

Differential Geometry

Amy Needham

October 2025

Contents

1	Introduction	1
2	Smooth Manifolds	2
2.1	Tangent and Cotangent Vectors	3
3	Submanifolds	5
4	Vector Bundles	6
4.1	Exterior Algebra	7
4.2	Exterior Differential	9
5	Integration	9
5.1	Stokes Theorem	10
6	Flows, Lie Derivatives and Cartan's Formula	11
6.1	Flows	11
6.2	Lie Derivatives	11
7	Riemannian Geometry	13
8	Connections and Curvature	16
8.1	Connections	16
8.2	Curvature	17
8.3	Levi-Civita Connection	18
8.4	Riemannian, Ricci, and Scalar curvature	20
9	Hodge Theory and Harmonic Representative	21

1 Introduction

Differential geometry provides a language for calculus on functions defined on manifolds. It tells you how to interpret ∂ and $\int$ on manifolds, allowing you to study PDEs on manifolds. Another motivation is the generalisation of the study

of curves and surfaces in $\mathbb{R}^3$, permitting us to study them (and hypersurfaces) in a more intrinsic manner. If we wish to study symmetry, sets of continuous symmetries frequently form a Lie group (a manifold with a group structure such that multiplication and inversion are smooth), for example matrix groups. It can also permit us to related local and global topology, via eg the Gauss-Bonnet theorem, which states that the integral of the curvature is 2π times the Euler characteristic.

2 Smooth Manifolds

We wish to set up the minimal topological structure that allows us to take derivatives and integrals.

Remark 2.1. We use the Einstein summation convention, where $x^i x_i = \sum_{i=1}^n x^i x_i$, and we always sum over exactly two indices, which are always one up and one down.

Definition 2.1. An n -dimensional smooth manifold M is a Hausdorff second countable topological space equipped with a collection of *coordinate charts* $\phi_\alpha: M \supseteq U_\alpha \rightarrow V_\alpha \subseteq \mathbb{R}^n$ where U_α, V_α are open such that

1. The open subsets U_α cover M
2. The maps ϕ_α are homeomorphisms
3. For any α, β with $U_\alpha, U_\beta \neq \emptyset$, the *transition function* $\phi_\beta \circ \phi_\alpha^{-1}: \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ is a smooth map (which is defined as the domain and codomain are subsets of $\mathbb{R}^n$)

Remark 2.2.

1. If we relax smooth to continuous, we get a *topological manifold*
2. If we require the Jacobian of the transition functions satisfy $\det \left(\frac{\partial \tilde{x}^i}{\partial x^j} \right) > 0$ (where x^i and $\tilde{x}^j$ are two different choices of coordinates), we have an *oriented manifold*
3. If you replace $\mathbb{R}^n$ by $\mathbb{C}^n$ and smooth by holomorphic.

Definition 2.2. A continuous map $F: M \rightarrow N$ is called a smooth map if for every pair of charts $(U, \phi) \subseteq M$ and $(V, \psi) \subseteq N$ containing p and $F(p)$ respectively, the composition $\psi \circ F \circ \phi^{-1}$ is smooth where defined

Remark 2.3.

1. Smooth functions $f: M \rightarrow \mathbb{R}$ form a vector space $C^\infty(M)$
2. If $F: M \rightarrow N$ has a smooth inverse $F^{-1}: N \rightarrow M$ then F is a *diffeomorphism*

Example 2.1.

1. $\mathbb{R}^n$ is a manifold, with charts the inclusions of open subsets, as is any open subset of it
2. The torus $\mathbb{R}^n/\mathbb{Z}^n$ is a manifold. As the quotient map is a covering map, we can endow sufficiently small opens with charts from $\mathbb{R}^n$
3. If f is a smooth function on $\mathbb{R}^n$ and $\nabla f \neq 0$ at all points, then $M = \{f = 0\}$ is a smooth manifold by implicit function theorem (eg. $S^n = \{|x|^2 = 1\}$)
4. $GL(n, \mathbb{R}) \subseteq \text{End}(\mathbb{R}^n)$ is open (defined by non-vanishing of a polynomial, the determinant) and hence a manifold
5. $SL_n(\mathbb{R}^n) = \{A \in GL_n(\mathbb{R}^n) \mid \det A = 1\}$ is a manifold by 3 as the partials of the determinant are $(n-1) \times (n-1)$ minors of the matrix, which are non-zero as the matrix is full rank
6. $O(n) = \{A \in GL_n(\mathbb{R}^n) \mid A^T A = I\}$ is a manifold, as the derivative of $A \rightarrow AA^T \in \text{Sym}^2(\mathbb{R}^n)$ is surjective (consider $DF(AY)$), so done by implicit function theorem
7. Similarly, $U(n) = \{A \in GL_n(\mathbb{C}^n) \mid A^\dagger A = I\}$ is a manifold
8. *Complex projective space*, $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ where the latter acts by scalar multiplication. This is the space of lines through 0. We define homogeneous coordinates $[z_0 : \dots : z_n]$ by the coordinates on $\mathbb{C}^{n+1}$, and we get local charts by taking $\left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right)$ on the open subset of points $U_i = \{[z_0 : \dots : z_n] \mid z_i \neq 0\}$ (which is well defined). The transition function ϕ_{ij} is multiplication by $\frac{z_i}{z_j}$, which is smooth (and in fact holomorphic) so defines a complex manifold.
9. An *elliptic curve* is a genus 1 complex manifold of dimension 1. Concretely, these are the zero loci of degree 3 homogeneous polynomials,

$$\{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid (z_1 - \lambda_1 z_0)(z_2 - \lambda_2 z_0)(z_2 - \lambda_3 z_1) = z_0 z_1^2\} \subseteq \mathbb{CP}^2$$

2.1 Tangent and Cotangent Vectors

Let M be a smooth manifold of dimension n . Consider a smooth curve $\gamma: (-1, 1) \rightarrow M$ such that $\gamma(0) = p \in M$. The velocity at time 0 should be a tangent vector. Let $\{x^i\}, \{\tilde{x}^i\}$ be local coordinates around $p \in M$. Then the curve γ is locally given by $x^i(\gamma(t))$ (resp. $\tilde{x}^i(\gamma(t))$), and the velocity is represented by $\frac{\partial}{\partial t} x^i(\gamma(t))$ (resp. $\frac{\partial}{\partial t} \tilde{x}^i(\gamma(t))$). These are related by the chain rule, $\frac{\partial}{\partial t} x^i(\gamma(t)) = \sum_j \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial}{\partial t} \tilde{x}^j(\gamma(t))$.

Definition 2.3. A *tangent vector* v at a given point p is a collection of vectors $(v^i) \in \mathbb{R}^n$ associated to each coordinate chart $\{x^i\}$ containing p such that they satisfy the *coordinate transformation rule* $\tilde{v}^i = \sum_j \frac{\partial \tilde{x}^i}{\partial x^j} v^j$. In particular, all the

tangent vectors at a given point p form a vector space isomorphic to $\mathbb{R}^n$, denoted $T_p M$

A more abstract (and better) definition can be given based on the notion of a *directional derivative*. If $\gamma: (-1, 1) \rightarrow M$ is a curve and f is a smooth function in a neighbourhood of p , we can compute $f \mapsto \frac{\partial f}{\partial t}|_{t=0} f(\gamma(t)) = \sum \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial t}$. This is a derivation on smooth functions near p , depending only on the germ at p (equivalence class of functions modulo agreement on a neighbourhood of p).

Definition 2.4. A *derivation* v at a point $p \in M$ is a linear map on the space of germs of smooth functions, such that the *Leibniz rule* holds, $v(fg) = f(p)v(g) + v(f)g(p)$

Lemma 2.1. Let $\{x^i\}$ be a coordinate system around $p \in M$. The derivations at p form an n -dimensional real vector space spanned by the basis $\frac{\partial}{\partial x^i}$

Proof. $\frac{\partial}{\partial x^i}: f \mapsto \frac{\partial f}{\partial x^i}$ is a derivation, and derivations also clearly form a vector space. Conversely, if v is a derivation at p , we want to show $v(f) = \sum v^i \frac{\partial f}{\partial x^i}$. By the Leibniz rule and linearity, $v(1) = v(1 \cdot 1) = v(1) + v(1)$ so $v(1) = 0$. Let $v^i = v(x^i)$. Again by the Leibniz rule, for any affine linear function f , $v(f) = \sum v^i \frac{\partial f}{\partial x^i}$, and inductively this holds for polynomials. It suffices to prove that if f is a smooth local function such that $f(p) = 0$ and $\frac{\partial f}{\partial x^i} = 0$ then $v(f) = 0$. This follows from the Hadamard lemma, by substituting in, as $g_i(x) = 0$ then. $\square$

Lemma 2.2. (Hadamard lemma) Let $\Omega \subseteq \mathbb{R}^n$ be a convex open subset around the origin, and $f: \Omega \rightarrow \mathbb{R}$. Then $f(x) = f(0) + \sum x^i g_i(x)$ for $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx) dt$.

Proof. Trivial $\square$

Now there is a canonical isomorphism between tangent vectors and derivations via $v = v^i \frac{\partial}{\partial x^i} \mapsto (v^i) \in T_p M$. By the chain rule this is well defined and injective, and is surjective for dimension reasons.

Definition 2.5. If we have a smooth map $F: M \rightarrow N$, we get a *pushforward* $F_* = DF: T_p M \rightarrow T_{F(p)} N$, defined on coordinates $\{x^i\}$ on M and $\{y^j\}$ on N via $\sum v^i \frac{\partial}{\partial x^i} \mapsto \sum v^i \frac{\partial F^j}{\partial x^i} \frac{\partial}{\partial y^j}$

Definition 2.6. The dual space of $T_p M$ is the *cotangent space* at p , denoted $T_p^* M$. Elements of $T_p^* M$ are called *cotangent vectors*.

Remark 2.4. A smooth function f induces a linear map $df: T_p M \rightarrow \mathbb{R}$ via $v \mapsto v(f)$ (with v seen as a derivation) called the *differential* of f .

Suppose we have a coordinate system $\{x^i\}$ near p and take $f = x^i, v = \frac{\partial}{\partial x^i}$. Then $\langle dx^i, \frac{\partial}{\partial x^j} \rangle = \frac{\partial x^i}{\partial x^j} = \delta_{ij}$. In particular dx^i is the dual basis of $\{\frac{\partial}{\partial x^i}\}$.

Exercise 2.1. $d(fg) = g(df) + f(dg)$, and $df = \sum \frac{\partial f}{\partial x^i} dx^i$

Remark 2.5. Tangent and cotangent vectors have different transformation rules. Given coordinates $\{x^i\}, \{\tilde{x}^i\}$ and $a = \sum a_i dx^i$, then $\tilde{a}_j = \sum \frac{\partial x^i}{\partial \tilde{x}^j} a_i$

Definition 2.7. Under a smooth map $F: M \rightarrow N$, we have *pullback*, F^* on cotangent vectors. In local coordinates, $\{x_i\}$ on M and $\{y_i\}$ on N , $\sum a_j dy^j \mapsto \sum a_j \frac{\partial F^j}{\partial x^i} dx^i$ is the dual map to F_* .

Definition 2.8. Using multivariable linear algebra, we can construct tensors as an element of the tensor product $T_p M \otimes \cdots \otimes T_p M \otimes T_p^* M \otimes \cdots \otimes T_p^* M$.

Example 2.2. Elements of $T_p^* M \otimes T_p^* M$ correspond to bilinear forms on $T_p M$, for example Riemannian metrics.

3 Submanifolds

Definition 3.1. A smooth map $\iota: M \rightarrow N$ is an *immersed submanifold* if $d\iota_p$ is injective for any $p \in M$. If ι is furthermore a homeomorphism onto its image, then it is an *embedded submanifold*.

Remark 3.1. These roughly correspond to parameterised submanifolds vs manifolds obtained as the zeros of an equation, considering two typical simple methods to get submanifold. Without qualification, ‘submanifold’ means embedded.

Example 3.1. Curves and surfaces in $\mathbb{R}^3$. An immersed curve in $\mathbb{R}^3$ is a smooth map $\gamma: (0, 1) \rightarrow \mathbb{R}^3$ so that $\gamma'(t) \neq 0$. A regularly parameterised surface, $x(u, v)$ such that $\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \neq 0$ is an immersed surface.

Remark 3.2. For immersed submanifolds, ι may not be injective, eg $(\sin t, \cos 3t)$, and in fact need not even be ‘tamely’ non-injective, ie $(\sin t, \cos \pi t)$. Inclusion of an open subset is an embedded submanifold, as is $\mathbb{R}^m \rightarrow \mathbb{R}^n$. Embedded submanifolds have nice local structure, and we usually identify ι with its image.

For embedded submanifolds, since $d\iota_p$ is injective we can use the inverse function theorem to find local coordinates $y^1, \dots, y^m$ in a neighbourhood U so that $\text{im } \iota \cap U = \{(y^i) \mid y^1, \dots, y^m = 0\}$.

Definition 3.2. Given a smooth map $F: M \rightarrow M' \ni q$, we say q is *regular value* for F if for any $p \in M$ such that $F(p) = q$, the map $dF|_p$ is surjective. If $F^{-1}(q) = \emptyset$, then q is regular.

Theorem 3.1. Let $F: M \rightarrow M' \ni q$ be a smooth map. Then if q is a regular value, $F^{-1}(q)$ is an embedded submanifold of dimension $\dim M - \dim M'$.

Proof. Let $n = \dim M, m = \dim M'$ and let $\{y^i\}$ be a local coordinate system around $q \in M'$ and $\{x^i\}$ local coordinates on $U \subseteq M$ such that there is $p \in U \cap F^{-1}(q)$.

Since q is a regular value, $dF: T_p M \rightarrow T_q M'$ is surjective, we note $n \geq m$, and $F^*: T_q^* M' \rightarrow T_p^* M$ is injective, so $dy^1, \dots, dy^m$ are linearly independent in $T_p^* M$. After reordering the x^i , we may assume $dy^1, \dots, dy^m, dx^{m+1}, \dots, dx^n$ is a basis of $T_p^* M$. We compute $\det \left(\frac{\partial(y^1, \dots, y^m, x^{m+1}, \dots, x^n)}{\partial(x^1, \dots, x^n)} \right) \neq 0$ at p . By implicit

function theorem, possibly after shrinking U , we can regard $y^1, \dots, y^m, x^{m+1}, \dots, x^n$ as the local coordinates, and $x^1, \dots, x^n$ as smooth functions of them. In the new coordinates on U , $F^{-1}(q) \cap U = \{y^1, \dots, y^m = 0\} \cap U$. We regard $F^{-1}(q) \cap U$ as a local chart for $F^{-1}(q)$. Suppose we have overlapping charts $\{x^i\}, \{\tilde{x}^i\}$ on M inducing overlapping charts on $F^{-1}(q)$. Because the $\tilde{x}^i$ are smooth functions of the x^i , they are also smooth functions of the $y^1, \dots, y^m, x^{m+1}, \dots$. Restricting to $y^i = 0$, we get that the transition function on $F^{-1}(q)$ is smooth and the result is proven. $\square$

Definition 3.3. Suppose $F: M \rightarrow M'$ is a smooth map and $\dim M \geq \dim M'$. We say $p \in M$ is a *critical point* if $dF_p: T_p M \rightarrow T_{F(p)} M'$ has rank less than $\dim M'$. The image of the set of critical points in M' is the set of *critical values*

Remark 3.3. The set of critical values is the complement of the set of regular values

Theorem 3.2. (*Sard's Theorem, non-examinable*) The set of critical values is a zero measure (intersection with every chart has zero measure) subset of M'

Proof. No Proof $\square$

So the implicit function theorem applies almost everywhere

Theorem 3.3. (*Whitney embedding theorem, non-examinable*) Any n -dimensional smooth manifold M can be embedded as a submanifold of $\mathbb{R}^{2n}$

Proof. No Proof $\square$

Remark 3.4. The dimension $2n$ cannot be improved (Algebraic topology)

Theorem 3.4. (*Nash embedding theorem, non-examinable*) If (M, g) is a Riemannian manifold, then M can be isometrically embedded into some $\mathbb{R}^N$ (and N can be taken to be a quadratic in n)

Proof. No Proof $\square$

4 Vector Bundles

(Co)Tangent vectors fit into a global object called the (co)tangent bundle

Definition 4.1. The *(co)tangent bundle* of a manifold M is, as a set, $TM = \bigcup_{p \in M} T_p M$ (resp. $T^*M = \bigcup_{p \in M} T_p^* M$).

Remark 4.1. These have maps to M given by projecting the p coordinate to p , and with these we wish to define smooth vector fields over M which are locally a cartesian product.

Definition 4.2. A *vector bundle* of rank r over a manifold M is a smooth map $\pi: E \rightarrow M$ such that

1. There is an r -dimensional vector space V , called the *typical fibre* of E such that for any $p \in M$, $E_p = \pi^{-1}(p)$ is linearly isomorphic to V
2. For any point $p \in M$, there is a neighbourhood U such that there is a *local trivialisation* Φ_U such that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi_U} & U \times V \\ & \searrow & \downarrow \\ & & U \end{array}$$

commutes.

3. Φ_U restricts to a linear isomorphism of vector spaces on $\pi^{-1}(p)$

Remark 4.2. On the intersection of overlapping trivialising neighbourhoods, $\Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (x, \psi_{\beta\alpha}(x)(v))$ where $\psi_{\beta\alpha}$ is a smooth map $U_\alpha \cap U_\beta \rightarrow GL(r, \mathbb{R})$, called the *transition function*

It is immediate that $\psi_{\alpha\alpha} = \text{id}_V$, $\psi_{\beta\alpha} \circ \psi(\alpha\beta) = \text{id}_V$ and $\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = \text{id}_V$ identically where defined.

For TM, T^*M , the transition functions will be given by some Jacobean matrices.

Remark 4.3. Vector bundles in general are not global products of the underlying space with the fibres.

Example 4.1. T^*S^2 and TS^2 will turn out to be not the same as $S^2 \times \mathbb{R}^2$ (algebraic topology, consider the Euler class)

Definition 4.3. Any smooth map $s: M \rightarrow E$ such that $\pi \circ s = \text{id}_M$ is called a *section* of E

Definition 4.4. Smooth sections of TM (resp. T^*M) are called *vector fields* (resp. *1-forms*)

Remark 4.4. We have a Lie bracket on the space of vector fields and an exterior differential on the space of 1-forms

We define the Lie bracket on vector fields X, Y defined in a neighbourhood of a point p . If f is a local smooth function, then $[X, Y]f = X(Y(f)) - Y(X(f))$. In coordinates, if $X = \sum v^i \frac{\partial}{\partial x^i}$, $Y = \sum w^i \frac{\partial}{\partial x^i}$ then (example sheet) $[X, Y] = \sum (v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}) \frac{\partial}{\partial x^j}$.

4.1 Exterior Algebra

Definition 4.5. We first define the *tensor product* of two vector spaces V, W . Concretely, if V has basis $v_1, \dots, v_n$ and W has basis $w_1, \dots, w_m$, then $V \otimes W$ has basis $v_i \otimes w_j$. Problem: proving this is independent of basis. We can instead define it abstractly as the vector space such that any bilinear $f: V \times W \rightarrow V'$

factors uniquely through a fixed bilinear map $V \times W \rightarrow V \otimes W$. By usual abstract nonsense, $V \otimes W$ is unique up to unique isomorphism.

We define the *tensor algebra* $T(V) = \mathbb{R} \oplus V \oplus V \otimes V \oplus \dots$ with multiplication by ‘concatenation’

Definition 4.6. We say the elements in the $V^{\otimes k}$ factor of $T(V)$ are the *degree k elements*.

Definition 4.7. The *exterior algebra* of V , $\Lambda(V)$, is $T(V)/\langle v \otimes v \mid v \in V \rangle$.

Remark 4.5. The motivation for this definition is that it is the universal graded skew-commutative algebra for the vector space V . We want it to be skew commutative because for, eg, differential forms, flipping the components is the same as a change of orientation.

We denote the k th degree component of $\Lambda(V)$ as $\Lambda^k(V)$

Exercise 4.1. Show that given a linear $f: V \rightarrow W$, there is a canonical homomorphism $\Lambda(f): \Lambda(V) \rightarrow \Lambda(W)$

Remark 4.6. Properties of $\Lambda(V)$ (fairly easy exercises)

1. $\Lambda(V)$ is an associative algebra generated by elements in degrees 0, 1, $V \cong \Lambda^1(V)$ and $\Lambda^k(V)$ is generated by elements of the form $v_1 \wedge \dots \wedge v_k$ where $v_i \in V$
2. $\wedge: \Lambda^a(V) \times \Lambda^b(V) \rightarrow \Lambda^{a+b}(V)$ is graded commutative, so $\alpha \wedge \beta = (-1)^{ab} \beta \wedge \alpha$
3. If $v_1, \dots, v_n$ is a basis of V , then the set of $v_{i_1} \wedge \dots \wedge v_{i_k}$ for i_k strictly increasing is a basis of $\Lambda^k(V)$
4. If $X \in V^*$, there is a linear map $\iota_X: \Lambda^k(V) \rightarrow \Lambda^{k-1}(V)$ by $v_1 \wedge \dots \wedge v_k \rightarrow \sum (-1)^{i-1} v_1 \wedge \dots \wedge \langle v_i, X \rangle v_{i+1} \wedge \dots \wedge v_k$. This satisfies a graded Leibniz rule $\iota_X(\alpha \wedge \beta) = (\iota_X \alpha) \wedge \beta + (-1)^a \alpha \wedge (\iota_X \beta)$

Our main application of exterior algebras will be for $V = T_p^*M$. As we vary $p \in M$, we get a vector bundle $\Lambda^k T^*M = \bigcup \Lambda_p^k T_p^*M$. Smooth sections of this are *k -forms*, and the space of k forms is denoted $\Omega^k(M)$, and we have a product $\wedge: \Omega^a(M) \times \Omega^b(M) \rightarrow \Omega^{a+b}(M)$. In local coordinates we can expand a k -form α in the basis $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$ as $\alpha = \sum \alpha_I dx^I$, and we say $\deg \alpha = k$. Given a vector field X we can define $\iota_X \alpha \in \Omega^{k-1}(M)$.

If F is a smooth map from M to N the pullback $T_{F(p)}^*N \rightarrow T_p^*M$ induces a map $\Lambda T_{F(p)}^*N \rightarrow \Lambda T_p^*M$ which respects the wedge product. For submanifolds, the pullback is called restriction.

Exercise 4.2. Prove that if $X \in T_p M$, then $\iota_X \iota_X = 0$, and $\iota_X \iota_Y + \iota_Y \iota_X = 0$

We define $\alpha(X_1, \dots, X_k) = \iota_{X_k} \dots \iota_{X_1} \alpha$

Example 4.2. Locally Λ^0 is a number, and globally it is a smooth global function. Locally Λ^1 is a vector, and globally it is a smooth 1-form. Λ^k is the space of antisymmetric k -linear forms

4.2 Exterior Differential

Theorem 4.1. *There is a unique $\mathbb{R}$ -linear operator $d: \Omega^k U \rightarrow \Omega^{k+1} U$ for any open $U \subseteq M$ with the following properties*

1. (Locality) *On any given open $V \subseteq M$, if $\alpha = \alpha'$ on V , then $d\alpha = d\alpha'$ on V*
2. *If f is a smooth local function, then df is the differential of f*
3. (Graded Leibniz Rule) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$
4. $d^2\alpha = 0$

Proof. (Uniqueness) In any local coordinate chart on U , $d(fdx^I) = df \wedge dx^I$ by (3), so by linearity, we can compute $d\alpha$ for any k -form α . (Existence) Within each chart we define d as above. This satisfies (1) and (2). For (3), this follows from $d(fg) = gdf + f dg$ with the minus signs from commuting terms in exterior product. We want to prove $d^2 = 0$. On functions, $d^2 f = d\left(\sum \frac{\partial f}{\partial x^i} dx^i\right) = \sum d\left(\frac{\partial f}{\partial x^i}\right) dx^i = \sum \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j = 0$ as second partial derivatives commute, and wedge products anticommute. For general α , it follows from the graded Leibniz rule and linearity. Finally check the definitions in different charts agree on the overlap (see uniqueness). $\square$

Example 4.3. The exterior derivative generalises the curl and 3d divergence from vector calculus

Exercise 4.3. If $F: M \rightarrow N$ is a smooth map then we have $F^*: \Omega^k N \rightarrow \Omega^k M$. Show F^* commutes with d .

If α is a 1-form, X and Y are vector fields, then $d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$

5 Integration

M will be a smooth oriented n -dimensional manifold. Let $\omega \in \Omega^n M$ be a top degree form with support in a compact subset of M . Special case: suppose $\text{supp } \omega \subseteq (U, \phi)$ a chart with coordinates $\{x^i\}$ (ordered compatibly with orientation). on $\phi(U) \subseteq \mathbb{R}^n$ there is a smooth function with $\omega = f(x) dx^1 \wedge \cdots \wedge dx^n$. Then $\int_M f(x) dx^1 \wedge \cdots \wedge dx^n$ is defined as $\int_{\phi(U)} f(x) dx^1 \wedge \cdots \wedge dx^n$. This is independent of choice of coordinates (exercise).

Now let ω be an n -form with compact support. We will integrate this with partitions of unity

Proposition 5.1. (Partition of unity) *For any open cover $M = \bigcup_{a \in A} U_a$, there is a countable collection of functions $\rho_i: M \rightarrow \mathbb{R}$ such that*

1. *For all i , $\text{supp}(\rho_i)$ is compact and contained in some U_a (the ρ_i are subordinate to the open cover)*

2. The collection is locally finite (at any point, there is a neighbourhood where only finitely many ρ_i are non-zero)

3. $\rho_i \geq 0$ and $\sum_{i=0}^{\infty} \rho_i(x) = 1$

Proof. No Proof □

Definition 5.1. Let ω be a compactly supported n -form on an oriented manifold M . Given an oriented system of charts (U_α, ϕ_α) covering M and a partition of unity subordinate to $\{(U_\alpha, \phi_\alpha)\}$, we define

$$\int_M \omega = \sum_i \int_{U_{\alpha_i}} \rho_i \omega$$

This is independent of the choice of ρ_i, U_α

Remark 5.1. The integral of n -forms depends on the orientation of M

5.1 Stokes Theorem

Theorem 5.1. Suppose $\omega \in \Omega^{n-1}(M)$ with compact support on a manifold without boundary. Then $\int_M \partial\omega = 0$

Proof. (1) Using a partition of unity, we reduce to the case where ω is supported in a coordinate chart.

(2) By linearity, we reduce to $\omega = f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$. By permuting coordinates, $i = 1$, and we already have f compactly supported.

In this special case, Stokes comes down to computing $\int_{\mathbb{R}^n} \frac{\partial f}{\partial x^1} dx^1 \wedge \cdots \wedge dx^n = \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x^1} dx^1 \right) dx^2 \wedge \cdots \wedge dx^n = 0$ □

Corollary 5.1.1. Suppose α, β are compactly supported differential forms on M and $\deg \alpha + \deg \beta = \dim M - 1$. Then $\int \alpha \wedge d\beta = (-1)^{1+\deg \alpha} \int (d\alpha) \wedge \beta$

Proof. Apply Stokes to $\alpha \wedge \beta$. □

The full version of Stokes treats oriented manifolds with boundary, $\int_M d\omega = \int_{\partial M} \omega$, by a near identical (but slightly messier) proof.

Definition 5.2. A form α is *closed* if $d\alpha = 0$ and *exact* if there is some β such that $\alpha = d\beta$

Remark 5.2. It is immediate that exact forms are closed as $d^2 = 0$. The converse does not necessarily hold.

Definition 5.3. The k th de Rham cohomology is $H_{dR}^k = \{\text{closed } k\text{-forms}\} / \{\text{exact } k\text{-forms}\}$. We call $H_{dR}^*(M) = \bigoplus_{k=0}^n H_{dR}^k(M)$ the *total de Rham cohomology*

Lemma 5.2. Given a smooth map $f: M \rightarrow N$, the pullback $f^*: \Omega^k(N) \rightarrow \Omega^k(M)$ descends to a map on cohomology $H_{dR}^k(N) \rightarrow H_{dR}^k(M)$.

Proof. As d and f^* commute. $\square$

Remark 5.3. The chain rule implies $(f \circ g)^* = g^* \circ f^*$ on cohomology too, as it holds on the level of forms. In particular, diffeomorphic spaces have isomorphic de Rham cohomology

Remark 5.4. The wedge product induces a ring structure on $H_{dR}^*(M)$, as the graded Leibniz rule ensures that the product of closed forms is closed, and if either factor is exact, then $\alpha \wedge \beta$ is exact. As pullbacks respect the wedge product ($f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$), the pullback is a graded ring homomorphism, so $H_{dR}^*(M)$ is a diffeomorphism invariant as a graded ring.

Remark 5.5. Let $M = \mathbb{C}^*$ and let $\alpha = \operatorname{Re}(\frac{1}{2\pi i} \frac{dz}{z})$, a closed 1-form. Show that along the unit circle, the integral of α is 1 and conclude that $[\alpha] \in H_{dR}^1(\mathbb{C}^*)$ is non-trivial by Stokes

Remark 5.6. (Non-examinable) In fact $H_{dR}^*(M)$ is canonically isomorphic as a graded ring to the singular cohomology, so the de Rham cohomology depends only on the homeomorphism type of M . (Morally, k forms can be integrated along k singular chains giving a pairing between $H_{dR}^k(M)$ and $H_k(M, \mathbb{R})$)

6 Flows, Lie Derivatives and Cartan's Formula

6.1 Flows

Suppose X is a compactly supported vector field on a manifold M . Consider the system of ODEs for $x(t) \in M$:

$$\frac{\partial x}{\partial t} = X(x(t))$$

In local coordinates this reads as $\frac{\partial x^i}{\partial t} = X^i(x(t))$ where $X = \sum X^i \frac{\partial}{\partial x^i}$. If we fix initial condition $x(0) = x \in M$, then there is a unique solution to the ODE, $x(t) = \phi_t(x)$ for $t \in (-\varepsilon, \varepsilon)$ depending on x, t smoothly. By compactness, we can expand this to be defined on all $\mathbb{R}$, not just $(-\varepsilon, \varepsilon)$. For fixed t , we may view ϕ_t as a map $M \rightarrow M$. It is easy to check that by uniqueness $\phi_t \phi_s = \phi_{t+s}$, and so ϕ_t is a one parameter family of maps $M \rightarrow M$. As $\phi_t \phi_{-t} = 1_M$, it is a family of diffeomorphisms.

Definition 6.1. We call ϕ_t the *flow generated by the vector field X* .

6.2 Lie Derivatives

We can differentiate functions, vector fields, forms, tensors etc along a given vector field X . The directional derivative along a vector in $\mathbb{R}^n$ is given by comparing $f(x), f(x + tX)$. In the case of a manifold, we replace $x + tX$ with $\phi_t(x)$. Caveat: for vectors, use $(\phi_{-t})_*$ to push everything to the same point (and for forms, use $(\phi_t)^*$)

Definition 6.2. If X is a compactly supported vector field and $x \in M$ is a given point

1. If f is a smooth function, $\mathcal{L}_X f(x) = \frac{\partial}{\partial t}|_{t=0} \phi_t^* f(x)$.
2. If Y is a vector field, $\mathcal{L}_X Y(x) = \frac{\partial}{\partial t}|_{t=0} ((\phi_{-t})Y)(x)$
3. If α is a k -form, $\mathcal{L}_X \alpha(x) = \frac{\partial}{\partial t}|_{t=0} \phi_t^* \alpha(x)$

Lemma 6.1. (*Leibniz rule for Lie derivative*)

1. If f is a smooth function, $\mathcal{L}_X f = X(f) = df(X)$
2. If Y is a vector field and f is a smooth function, $\mathcal{L}_X(Yf) = Y(\mathcal{L}_X f) + \mathcal{L}_X(Y)(f)$
3. If Y is a vector field and α is a 1-form, then $\mathcal{L}_X(\alpha(Y)) = \mathcal{L}_X(\alpha)(Y) + \alpha(\mathcal{L}_X(Y))$
4. If α, β are differential forms, $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta$

Proof. Trivial □

Corollary 6.1.1. (*Lie Bracket and Cartan's magic formula*)

1. If V is a vector field, then $\mathcal{L}_X Y = [X, Y]$
2. If α is a k -form, then $\mathcal{L}_X \alpha = (d\iota_X + \iota_X d)\alpha$

Proof. By (1) and (2) of the above lemma, $(\mathcal{L}_X Y)f = \mathcal{L}_X(Yf) - Y(\mathcal{L}_X f) = XYf - YXf$ as desired.

Suppose α is a 1-form. By (3), $(\mathcal{L}_X \alpha)Y = \mathcal{L}_X(\alpha(Y)) - \alpha(\mathcal{L}_X(Y)) = X(\alpha(Y)) - \alpha([X, Y])$. Recalling that $d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$, the above becomes $(\mathcal{L}_X \alpha)Y = (\iota_X d\alpha)(Y) + d(\alpha(X))(Y)$ as desired for 1-forms. To extend this to k -forms, let $\alpha = \alpha_1 \wedge \dots \wedge \alpha_k$ and verify the Leibniz rule for $d\iota_X + \iota_X d$ □

Example 6.1. Let M be a smooth manifold of dimension $2n$ and ω be a closed 2-form such that $X \mapsto \iota_X \omega$ is an isomorphism between the tangent and cotangent spaces. We call (M, ω) a *symplectic manifold*. Suppose H is a smooth function on M with compact support. Then we can define a vector field X_H such that $\iota_{X_H} \omega = dH$. Let ϕ_t be the one parameter family of diffeomorphisms generated by X_H (Hamiltonian flow). By Cartan's formula, $\frac{\partial}{\partial t}|_{t=0} \phi_t^* \omega = \iota_{X_H} d\omega + d(\iota_{X_H} \omega) = ddH = 0$. By time translation, $\frac{\partial}{\partial t}(\phi_t^* \omega) = 0$, so $\phi_t^* \omega = \omega$, ie Hamiltonian flow preserves the symplectic structure ω . The volume form ω^n is also preserved (Liouville's theorem)

7 Riemannian Geometry

On curves and surfaces, we have a first fundamental form given by (where the surface is parameterised by (u, v)) $g = g_{uu}du^2 + 2g_{uv}dudv + g_{vv}dv^2$, where $g_{ab} = \vec{x}_a \cdot \vec{x}_b$. This is an intrinsic quantity (doesn't depend on the embedding). We generalise this with the following notion

Definition 7.1. Let M be a smooth manifold. A *Riemannian metric* g on M is a positive definite symmetric bilinear form $g: T_p M \times T_p M \rightarrow \mathbb{R}$ for all $p \in M$ which varies smoothly with p (ie for any smooth vector fields X, Y , the function $g(X, Y)$ is smooth)

Remark 7.1. In local coordinates, $g_{ij} = g(\partial_i, \partial_j)$ (where $\partial_i = \frac{\partial}{\partial x^i}$) gives a matrix for the inner product. We write $g = g_{ij}dx^i \otimes dx^j$. g is a section of $T^*M \otimes T^*M \rightarrow M$

Definition 7.2. A diffeomorphism of Riemannian manifolds $f: M \rightarrow N$ is an *isometry* if $F^*g_N = g_M$

Example 7.1.

1. $\mathbb{R}^n$ with the standard Euclidean metric $g_{Euc} = \sum_i dx^i \otimes dx^i$ is a Riemannian manifold. This also descends to a Riemannian metric on $T^n = \mathbb{R}^n/\mathbb{Z}^n$ as translation by $\mathbb{Z}^n$ preserves the metric.
2. The unit sphere $\iota: S^n \subseteq \mathbb{R}^{n+1}$ has a canonical round metric $g_{S^n} = \iota^*g_{Euc}$
3. If $\Gamma \leq O(n)$ is a finite group which acts freely (and trivially acting properly discontinuously) on S^n , we get a quotient manifold S^n/Γ which inherits a round metric (eg $\mathbb{RP}^2 = S^2/\{\pm 1\}$)
4. (Hyperbolic space) Consider the Riemannian manifold with underlying manifold the open unit ball in $\mathbb{R}^n$, called B^n and hyperbolic Riemannian metric $g_{hyp} = \frac{4}{(1-|x|^2)^2} \sum dx^i \otimes dx^i$. In the $n = 2$ case, this is the Poincaré disk model.
5. If Γ is a discrete subgroup of $\text{Isom}(\mathbb{H})$ which acts freely and properly discontinuously. Then $M = \mathbb{H}/\Gamma$ inherits g_{hyp} . These are called *hyperbolic manifolds*

There are many things a Riemannian metric allows us to construct

1. The pointwise inner product induces an isomorphism $T_p M \cong T^p M$ which in index notation we write $v = v^i \partial_i \mapsto v_j dx^j$ where $v_j = g_{ij}v^i$ with inverse $v^i = g^{ij}v_j$ where (g^{ij}) is the inverse matrix to (g_{ij}) . This is sometimes called the musical isomorphism, as sometimes $\sharp$ and $\flat$ are used to denote it.
2. Suppose we have a smooth function f on M . Then we have df which is a 1-form. Under the musical isomorphism this corresponds to a vector field ∇f , which we call the *gradient* vector field. The flow generated by ∇f is called the gradient flow.

3. Suppose we have a submanifold $\iota: N \rightarrow M$. Then g pulls back to a Riemannian metric on N (and this even works for immersed submanifolds). Check that for surfaces in $\mathbb{R}^3$, this is the first fundamental form.
4. Given a smooth curve $\gamma: [0, 1] \rightarrow M$ and $\dot{\gamma} = \frac{\partial \gamma}{\partial t}$, the *length* of the curve is $L(\gamma) = \int_0^1 \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt$ (the square root makes it invariant under transformation). The *energy* is $E(\gamma) = \frac{1}{2} \int_0^1 g(\dot{\gamma}, \dot{\gamma}) dt$. The distance between $p, p' \in M$ is given by the infimum of lengths of paths between them. This induces a metric structure.
5. The volume element $dvol_g = \sqrt{\det(g_{ij})} dx^1 \dots dx^n$. This is independent of coordinate system. If M is oriented, then $dvol_g$ can be regarded as an n -form. More generally, this defines a measure and this can be regarded as a measure density regardless of orientability. If M is compact then we can define $\text{Vol } M = \int_M dvol$. This depends on (M, g) but not on the choice of coordinates. More generally for compact immersed submanifolds we have the pullback metric and we can take the volume with respect to this. This is sometimes called the *area* of N .
6. Suppose we have a smooth function f on M with compact support. Then the *Dirichlet energy* of f is $E(f) = \frac{1}{2} \int_M (\nabla f, \nabla f)_g dvol_g$.
7. The inner product on $T_p M$ induces an inner product on $T_p^* M$, either via the musical isomorphism or by g^{ij} in coordinates (they induce the same inner product). More generally we get an inner product on $\bigwedge^k T_p^* M$ characterised in the following way. Given an orthonormal basis e_i of $T_p^* M$, $e_{i_1} \wedge \dots \wedge e_{i_k}$ is a basis of $\bigwedge^k T_p^* M$ which we declare to be orthonormal. You can check that this does not depend on basis.
8. Suppose M is oriented so $dvol_g$ makes sense as an n -form. We can define the *hodge ** isomorphism $*$: $\bigwedge^k T_p^* M \rightarrow \bigwedge^{n-k} T_p^* M$, given by $\alpha \wedge *\beta = (\alpha, \beta)_g dvol_g$ for all k -forms α, β . In coordinates, $*(e_{i_1} \wedge \dots \wedge e_{i_k}) = \pm e_{j_1} \wedge \dots \wedge e_{j_{n-k}}$ (exercise: compute the sign) where $\{j_1, \dots, j_{n-k}\}$ is the complement of $\{i_1, \dots, i_k\}$. You can check that $*^2 \alpha = (-1)^{k(n-k)} \alpha$ on k -forms α . On $\mathbb{R}^3$, we let e_1, e_2, e_3 be an orthonormal basis of cotangent vectors. Then $*e_1 = e_2 \wedge e_3, *e_2 = e_3 \wedge e_1, *e_3 = e_1 \wedge e_2$.

Definition 7.3. A smooth immersed curve $\gamma: [0, 1] \rightarrow M$ is called a *geodesic* if it is a critical point of the energy $E(\gamma)$.

Remark 7.2. Length minimising curves are geodesics under arclength parameterisation. Conversely geodesics do not need to be length minimising by our definition (eg the longer path around a great circle).

Proposition 7.1. The curve $\gamma: [0, 1] \rightarrow M$ is a geodesic if and only if it satisfies the geodesic equation, which reads as follows in local coordinates

$$\frac{\partial^2 x^i}{\partial t^2} + \Gamma_{jk}^i \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial t} = 0$$

where $\Gamma_{jk}^i = \Gamma_{kj}^i = \frac{1}{2}g^{is} \left(\frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s} \right)$, the Christoffel symbol

Proof. The proof proceeds using the calculus of variations. We focus on one coordinate chart by cutting up the curve.

$$E(\gamma) = \frac{1}{2} \int_0^1 g_{ij}(x) \frac{\partial x^i}{\partial t} \frac{\partial x^j}{\partial t} dt$$

We take a first order perturbation $\gamma = x^i(t) \mapsto x^i(t) + \delta x^i(t)$. Then

$$\delta E(\gamma) = \frac{1}{2} \int_0^1 \partial_s g_{ij} \frac{\partial x^i}{\partial t} \frac{\partial x^j}{\partial t} \delta x^s(t) + g_{ij} \frac{\partial x^i}{\partial t} \frac{\partial \delta x^j}{\partial t} + g_{ij} \frac{\partial \delta x^i}{\partial t} \frac{\partial x^j}{\partial t} dt$$

As the end points are fixed, if we integrate by parts (to move the derivatives away from the first variation term) and relabelling indices, we get

$$\begin{aligned} \delta E(\gamma) &= \frac{1}{2} \int_0^1 \left(\frac{1}{2} \partial_s g_{kj} \frac{\partial x^k}{\partial t} \frac{\partial x^j}{\partial t} - \frac{\partial}{\partial t} \left(g_{ks} \frac{\partial x^k}{\partial t} \right) \right) \delta x^s(t) dt \\ &= \int_0^1 \left(\frac{1}{2} \partial_s g_{kj} \frac{\partial x^k}{\partial t} \frac{\partial x^j}{\partial t} - \partial_j g_{ks} \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial t} - g_{ks} \frac{\partial^2 x^k}{\partial t^2} \right) \delta x^s(t) dt \\ &= \int_0^1 \left(\frac{1}{2} \partial_s g_{kj} \frac{\partial x^k}{\partial t} \frac{\partial x^j}{\partial t} - \frac{1}{2} \partial_j g_{ks} \frac{\partial x^k}{\partial t} \frac{\partial x^j}{\partial t} - \frac{1}{2} \partial_k g_{js} \frac{\partial x^k}{\partial t} \frac{\partial x^j}{\partial t} - g_{is} \frac{\partial^2 x^i}{\partial t^2} \right) \delta x^s(t) dt \\ &= - \int_0^1 g_{is} \left(\frac{\partial^2 x^i}{\partial t^2} + \Gamma_{jk}^i \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial t} \right) \delta x^s(t) dt = 0 \end{aligned}$$

So the term in brackets must identically vanish, and we are done (as the converse holds by the same computation) $\square$

Remark 7.3. The geodesic equation is second order, so we need an initial point and direction of the geodesic to solve it, and the solution is unique for sufficiently short timescales. The Hopf-Rinow theorem states that the following are equivalent:

1. The metric space induced by (M, g) is complete
2. For every $p \in M, v \in T_p M$, the geodesic equation can be solved for all time

Given $v \in T_p M$, we define $\exp_p(v) = \gamma(1)$ for γ the geodesic starting at p in direction v . This is called the exponential map, and gives geodesic coordinates on a neighbourhood of p , but such coordinates may not be unique.

(Note-taker's remark: Hopf-Rinow generalises to length spaces, where we can state that every complete locally compact length space is a proper geodesic space. I will not define these terms.)

Important caveat: Γ_{jk}^i is not a tensor. Instead, it is a Levi-Civita connection

Lemma 7.1. (*Geodesic coordinate trick*) Around a given $p \in M$ there are local coordinates $\{x^i\}$ such that $g_{ij}(p) = \delta_{ij}$ and $\partial_k g_{ij}(p) = 0$. In this situation, $\Gamma_{jk}^i = 0$.

Proof. We transform coordinates to $\{\tilde{x}^i\}$. Then $\tilde{g}_{ab} = g_{ij} \frac{\partial x^i}{\partial \tilde{x}^a} \frac{\partial x^j}{\partial \tilde{x}^b}$. By a linear change of coordinates, we can make $g_{ij} = \delta_{ij}$. Now we perform the non-linear transformation $x^i = \tilde{x}^i + \frac{1}{2} c_{jk}^i \tilde{x}^j \tilde{x}^k$ for some $c_{jk}^i = c_{kj}^i$ to be determined.

$$\frac{\partial g_{ab}}{\partial \tilde{x}^k}(0) = \frac{\partial g_{ij}}{\partial x^k} \delta_{ia} \delta_{jb} + \delta_{ij} \delta_{jb} c_{ak}^i + c_{bk}^j \delta_{ij} \delta_{ia} = \frac{\partial g_{ab}}{\partial x^k} + c_{ak}^b + c_{bk}^a$$

To set this equal to 0, we have $n \frac{1}{2} n(n+1)$ equations in $\frac{1}{2} n^2(n+1)$ variables, and the homogeneous related equation has only the 0 solution, so this has a unique solution. $\square$

Remark 7.4. We use these as we wish to have local coordinates where $g_{ij} = \delta_{ij} + O(|x|^2)$. In general we can't remove the higher order term (because of curvature)

8 Connections and Curvature

8.1 Connections

A connection is a device to take derivatives of sections of vector bundles. Notation: In this section $E \rightarrow M$ is a vector bundle of rank r . $E \otimes \bigwedge^k T^*M \rightarrow E$ is a vector bundle with fibres $E_p \otimes \bigwedge^k T_p^*M$ and global sections $\Omega^k(M, E)$

Definition 8.1. A *connection* on E is an operator $D: \Omega^0(U, E) \rightarrow \Omega^1(U, E)$ for open $U \subseteq M$ satisfying

1. (Locality) If two sections s, s' agree on a given open subset V , then $Ds = Ds'$ on V
2. (Leibniz rule) $D(fs) = df \otimes s + fDs$ for any local smooth function f

Equivalent view: For any vector field X , we can contract X with the T^*M factor of Ds to get $D_X s \in \Omega^0(U, E)$, called the *covariant derivative* and the Leibniz rule for all such X is equivalent to the Leibniz rule above

Remark 8.1. (Alternative definition) There is an alternative definition via $\begin{cases} \text{Horizontal Distributions} \\ \text{Connections on Principal bundles} \end{cases}$ see Kovalev's notes.

In local coordinates, if we take a trivialising basis of sections $\{e_a\}: E \rightarrow U$, so for a section s , $s = \sum s^a(x) e_a$. Then

$$Ds = \sum_a ds^a \otimes e_a + s^a D e_a$$

by the Leibniz rule, or in the covariant terms,

$$D_X s = \sum ds^a(X) \otimes e_a + s^a D_X e_a$$

We use the shorthand $D_i = D_{\partial_i}$. We define the matrix of local 1-forms, $De_a = \sum A_a^b e_b$ where A_a^b are local 1-forms, $A_a^b = \sum A_{ak}^b dx^k$. Then A_{ak}^b are functions. Now $D_k e_a = \sum_b A_{ak}^b e_b$. Then we have a concrete expression

$$Ds = \sum_a ds^a \otimes e_a + \sum_b s^b \otimes De_b = \sum_{a,b} (ds^a + A_b^a s^b) \otimes e_a$$

We can therefore write

$$D_k s = \sum_{a,b} (\partial_k s^a + A_{bk}^a s^b) e_a$$

The A_{ak}^b are called the connection coefficients

8.2 Curvature

Curvature measures in some sense the non-commutativity of the covariant derivative.

Lemma 8.1. $(D_i D_j - D_j D_i)s = \sum_{a,b} F_{bij}^a s^b e_a$ where

$$F_{bij}^a = \partial_i A_{bj}^a - \partial_j A_{bi}^a + A_{ci}^a A_{bj}^c - A_{cj}^a A_{bi}^c$$

Proof. We compute

$$\begin{aligned} D_i D_j s &= D_i \left(\sum_{a,b} (\partial_j s^a + A_{bj}^a s^b) e_a \right) \\ &= \sum_a \left(\partial_i \left(\partial_j s^a + \sum_b A_{bj}^a s^b \right) \right) + \sum_c A_{ci}^a (\partial_j s^c + A_{bj}^c s^b) e_a \end{aligned}$$

Likewise we swap the indices to get $D_j D_i s$. Taking the difference we get the result. $\square$

Remark 8.2. Notation: $F_b^a = \frac{1}{2} \sum_{i,j} F_{bij}^a dx^i \wedge dx^j$. This is a matrix-valued 2-form with matrix indices a, b . It is called the *curvature matrix*.

You can view A_b^a as a matrix of 1-forms. With this notation, we have $F_b^a = dA_b^a + A_c^a \wedge A_b^c$. More memorably, $F = dA + A \wedge A$

Exercise 8.1. (On example sheet 2) For more general vector fields X, Y we have

$$(D_X D_Y - D_Y D_X - D_{[X,Y]})s = \sum_{a,b} F_b^a(X, Y) s^b e_a$$

and so the curvature is independent of the coordinates.

We give a more ‘elegant’ approach. We have a connection operator $D: \Omega^0(M, E) \rightarrow \Omega^1(M, E)$ and want to extend this to $D: \Omega^0(M, E) \rightarrow \Omega^1(M, E)$. We have the Leibniz rule $D(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg \omega} \omega \otimes Ds$ and this gives us an extension

for k -forms. This is subject to a ‘consistency check’ $D(f\omega \otimes s) \stackrel{?}{=} D(\omega \otimes fs)$. We compute

$$\begin{aligned} D(f\omega \otimes s) &= d(f\omega) \otimes s + (-1)^{\deg \omega} f \otimes Ds \\ &= d\omega \otimes fs + (df \wedge \omega) \otimes \omega + (-1)^{\deg \omega} f\omega \otimes Ds \\ &= d\omega \otimes fs + (-1)^{\deg \omega} \omega \wedge D(fs) \end{aligned}$$

so it works. Next we define $F = D^2: \Omega^0(U, E) \rightarrow \Omega^2(U, E)$. Then

$$\begin{aligned} F(fs) &= DD(fs) \\ &= D(df \otimes s + fDs) \\ &= d(df) \otimes s - df \wedge Ds + df \wedge Ds + fD^2s \\ &= fD^2s \end{aligned}$$

as $d^2 = 0$. Therefore F is a differential operator of order at most 2 with $F(fs) = fF(s)$, and hence F is 0th order, which therefore acts independently at each point of M . Therefore F is represented by a section in $\Omega^2(\text{End}(E))$. The curvature matrix F_b^a is simply the coefficient of F with respect to the basis $\{e_a\}$

Remark 8.3. Curvature is a local obstruction to the existence of a basis of parallel sections (sections with $Ds = 0$). Suppose there is a local basis of parallel sections $\{e_i\}$. Then $D^2e_i = 0$, so $F = 0$. Conversely if $D^2 = 0$, then (using some ODE theory) on any contractible open U there is a basis of parallel sections

8.3 Levi-Civita Connection

Let $E = TM \rightarrow M$. Then in a coordinate chart suppose we have a connection ∇ on TM with coefficients Γ_{jk}^i (not necessarily the Christoffel symbols) defined by

$$\nabla_k \partial_j = \sum_i \Gamma_{jk}^i \partial_i$$

On TM we have a Lie bracket, and this structure allows us to define the torsion

Definition 8.2. The *torsion* of ∇ is $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$

Exercise 8.2. In local coordinates, $T(X, Y) = (\Gamma_{jk}^i - \Gamma_{kj}^i)X^k Y^j$, so ∇ is torsion free if and only if $\Gamma_{jk}^i = \Gamma_{kj}^i$

Theorem 8.2. *On any Riemannian manifold (M, g) there is a unique torsion free connection ∇ which is compatible with the metric in the sense that we have a “Leibniz rule”*

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

Proof. (Uniqueness) We wish to determine Γ_{jk}^i . Let $X = \partial_i, Y = \partial_j, Z = \partial_k$. Then metric compatibility reads

$$\partial_k g_{ij} = \Gamma_{ik}^p g_{pj} + \Gamma_{jk}^p g_{ip}$$

Permuting i, j, k cyclically, we get another two similar equations. We recall that (g^{iq}) is the inverse matrix to (g^{iq}) so $\Gamma_{jk}^p g_{qp} g^{iq} = \Gamma_{jk}^i$. Adding the first two equations and subtracting the third from before, we get

$$\Gamma_{jk}^p g_{ip} = \frac{1}{2} (\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk})$$

and multiplying with the inverse matrix gives

$$\Gamma_{jk}^i = \frac{1}{2} g^{iq} (\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk})$$

, and in fact this is exactly the Christoffel symbol (justifying our use of Γ for them).

For existence, we use the formula above to define the coefficients, and verify that this is a well-defined (ie coordinate independent - by uniqueness on the overlaps) torsion-free connection compatible with g . $\square$

Definition 8.3. The ∇ above is called the *Levi-Civita* connection.

Remark 8.4. Intuitively, using the geodesic coordinates, $\Gamma_{jk}^i(p) = 0$, and $\nabla = d$, $\nabla_i = \partial_i$. ∇ on TM also induces a connection on T^*M via the Leibniz rule $\nabla_i(\omega, X) = (\nabla_i \omega, X) + (\omega, \nabla_i X)$, so we can apply this to dx^i, ∂_j to define it, and this turns out to be $\nabla_k dx^j = -\Gamma_{ki}^j dx^i$.

Suppose T is a tensor, eg $T = T_{ij} dx^i \otimes dx^j$ valued in $E = T^*M \otimes T^*M$, this can also be denoted T_{ij} . $\nabla_k \nabla_l T_{ij}$ is valued in $(T^*M)^{\otimes 4}$

Remark 8.5. $\nabla g = 0$. $g = g_{ij} dx^i \otimes dx^j$, so $\nabla_Z(g(X, Y)) = (\nabla_Z g)(X, Y) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$, but then metric compatibility says $\nabla_Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$, so we are done.

Example 8.1. (Divergence operator on a vector field) Let X be a vector field on M . The divergence of X is $\text{div } X = \sum dx^i (\nabla_i X)$, a function independent of local coordinates. Suppose M is oriented. Then $\text{div}(X) d\text{vol}_g = d(\iota_X d\text{vol}_g)$ (use geodesic coordinates to assume $g_{ij} = \delta_{ij} + O(|x|^2)$, then $(\nabla_i X)(p) = (\partial_i X)(p)$ and $\text{div}(X) = \partial_i X^i$)

This trick is quite general - if you want to prove first order identities you can pretend g is Euclidean

Remark 8.6. Submanifold connection: If $\iota: M \rightarrow N$ is an embedded submanifold and $\bar{g}$ is a Riemannian metric on N with $g = \iota^* \bar{g}$ on M , at any point on M we let π be the orthogonal projection of $T_p N$ onto $T_p M$. Let X, Y be local vector fields on M (which extend smoothly to a local vector field on N). Let $\bar{\nabla}$ be the Levi-Civita connection on N . Then the vector field on M defined by $\nabla_X Y = \pi(\bar{\nabla}_X Y)$ and you can check this is a connection on M . You can further check that it is compatible with g and torsion free, so ∇ is the Levi-Civita connection on M

8.4 Riemannian, Ricci, and Scalar curvature

Definition 8.4. The *curvature 2-form* is

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

Remark 8.7. In coordinate form, this is

$$R_{bij}^a = \partial_i \Gamma_{bj}^a - \partial_j \Gamma_{bi}^a + \Gamma_{ci}^a \Gamma_{bj}^c - \Gamma_{cj}^a \Gamma_{bi}^c$$

After lowering indices, we get the *Riemannian Curvature*

$$R_{klij} = g_{ka} R_{lij}^a$$

, ie $Rm(Z, W, X, Y) = g(R(X, Y)W, Z)$

Remark 8.8. In geodesic coordinates around p , $\partial_i g_{jk} = 0$ so $\Gamma_{ij}^k = 0$, and $R_{klij} = \frac{1}{2}(\partial_i \partial_l g_{kj} - \partial_i \partial_k g_{jl} - \partial_j \partial_l g_{ki} + \partial_j \partial_k g_{il})$, so if Rm doesn't vanish at a point, you can't eliminate second order terms in the geodesic coords

Corollary 8.2.1.

1. $R_{ijkl} = -R_{ijlk}$ (and same in the first two indices)
2. $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ (Bianchi identity)
3. $R_{ijkl} = R_{klij}$

Proof. This is coordinate independent, so do it in geodesic coords $\square$

Remark 8.9. So the Riemmanian curvature is a symmetric bilinear form on $\bigwedge^2 T_p M$.

Definition 8.5. If X, Y are linearly independent, they span a plane corresponding to $X \wedge Y$, called Π . The *sectional curvature* along the 2-plane Π is $\sec(\Pi) = \frac{Rm(X, Y, X, Y)}{|X \wedge Y|_g} = \frac{Rm(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}$

Remark 8.10. This is invariant under change of choice of X, Y

Definition 8.6. The *Ricci curvature* is a symmetric bilinear form on $T_p M$ defined by $\text{Ric}(X, Y) = \sum_i Rm(e_i, X, e_i, Y)$ for a basis e_i . The *scalar curvature* is $\text{scal}_g = \sum_i \text{Ric}(e_i, e_i)$

Exercise 8.3. Compute the Riemannian curvature of $S^n, \mathbb{R}^n, \mathbb{H}^n$.

Remark 8.11. Symmetric bilinear forms on $T_p M$ have a partial order by their induced “norm”, so we can talk about $\text{Ric} \geq \Lambda g$

9 Hodge Theory and Harmonic Representative

Let (M, g) be an n dimensional compact orientable Riemannian manifold. For $k = 0, 1, \dots, n$, the vector bundle $E = \bigwedge^k T^*M \rightarrow M$ has a fibrewise inner product structure and so we can define an inner product for k -forms as

$$(\alpha, \beta)_{L^2} = \int_M (\alpha, \beta)_g \text{dvol}_g$$

Likewise $\bigwedge^k T^*M \otimes T^*M \rightarrow M$ has a fibrewise inner product inducing the L^2 inner product on $\Omega^k(M, T^*M)$.

First order differential operators look like $D = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha$ for $k = 1$. The formal adjoint is the operator D^* so $(D\alpha, \beta) = (\alpha, D^*\beta)$. In our context we know of the differential operators d, ∇ . The formal adjoints are $d^*: \Omega^{k+1}(M) \rightarrow \Omega^k(M)$, and $\nabla^*: \Omega^k(M, T^*M) \rightarrow \Omega^k(M)$ so that $(d\alpha, \beta) = (\alpha, d^*\beta)$ for any k -form α and $k+1$ form β and similar for ∇ . These have simple formulae:

$$\begin{aligned} d\alpha &= \sum_i dx^i \wedge \nabla_i \alpha \\ d^*\alpha &= -g^{ij} \iota_{\partial_i} (\nabla_j \alpha) \\ \nabla \alpha &= \sum_i dx^i \otimes \nabla_i \alpha \\ \nabla^* \alpha &= g^{ij} \iota_{\partial_i} \nabla_j \alpha \end{aligned}$$

where the ι means interior multiplication for d^* , and contraction with the T^*M factor for ∇^* . That these are adjoints can be checked by Stokes formula/integration by parts.

Remark 9.1. (d^* vs $*$) d^* can be expressed as $d^* = (-1)^{n(k+1)+1} * \circ d \circ *$

The Laplacian is a second order differential operator on (M, g) .

Definition 9.1. The *Hodge Laplacian* on k -forms is $\Delta_{Hodge} \alpha = (dd^* + d^*d)\alpha$. The *rough Laplacian* is $\Delta \alpha = \nabla^* \nabla \alpha$

Remark 9.2. On $\mathbb{R}^2$, both of these agree and are the usual Laplacian

Lemma 9.1. Suppose α is a smooth k -form on compact oriented M . Then

$$(\Delta_{Hodge} \alpha, \alpha) = (d\alpha, d\alpha) + (d^*\alpha, d^*\alpha) = \|d\alpha\|^2 + \|d^*\alpha\|^2 \geq 0$$

with equality if and only if $d\alpha = d^*\alpha = 0$, ie $\Delta_{Hodge} \alpha = 0$

Proof. Trivial □

Definition 9.2. Such $\alpha \in \Omega^k(M)$ are called *harmonic*, and form a vector space $\mathcal{H}_k$

Example 9.1. If $M = T^n$ and $g = \sum dx^i \otimes dx^i$, the k -form α can be written as a Fourier series $\alpha = \sum \alpha_I dx^I$, $\alpha_I = \sum_{\omega \in \mathbb{Z}^n} \alpha_{I,\omega} e^{2\pi i \omega x}$ and the L^2 norm is (by Parseval) $\sum_I \sum_{\omega \in \mathbb{Z}^n} |\alpha_{I,\omega}|^2$, and $d\alpha = \sum dx^i \wedge \partial_i \alpha$, $d^* \alpha = -\sum \iota_{\partial_i} \partial_i \alpha$, so $\Delta = \Delta_{Hodg} = -\sum_i \partial_i^2$ and $(\Delta \alpha, \alpha) = \sum_I \sum_{\omega \in \mathbb{Z}^n} |2\pi \omega|^2 |\alpha_{I,\omega}|^2 \geq 0$ with equality if and only if $\alpha_{I,\omega} = 0$ for any non-zero ω , so $\mathcal{H}_k = \text{span}\{dx^I\}$

Example 9.2. If $k = 0$, $\Delta = \Delta_{Hodg}$,

$$\int_M (\Delta f) h \text{dvol}_g = (df, dh) = \int_M g^{ij} (\partial_i g) (\partial_j h) (\sqrt{\det g}) dx^1 \dots dx^n$$

as $d^* f = 0$ for degree reasons. Integrating by parts we get

$$- \int_M h \partial_j (g^{ij} \partial_i f \sqrt{\det g}) dx^1 \dots dx^n$$

As these hold for any h , we must have

$$\Delta f = \frac{1}{\sqrt{\det g}} \partial_j (g^{ij} \partial_i f \sqrt{\det g})$$

Example 9.3. If $k = 1$, then $\Delta_{Hodg} \alpha = \Delta \alpha + \text{Ric}_{ij} \alpha^j dx^i$

Now Δ, Δ_{Hodg} are 2nd order differential operators, and their difference is 0th order. (In general, the difference depends on the Riemannian curvature, and for degree 1 it only depends on the Ricci curvature)

Theorem 9.2. (*Harmonic Representation*) *On any smooth compact oriented Riemannian manifold (M, g) , each de Rham cohomology class has a unique harmonic representative. Equivalently there is a well defined map from harmonic k -forms to de Rham classes which is an isomorphism.*

Proof. No Proof □

(The heuristic idea is to minimise L^2 norm within the class - messy PDE theory stuff)