in U$ for some $i \in I$. Let j be such that $m_{ij} > 2$. Then $(e_i, e_j) \neq 0$ but $x_j e_i = e_i - 2(e_i, e_j)e_j$ and so $e_j \in U$. But $\Gamma(I)$ is connected, so all $e_j, j \in I$ are in U , hence $U = V$
3. V is completely reducible if and only if each V_j is completely reducible, if and only if every V_j is irreducible (by 2), if and only if each $V_j^\perp = 0$ if and only if $V^\perp = 0$ if and only if $(\cdot, \cdot)_M$ is non-degenerate
4. If W is finite, V is completely reducible by Maschke, so it follows by 3 $\square$

Lemma 4.4. *If W is irreducible, $\phi \in \text{End}_{\mathbb{R}}(V)$ and ϕ commutes with everything in $\sigma(W)$, then $\phi = \lambda \text{Id}_V$ for some $\lambda \in \mathbb{R}$*

Proof. Since $\phi\sigma(w) = \sigma(w)\phi$, $\phi(wv) = w\phi(v)$ and so ϕ is a homomorphism of W representations. Given $i \in I$,

$$-\phi(e_i) = \phi(-e_i) = \phi(s_{e_i}e_i) = x_i\phi(e_i) = s_{e_i}\phi(e_i)$$

and hence $\phi(e_i)$ is in the -1 eigenspace of s_{e_i} , and hence $\phi(e_i) = \lambda_i e_i$. Fix i and let $\lambda = \lambda_i$. If we let $\theta = \phi - \lambda \text{Id}$. Then θ is a W module homomorphism. Let $U = \ker \theta \leq V$ so $U = V$ or $U \leq V^\perp$. If $U = V$ then we are done. Assume $U \leq V^\perp$. But $\phi(e_i) = \lambda_i e_i$ so $\theta(e_i) = 0$ so $e_i \in U \subset V^\perp$ so $0 = (e_i, e_i) = 1$ contradiction $\square$

Remark. $(\cdot, \cdot)_M$ is W -invariant: $(wu, wv)_M = (u, v)_M$ for all $w \in W$

Proposition 4.2. *Suppose W is irreducible and $\beta: V \times V \rightarrow \mathbb{R}$ is symmetric, non-degenerate, W -invariant and bilinear. Then*

1. $\phi_\beta: V \rightarrow V^*$ by $v \mapsto \beta(v, \cdot)$ is an isomorphism of W -representations
2. If $(\cdot, \cdot)_M$ is non-degenerate then $\beta = \lambda(\cdot, \cdot)_M$ for some $0 \neq \lambda \in \mathbb{R}$

Proof. (1 implies 2) Let $\gamma = (\cdot, \cdot)_M$. Then $\phi_\gamma, \phi_\beta: V \rightarrow V^*$ are isomorphisms so $\phi = \phi_\gamma^{-1}\phi_\beta \in \text{End}(V)$ and $\phi(wv) = w\phi(v)$ so by lemma 4.4, $\phi = \lambda \text{Id}$ so $\phi_\beta = \lambda\phi_\gamma$ so $\lambda\gamma(u, v) = \lambda\phi_\gamma(u)(v) = \phi_\beta(u)(v) = \beta(u, v)$

(1) $\phi_\beta(wv)(u) = \beta(wv, u) = \beta(v, w^{-1}u) = (w\phi_\beta(v))(u)$ so ϕ_β is a homomorphism of W representations. If $\phi_\beta(v) = 0$ then $\beta(v, u) = 0$ for all u so $v \in V^\perp = 0$ so ϕ_β is injective, so it is surjective for dimension reasons $\square$

Theorem 4.5. *If W is finite then $(\cdot, \cdot)_M$ is positive definite*

Proof. Since $V = V_1 \oplus \cdots \oplus V_r$ is an orthogonal decomposition, it suffices to prove it for each component, and hence we may assume W is irreducible. We will show that there is a symmetric W -invariant positive definite bilinear form $\beta: V \times V \rightarrow \mathbb{R}$. If this holds, by proposition 4.2, because W is finite and hence

$(\cdot, \cdot)_M$ is non-degenerate, $\beta = \lambda(\cdot, \cdot)_M$ for $0 \neq \lambda \in \mathbb{R}$. If $\lambda < 0$, we get $\beta(e_i, e_i) = \lambda(e_i, e_i)_M = \lambda < 0$ contradiction so $\lambda > 0$ and we will be done (as positive scalar multiples of positive definite forms are easily seen to be positive definite). Let $\beta_0: V \times V \rightarrow \mathbb{R}$ be the standard bilinear form $(e_i, e_j) \mapsto \delta_{ij}$. This is symmetric positive definite. Define $\beta: V \times V \rightarrow \mathbb{R}$ by $(u, v) \mapsto \sum_{w \in W} \beta_0(wu, wv)$. This is W -invariant, symmetric and positive definite so we have proven the claim and are done. $\square$

Corollary 4.5.1. *If (W, I, M) is a Coxeter system and W is finite then W is naturally isomorphic to a finite reflection group via the geometric representation*

Proof. Follows from theorem 4.5 and corollary 3.2.2 $\square$

4.2 Classification of Coxeter Groups of Positive Type

Definition 4.3. We say that (W, I, M) has *positive type* if $(\cdot, \cdot)_M$ is positive semi-definite. W is *positive* if $(\cdot, \cdot)_M$ is positive definite

The classification of graphs $\Gamma(W)$ for W a positive Coxeter group can be given as follows

1. Type $A_n, n \geq 1$ $1 \text{ --- } 2 \text{ --- } \dots \text{ --- } n-1 \text{ --- } n$ $W = S_{n+1}$

2. Type $B_n, n \geq 2$ $1 \text{ ---}^4 2 \text{ --- } \dots \text{ --- } n-1 \text{ --- } n$ $W = SS_n$

3. Type $D_n, n \geq 4$

$$\begin{array}{c}
 1 \\
 \diagdown \\
 2 \text{ --- } \dots \text{ --- } n-1 \\
 \diagup \\
 -1
 \end{array}
 \quad W = ESS_n$$

4. Type E_6

$$\begin{array}{c}
 \bullet \\
 | \\
 \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet
 \end{array}$$

5. Type E_7

$$\begin{array}{c}
 \bullet \\
 | \\
 \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet
 \end{array}$$

6. Type E_8

$$\begin{array}{c}
 \bullet \\
 | \\
 \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet
 \end{array}$$

7. Type F_4 $\bullet \text{ --- } \bullet \text{ ---}^4 \bullet \text{ --- } \bullet$

8. Type H_3 $\bullet \xrightarrow{5} \bullet \text{---} \bullet$ $W = \text{icosahedronal symmetry group}$
9. Type H_4 $\bullet \xrightarrow{5} \bullet \text{---} \bullet \text{---} \bullet$
10. Type $I_2(m), m \geq 3$ $\bullet \xrightarrow{m} \bullet$

Remark. We have $I_2(3) = A_2, I_2(4) = B_2, I_2(5) = H_2, I_2(6) = G_2$ where the latter two are definitions/natural extensions.

The only rank 3 symmetry groups are A_3, B_3, H_3 which are the symmetry groups of the platonic solids

Remark. Suppose $I = \{1, \dots, n\}$ so that $V = \mathbb{R}^n$ and $(\cdot, \cdot)_M$ is defined on $\mathbb{R}^n$ by $(u, v)_M = u^T A v$ where $A \in M_n(\mathbb{R})$ has $A_{ij} = -\cos \frac{\pi}{m_{ij}}$. If (W, I, M) is irreducible then A is *indecomposable*: there are no $J, K \subset I$ such that $a_{jk} = 0$ for all $j \in J, k \in K$

Proposition 4.3. *If $A \in M_n(\mathbb{R})$ is symmetric, indecomposable, $A_{ij} < 0$ for $i \neq j$ and $u^T A u \geq 0$ for all $u \in \mathbb{R}^n$ then $N = \{v \in \mathbb{R}^n \mid Av = 0\}$ is equal to $\{v \in \mathbb{R}^n \mid v^T A v = 0\}$ and either $N = 0$ or $N = \langle z \rangle$ such that $z_i > 0$ for all $1 \leq i \leq n$*

Proof. Linear Algebra Exercise □

Corollary 4.5.2. *If (W, I, M) is irreducible of positive type then for all $j \in I$, if we let $W_j = \langle x_i \mid i \neq j \rangle$ then W_j is a positive Coxeter group, that is $W = W_{j,1} \times \dots \times W_{j,m}$ where $W_{j,i}$ is one of the irreducible positive groups for all $1 \leq i \leq m$. In other words, if we remove any vertex from $\Gamma(W)$ we get a disjoint union of $A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_2(m)$*

Proof. $N = \{v \in V \mid (v, v)_M = 0\}$ is a subspace of dimension 0, 1 by 4.3 so $N = 0$ and $(\cdot, \cdot)_M$ is positive definite or $N = \langle \alpha_1 e_1 + \dots + \alpha_n e_n \rangle, \alpha_i > 0$ so removing any e_i makes N become 0 □

4.3 The Coxeter Complex

Let $(V, (\cdot, \cdot))$ be a finite dimensional inner product space. Let $\mathcal{H}$ be a set of affine hyperplanes in V and let $s_H \in \mathcal{I}(V)$ be the reflection through the hyperplane $H \in \mathcal{H}$. That is, if $H = T(H_0)$ for $T = T_{\alpha_H}$ a translation and the linear hyperplane $H_0 = \{v \in V \mid (v, u_H) = 0\}$ for u_H normal to H . Therefore $H = \{v \in V \mid (v, u_H) = (u_H, \alpha_H) = \lambda_H\}$. For each $H \in \mathcal{H}$ we can divide V into half-spaces $A_{1,H} = \{v \in V \mid (v, u_H) > \lambda_H\}$, $A_{-1,H} = \{v \in V \mid (v, u_H) < \lambda_H\}$ and $V = H \sqcup A_{1,H} \sqcup A_{-1,H}$

Remark. We may assume $\|u_H\| = 1$ and replacing u_H with $-u_H$ swaps the upper and lower half-spaces

Let $W(\mathcal{H}) = \langle s_H \mid H \in \mathcal{H} \rangle \leq \mathcal{I}(V)$ be the reflection group of $\mathcal{H}$. If $\mathcal{H}' = \{w(H) \mid w \in W(\mathcal{H}), H \in \mathcal{H}\}$ then $W(\mathcal{H}) = W(\mathcal{H}')$ as $s_{w(H)} = w s_H w^{-1}$ so we may assume $\mathcal{H}$ is $W(\mathcal{H})$ invariant

Definition 4.4. We say that $\mathcal{H}$ is *locally finite* if there is some R such that for all $V \in V$, $B(v, R)$ intersects finitely many $H \in \mathcal{H}$.

$W \leq \mathcal{I}(V)$ is an *affine reflection group* if $W = W(\mathcal{H})$ for some locally finite W -invariant set of affine hyperplanes $\mathcal{H}$

Fix $W = W(\mathcal{H})$ an affine reflection group and let $X = V \setminus \bigcup_{H \in \mathcal{H}} H$. We define an equivalence relation $\equiv$ on X by $x \equiv y$ if for all $H \in \mathcal{H}$ either $x, y \in A_{1,H}$ or $x, y \in A_{-1,H}$.

Definition 4.5. Let $\mathcal{C}$ be the set of all equivalence classes of X under $\equiv$. We call the elements of $\mathcal{C}$ *chambers* and for each chamber $C \in \mathcal{C}$ and $H \in \mathcal{H}$ we say H is a *wall* of C if $\bar{C} \cap H = \bar{U}$ for some open subset U of H and we define $\text{wall}(C)$ be the set of all walls of C .

Given chambers $C \neq D \in \mathcal{C}$, we say they are *adjacent* if there is some hyperplane $H \in \text{wall}(C) \cap \text{wall}(D)$ such that $\bar{C} \cap H = \bar{D} \cap H$ and we write $C \sim D$ or $C \sim_H D$. We call $\mathcal{C}$ the Coxeter complex

Example 4.1. If $V = \mathbb{R}$, $\mathcal{H} = \frac{1}{2}\mathbb{Z}$ then $W(\mathcal{H}) = D_\infty$. If $C = (-1/2, 0)$ and $D = (0, 1/2)$ then C, D are chambers and $C \sim_0 D$

If $V = \mathbb{R}^2$ and $\mathcal{H}$ is the axes and the lines at 45 degrees to them, then the set $C = \{0 < x < y\}$ is a chamber and is adjacent to $D = \{0 < y < x\}$

Proposition 4.4. 1. For all $C \in \mathcal{C}$, $\text{wall}(C)$ is finite and C is an intersection of half spaces corresponding to hyperplanes in $\text{wall}(C)$ and so chambers are open in V

2. For all $C \in \mathcal{C}$ and $H \in \text{wall}(C)$ there is a unique $D \in \mathcal{C}$ such that $C \sim_H D$

3. For all $C, D \in \mathcal{C}$ there is a gallery $C = C_0 \sim C_1 \sim \dots \sim C_n = D$

Proof. 2, 3 are exercises.

For 1, for all $C \in \mathcal{C}$ and $H \in \mathcal{H}$ either $C \subset A_{1,H}$ or $C \subset A_{-1,H}$ as $C \cap H = \emptyset$ for all $H \in \mathcal{H}$, so define the two sets $\mathcal{H}_{\pm 1} = \{H \in \mathcal{H} \mid C \subset A_{\pm 1,H}\}$. Then $C = \bigcap_{H \in \mathcal{H}_1} A_{1,H} \cap \bigcap_{H \in \mathcal{H}_{-1}} A_{-1,H}$. Now $\mathcal{H}_1 \sqcup \mathcal{H}_{-1} = \mathcal{H}$. If $H, H' \in \mathcal{H}'$ and $u_H = u_{H'}$ then if $\lambda_H \geq \lambda_{H'}$ then $A_{1,H} \subset A_{1,H'}$ and similarly the other way around. Fix $H \in \mathcal{H}_1$ and let $\lambda^{(H)} = \sup\{\lambda_{H'} \mid H' \in \mathcal{H}_1, u_H = u_{H'}\}$. This is finite as if $\lambda^{(H)} = \infty$ then $C \subset \bigcap_{H' \in \mathcal{H}_1, u_{H'} = u_H} A_{1,H} = \emptyset$ contradiction. Moreover, $\lambda^{(H)} = \lambda_{H'}$ for some $H' \in \mathcal{H}_1$ with $u_{H'} = u_H$ by local finiteness. We may assume that $\lambda_H = \max\{\lambda_{H'} \mid u_H = u_{H'}\}$ and $\bigcap_{H' \in \mathcal{H}, u_{H'} = u_H} A_{1,H'} = A_{1,H}$. Let

$$\mathcal{H}'_1 = \{H \in \mathcal{H}_1 \mid \lambda_H = \max\{\lambda_{H'} \mid u_H = u_{H'}\}\}$$

and

$$\mathcal{H}'_{-1} = \{H \in \mathcal{H}_{-1} \mid \lambda_H = \min\{\lambda_{H'} \mid u_H = u_{H'}\}\}$$

. Then $C = \bigcap_{H \in \mathcal{H}'_1} A_{1,H} \cap \bigcap_{H \in \mathcal{H}'_{-1}} A_{-1,H}$. But $\{u_H \mid H \in \mathcal{H}\}$ is a finite set by local finiteness so $\mathcal{H}'_1, \mathcal{H}'_{-1}$ are finite sets. If $H \in \text{wall}(C) \cap \mathcal{H}_1$ then if $\lambda_H < \lambda_{H'}$ for some $H' \in \mathcal{H}_1$ parallel to H with $\lambda_{H'} = \lambda_H + \epsilon$, then $C \subset A_{1,H'}$ and so $(v, u_H) = (v, u_{H'}) > \lambda_{H'} = \lambda_H + \epsilon$ so for all $v \in \bar{C}$, $(v, u_H) \geq \lambda_H + \epsilon > \lambda_H$ so

$\bar{C} \cap H = \emptyset$ and so $\lambda_H = \max\{\lambda_{H'} \mid u_H = u_{H'}, H' \in \mathcal{H}_1\}$ so $\text{wall}(C) \cap \mathcal{H}_1 \subset \mathcal{H}'_1$ and symmetrically the same holds for -1 so

$$\begin{aligned} C &\subset \bigcap_{H \in \text{wall}(C) \cap \mathcal{H}'_1} A_{1,H} \cap \bigcap_{H \in \text{wall}(C) \cap \mathcal{H}'_{-1}} A_{-1,H} \\ &\subset \bigcap_{H \in \mathcal{H}'_1} A_{1,H} \cap \bigcap_{H \in \mathcal{H}'_{-1}} A_{-1,H} \\ &= C \end{aligned}$$

□

Remark. By flipping some unit vectors (ie replacing u_H with $-u_H$), we may in fact assume that for some $C \in \mathcal{C}$, all the chambers in its intersection are upper half chambers. We call such a chamber a *fundamental chamber*

Note that if $u, v \in X = V \setminus \bigcup_{H \in \mathcal{H}} H$ and $u \equiv v$, given $w \in W = W(\mathcal{H})$, $w(u), w(v)$ are in the same half space for any given hyperplane as $w\mathcal{H} = \mathcal{H}$ and so $w(u) \equiv w(v)$. Thus $W \curvearrowright \mathcal{C}$ via $w \cdot C = w(C)$

Proposition 4.5. *Let $C_0 \in \mathcal{C}$ be a fundamental chamber and $I = \text{wall}(C_0)$. Let $W' = \langle s_H \mid H \in I \rangle \leq W$*

1. W' acts transitively on the set of chambers C
2. $\text{Stab}_{W'}(C) = \{1\}$ for all $C \in \mathcal{C}$

Proof. 1. For all $C \in \mathcal{C}$ and $H \in \text{wall}(C)$, we know $C \sim_H s_H C$. Therefore if $C = wC_0$ and $H \in I$, then $\overline{s_H C_0} \cap H = \bar{C}_0 \cap H = \bar{U}$ for some non-empty open $U \subset H$. So we have

$$\begin{aligned} \overline{ws_H C_0} \cap wH &= ws_H \bar{C}_0 \cap wH \\ &= w(\overline{s_H C_0} \cap H) \\ &= w(\bar{C}_0 \cap H) \\ &= \overline{wC_0} \cap wH \\ &= \bar{C} \cap wH \\ &= \overline{w(U)} \end{aligned}$$

further we have that

$$\begin{aligned} \bar{C} \cap wH &= \overline{wC_0} \cap wH \\ &= \overline{ws_H C_0} \cap wH \\ &= \overline{wU} \end{aligned}$$

and so $wC_0 \sim_{wH} ws_H C_0$ and $wH \in \text{wall}(wC_0)$. Now we know w is an isomorphism from walls of C_0 to walls of wC_0 , so for all $C \in \mathcal{C}$ there is a gallery $C_0 \sim C_1 \sim \dots \sim C_m = C$.

We proceed by induction, with $m = 0$ clear as $\text{Id} \in W$. Assume by induction $C_i = w_i C_0$ for some $w_i \in W'$. Now $w_i C_0 = C_i \sim_{H_i} C_{i+1}$ where $H_i \in \text{wall } w_i C_0 = w_i \text{wall } C_0$ and so $H_i = w_i H'$ for some $H' \in I$. Now $w_i C_0 \sim_{w_i H'} w_i s_{H'} C_0$ so by uniqueness of $w_i H'$ adjacency, we have that $w_i s_{H'} C_0 = C_{i+1}$ and we are done

2. Suppose $w \in W'$ and $w C_0 = C_0$. It suffices to prove $w = \text{Id}$. Suppose $w = s_1 \dots s_d$ with $s_i = s_{H_i}, H_i \in I$ with d minimal and suppose for contradiction that $d \geq 1$. Let $w_i = s_1 \dots s_i$ and we have a gallery of the form $C_0 \sim w_1 C_1 \sim \dots \sim w_d C_0 = C_0$. Now $w_{i+1} C_0 = w_i s_{i+1} C_0 \sim_{w_i H_{i+1}} w_i C_0$. But if $C \sim_H D$ and $C \subset A_{-1, H'}$ then either $H = H'$ or $D \subset A_{-1, H'}$ (exercise). Since $C_0 \subset A_{1, H_1}$ and $w_1 C_0 = s_{H_1} C_0 \in A_{-1, H}$ we have for some $i > 1, w_i H_{i+1} = H_1$. Hence $s_{H_1} = s_{w_i H_{i+1}} = w_i s_{H_{i+1}} w_i^{-1}$ and so we have $s_1 w_i = w_i s_{i+1}$ as $s_2 \dots s_i = s_1 \dots s_{i+1}$ and so we can rewrite w removing s_1, s_{i+1} contradiction by minimality of d

□

Theorem 4.6. *If C_0 is a fundamental chamber and $I = \text{wall } C_0$ then $W = W(\mathcal{H}) = \langle s_H \mid H \in I \rangle$*

Proof. We will show that for all $H \in \mathcal{H}$, that H is the wall of some $C \in \mathcal{C}$. We then have $C = w C_0$ for some $w \in W' = \langle s_H \mid H \in I \rangle$. Therefore we will have $H = w H'$ for some $H' \in \text{wall } C_0 = I$ so $s_H = s_{w H'} = w s_{H'} w^{-1} \in W'$ so that $W = W'$. Define $\equiv_H$ on $H \setminus \bigcup_{H' \in \mathcal{H}, H' \neq H} H' \cap H$ by $x \equiv_H y$ if for all $H' \neq H$, $x, y \in A_{1, H}$ or $x, y \in A_{-1, H}$. If U is an equivalence class of this relation then $U = A_{1, H} \cap \dots \cap A_{1, H_r} \cap A_{-1, H_{r+1}} \cap \dots \cap A_{-1, H_m} \cap H$. So $\emptyset \neq U \subset H$ is open. Let S be the set of all half spaces in V which contain U and let $C = \bigcap_{A \in S} A \cap A_{1, H}$. We know that $U = \bigcap_{A \in S} A \cap H$ so $\bar{C} \cap H = \bar{U}$. If $u \neq v \in C$ then there is H such that $C \not\subset A_{1, H'}, A_{-1, H'}$ so both are in S and $H' \cap U \neq \emptyset$ contradiction. So $u \equiv v$ for all $u, v \in C$ and if $v \in V \setminus \bigcup_{H' \in \mathcal{H}} H'$, $u \equiv v$ for some $u \in C$ then $\{v\} \cup C \subset A_{1, H'}$ or $A_{-1, H'}$ for all $H' \in \mathcal{H}$ and hence $v \in \bigcap_{A \in S} A \cap A_{1, H} = C$ so $C \in \mathcal{C}$

□

4.4 Fundamental Domains

Definition 4.6. Let G be a group acting on a set X . We say $D \subseteq X$ is a *fundamental domain* for the action of G on X if for all $x \in X$, $|\text{Orb}_G x \cap D| = 1$

Lemma 4.7. 1. *If D is a fundamental domain for the action of G on X then for all $g \in G$, $gD \cap D = \{x \in D \mid gx = x\}$*

2. *If $\mathcal{X} = \{gD \mid g \in G\}$ then G acts transitively on $\mathcal{X}$ and we have that $\ker(G \curvearrowright \mathcal{X}) = \ker(G \curvearrowright X)$*

Proof. Trivial

□

Example 4.2. If $W = W(\mathcal{H})$ for $\mathcal{H}$ a locally finite W -invariant set of hyperplanes, and $\mathcal{C}$ is the Coxeter complex with fundamental chamber $C_0 \in \mathcal{C}$, then $D = \bar{C}_0$ is a fundamental domain for $W \curvearrowright V$ (proof later)

Let V be an inner product space. $\mathcal{I}(V)$ can be given a topology with open subsets $C(K, U) = \{f \in \mathcal{I}(V) \mid f(K) \leq U\}$ where $K, U \subseteq V$ are compact and open respectively. This is the *compact-open* topology. Under the isomorphism $O(V) \cong O_n(\mathbb{R})$, the topology on $O(V) \leq \mathcal{I}(V)$ induced by d agrees with the matrix topology.

A subset $X \subseteq \mathcal{I}(V)$ is *discrete* if for all $x \in X$, there is an $\epsilon > 0$ such that $B(x, \epsilon) \cap X = \{x\}$

Proposition 4.6. *If $G \leq \mathcal{I}(V)$ and $C \subseteq V$ is open such that $D = \bar{C}$ is a fundamental domain for the action of G on $\bigcup_{g \in G} gD$, then G is discrete*

Proof. For $v \in C$, define $\theta_v: \mathcal{I}(V) \rightarrow V$ by $f \mapsto f(v)$. This is continuous, and so $C_v = \theta_v^{-1}(C)$ is an open subset of $\mathcal{I}(V)$. If $g \in C_v \cap G$, then $g(v) \in C \cap gC$. But $C \cap gC \subseteq D \cap gD = \{x \in D \mid gx = x\}$ and so $C \cap gC = \{x \in C \mid gx = x\}$ is an open subset of V , but also $S = \{x \in V \mid gx = x\}$ is a closed subset of V and in fact since $\mathcal{I}(V) = \langle O(V), T(V) \rangle$, it follows that S is isometric to a linear subspace, and $C \cap S \subseteq V$ is open, hence $C \leq S = V$ and so $gx = x$ for all $x \in V$ and so g is the identity. Hence $C_v \cap G = \{\text{Id}\}$, but C_v is open and we can translate it to get such opens for the other points of G and so we are done $\square$

4.5 Finite Coxeter Groups

Let (W, I, M) be a Coxeter system with $|I| < \infty$ such that $(\cdot, \cdot)_M = (\cdot, \cdot)$ is positive definite and let $\sigma: W \rightarrow GL(V_M)$ be the geometric representation, and $V_M = V$ is a finite dimensional inner product space.

For each $i \in I$, let $H_i = \{x \in V \mid (v, e_i) = 0\}$ and $A_i = \{v \in V \mid (v, e_i) > 0\}$, so $-A_i = \{v \in V \mid (v, e_i) < 0\}$. Let $C = \bigcap_{i \in I} A_i$, which is an open subset of V and let $D = \bar{C}$

Lemma 4.8. *For all $w \in W, i \in I$ we have*

1. *If $l(x_i w) > l(w)$ then $w \cdot C \subset A_i$*
2. *If $l(x_i w) < l(w)$ then $w \cdot C \subset -A_i$*

Proof. 1 implies 2: If $l(x_i w) < l(w)$, let $w' = x_i w$ and then $l(x_i w') > l(w)$ and so $w' C \subseteq A_i$ and so $w C = x_i w' C \subseteq x_i A_i \subseteq -A_i$

1: If $l(x_i w) > l(w)$ then $l(w^{-1} x_i) > l(w^{-1})$ and so $w^{-1} e_i > 0$. This means that $w^{-1} e_i = \sum_{j \in I} \alpha_j e_j$ where $\alpha_j \geq 0$ for all $j \in I$. So if $v \in C = \bigcap_{i \in I} A_i$ then $(e_i, v) > 0$ for all $j \in I$. We therefore know that

$$(wv, e_i) = \left(v, \sum_{j \in I} \alpha_j e_j \right) = \sum_{i \in J} \alpha_j (v, e_j) > 0$$

so $v \in A_i$ $\square$

Theorem 4.9. *$D = \bar{C}$ is a fundamental domain for $W \curvearrowright \bigcup_{w \in W} wD$*

Proof. First note that $D = \{v \in V \mid (v, e_i) \geq 0\}$

If $wC = C$ then $wC \subseteq A_i$ for all $i \in I$ and so $l(x_i w) > l(w)$ for all $i \in I$. Therefore $w = 1$.

Now $|\text{Orb}_W(x) \cap D| \geq 1$ for all $x \in \bigcup_{w \in W} wD = X$ by definition. Suppose $y, z \in \text{Orb}_w x \cap D$ for some $x \in X$. Then $y, z \in D$ and $z = wy$ for some $w \in W$. We want to show that $y = z$, and then we are done. Let $J, K \subseteq I$ be defined as $J = \{i \in I \mid (y, e_i) = 0\}$ and $K = \{i \in I \mid (z, e_i) = 0\}$. We will prove by induction on $l(w)$ that $J = K$ and $w \in W_J = \langle x_i \mid i \in J \rangle$. It immediately follows from the definition of how x_i acts that $wy = y$ so $y = z$.

If $l(w) = 0$ then $w = 1$ so $z = wy = y$ and $1 \in W_J$, $J = K$ trivially.

Assume $l(w) > 0$. Then there is $i \in I$ such that $l(x_i w) < l(w)$. We deduce that $wC \subseteq -A_i$ so $wD = \overline{wC} \subset -A_i = -A_i \sqcup H_i$. But

$$z = wy \in D \cap wD \subseteq (A_i \sqcup H_i) \cap (-A_i \sqcup H_i) \subseteq H_i$$

and so $(z, e_i) = 0$ and $i \in K$. So $x_i z = z$ and $z = (x_i w)y$. By induction, we are done as $(x_i w) \in W_J = W_K$ so $w \in W_J$ $\square$

Corollary 4.9.1. *If (W, I, M) is a Coxeter system, $|I| < \infty$ and $(\cdot, \cdot)_M$ positive definite, then W is finite. So the following are equivalent*

1. W is a finite Coxeter group
2. W is a positive Coxeter group
3. W is a finite reflection group
4. W is the reflection group of a root system Φ
5. W is composed of types $A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_2(m)$

Proof. By the previous theorem, $D = \bar{C}$ is a fundamental domain for the action $W \curvearrowright \bigcup_{w \in W} wD$ and so $W \subset O(V) \cong O_n(\mathbb{R}^n)$ is discrete. But $O_n(\mathbb{R}^n)$ is compact, and so W is finite $\square$

It also follows that C is a fundamental chamber for $W(\mathcal{H})$ where $\mathcal{H} = \text{wall } C$

5 Affine Coxeter Groups

5.1 Affine Weyl Groups

Let V be an inner product space and $\Phi \subseteq V$ be a crystallographic root system. For each $\alpha \in \Phi, k \in \mathbb{Z}$ define $H_{\alpha, k} = \{v \in V \mid (v, \alpha) = k\}$. This is an affine hyperplane, and define $H_\alpha = H_{\alpha, 0}$, a linear hyperplane. Therefore $s_\alpha = s_{H_\alpha}$. We also define $s_{\alpha, k} = s_{H_{\alpha, k}}$ which sends $v \mapsto v - \frac{2((v, \alpha) - k)}{(\alpha, \alpha)}\alpha$. For each root α , let $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$ be the *coroot* of α . Note that $W(\Phi) = \langle s_\alpha \mid \alpha \in \Phi \rangle$. If we let $\mathcal{H} = \langle H_{\alpha, k} \mid \alpha \in \Phi, k \in \mathbb{Z} \rangle$, then $\mathcal{H}$ is locally finite and $W(\Phi) \subseteq W(\mathcal{H})$

Definition 5.1. The *coroot lattice* of Φ is the subgroup of V defined by

$$L'(\Phi) = \text{Span}_{\mathbb{Z}}\{\alpha^\vee \mid \alpha \in \Phi\}$$

Under the isomorphism $V \rightarrow T(V)$ the image of $L'(\Phi)$ is $L(\Phi) \leq T(V)$ which is a discrete subgroup of $T(V)$.

Moreover, $W(\Phi)$ normalises $L(\Phi)$ since $wT_vw^{-1} = T_{wv}$ and if $v \in L'(\Phi)$ then $v = \sum_{\gamma \in \Phi} n_\gamma \gamma^\vee$ so $wv = \sum_{\gamma \in \Phi} n_\gamma w(\gamma^\vee) \in L'(\Phi)$. Therefore we have that $\langle L(\Phi), W(\Phi) \rangle = L(\Phi) \rtimes W(\Phi) \leq \mathcal{I}(V)$

Remark.

1. If Φ is not crystallographic then $L'(\Phi)$ is not discrete, and is in fact dense in V
2. If Δ is a fundamental system in Φ then $\Delta^\vee = \{\delta^\vee \mid \delta \in \Delta\}$ is a $\mathbb{Z}$ basis for $L'(\Phi)$

Let $\widetilde{W(\Phi)} = W(\mathcal{H})$ be the affine reflection group

Proposition 5.1. 1. $W = \widetilde{W(\Phi)} = L(\Phi) \rtimes W(\Phi)$

2. $\mathcal{H}$ is W -invariant and $\widetilde{W(\Phi)}$ is an affine reflection group

Proof. 1. If $\alpha \in \Phi$ then given $k \in \mathbb{Z}$, consider $T_{\alpha^\vee}^k s_\alpha$. This sends

$$v \mapsto v - \frac{2(v, \alpha)}{(\alpha, \alpha)} \alpha \mapsto v - \frac{2(v, \alpha)}{(\alpha, \alpha)} + \frac{2k}{(\alpha, \alpha)} \alpha = s_{\alpha, k}(v)$$

and so $T_{\alpha^\vee}^k s_\alpha = s_{\alpha, k} \in \widetilde{W(\Phi)}$ and so $T_{\alpha^\vee}^k \in \widetilde{W(\Phi)}$ and so if we have that $v = \sum_{\gamma \in \Phi} n_\gamma \gamma^\vee$ then $T_v = \prod_{\gamma \in \Phi} T_{\gamma^\vee}^{n_\gamma} \in \widetilde{W(\Phi)}$ hence $L(\Phi) \subseteq \widetilde{W(\Phi)}$. Moreover $s_{\alpha, k} = T_{\alpha^\vee}^k s_\alpha \in L(\Phi) \rtimes W(\Phi)$ and so $\widetilde{W(\Phi)} = L(\Phi) \rtimes W(\Phi)$

2. For all $\alpha \in \Phi$ and $k \in \mathbb{Z}$ we want to show that if $w \in W(\Phi)$ and $\beta \in \Phi$ then $w(H_{\alpha, k}) \in \mathcal{H}$ and $T_{\beta^\vee}(H) \in \mathcal{H}$
 If $v \in H_{\alpha, k}$ then $(v, \alpha) = k$ and so $(wv, w\alpha) = (v, \alpha) = k$. We deduce that $wv \in H_{w(\alpha), k} \in \mathcal{H}$ and so $w(H_{\alpha, k}) = H_{w(\alpha), k} \in \mathcal{H}$
 Now for translations, $T_{\beta^\vee}(v) = v + \beta^\vee = v + \frac{2\beta}{(\beta, \beta)}$ so that

$$(T_{\beta^\vee}(v), \alpha) = (v + \beta^\vee, \alpha) = (v, \alpha) + \frac{2(\beta, \alpha)}{(\beta, \beta)} = k + \frac{2(\beta, \alpha)}{(\beta, \beta)} \in \mathbb{Z}$$

and so $T_{\beta^\vee}(H_{\alpha, k}) = H_{\alpha, k + \frac{2(\beta, \alpha)}{(\beta, \beta)}} \in \mathcal{H}$

□

Example 5.1. If $V = \mathbb{R}$ and $\Phi = \{\frac{1}{2}, -\frac{1}{2}\}$. Then $\mathcal{H} = \frac{1}{2}\mathbb{Z}$ and the affine reflection group is $\widetilde{W(\Phi)} = D_\infty = \mathbb{Z} \rtimes C_2$.
Let $V = \mathbb{R}^2$, $\Phi = \{\alpha, \beta, \alpha + \beta, -\alpha, -\beta, -\alpha - \beta\}$ where the angle between α and β is $\frac{2\pi}{3}$. The affine reflection group is $\widetilde{W(\Phi)} = (\mathbb{Z}\alpha^\vee \oplus \mathbb{Z}\beta^\vee) \rtimes S_3$. This is also called $\tilde{A}_2$ and has fundamental domain a triangle

It is a fact that if $\Delta \subseteq \Phi$ is a fundamental system, then there is a chamber in the Coxeter complex with walls determined by H_δ for $\delta \in \Delta$ and $T_{\alpha^\vee}H_\alpha$ for some $\alpha \in \Phi$. We therefore have that

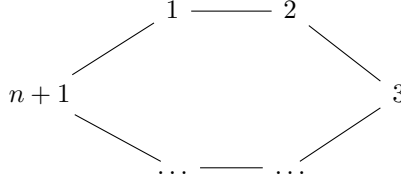
$$\widetilde{W(\Phi)} = \langle s_\delta, s_{T_{\alpha^\vee}H_\alpha} \mid \delta \in \Delta \rangle$$

and removing any generator gives a finite group

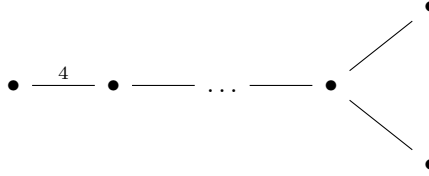
5.2 Groups of Positive Type

Let (W, I, M) be a Coxeter system, $(\cdot, \cdot)_M$ is a positive semidefinite non positive definite form and W is irreducible. Then W has one of the following forms

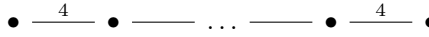
1. $\tilde{A}_n$:



2. $\tilde{B}_n$



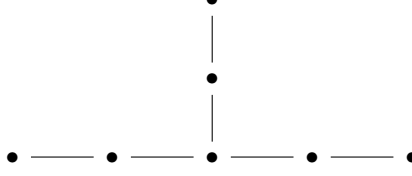
3. $\tilde{C}_n$



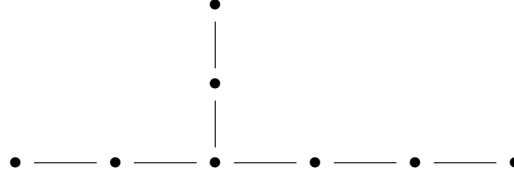
4. $\tilde{D}_n$



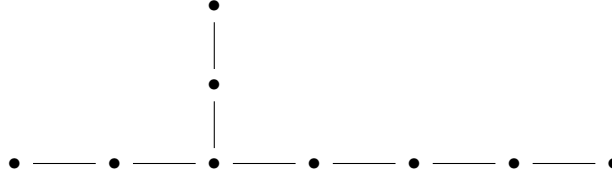
5. $\tilde{E}_6$



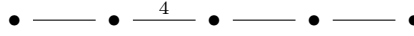
6. $\tilde{E}_7$



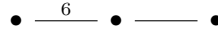
7. $\tilde{E}_8$



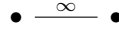
8. $\tilde{F}_4$



9. $\tilde{G}_2$



10. $I_2(\infty)$



If $\Phi \leq V$ is a crystallographic root system of type $A_n, D_n, E_6, E_7, E_8, F_4$ or G_2 then $\widetilde{W(\Phi)}$ has the corresponding type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4$ or $\tilde{G}_2$. If Φ has type B_n then $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ is a crystallographic root system of type B_n but $\widetilde{W(\Phi)}, \widetilde{W(\Phi^\vee)}$ have type $\tilde{B}_n, \tilde{C}_n$ in some order.

In general if the Coxeter system (W, I, M) has positive type then we know that $V_M^\perp = \langle \alpha_1 v_1 + \cdots + \alpha_n v_n \rangle$ is one dimensional with all $\alpha_i > 0$.

If $V_i = \text{Span}\{e_j \mid i \neq j\}$, $W_j = \langle x_j \mid j \neq i \rangle$ is finite, let

$$\Phi = \{we_j \mid w \in W_i, j \neq i\}$$

be a root system in V . We show that $W = \widetilde{W(\Phi)}$ by the isomorphism $x_j \mapsto Se_i$ for $i \neq j$ and $x_i \mapsto s_{T_\alpha \vee H_\alpha}$ where $\alpha = \alpha_1 e_1 + \cdots + \bar{\alpha}_i e_i + \cdots + \alpha_n e_n$

5.3 The Fundamental Domain

Let $\mathcal{H}$ be a locally finite $W(\mathcal{H})$ -invariant set of hyperplanes in an inner product space V and let $W = W(\mathcal{H})$.

Recall that $W = \langle s_H \mid H \in \text{wall } C_0 \rangle$ for $C_0 \in \mathcal{C}$ a fundamental chamber, and $I = \text{wall } C_0$ is finite. For $H, H' \in I$, let $m_{H,H'}$ be the order of $s_H s_{H'}$. Let $M_{\mathcal{H}} = (m_{H,H'})_{H,H' \in I}$ be the associated Coxeter matrix. Let $(\widetilde{W(\mathcal{H})}, I, M_{\mathcal{H}})$ be the Coxeter system with generators $x_H, H \in I$. Clearly there is a surjective map $\theta: \widetilde{W(\mathcal{H})} \rightarrow W(\mathcal{H})$ via $x_H \mapsto s_H$

Theorem 5.1. *For all $w \in \widetilde{W(\mathcal{H})}, H \in I$ we have*

1. *If $l(x_H w) > l(w)$, then $\theta(w)(C_0) \subset A_{1,H}$*
2. *If $l(x_H w) < l(w)$, then $\theta(w)(C_0) \subset A_{-1,H}$*
3. *θ is an isomorphism and $W(\mathcal{H}) \cong \widetilde{W(\mathcal{H})}$ is a Coxeter group*

Proof. • 2 implies 3: If $\theta(w) = 1$ and $w \neq 1$ then there is $H \in I$ such that $l(x_H w) < l(w)$ and so $\theta(w)(C_0) \subseteq A_{-1,H} \cap A_{1,H}$

• 1 implies 2: Same as when proven before

• Proof of 1: First assume $|I| = 2$ so that $\text{wall } C_0 = \{H_1, H_2\}$. If $H_1 \cap H_2 \neq \emptyset$ choose $x \in H_1 \cap H_2$ and replace $\mathcal{H}$ with $T_x^{-1}\mathcal{H}$ to assume $0 \in H_1 \cap H_2$ and so H_1, H_2 are linear hyperplanes, and this is exactly what was proven before.

If $H_1 \cap H_2 = \emptyset$ then $T = s_{H_1} s_{H_2}$ is a translation and since $l(x_{H_1} w) > l(w)$, we know that $\theta(w) = (s_{H_2} s_{H_1})^k = T^k$ or $\theta(w) = T^k s_{H_2}$, but in both cases the result is clear, and (1) holds when $|I| = 2$

If $|I| > 2$ we proceed by induction on $l(w)$. We are done if $l(w) = 0$. If $l(w) > 0$ there is some $H' \in I$ such that $l(x_{H'} w) < l(w)$. Choose some $w_0 \in \widetilde{W(\mathcal{H})}, v \in \langle x_H, x_{H'} \rangle$ such that $w = vw_0$ with $l(w) = l(v) + l(w_0)$, and such that $l(w_0) < l(w)$ is minimal such that such a decomposition exists. We can show as before that $l(x_H w_0), l(x_{H'} w_0) > l(w_0)$ and $l(x_H v) > l(v)$. By the case of $|I| = 2$, $\theta(v)(A_{1,H} \cap A_{1,H'}) \subseteq A_{1,H}$ and by induction $\theta(w_0)(C_0) \subseteq A_{1,H} \cap A_{1,H'}$ so that

$$\theta(w)(C_0) = \theta(v)(\theta(w_0)(C_0)) \subseteq \theta(v)(A_{1,H} \cap A_{1,H'}) \subseteq A_{1,H}$$

□

Corollary 5.1.1. *If $D = \bar{C}_0$, then D is a fundamental domain for the action of $W(\mathcal{H})$ on V , and so $W(\mathcal{H})$ is a discrete subgroup of $\mathcal{I}(V)$*

Proof. Firstly, $\bigcup_{w \in W(\mathcal{H})} wD = V$ as every $H \in \mathcal{H}$ is the wall of some $C \in \mathcal{C}$ so for all $x \in V$, $|\text{orb}_w(x) \cap D| \geq 1$.

If $y, z \in \text{orb}_w(x) \cap D$ then $y, z \in D$ and $z = wy$ for some $w \in W$. We induct

on $l(w)$ to show $y = z$. If $l(w) = 0$ we are done. If $l(w) > 0$ then there is some $H \in I$ such that $l(s_H w) < l(w)$ and so $w(C_0) \subseteq A_{-1,H}$. Then we have

$$\overline{wD} = w\bar{C}_0 = \overline{wC_0} \subseteq \overline{A_{-1,H}} = A_{-1,H} \sqcup H$$

We deduce that

$$z = wy \in D \cap wD \subseteq (A_{1,H} \sqcup H) \cap (A_{-1,H} \sqcup H) \subseteq H$$

Therefore $z \in H$ so $s_H wy = s_H z = z$ and $l(s_H w) < l(w)$ so $y = z$ by induction $\square$

5.4 Infinite Reflection Groups

Proposition 5.2. *If $\bigcap_{H \in \text{wall } C_0} H \neq \emptyset$ then $W(\mathcal{H})$ is finite and there is some translation $T \in T(V)$ such that $TW(\mathcal{H})T^{-1}$ is a finite reflection group*

Proof. Choose $x \in \bigcap_{H \in \text{wall } C_0} H$ and since $W(H) = \langle s_H \mid H \in \text{wall } C_0 \rangle$, we see that $w x = x$ for all $w \in W(H)$ so that $s_H x = x$ for all $H \in \mathcal{H}$ so $x \in \bigcap_{H \in \mathcal{H}} H$ and $\mathcal{H}$ is finite by local finiteness. We deduce that $\mathcal{C}$ is finite, and we know that $W(\mathcal{H})$ acts transitively on $\mathcal{C}$ with the stabiliser of any chamber being trivial, and so $W(H)$ is finite as it embeds into $\text{Sym}(\mathcal{C})$. If we let $T = T_x$ and consider $T^{-1}W(\mathcal{H})T$, this fixes zero and so is a linear reflection group. By finiteness it is also a finite reflection group $\square$

We will assume that $\bigcap_{H \in \text{wall } C_0} H = \emptyset$. Now $W = \langle H \mid H \in \text{wall } C_0 \rangle$ is a Coxeter group and we aim to prove that

Theorem 5.2. *$W(\mathcal{H})$ can be written as $\widetilde{W(\Phi)}$ for some crystallographic root system $\Phi \subseteq V$*

Proof. The whole rest of this section is the proof $\square$

Lemma 5.3. *The map $p: \mathcal{I}(V) \rightarrow O(V)$ by $\alpha \mapsto T_{\alpha(0)}^{-1}\alpha$ is a surjective group homomorphism and $p^2 = p$. $\ker p = T(V)$ and thus*

$$\mathcal{I}(V) = T(V) \rtimes O(V)$$

Proof. Trivial $\square$

From now on, we set $\bar{W} = p(W) \leq O(V)$, and for all $x \in V$, define the stabiliser $W_x = \{w \in W \mid wx = x\}$. Therefore $W_0 \subseteq \bar{W}$. Let $L(\mathcal{H}) = W \cap T(V)$ and $\mathcal{H}_L = \{\text{Linear hyperplanes } H \leq V \mid H \text{ is parallel to some } H' \in \mathcal{H}\}$. The idea of the proof is to proceed as follows

- $W_0 = \bar{W}$
- $L(\mathcal{H})$ is a lattice in $T(V) \cong V$
- $W_0 = W(\Phi)$, $L(\mathcal{H}) = \text{span}\{\alpha^\vee \mid \alpha \in \Phi\}$ where Φ is crystallographic

- $W(\mathcal{H}) = L(H) \rtimes W_0 = \widetilde{W(\Phi)}$

Lemma 5.4. $\bar{W}$ is a finite reflection group equal to $\langle s_H \mid H \in \mathcal{H}_L \rangle$ and if $s_v \in \bar{W}$ for some $v \in V$ then $H_v \in \mathcal{H}_L$

Proof. For each $H \in \mathcal{H}$, let u_H be a normal vector to H . Then

$$p(s_H) = T_{s_H(0)}^{-1} s_H = s_{u_H} = s_{H'}$$

for some $H' \in \mathcal{H}_L$. Then

$$\bar{W} = p(W) = \langle p(s_H) \mid H \in \mathcal{H} \rangle = \langle s_{H'} \mid H' \in \mathcal{H}_L \rangle$$

is a linear reflection group equal to $W(\mathcal{H}_L)$. But $\mathcal{H}_L$ is a $W(\mathcal{H}_L)$ invariant finite set as $s_{u_H}(H') = T_{s_H(0)} s_H(H')$ is a translate of a hyperplane in $\mathcal{H}$. We deduce that $\bar{W} = W(\mathcal{H}_L)$ is a finite reflection group. If $s_v \in \bar{W}$ and H_v is the linear hyperplane orthogonal to v , then $s_v = p(s_H) = T_{s_H(0)}^{-1} s_H$ and so $H = T_{s_H(0)} H_v$ hence $H_v \in \mathcal{H}_L$ $\square$

Define

$$U = \text{span}\{u_H \mid H \in \mathcal{H}\} = \text{span}\{u_H \mid H \in \mathcal{H}_L\}$$

. Let $\{v_1, \dots, v_n\} \subseteq \{u_H \mid H \in \mathcal{H}\}$ be a basis for U such that

$$\bar{W} = \langle s_{v_i} \mid 1 \leq i \leq m \rangle$$

(ie a fundamental system) and for $1 \leq i \leq m$ choose $H_i \in \mathcal{H}$ such that H_i is orthogonal to v_i . Then $\bigcap_{i=1}^m H_i \neq \emptyset$ (exercise) so choose $x \in \bigcap_{i=1}^m H_i$. Then $T_x^{-1} H_i \in \mathcal{H}_L$ so that

$$\bar{W} = \langle s_{v_i} \mid 1 \leq i \leq m \rangle = \langle s_{T_x^{-1} H_i} \mid 1 \leq i \leq m \rangle = T_x^{-1} \langle s_{H_i} \mid 1 \leq i \leq m \rangle T_x$$

Letting $W' = \langle s_{H_i} \mid 1 \leq i \leq m \rangle \leq W(\mathcal{H})$, we have $T_x^{-1} W' T_x = \bar{W}$ which implies $|\bar{W}| = |W'|$. Translating $\mathcal{H}$ by $-x$, we may assume that $x = 0$ so that each H_i is linear, and $W' \leq W_0 \leq \bar{W}$ which forces equality throughout for size reasons and $\bar{W} = W_0 = W'$. If $s_v \in \bar{W} = W_0$ then $H_v \in \mathcal{H}_L$ and $H_v \in \mathcal{H}$ so that $\mathcal{H}_L \subseteq \mathcal{H}$. In other words $\mathcal{H}_L = \{H \in \mathcal{H} \mid H \text{ is linear}\}$. Also $\bigcap_{H \in \text{wall } C_0} H = \emptyset$ so there is some $H \in \text{wall } C_0$ such that $0 \notin H$ and some $H' \in \mathcal{H}_L$ such that $s_{H'} = p(s_H)$ so H, H' are parallel, so $s_H s_{H'} \in L(\mathcal{H}) \setminus \{\text{Id}\}$ so $L(\mathcal{H})$, and hence $W(\mathcal{H})$ is infinite. Note that $W \rightarrow \mathcal{I}(U)$ by restriction is well-defined and injective (as $H \cap U \neq \emptyset$ for all $H \in \mathcal{H}$) so we may assume $U = V$

Proposition 5.3. $L(\mathcal{H})$ is a $\mathbb{Z}$ -lattice in $T(V) \cong V$. $L(\mathcal{H})$ is generated as an abelian group by $\{s_H s_{H'} \mid H \in \mathcal{H}, H \in \mathcal{H}_L \text{ are parallel}\}$ and $W(\mathcal{H}) = L(\mathcal{H}) \rtimes \bar{W}$

Proof. $L(\mathcal{H}) \leq W(\mathcal{H})$ by definition and $W(\mathcal{H})$ is a discrete subgroup of $\mathcal{I}(V)$, so $L(\mathcal{H})$ is a discrete subgroup of $T(V) \cong V$ and it is well known that such subgroups are lattices. Let

$$L'(\mathcal{H}) = \langle \{s_H s_{H'} \mid H \in \mathcal{H}, H \in \mathcal{H}_L \text{ are parallel}\} \rangle \leq L(\mathcal{H})$$

If we have $H \in \mathcal{H}, H' \in \mathcal{H}_L$ which are parallel then

$$p(s_H s_{H'}) = p(s_H) s_{H'} = T_{s_H(0)}^{-1} s_H s_{H'} \in \bar{W} \subseteq W$$

so $T_{s_H(0)} \in L(\mathcal{H})$ and $T_{s_H(0)} = s_H s_{u_H} \in L'(\mathcal{H})$. Hence

$$W = \langle L'(\mathcal{H}), W_0 \rangle = L'(\mathcal{H}) \rtimes W_0 = L(\mathcal{H}) \rtimes W_0$$

as the former is generated by translations □

Theorem 5.5. *We can realise $W(\mathcal{H}) = \widetilde{W(\Phi)}$ for $\Phi \subseteq V$ a crystallographic root system*

Proof. For each $H \in \mathcal{H}_L$, let $\lambda_H > 0$ be the minimal number such that $\lambda_H u_H$ lies in a hyperplane $H' \in \mathcal{H}$ parallel to H (which exists as $L(\mathcal{H})$ is discrete). Now $H' \notin \mathcal{H}_L$ and let $\alpha_H = \lambda_H u_H$. Then $s_{T_{\alpha_H}} s_H = T_{2\alpha_H}$. By the previous proposition, $L(\mathcal{H}) = \text{span}_{\mathbb{Z}}\{\mathbb{Z}\alpha_H \mid H \in \mathcal{H}_L\}$, but $W_0 = \bar{W} = W(\Phi)$ where $\Phi = \{\pm\alpha_H / \|\alpha_H\|^2 \mid H \in \mathcal{H}_L\}$ by the proof of the original theorem in section 2. We know that

$$\left(\frac{\alpha_H}{\|\alpha_H\|^2} \right)^{\vee} = 2\alpha_H$$

so $L(\mathcal{H}) = \text{span}_{\mathbb{Z}}\{\alpha_H^{\vee} \mid H \in \mathcal{H}\} = L(\Phi)$ and $W(\mathcal{H}) = L(\Phi) \rtimes W(\Phi)$. If $\alpha, \beta \in \Phi^{\vee}$ then $s_{\alpha}(\beta) \in \Phi^{\vee} \leq L(\Phi)$ so $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \alpha \in L(\Phi)$, and hence $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$ as α is minimal above 0 on $\mathbb{R}\alpha \cap L(\Phi)$ so $\Phi^{\vee}$ is crystallographic and hence Φ is and so $W(\mathcal{H}) = \widetilde{W(\Phi)}$ □

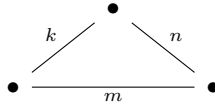
5.5 The Hyperbolic Case

Definition 5.2. Let (W, I, M) be an irreducible Coxeter system. It is (*compact*) *hyperbolic* if

1. $(\cdot, \cdot)_M$ is non-degenerate
2. If we remove any vertex from the Coxeter graph, we get a graph of (strictly) positive type

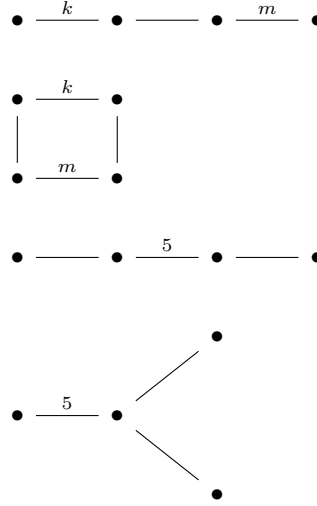
Remark. It is a fact that compact hyperbolic groups only exist in ranks 3, 4, 5:

1. If $|I| = 3$, we have



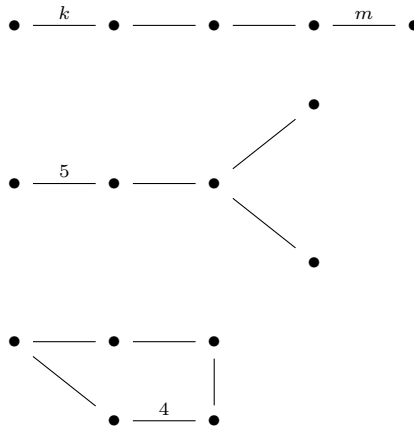
with $k, m, n \geq 3$ and not all of them are 3

2. If $|I| = 4$, we have



with $k, m \geq 4$

3. If $|I| = 5$ we have



for $3 \leq k, m \leq 5$

For general hyperbolic groups, these only exist in rank less than 11, and we will not list these.

Let V be a finite dimensional vector space and B a bilinear form on it with orthogonal group G . G is a real Lie group with a Haar measure, and if B has signature $(n-1, 1)$, we can define a model for $\mathbb{H}^{n-1}$ as those vectors with $B(v, v) = -1$. G acts on this space, and if $W \leq G$ is generated by hyperbolic reflections, then W is isomorphic to a hyperbolic Coxeter group, and is compact if and only if G/W is.

6 Hecke Algebras and Representation Theory

6.1 Generic Hecke Algebras

Hecke algebras can refer to

- Algebras of Hecke Operators on the space of modular forms of fixed weight
- For G a locally compact topological group with a Haar measure, and $K \leq G$ a compact and closed subgroup, we define the *Hecke algebra* of the pair (G, K) to be $H(G, K) = \{f: G \rightarrow \mathbb{C} \mid f \text{ cts compactly supported, } f(gk) = f(g) = f(kg)\}$ with multiplication by convolution, ie $\int_G f(gh)f'(h^{-1})dh$. For all $g \in G$ we can define $KgK: G \rightarrow \mathbb{C}$ via 1_{KgK} .
- If G is any group with subgroup K , the Hecke algebra over k of (G, K) should be $k[K \backslash G/K] = \text{Span}\{KgK \mid g \in G\}$. Let $\mathbb{X}_k = \text{Span}\{gK \mid g \in G\}$, which is a G module, and we take $H_k(G, K) = (\text{End}_G \mathbb{X}_k)^{op}$. If we have $g \in G$, then we define KgK to be the endomorphism of $\mathbb{X}_k$ given by $K \mapsto \sum_{h \in K/K \cap gKg^{-1}} hgK$ if well defined
- If (W, I, M) is a Coxeter system with generators $X = \{x_i \mid i \in I\}$. Let $\{a_i \mid i \in I\}$ be a set of parameters where $a_i = a_j$ if x_i and x_j are conjugate. Let $A = \mathbb{Z}[a_i \mid i \in I]$

Definition 6.1. We define the *generic Hecke algebra* of $(W, X, \{a_i \mid i \in I\})$ is the A algebra H_{gen} the unique A algebra with basis $\{t_w \mid w \in W\}$ satisfying

1. $t_1 = 1$
2. For $w \in W, x \in X$, if $l(xw) > l(w)$, then $t_x t_w = t_{xw}$
3. For all $x \in X$, $t_x^2 = a_x + (a_x - 1)t_x$

Remark. This exists and is unique up to isomorphism. We can deduce the case of $l(xw) < l(w)$ from 2 and 3 above. If w is written in reduced form, t_w is the product of the t_x in the form.

Let k be a commutative ring and $\theta: A \rightarrow k$ a ring homomorphism. The specialisation of H_{gen} with respect to θ is the k -algebra $H_\theta = H(W, \{q_i\}) = H_{gen} \otimes_A k$

Example 6.1. If every $q_i = 1$, we get the group algebra. If $k = \mathbb{C}(a_i \mid i \in I), \theta: A \rightarrow k$, then $KH_{gen} = H_\theta$

6.2 BN-pairs

Definition 6.2. Let G be a group. A *BN-pair* in G is a triple (B, N, S) where $B, N \leq G$, $S \subseteq N$ such that $G = \langle B, N \rangle, T = B \cap N \trianglelefteq N$, $W = N/T$ is generated by the image of S and the following axioms hold:

1. For all $s \in S, w \in W$, we have $BsBwB \subseteq BwB \cup BswB$

2. For all $s \in S$, $sBs^{-1} \not\subseteq B$

Remark. We call W the *Weyl group* of (B, N, S) . For all $w, w' \in N$, $BwB \cap Bw'B \neq \emptyset$ if and only if $wT = w'T$ (as we can use axiom 1 to ‘commute’ elements of B and N). We also therefore have $G = \bigsqcup_{w \in W} BwB$. For all $S' \subseteq S$, if $W' = \langle S' \rangle \leq S$, then $G' = \bigsqcup_{w \in W'} BwB$ is a parabolic subgroup.

Example 6.2. If $G = GL_n(k)$, $B = U_n(k)$ (upper triangular matrices), T the diagonal matrices and N the normaliser of T (ie the group generated by T and all permutation matrices). Take S to be the set of transposition matrices. The axioms are exercises.

Example 6.3. If G is any connected algebraic group over an algebraically close field k which is reductive (has no unipotent normal subgroups). Let T be a split maximal connected torus in G , let B be a Borel subgroup containing T , and N the normaliser of T . If Φ is the root system of the Lie algebra of G and Δ a fundamental system. Then $S = \{x_\alpha \mid \alpha \in \Delta\}$. This is a BN-pair.

Example 6.4. Let W be an affine Coxeter group, let C be the Coxeter complex, a simplicial complex of dimension the rank of W . More generally let Δ be a simplicial complex of dimension d everywhere, which is a union of Coxeter complexes $A_j \cong C$ for $j \in J$ (called *apartments*) where for all chambers (d -simplices) $C, D \in \Delta$ there is some j with $C, D \in A_j$ and for all $j, k \in J$ there is an isometry $A_j \rightarrow A_k$ fixing every chamber in $A_j \cap A_k$. We call such a complex a *building* associated to W . Let G be a group acting faithfully and transitively on the sets of chambers and apartments in Δ . Let C be a chamber in Δ , A_j an apartment containing C , and B the stabiliser of C in G , and N the stabiliser of A_j . Then (B, N, S) is a BN-pair for some S , and $N/T = W$.

Lemma 6.1. *If (B, N, S) is a BN pair in G , then for all $s \in S \subseteq W$, $s^2 = 1$*

Proof. We know $sBs^{-1} \not\subseteq B$ and $BsBs^{-1}B \subseteq Bss^{-1}B \cup Bs^{-1}B = B \cup Bs^{-1}B$. But $BsBs^{-1}B$ is a union of double cosets, so we have equality. Now $(BsBs^{-1}B)^{-1} = B \cup BsB$, so $BsB = Bs^{-1}B$ and so $B \cup BsB = BsBs^{-1}B = BsBsB \subseteq BsB \cup Bs^2B$, so $B = Bs^2B$ and we are done. $\square$

Remark. It follows that for all $s \in S$, $B \sqcup BsB$ is a parabolic subgroup of G . Also $BwBsB = (BsBw^{-1}B)^{-1} \subseteq (Bsw^{-1}B \cup Bw^{-1}B)^{-1} = BwB \sqcup BwsB$.

Proposition 6.1. *For all $s \in S, w \in W$,*

1. *If $l(sw) \geq l(w)$, then $BsBwB = BswB$*
2. *If $l(sw) \leq l(w)$ then $BsBwB = BswB \cup BwB$*

In particular, $l(sw) \neq l(w)$

Proof.

1. If $l(w) = 0$ then $w = 1$ so the result is trivial. We proceed by induction on $l(w)$. If $l(w) = n > 0$. Then there is $t \in S$ such that $l(wt) < n$. Let $w' = wt$. If $BsBwB \neq BswB$, then $BwB \cap BsBwB \neq \emptyset$, and hence $sBw \cap BwB \neq \emptyset$, so $sBw' \cap BwBt \neq \emptyset$. Now $BwBtB \subseteq BwB \cup BwtB = BwB \cup Bw'B$, so $sBw' \cap BwB \neq \emptyset$ or $sBw' \cap Bw'B \neq \emptyset$. As $l(sw') = l(swt) \geq l(wt) = l(w')$, we have $BsBw'B = Bsw'B$ by induction, so $sw' = w$ (as we can't have $sw' = w'$), so $swt = w$, so $sw = wt$ has length less than w , contradiction.
2. Let $w' = sw$ and then $l(sw') \geq l(w)$, so $BsBswB = BsBw'B = Bsw'B = BwB$. Now $BsBsB \subseteq B \cup BsB$, and we have equality as $sBs \neq B$. Now $BsBwB = BsBsBswB = BsBswB \cup BswB = Bw'B \cup BsBw'B = Bw'B \cup Bsw'B = BswB \cup BwB$ as desired.

□

Theorem 6.2. $W = N/T$ is a Coxeter group with generators S

Proof. We only need to show that for all $w \in W$, $s, t \in S$ and $l(sw) = l(wt) = l(w) + 1$ then either $l(swt) = l(w) + 2$ or $swt = w$. If $l(swt) < l(w) + 2$ then $l(swt) = l(w) < l(sw), l(wt)$, so $BsBwtB = BwtB \cup BswtB$ and $BswBtB = BswB \cup BswtB$, so $BsBwBtB = BwtB \cup BswtB = BswB \cup BswtB$, so $wt = sw \in N/T$ □

6.3 The Building of (B, N)

Definition 6.3. Let G be a group and (B, N, S) a BN pair in G . We define $\mathcal{X} = G/B$, the set of left cosets, which we call the *building* of (B, N, S)

Remark. Given $x, y \in \mathcal{X}$ and $s \in S$ we say x, y are s -adjacent, $x \sim_s y$ if $x = gB, y = hB$ and $Bg^{-1}hB = BsB$ or $x = y$. More generally for any $x = gB, y = hB$ we let $\delta(x, y)$ be the unique element of W representing $Bg^{-1}hB$. We write $\delta(x) = \delta(x, B)$. G acts on $\mathcal{X}$ by left translation, and $\delta(x, y)$ is g -invariant. $x \sim_s y$ is a G -invariant equivalence relation, and if $x \sim_s y$ and $x \neq y$, and $l(\delta(x)s) > l(\delta(x))$ then $\delta(y) = \delta(x)s$

Remark. (Geometric realisation) We realise $\mathcal{X}$ as a simplicial complex of dimension $d = |S|$ where the d simplices are the cosets gB with $d - 1$ dimensional faces P_s/B where $P_s = B \cup BsB$, and further dimensions are given by k -size subsets of S .

Proposition 6.2. All subgroups $B \leq P \leq G$ can be realised as some P_S

Proof. Exercise. (Use proposition 6.1, and consider $Bw^{-1}BwB$) □

Theorem 6.3. For all $w \in W, s \in S$ such that $l(ws) > l(w)$ there is a bijection $B/(B \cap wBw^{-1}) \times B/(B \cap sBs) \cong B/(B \cap wsBsw^{-1})$

Proof. The stabiliser of wB is $B \cap wBw^{-1}$ so we will define a bijection from the orbit of wB times $B/(B \cap sBs)$ bijects with the orbit of wsB . Let $A = \{(z, u) \mid z \in \text{Orb}_B wB, u \neq z, u \sim_s z\}$. We define $\phi: A \rightarrow \text{Orb}_B wsB$ by $(z, u) \mapsto u$. This is well defined as if $z \in \text{Orb}_B wB$ with $u \sim_s z$ and $u \neq z$ then $z = bwB$ and $\delta(z) = \delta(zB) = w$. We know $l(\delta(z)s) > l(\delta(z))$ so $\delta(u) = \delta(z)s = ws$, so $u = bwsB$ and $u \in \text{Orb}_B wsB$. To prove ϕ is injective, suppose $(z_1, u_1), (z_2, u_2) \in A$ such that $\phi(z_1, u_1) = \phi(z_2, u_2)$, ie $u_1 = u_2$. Then $z_1 \sim_s u_1 = u_2 \sim_s z_2$, and so $z_1 = z_2$ or $\delta(z_1) = \delta(z_2)s$. The second is equivalent to $w = ws$, so $z_1 = z_2$ and we are done. To prove surjectivity, if $u \in \text{Orb}_B wsB$, then $u = bwsB$ and we let $z = bwB \in \text{Orb}_B wB$. Then $\delta(z, u) = s$, so $z \sim_s u$ and $u \neq z$ by definitio, so we have proven ϕ is a bijection. Finally if $z = gB \in \mathcal{X}$,

$$\{u \in \mathcal{X} \mid u \neq z, u \sim_s z\} = \{u \in \mathcal{X} \mid \delta(u, z) = s\} = \{u \in \mathcal{X} \mid \delta(g^{-1}u) = s\}$$

which is in bijection with $\{u \in \mathcal{X} \mid \delta(u) = s\} = \text{Orb}_B sB$. We define a bijection $\psi: A \rightarrow \text{Orb}_B wB \times \text{Orb}_B sB$ by the above correspondences, and we are done. $\square$

Remark. Geometrically, this states that if we fix a chamber B , then the orbit of wB corresponds to the orbit of wB around B and the orbit of swB around wB ,

6.4 The Iwahori-Hecke Algebra

For $H \leq G$, let $\mathbb{X}_H$ be the standard module from the start of section 6.1 and assume for all $g \in G$, $H/(H \cap gHg^{-1})$ is finite

Lemma 6.4. *Let $\mathbb{X}^H = \{x \in \mathbb{X} \mid hx = x, h \in H\}$. Then there is an isomorphism of vector spaces $H_k(G, H) \cong \mathbb{X}^H$ via $\theta: \phi \mapsto \phi(H)$*

Proof. θ is linear and is well defined as $h\phi(H) = \phi(hH) = \phi(H)$. Given $v \in \mathbb{X}^H$, we define $\phi_v: \mathbb{X} \rightarrow \mathbb{X}$ via $\sum \alpha_g hH = \sum \alpha_g gv$. Then $\phi_v \in \text{End}_G \mathbb{X} = H_k(G, H)$ and $\theta(\phi_v) = v$. The other composition is also the identity, so we are done. $\square$

If $v \in \mathbb{X}^H$, then $v = \sum \alpha_g gH = hv$ so for all $h \in H$, $\alpha_{h^{-1}g} = \alpha_g$, so for all $g \in G$, let $v_g = \sum_{h \in H/(H \cap gHg^{-1})} hgH \in \mathbb{X}^H$. Then for all $x \in \mathbb{X}^H$, $v = \sum_{g \in H \backslash G/H} \alpha_g v_g$ for some $\alpha_g \in k$. Thus $H_k(G, H)$ has a basis $\{HgH \mid g \in G\}$ where $HgH \in H_k(G, H)$ is defined by $H \mapsto v_g$. Let (B, N, S) be a BN pair for G , and further assume that $B/(B \cap wBw^{-1})$ is finite for $w \in N$. We write $\mathbb{X} = \mathbb{X}_B$ and use this to define $H_k(G, B) \cong \mathbb{X}^B$ with k -basis $\{T_w \mid w \in W\}$ where $T_w(H) = v_w$. We call this $H_k(G, B)$ the *Iwahori algebra* of the BN pair.

Theorem 6.5. *The basis T_w satisfies the following properties*

1. $T_1 = \text{Id}$
2. If $w \in W, s \in S$ and $l(sw) > l(w)$ then $T_{sw} = T_s T_w$
3. If $s \in S$ then $T_s^2 = q_s + (q_s - 1)T_s$ where $q_s = |B/(B \cap sBs)| \in k$

Proof.

1. $T_1(B) = B$ so we are done.
2. We only need to show $T_s T_w(B) = T_{sw}(B)$. Now $T_s T_w(B) = T_w(T_s(B)) = T_{sw}(B)$ so we are done (as $bsb'w \in BsBwB = BswB$ by the length condition, and $|O(sw)| = |O(s)||O(w)|$ where $O(w) = B/(B \cap wBw^{-1})$).
3. We can write $BsBsB = B \cup BsB$, so $T_s^2(B) = T_s(\sum_{b \in B \cap sBs} bsB) = \sum_{b' \in B/(B \cap sBs)} \sum_{b \in B/(B \cap sBs)} bsb'sB$. Note $bsb'sB = B$ if and only if $b' \in B \cap sBs$, so the trivial coset is represented q_s times, and considering the rest, $\sum_{b' \neq 1} bsb'sB$ which is a sum of $q_s(q_s - 1)$ cosets in BsB . If sB appears n times, then by g -invariance, so does every other coset in BsB , so the result follows as there are q_s many cosets.

□

Remark. The remaining property to prove this is a specialisation of the generic Hecke algebra is an exercise. In particular if $w = \prod s_i$ is a reduced expression, $T_w = \prod T_{s_i}$ and the generators satisfy the braid relations.

Corollary 6.5.1. $q_s = q_t$ if s, t are conjugate in W . Hence the Iwahori algebra is a specialisation of the generic Hecke algebra.

Proof. Exercise.

□