

Functional Analysis

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1 Hahn-Banach Extension Theorems

Definition 1.1. Let X be a normed space over $\mathbb{R}$ or $\mathbb{C}$. The *dual space* of X , denoted X^* , is the space of all bounded linear functionals on X .

Remark 1.1. This is a Banach space in the operator norm $\|f\| = \sup\{|f(x)| \mid x \in B_X\}$, where B_X is the closed unit ball of X . We also denote by S_X the unit sphere

Example 1.1.

1. $l_p^* \cong l_q$ where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p < \infty$, and $c_0^* \cong l_1$
2. If H is a Hilbert space, then $H^* \cong H$ (this map is conjugate linear in the complex case)

Remark 1.2.

1. $X \sim Y$ means the normed spaces X and Y are isomorphic - there is a linear bijection between them which is continuous with continuous inverse.
2. $X \cong Y$ means X and Y are isometrically isomorphic, ie there is a bijective linear isometry between X and Y .
3. For $x \in X$ and $f \in X^*$, we write $\langle x, f \rangle = f(x)$. Note that $|\langle x, f \rangle| = |f(x)| \leq \|f\| \cdot \|x\|$

Definition 1.2. Let X be a real vector space. A functional $p: X \rightarrow \mathbb{R}$ is *positive homogeneous* if $p(tx) = tp(x)$ for all $x \in X$ and $t \geq 0$. It is *subadditive* if $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

We recall:

Definition 1.3. Let $(P, \leq)$ be a poset. We say $x \in P$ is an *upper bound* for $A \subseteq P$ if $a \leq x$ for all $a \in A$. For $A \subseteq P$, it is a *chain* if it is linearly ordered by $\leq$. $x \in P$ is *maximal* if for all $y \in P$, $x \leq y$ implies $x = y$.

Proposition 1.1. (*Zorn's lemma*) If $P \neq \emptyset$ is a poset such that every non-empty chain has an upper bound, then P has a maximal element.

Proof. See your favourite logic course □

Theorem 1.1. (*Hahn-Banach*) Let p be a positive homogeneous subadditive functional on the real vector space X and $g: Y \rightarrow \mathbb{R}$ be a linear functional such that for all $y \in Y$, $g(y) \leq p(y)$. Then there is a linear $f: X \rightarrow \mathbb{R}$ which restricts to g on Y and for all $x \in X$, $f(x) \leq p(x)$.

Proof. Let $P = \{(Z, h) \mid Z \leq X, h: Z \rightarrow \mathbb{R} \text{ is linear, } h|_Y = g, \forall z \in Z. h(z) \leq p(z)\}$. $(Z_1, h_1) \leq (Z_2, h_2)$ if $Z_1 \leq Z_2$ and h_2 extends h_1 . The conditions of Zorn are satisfied, so we get a maximal element (X_0, f_0) . If $X_0 = X$, we are done. Otherwise, fix $x_1 \in X \setminus X_0$ and set $X_1 = X_0 + \mathbb{R}x_1$. Fix $\alpha \in \mathbb{R}$ and define $f_1: X \rightarrow \mathbb{R}$ by $f_1(x + \lambda x_1) = f_0(x) + \lambda\alpha$. Then f_1 is linear, extends f_0 . We wish to find α so $f_1 \leq p$, and the result follows as $(X_0, f_0) < (X_1, f_1)$. We want $\alpha \in \mathbb{R}$ such that

$$f_0(x) + \lambda\alpha \leq p(x + \lambda x_1)$$

This is equivalent to

$$f_0(x) + \alpha \leq p(x + x_1)$$

and

$$f_0(x) - \alpha \leq p(x - x_1)$$

by positive homogeneity. This happens if and only if

$$f_0(y) - p(y - x_1) \leq \alpha \leq p(x + x_1) - f_0(x)$$

which can be done if and only if the outer terms satisfy that inequality for all x, y (as then we set α to be the supremum of the left hand side). This follows as

$$f_0(x) + f_0(y) = f_0(x + y) \leq p(x + y) \leq p(x + x_1) + p(y - x_1)$$

□

Definition 1.4. Let X be a real or complex vector space. A *seminorm* on X is a functional $p: X \rightarrow \mathbb{R}$ such that $p(x) \geq 0$ for all $x \in X$, $p(\lambda x) = |\lambda|x|$ for all scalar λ and $x \in X$, and $p(x + y) \leq p(x) + p(y)$.

Remark 1.3. All norms are seminorms, and all seminorms are positive homogeneous and subadditive.

Theorem 1.2. (*Hahn-Banach 2*) Let p be a seminorm on X and $g: Y \rightarrow \text{scalars}$ be a linear functional such that for all $y \in Y$, $|g(y)| \leq p(y)$. Then there is a linear $f: X \rightarrow \text{scalars}$ which restricts to g on Y and for all $x \in X$, $|f(x)| \leq p(x)$

Proof. In the real case, this is just a special case of the first Hahn-Banach theorem, as $f(x) \leq p(x)$ and $-f(x) = f(-x) \leq p(-x) = p(x)$, so $|f(x)| \leq p(x)$. In the complex case, let $g_1 = \text{Re } g: Y \rightarrow \mathbb{R}$. This is dominated by p , so by the real case there is a real-linear extension $f_1: X \rightarrow \mathbb{R}$ which is dominated by p . Let $f: X \rightarrow \mathbb{C}$ be the unique complex linear functional whose real part is f_1 . Uniqueness shows that f extends g . Fix $x \in X$. We need $|f(x)| \leq p(x)$. Choose λ of modulus 1 such that $|f(x)| = \lambda f(x)$. Then

$$|f(x)| = f(\lambda x) = f_1(\lambda x) \leq p(\lambda x) = p(x)$$

so we are done. $\square$

Corollary 1.2.1. Let X, p as in the above theorem. Let $x_0 \in X$. Then there is a linear functional f on X such that $|f(x)| \leq p(x)$ and $f(x_0) = p(x_0)$

Proof. Define $g: \text{span}(x_0) \rightarrow \text{scalars}$ by $g(\lambda x_0) = \lambda p(x_0)$. Then apply the above theorem. $\square$

Theorem 1.3. (*Hahn-Banach, again*) Let X be a normed space.

1. Given $Y \leq X$ and $g \in Y^*$ there is $f \in X^*$ such that $f|_Y = g$ and $\|f\| = \|g\|$.
2. Let $x_0 \in X \setminus \{0\}$. There is $f \in S_{X^*}$ such that $f(x_0) = \|x_0\|$.

Proof. 1. Let $p(x) = \|g\| \cdot \|x\|$. This is a seminorm which bounds g by definition of $\|g\|$. By theorem 2, there is a linear functional f on X such that $f|_Y = g$ and $|f(x)| \leq p(x) = \|g\| \cdot \|x\|$, ie f is a bounded linear functional of norm at most $\|g\|$, with equality as $f|_Y = g$.

2. Let $p(x) = \|x\|$. By the above corollary, there is a linear functional f on X such that $f(x_0) = \|x_0\|$ and for all $x \in X$, $\|f(x)\| \leq \|x\|$. Hence f has norm 1. $\square$

Remark 1.4.

1. (1) of the above is a linear version of Tietze's extension theorem.
2. (2) of the above is analogous to Urysohn's lemma, as it gives an f that separates points (by considering $x - y$).
3. The f of part (2) is called a *norming functional* at x_0 , it tells you $\|x_0\| = \max\{|g(x_0)| \mid g \in B_{X^*}\}$. It is also called a *support functional* at x_0 , as if $\|x_0\| = 1$ and X is real, then $B_X \subseteq \{x \in X \mid f(x) \leq 1\}$. The hyperplane $\{x \in X \mid f(x) = 1\}$ is tangent to B_X at x_0 .

1.1 Bidual

Definition 1.5. Let X be a normed space. Then $X^{**} = (X^*)^*$ is the *bidual* (or second dual) of X

For $x \in X$, define $\hat{x}: X^* \rightarrow \text{scalars}$ via $f \mapsto f(x)$. Then $\hat{x}$ is linear and bounded, as $|\hat{x}(f)| = |f(x)| \leq \|f\| \cdot \|x\|$ for all $f \in X^*$, with norm at most $\|x\|$, so $\hat{x} \in X^{**}$.

Theorem 1.4. The map $x \mapsto \hat{x}$ from $X \rightarrow X^{**}$, called the *canonical embedding* is an isometric linear map.

Proof. Remark 1.4(3) shows it is isometric, and it is trivially linear, so we are done. $\square$

Remark 1.5.

1. $\langle f, \hat{x} \rangle = \langle x, f \rangle = f(x)$ for $x \in X, f \in X^*$
2. $X \cong \{\hat{x} \mid x \in X\} = \hat{X} \subseteq X^{**}$, and $\hat{X}$ is closed if and only if X is complete
3. $\bar{\hat{X}}$ (the closure in X^{**}) is a Banach space containing a dense isometric copy of X (a completion)

Definition 1.6. A normed space X is *reflexive* if the canonical embedding $X \rightarrow X^{**}$ is surjective.

Example 1.2.

1. $l_p, 1 < p < \infty, L_p(\mu), 1 < p < \infty$, Hilbert spaces and finite dimensional spaces are reflexive
2. $c_0, l_1, L^1[0, 1]$ are not reflexive

Remark 1.6. If X is reflexive, then $X \cong X^{**}$, so X is complete, but there exist Banach spaces with $X \cong X^{**}$ and X not reflexive.

1.2 Dual Operators

Let X, Y be normed space and recall that $\mathcal{B}(X, Y)$ is the set of bounded linear maps from X to Y . This is a normed space in the operator norm, $\|T\| = \sup\{\|Tx\| \mid x \in B_X\}$. If Y is complete then $\mathcal{B}(X, Y)$ is too.

Definition 1.7. For $T \in \mathcal{B}(X, Y)$, its *dual operator* is $T^*: Y^* \rightarrow X^*$ by $T^*(g) = g \circ T$.

T^* is linear and bounded.

$$\begin{aligned}
\|T^*\| &= \sup_{g \in B_{Y^*}} \|T^*(g)\| \\
&= \sup_{g \in B_{Y^*}} \sup_{x \in B_X} |\langle x, T^*g \rangle| \\
&= \sup_{x \in B_X} \sup_{g \in B_{Y^*}} |\langle Tx, g \rangle| \\
&= \sup_{x \in B_X} \|Tx\| \\
&= \|T\|
\end{aligned}$$

Remark 1.7. If X, Y are Hilbert spaces, T^* is the adjoint of T

Example 1.3. Let $R: l^p \rightarrow l^p$ be the right shift. $R^*: l^q \rightarrow l^q$ is the left shift.

Remark 1.8. We list some properties of dual operators

1. $\text{Id}_X = \text{Id}_X^*$
2. $(\lambda S + \mu T)^* = \lambda S^* + \mu T^*$
3. $(ST)^* = T^*S^*$
4. $T \mapsto T^*: \mathcal{B}(X, Y) \rightarrow \mathcal{B}(Y^*, X^*)$ is a linear isometry
5. The following diagram commutes

$$\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow & & \downarrow \\
X^{**} & \xrightarrow{T^{**}} & Y^{**}
\end{array}$$

Remark 1.9. By the above properties, if $X \sim Y$ then $X^* \sim Y^*$

1.3 Quotient Spaces

Definition 1.8. Let X be a normed space and Y be a closed subspace of X . The *quotient space*, X/Y is a normed space in the *quotient norm*, ie

$$\|x + Y\| = \inf_{y \in Y} \|x + y\| = d(x, Y)$$

It is easy to check that this is a norm (we need Y closed for positivity)

Definition 1.9. The *quotient map* $q: X \rightarrow X/Y$ by $q(x) = x + Y$ is a surjective linear map which is norm-decreasing (and hence $\|q\| \leq 1$)

Remark 1.10. The quotient map maps the open unit ball D_X onto $D_{X/Y}$. Mapping into is immediate from norm decreasing, and if $z \in D_{X/Y}$ then $z = q(x)$ for some $x \in X$, and $\|z\| = \|x + Y\| < 1$, so there is $y \in Y$ with $\|x + y\| < 1$ and $q(x + y) = z$, so $\|q\| = 1$ provided $X \neq Y$. It follows that q is an open map, by translating and dilating D_X

Let X, Z be normed spaces, $Y \subseteq X$ a closed subspace and $T \in \mathcal{B}(X, Z)$ such that $Y \leq \ker T$. Then there is a unique map $\tilde{T}: X/Y \rightarrow Z$ such that

$$\begin{array}{ccc} X & \xrightarrow{T} & Z \\ & \searrow q & \uparrow \tilde{T} \\ & & X/Y \end{array}$$

commutes. This map is linear (automatically) and bounded with $\|\tilde{T}\| = \|T\|$, as $\tilde{T}(D_{X/Y}) = \tilde{T}(q(D_X)) = T(D_X)$.

Theorem 1.5. *Let X be a normed space. If X^* is separable, then so is X .*

Proof. Fix a countable dense subset $\{x_n^* \mid n \in \mathbb{N}\}$ of S_{X^*} . For all n , choose $x_n \in B_X$ such that $x_n^*(x_n) > \frac{1}{2}$. Let $Y = \overline{\text{span}}\{x_n \mid n \in \mathbb{N}\}$. Assume $Y \neq X$. Then choose $x^* \in S_{X^*}$ such that $x^*|_Y = 0$ by Hahn-Banach on X/Y . Choose x_n^* such that $\|x^* - x_n^*\| < \frac{1}{10}$. Then $\frac{1}{2} < x_n^*(x_n) = (x_n^* - x^*)(x_n) + x^*(x_n) < \frac{1}{10}$ contradiction. $\square$

Remark 1.11. The converse is false, eg l^1 .

Theorem 1.6. *Every separable normed space X embeds isometrically into l^∞ .*

Proof. Let $\{x_n \mid n \in \mathbb{N}\}$ be a countable dense subset of X and let x_n^* be a norming functional for x_n . We define $T: X \rightarrow l^\infty$ by $x \mapsto (x_n^*(x))$. This is easily seen to be well defined, linear and bounded with norm at most one. For x_n , $\|x_n\| \geq \|Tx_n\| \geq |x_n^*(x_n)| = \|x_n\|$, so by density, it is isometric everywhere $\square$

Remark 1.12. 1. We say l^∞ is *isometrically universal* for the class $\mathcal{SB}$ of separable Banach spaces. Later we will show there is a separable Banach space Z with this property.

2. There is a dual version of theorem 7 that states every separable Banach space is isometrically isomorphic to a quotient of l^1 .

Theorem 1.7. (*Vector-valued Liouville's theorem*) *Let X be a complex Banach space and $f: \mathbb{C} \rightarrow X$ be holomorphic and bounded. Then f is constant.*

Proof. f being holomorphic means that for every $z \in \mathbb{C}$, $\lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z}$ exists. Fix $x^* \in X^*$. Then $\lim_{w \rightarrow z} \frac{x^*f(w) - x^*f(z)}{w - z} = x^*f'(z)$ as x^* is linear and continuous. Also $|x^*f(z)| \leq \|x^*\| \cdot \|f(z)\| \leq M\|x^*\|$ so by usual Liouville, x^*f is constant. Now fix $z \in \mathbb{C}$. We know $x^*f(z) = x^*f(0)$. As x^* separates points (varying over X^*), $f(z) = f(0)$, so we are done. $\square$

Definition 1.10. A *locally convex space* (LCS) is a pair $(X, \mathcal{P})$ where X is a vector space, and $\mathcal{P}$ is a family of seminorms on X that separates points of X in the sense that for all $x \neq 0$ in X , there is $p \in \mathcal{P}$ such that $p(x) \neq 0$. Such a family $\mathcal{P}$ defines a topology on X . $U \subseteq X$ is open if for all $x \in U$ there are $p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon > 0$ with $\{y \in X \mid p_k(y - x) < \varepsilon, 1 \leq k \leq n\} \subseteq U$

Remark 1.13. We give some properties of LCSs

1. Addition and scalar multiplication are constant
2. The topology is hausdorff as $\mathcal{P}$ separates points
3. $x_n \rightarrow x$ in X if and only if $p(x_n - x) \rightarrow 0$ for every $p \in \mathcal{P}$
4. Let $Y \subseteq X$ be a subspace and $\mathcal{P}_Y$ be the set of restrictions of $p \in \mathcal{P}$ to Y . Then $(Y, \mathcal{P}_Y)$ is a LCS with the subspace topology induced from X
5. Let $\mathcal{P}, \mathcal{Q}$ be two families of seminorms on X which both separate points, we say they are *equivalent*, denoted $\mathcal{P} \sim \mathcal{Q}$ if they induce the same topology. In fact it is the case that $(X, \mathcal{P})$ is metrisable if and only if there is a countable $\mathcal{Q}$ equivalent to $\mathcal{P}$

Definition 1.11. A *Fréchet space* is a complete metrisable LCS

Example 1.4.

1. Any normed space is a locally convex space with $\mathcal{P} = \{\|\cdot\|\}$
2. Fix a non-empty open set $U \subseteq \mathbb{C}$ and let $\mathcal{O}(U)$ be the set of holomorphic functions from U to $\mathbb{C}$. For a compact $K \subseteq U$, let $\|f\|_K = \sup\{|f(z)| \mid z \in K\}$. Let $\mathcal{P} = \{\|\cdot\|_K \mid K \subseteq U \text{ compact}\}$. Then $(\mathcal{O}(U), \mathcal{P})$ is a locally convex space, with the topology of locally uniform convergence. As there is an increasing sequence of compact set which exhausts U , ie $U = \bigcup K_n$, such that $K_n \subseteq \text{int } K_{n+1}$, $\{\|\cdot\|_{K_n} \mid n \in \mathbb{N}\} \sim \mathcal{P}$ so $(\mathcal{O}(U), \mathcal{P})$ is metrisable, and in fact a Fréchet space. In fact it is not normable (provable via Montel's theorem: given $f_n \in \mathcal{O}(U)$ such that for all compact $K \subseteq U$, the sequence $f_n|_K$ is bounded in $C(K)$, then f_n has a locally uniformly convergent subsequence. If it were normable, then this would say it had compact unit ball, but the space is infinite dimensional.)
3. Fix $d \in \mathbb{N}$ and $\emptyset \neq \Omega \subseteq \mathbb{R}^d$. Then we have $C^\infty(\Omega)$, the smooth real valued functions on Ω . For a multiindex $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{Z}_{\geq 0})^d$, let $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$. Let $p_{K,\alpha}(f) = \|(D^\alpha f)|_K\|_\infty$ and let $\mathcal{P} = \{p_{K,\alpha} \mid K \text{ compact}, \alpha \text{ a multi index}\}$. $(C^\infty(\Omega), \mathcal{P})$ is a LCS and in fact a Fréchet space, but isn't normable

Lemma 1.8. Let $(X, \mathcal{P}), (Y, \mathcal{Q})$ be LCSs and let $T: X \rightarrow Y$ be linear. The following are equivalent

1. T is continuous
2. T is continuous at 0
3. For all $q \in \mathcal{Q}$ there is $n \in \mathbb{N}, p_1, \dots, p_n \in \mathcal{P}$ and $C \geq 0$ such that for all $x \in X$, $q(Tx) \leq C \max_{1 \leq k \leq n} p_k(x)$

Proof. 1 and 2 are equivalent as translation is a homeomorphism.

2 implies 3: Fix a $q \in \mathcal{Q}$ and let $V = \{y \in Y \mid q(y) \leq 1\}$. This is a neighbourhood of 0 in Y , so there is a neighbourhood U of 0 in X mapping into V . Without loss of generality, $U = \{x \in X \mid p_k(x) \leq \varepsilon, 1 \leq k \leq n\}$. Put $p(x) = \max_{1 \leq k \leq n} p_k(x)$. Fix $x \in X$. We want to show $q(Tx) \leq \frac{1}{\varepsilon} p(x)$. If $p(x) \neq 0$ then $\frac{\varepsilon x}{p(x)} \in U$ so

$T\left(\frac{\varepsilon x}{p(x)}\right) \in V$, so $q(Tx) \leq \frac{1}{\varepsilon} p(x)$. If $p(x) = 0$, then $\lambda x \in U$ for all scalars λ , so $q(T(\lambda x)) \leq 1$, so $q(Tx) = 0$.

3 implies 2: Let V be a neighbourhood of 0 in Y . Without loss of generality, $V = \{y \in Y \mid q_k(y) \leq \varepsilon, 1 \leq k \leq m\}$ for some $q_1, \dots, q_m \in \mathcal{Q}, \varepsilon > 0$. For each $1 \leq k \leq m$, there is $n_k \in \mathbb{N}$ and $p_{k,1}, \dots, p_{k,n_k}$ and $C_k \geq 0$ such that for every $x \in X$, $q_k(Tx) \leq C_k \max_{1 \leq j \leq n_k} p_{k,j}(x)$. Let $U = \{x \in X \mid p_{k,j} \leq \frac{\varepsilon}{C_k+1}, 1 \leq j \leq n_k, 1 \leq k \leq m\}$. Then U is a neighbourhood of 0 in X with image in V so we are done. $\square$

Definition 1.12. Let $(X, \mathcal{P})$ be a LCS. The *dual space*, X^* , of X is the set of linear functionals $f: X \rightarrow \mathbb{R}$ that are continuous with respect to $\mathcal{P}$

By the above lemma, a linear functional f on X is continuous if and only if there is $n \in \mathbb{N}$ and $p_1, \dots, p_n \in \mathcal{P}, C \geq 0$ such that for all $x \in X$, $|f(x)| \leq C \max_{1 \leq k \leq n} p_k(x)$.

Lemma 1.9. Let f be a linear functional on a LCS X . Then $f \in X^*$ if and only if $\ker f$ is closed.

Proof. The forwards direction is easy.

In the backwards direction, without loss of generality $f \neq 0$ and fix $x_0 \in X \setminus \ker f$. Take a neighbourhood U of 0 in X such that $(x_0 + U) \cap \ker f = \emptyset$. Without loss of generality, $U = \{x \in X \mid p_k(x) \leq \varepsilon, 1 \leq k \leq n\}$ for some $n \in \mathbb{N}$ and $p_1, \dots, p_n \in \mathcal{P}, \varepsilon \geq 0$. Then U is convex and balanced (if $x \in U, |\lambda| = 1$ then $\lambda x \in U$) and $0 \notin f(x_0 + U)$. So $f(U)$ is not all the scalar field. Since f is linear $f(U)$ is convex and balanced, so it is bounded, say by M . Then $f(\frac{\varepsilon}{M}U) \subseteq \{\lambda \mid |\lambda| < \varepsilon\}$, so we are done as we have shown it is continuous at 0. $\square$

Theorem 1.10. (Hahn Banach for LCS) Let X be a LCS

1. Given $Y \leq X$ and $g \in Y^*$ there is $f \in X^*$ such that $f|_Y = g$.
2. Given closed $Y \leq X$ and $x_0 \in X \setminus Y$ there is $f \in X^*$ such that $f(Y) = 0$ and $f(x_0) \neq 0$

Proof. (1): By lemma 1.8, there is $n \in \mathbb{N}, p_1, \dots, p_n \in \mathcal{P}$ and $C \geq 0$ such that for all $x \in X$, $|g(y)| \leq C \max_{1 \leq k \leq n} p_k(x)$. Let $p(x) = C \max_{1 \leq k \leq n} p_k(x)$. This is a seminorm on X . By Theorem 1.2, there is a linear functional f on X restricting to g dominated by $p(x)$, so by Lemma 1.8, $f \in X^*$

(2): Let $Z = \text{Span}(Y \cup \{x_0\})$ and define $g: Z \rightarrow \text{scalars}$ by $g(y + \lambda x_0) = \lambda$. This is linear with kernel Y , so is continuous by Lemma 1.9. By part (1) we can extend this to the whole space, so are done. $\square$

Remark 1.14. (2) shows that X^* separates points of X

1.4 The Dual Space of L_p

Fix a measure space $(\Omega, \mathcal{F}, \mu)$. Fix $1 \leq p < \infty$ and let q be the conjugate index, $\frac{1}{p} + \frac{1}{q} = 1$. Given $g \in L_q$, define $\phi_g: L_p \rightarrow \text{scalars}$ by $\phi_g(f) = \int_{\Omega} f g d\mu$. This is well defined by Hölder, is linear and $|\phi_g(f)| \leq \|f\|_p \|g\|_q$, so $\phi_g \in L_p^*$ with $\|\phi_g\| \leq \|g\|_q$. We have a map $\phi: L_q \rightarrow L_p^*$ by $g \mapsto \phi_g$ which is linear and bounded with $\|\phi\| \leq 1$

Theorem 1.11. *Let p, q be conjugate indices with $p < \infty$ and ϕ be as above.*

1. *If $1 < p < \infty$ then ϕ is isometric and surjective, so $L_p^* \cong L_q$*
2. *If $p = 1$ and μ is σ -finite, then ϕ is isometric and surjective, so $L_1^* \cong L_{\infty}$*

Proof. Surjectivity will be proven later. We prove ϕ is isometric.

First assume $1 < p < \infty$. Fix $g \in L_q$. We need to show $\|\phi_g\| \geq \|g\|_q$. Let

$$f(\omega) = \begin{cases} |g(\omega)|^q / g(\omega) & \text{if } g(\omega) \neq 0 \\ 0 & \text{otherwise} \end{cases}. \text{ Then } f \text{ is measurable with } \int |f|^p d\mu = \int |g|^q d\mu, \text{ so } f \in L_p \text{ and } \|f\|_p = \|g\|_q^{q/p}. \text{ Hence}$$

$$\phi_g(f) = \int |g|^q d\mu = \|g\|_q^q = \|g\|_q \cdot \|g\|_q^{q-1} = \|g\|_q \cdot \|f\|_p$$

so $\|\phi_g\| \geq \|g\|_q$ and we are done.

Now if $p = 1$, without loss of generality $\|g\|_{\infty} > 0$ and fix $0 \leq s < \|g\|_{\infty}$, so $\{\omega \in \Omega \mid |g(\omega)| > s\}$ has non-zero measure. By σ -finiteness, this set is a countable union of finite measure sets, at least one of which has positive measure. Call

it A , this has finite non-zero measure. Let $f = \begin{cases} |g(x)|/g(x) & x \in A \\ 0 & x \notin A \end{cases}$. Then

$f \in L_1$ and $\|f\|_1 = \mu(A)$. Then $\phi_g(f) = \int |g| d\mu \geq s\mu(A) = s\|f\|_1$, so $\|\phi_g\| \geq s$ as $\|f\|_1 = \mu(A) > 0$, so the result follows. $\square$

Remark 1.15. One approach to the surjectivity is via the Radon-Nikodym theorem

1.5 $C(K)$ spaces

Definition 1.13. 1. Let Ω be a set and $\mathcal{F}$ a σ -field on Ω . A *complex measure* on $(\Omega, \mathcal{F})$ is a countably additive set function $\nu: \mathcal{F} \rightarrow \mathbb{C}$

2. The *total variation measure* $|\nu|$ of ν is defined by $|\nu|(A) = \sup\{\sum_{k=1}^n |\nu(A_k)| \mid A = \bigcup_{k=1}^n A_k \text{ is a measurable partition}\}$. (ie A_k are all measurable and disjoint). Then $|\nu|: \mathcal{F} \rightarrow [0, \infty]$ is a (positive) measure. In fact, $|\nu|(\Omega) < \infty$ (no proof). The *total variation* $\|\nu\|_1$ of ν is $|\nu|(\Omega)$.
3. A *signed measure* is a countably additive set function $\nu: \mathcal{F} \rightarrow \mathbb{R}$.

Theorem 1.12. (*proof non-examinable*) For a signed measure ν , there are unique finite positive measures ν^+, ν^- such that $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$. This is called the *Jordan decomposition* of ν

Proof. No Proof □

Remark 1.16. If ν is a complex measure, then $\operatorname{Re}(\nu), \operatorname{Im}(\nu)$ are signed measures, so have Jordan decompositions $\nu_1 - \nu_2$ and $\nu_3 - \nu_4$ respectively. It follows that $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$, the *Jordan decomposition* of ν . We have $\nu_k \leq |\nu| \leq \nu_1 + \nu_2 + \nu_3 + \nu_4$. It follows that $|\nu|$ is finite

Definition 1.14. Let ν be a complex measure on $\mathcal{F}$. A measurable function $f: \Omega \rightarrow \mathbb{C}$ is said to be ν -integrable if $\int_{\Omega} |f| d|\nu| < \infty$. For such functions we define

$$\int f d\mu = \int f d\nu_1 - \int f d\nu_2 + i \int f d\nu_3 - i \int f d\nu_4$$

Remark 1.17. 1. $\int_{\Omega} 1_A d\nu = \nu(A)$ for all $A \in \mathcal{F}$

2. The integral is linear

3. Dominated convergence. If f_n is a sequence of complex valued measurable functions and f is also complex valued measurable functions, and g is measurable non-negative real valued. If $g \in L_1(|\nu|)$, $|f_n| \leq |g|$, and $f_n \rightarrow f$ almost everywhere. Then f_n, f are ν -integrable and $\int f_n d\nu \rightarrow \int f d\nu$

4. $|\int f d\mu| \leq \int |f| d|\nu|$ (Do for simple functions, then use dominated convergence.)

Proof. Trivial □

Now K is a compact Hausdorff space, and $C(K)$ is the space of continuous complex valued functions. We denote the space of continuous real valued functions by $C^{\mathbb{R}}(K)$, and the positive functions by $C^+(K)$. We write $M(K) = C(K)^*$, $M^{\mathbb{R}}(K)$ is the set of $\phi \in M(K)$ such that $\phi(f) \in \mathbb{R}$ for all $f \in C^{\mathbb{R}}(K)$. The latter is a closed real-linear subspace of $M(K)$, and is isometrically isomorphic to the dual of $C^{\mathbb{R}}(K)$ by restriction. $M^+(K)$ is the set of linear not-necessarily-continuous functions from $C(K)$ to $\mathbb{C}$ such that $\phi(f) \geq 0$ for $f \in C^+(K)$.

Remark 1.18. It is a fact that $M^+(K) = \{\phi \in M(K) \mid \|\phi\| = \phi(1_K)\}$

Definition 1.15. Let X be a topological space and $\mathcal{G}$ the set of all open subsets of X . Let $\mathcal{B} = \sigma(\mathcal{G})$. These are the *Borel sets*. A *Borel measure* is a positive measure μ on $\mathcal{B}$. This is *regular* if

1. $\mu(E) < \infty$ for all compact E
2. $\mu(A) = \inf\{\mu(U) \mid A \subseteq U \in \mathcal{G}\}$ for all $A \in \mathcal{B}$
3. $\mu(U) = \sup\{\mu(E) \mid E \subseteq U \text{ is closed}\}$ for all $U \in \mathcal{G}$

A complex Borel measure is *regular* if $|\nu|$ is regular.

Remark 1.19. If X is compact, then a Borel measure μ on X is regular if and only if $\mu(X) < \infty$ and (2) in the definition of regularity holds, which is equivalent to $\mu(X) < \infty$ and (3) holding for all Borel sets.

Example 1.5. The lebesgue measure on $\mathbb{R}$ is a regular measure.

Now let ν be a complex borel measure on K . For $f \in C(K)$, $\int_K |f|d|\nu| \leq \|f\|_\infty |\nu|(K)$ so f is ν -integrable. We define $\phi: C(K) \rightarrow \mathbb{C}$ by $\phi(f) = \int f d\nu$. This is linear and bounded with $\|\phi\| \leq \|\nu\|_1$, so $\phi \in M(K)$. If ν is a signed measure, $\phi \in M^{\mathbb{R}}(K)$. If ν is a positive measure then $\phi \in M^+(K)$

Theorem 1.13. (*Riesz Representation Theorem*) Let $\phi \in M(K)$. Then there is a unique regular complex borel measure ν on K that represents ϕ in the above way. Moreover $\|\phi\| = \|\nu\|_1$. If $\phi \in M^{\mathbb{R}}(K)$ then ν is a signed measure and if $\phi \in M^+(K)$ then ν is a positive measure.

Proof. No Proof □

Corollary 1.13.1. The space of regular complex Borel measures on K is a complex Banach space in $\|\cdot\|_1$, and is isometrically isomorphic to $M(K)$. The same holds for signed measures being a real banach space isomorphic to $M^{\mathbb{R}}(K)$

2 Weak Topologies

Definition 2.1. Let X be a set and $\mathcal{F}$ be a family of functions where each $f \in \mathcal{F}$ where each $f \in \mathcal{F}$ is a function $f: X \rightarrow Y_f$ where Y is a topological space. The *weak topology* $\sigma(X, \mathcal{F})$ on X generated by $\mathcal{F}$ is the smallest topology on X such that each $f \in \mathcal{F}$ is continuous

Remark 2.1.

1. $S = \{f^{-1}(U) \mid f \in \mathcal{F}, U \subseteq Y_f, U \text{ open}\}$ is a subbase of $\sigma(X, \mathcal{F})$, so the collection of all finite intersections of elements of S is a base.
2. If S_f is a subbase for the topology of Y_f for all $f \in \mathcal{F}$, then $\{f^{-1}(U) \mid f \in \mathcal{F}, U \in S_f\}$ is a subbase of $\sigma(X, \mathcal{F})$
3. If $\mathcal{F}$ separates the points of X and each Y_f is Hausdorff, then $\sigma(X, \mathcal{F})$ is Hausdorff
4. Let $Y \leq X$ and $\mathcal{F}_Y = \{f|_Y \mid f \in \mathcal{F}\}$. Then $\sigma(X, \mathcal{F})|_Y = \sigma(Y, \mathcal{F}_Y)$
5. Let Z be a topological space and $g: Z \rightarrow X$ a function. g is continuous with respect to the $\sigma(X, \mathcal{F})$ topology if every $Z \rightarrow X \rightarrow Y_f$ is continuous.

Example 2.1.

1. If $\mathcal{F} = \{\iota: Y \rightarrow X\}$ is the inclusion, then $\sigma(Y, \mathcal{F})$ is the subspace topology.

2. Given a set Γ and a Γ -indexed set of topological spaces X_γ . Let $X = \prod_{\gamma \in \Gamma} X_\gamma$. We write x_γ for the γ component of x . We define maps $\pi_\gamma: X \rightarrow X_\gamma$ by $x \mapsto x_\gamma$. The product topology on X is the weak topology $\sigma(X, \{\pi_\gamma \mid \gamma \in \Gamma\})$

Proposition 2.1. *Let X be a set and for each $n \in \mathbb{N}$, let (Y_n, d_n) be a metric space and let $f_n: X \rightarrow Y_n$ be a function. If $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ separates points of X . Then $\sigma(X, \mathcal{F})$ is metrisable.*

Proof. Define

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min(d_n(f_n(x), f_n(y)), 1)$$

Each f_n is uniformly continuous with respect to d so $\sigma(X, \mathcal{F}) \subseteq \tau_d$, and by the Weierstrass M test, d is continuous with respect to the product $\sigma(X, \mathcal{F})$ topology, so done. $\square$

Theorem 2.1. (*Tychonoff's Theorem*) *If X_γ is a compact topological space for each γ then $X = \prod_{\gamma \in \Gamma} X_\gamma$ is compact in the product topology.*

Proof. Let $\mathcal{F}$ be a non-empty family of closed subsets of X with the finite intersection property. We need to show $\bigcap \mathcal{F} \neq \emptyset$. By Zorn there is a maximal family $\mathcal{A}$ of subsets of X with the finite intersection property such that $\mathcal{F} \subseteq \mathcal{A}$. It suffices to prove $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$. From maximality, $\mathcal{A}$ is closed under intersection, and if A meets every member of $\mathcal{A}$ then $A \in \mathcal{A}$ (from closure under intersection). For $\gamma \in \Gamma$, the set $\pi_\gamma(\mathcal{A})$ has finite intersection property, so the intersection of the $\bar{A}$ is non-empty. Take x_γ in it and set $x = (x_\gamma) \in X$. We show $x \in \bigcap_{A \in \mathcal{A}} \bar{A}$. Let V be a neighbourhood of $x \in X$. Without loss of generality V is an intersection of sets of the form $\pi_\gamma^{-1}(V_\gamma)$ where V_γ is a neighbourhood of x_γ . Now $V_j \cap \pi_\gamma(A) \neq \emptyset$ for all $A \in \mathcal{A}$, so $V \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, so $x \in \bar{A}$. $\square$

We now consider a real or complex vector space E and let F be a vector space of linear functionals on E that separates the points of E . We consider the weak topology $\sigma(E, F)$ on E