

Geometric Group Theory

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1 Combinatorial Group Theory

1.1 Free Groups and Presentations

Let $A = \{a_1, a_2, a_3, \dots\}$ be an alphabet (ie a set).

Definition 1.1. A group F is *free on A* if

1. there is a map of sets $A \rightarrow F$
2. for any map of sets $A \rightarrow G$ where G is a group, there is a unique group homomorphism $FA \rightarrow G$ such that $A \rightarrow G = A \rightarrow FA \rightarrow G$

Remark 1.1. This is a “universal property”

Exercise 1.1. If F satisfies the above definition, it is unique up to unique isomorphism. Hence we may write $F = F(A)$

The question of existence can be resolved in at least two ways we will give.
Topological method: Let $X = \bigvee_{a \in A} S^1$. Then $\pi_1(X) \cong F(A)$ by Seifert van Kampen

Combinatorial method: Let $A^* = \{\text{words in } a, a^{-1} \text{ for } a \in A\}$

Example 1.1. If $A = \{a, b\}$, then $ab, aba^{-1}b^{-1}, \emptyset, aa^{-1}$ are in A^*

Definition 1.2. A word $w \in A^*$ is *reducible* if $w = uaa^{-1}v$ or $w = ua^{-1}av$ for some $u, v \in A^*$, ie aa^{-1} or $a^{-1}a$ is a *subword*. Otherwise w is *reduced*

As a set, $F(A)$ is the set of reduced words, we multiply by concatenation with identity $\emptyset$. We reduce by removing aa^{-1} or $a^{-1}a$ to obtain a reduced result.

Example 1.2. $(aba^{-1}b^{-1})(b^2a) = aba^{-1}ba$

Inverses are the obvious thing. Well definedness of multiplication is the reason we use the topological method.

Exercise 1.2. What we have constructed is indeed an $F(A)$

Definition 1.3. A *presentation* consists of a set of generators A and a set of relations $R \subseteq F(A)$. We write $\langle A \mid R \rangle$ which generates the group $F(A)/\langle\langle R \rangle\rangle$ where the double angle brackets mean “the smallest normal subgroup containing”

Example 1.3.

1. $\langle a \mid a^n \rangle = \mathbb{Z}/n\mathbb{Z}$
2. $\langle r, s \mid r^n, s^2, sr sr \rangle$
3. $\langle A \mid \rangle = F(A)$
4. $\langle a_1, \dots, a_n, b_1, \dots, b_n \mid \prod_{i=1}^n [a_i, b_i] \rangle$, the fundamental group of the genus G surface
5. The fundamental group of the complement of the trefoil knot is $\langle a, b \mid a^2 b^{-3} \rangle$

Remark 1.2. Every group has a presentation (take the generators to be the group, and the relations the kernel of the obvious map from the free group on the underlying set of the group to the group)

Presentations became important in early 20th century mathematics as they arise naturally from topology. Given a cell complex X , the SvK theorem enables us to write down a presentation for $\pi_1(X)$. Indeed, every group presentation arises this way

Corollary 1.0.1. (Of SvK theorem) For $G = \langle a_1, a_2, \dots \mid r_1, r_2, \dots \rangle$, let X be the wedge of circles corresponding to the a_i s and attach discs to the wedge, corresponding to each r_i , along the path in the wedge given by r_i . Then $\pi_1(X) = G$

Proof. Trivial □

The beginnings of combinatorial and geometric group theory are often identified with a 1911 paper of Max Dehn. He formulated 3 questions about a finite presentation $\langle A \mid R \rangle = G$ (ie A, R are both finite)

1. The Word Problem (WP): Given a word $w \in A^*$, determine whether or not w represents the identity in G .
2. The Conjugacy Problem (CP): Given $u, v \in A^*$, determine whether $u \sim v$ (are conjugate) in G .
3. The Isomorphism Problem (IP): Given two presentations $\langle A_1 \mid R_1 \rangle, \langle A_2 \mid R_2 \rangle$, determine whether the groups they present are isomorphic.

Remark 1.3.

1. The conjugacy problem is stronger than the word problem, as $w = e$ if and only if $w \sim e$

2. Dehn was motivated by topology. All these problems ask for algorithms (in 1911!)
3. All three are undecidable in full generality
 - (a) Novikov (1955) and Boone (1958) unsolved the word problem (and hence the conjugacy problem): they exhibited $\langle A \mid R \rangle$ finite with unsolvable word problem.
 - (b) Adyan (1955) and Rabin (1958) unsolved the isomorphism problem. They exhibited a sequence of presentations $\langle A_n \mid R_n \rangle$ such that you can't decide for which n we have $\langle A_n \mid R_n \rangle = e$

Nevertheless, positive solutions exist for reasonable classes of groups.

Example 1.4. (Word Problem in Free Groups) Let $w \in A^*$. If w is reduced, then $w = e \in F(A)$ if and only if w is the empty word. Otherwise w contains a cancelling pair of the form aa^{-1} or $a^{-1}a$. Removing this pair, we get a shorter word w' in A^* such that $w = w' \in F(A)$, and we are done by induction.

We can also solve the conjugacy problem in a similar way.

Definition 1.4. There is a natural action of $\mathbb{Z}$ on A^* given by cyclically permuting the letters of words, ie $a_1^{i_1} a_2^{i_2} \dots a_n^{i_n} \mapsto a_2^{i_2} \dots a_n^{i_n} a_1^{i_1}$ where $i_j \in \pm 1$. The orbits of this action are called *cyclic words*. Its elements are called the *cyclic conjugates*. A word $w \in F(A)$ is called *cyclically reduced* if all its cyclic conjugates are reduced

Example 1.5. $w = aba^{-1}$ is reduced but not cyclically reduced

Remark 1.4. If $w \in A^*$ is reduced but not cyclically reduced, then $w = aw'a^{-1}$ for some $a \in A \cup A^{-1}$. w' is conjugate to w and shorter than w . Hence we may assume w is cyclically reduced to solve the conjugacy problem

Lemma 1.1. (*The Conjugacy Problem in Free Groups*) If $u, v \in F(A)$ are cyclically reduced, then $u \sim v$ if and only if the corresponding cyclic words are equal.

Proof. Since cyclic conjugates are conjugate, the reverse direction is immediate. For the forwards direction, suppose $u = gvg^{-1}$. We induct on the length of g , and hence only need to prove the result for $g \in A \cup A^{-1}$. We know u is cyclically reduced, and therefore either the g or the g^{-1} cancels into v , and hence v is either $g^{-1}v'$ or $v'g$. In these cases, $u = v'g^{-1}$ or $u = gv'$ so u is a cyclic conjugate of v . $\square$

2 Historical Case Study

To understand why Dehn formulated these problems, we should understand the state of topology in the early 20th century. Knowing that compact surfaces are

classified by their homology groups, Poincaré wondered if the same was true for 3d manifolds. We thus get the Poincaré conjecture (version 1, not the famous one). Let M be a closed 3-manifold. If $H_*(M) = \begin{cases} \mathbb{Z} & * = 0, 3 \\ 0 & \text{else} \end{cases}$ (ie, M is a homology sphere), then $M \cong S^3$

Theorem 2.1. (*Poincaré, 1905*) *There is a homology sphere P such that there is a surjection $\pi_1(P) \rightarrow A_5$, in particular, $P \not\cong S^3$*

Proof. No Proof □

Theorem 2.2. (*Poincaré Conjecture, proved by Perelman, 2003*) *Let M be a closed 3-manifold. If $\pi_1(M) \cong 1$, then $M \cong S^3$*

Proof. No Proof □

In 1910, motivated by Poincaré, Dehn was interested in understanding homology spheres

Theorem 2.3. (*Dehn, 1910*) *There are infinitely many pairwise non-homeomorphic 3d homology spheres*

Proof. Let K be the trefoil knot

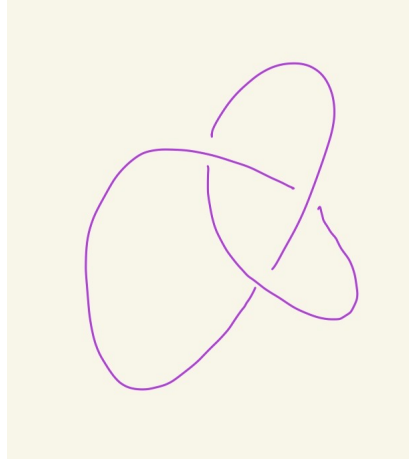


Figure 1: Trefoil Knot

in S^3 . We look at $N = S^3 \setminus N(K)$ where $N(K)$ is a small open neighbourhood of K . This has boundary T a torus. Now $\pi_1(N) = \langle x, y, z \mid x^2 = y^3 = z \rangle$ and $H_1(N) = \pi_1(N)_{ab} \cong \mathbb{Z}^2 / \langle (2, -3) \rangle \cong \mathbb{Z}$ (more generally, $H_*(N) = \mathbb{Z}$ for 0, 1 and 0 otherwise. It turns out that $\pi_1(T) = \langle xy, z \rangle = \mathbb{Z}^2 \leq \pi_1(N)$). By abelianising, note that $H_1(T) \cong \mathbb{Z}^2 \rightarrow H_1(N) \cong \mathbb{Z}$ via $xy \mapsto 5, z \mapsto 6$. We now build infinitely many closed 3-manifolds by Dehn filling. Let $U = D^2 \times S^1$ be a solid

torus. We call the boundary of one of the D_2 the meridian, μ and we call the S^1 loop λ . For any homeomorphism $\phi: \partial U \rightarrow T$, we construct $M_\phi = (N \sqcup U)/\sim$ where $x \sim \phi(x)$ for all $x \in \partial U$. μ gets sent to some curve $\phi_*(\mu)$ which we denote as g . Then $\pi_1(M_\phi) = \pi_1(N)/\langle\langle g \rangle\rangle$ by SvK. Likewise $H_1(M_\phi) = \mathbb{Z}/\langle[g]\rangle$. To produce an integral homology sphere, we need $[g] = \pm 1$ (we will focus on $+1$). In general, $g = (xy)^a z^b \in \pi_1(N)$, so in $H_1(N)$, $[g] = 5a + 6b$. Dehn chooses $a = 6n + 5, b = -(5n + 4)$ for $n \geq 1$ and we call this element g_n . It is a fact that the choices of ϕ correspond to elements of $SL_2(\mathbb{Z})$, and hence for every $n \geq 1$, there is a choice realising this g under the a, b above. We set M_n to be the resulting M_{ϕ_n} . Finally, we need to prove these are different. We know $G_n = \pi_1(M_n) = \langle x, y, z \mid x^2 = y^3 = z, (xy)^{6n+5} = z^{5n+4} \rangle$. We need to prove these are pairwise non-isomorphic. We will not for about 20 lectures. $\square$

Note also that if $g_m \sim g_n$, then the normal subgroups they generate are equal, so $G_n = G_m$, so we also care about the conjugacy problem.

3 Van Kampen Diagrams

Definition 3.1. A map of cell complexes $Y \xrightarrow{f} X$ is called *combinatorial* if for all dimensions k and for every k -cell e^k of Y , f maps the interior of e^k homeomorphically to the interior of a k -cell of X (the interior of a 0 cell is itself).

Consider a presentation $G = \langle a_i \mid r_i \rangle$ and the associated presentation complex X

Definition 3.2. A (*singular*) *disc diagram* is a compact contractible 2-complex D equipped with an embedding $D \rightarrow \mathbb{R}^2$. This is *over* X if it is equipped with a combinatorial map $D \rightarrow X$.

Remark 3.1. The combinatorial map is equivalent to labelling each 1-cell with a generator, such that each boundary of a 2-cell reads a cyclic conjugate of a relation. The boundary cycle, ∂D reads a cyclic word $w \in A^*$, and w is trivial in G . We say D is a *van Kampen diagram* for w

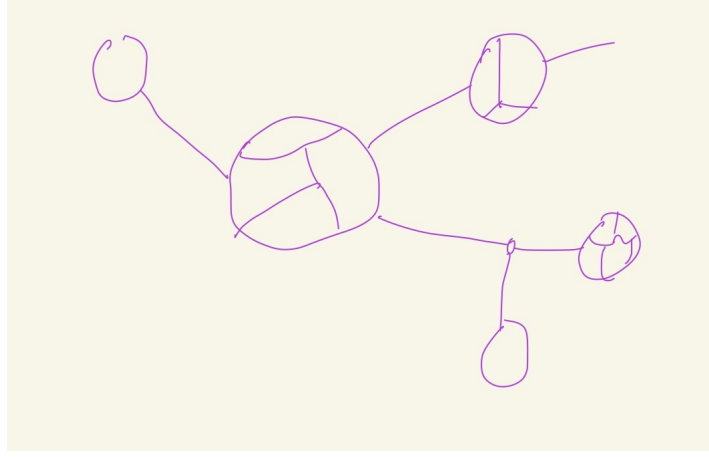


Figure 2: Disc Diagram

Lemma 3.1. (*van Kampen's Lemma*) If $w \in \langle\langle R \rangle\rangle$ (or even $w \in A^*$ reduces to $w' \in \langle\langle R \rangle\rangle$) then there is a van Kampen diagram for w

Proof. Since $w \in \langle\langle R \rangle\rangle$, there are $h_i \in F(A)$ and $r_i \in R$ such that $w = \prod_{i=1}^k h_i r_i^{\pm 1} h_i^{-1}$. We now construct a singular disc diagram for the right hand side. Note that the boundary of D_0 is the right hand side, $w_0 \in A^*$.

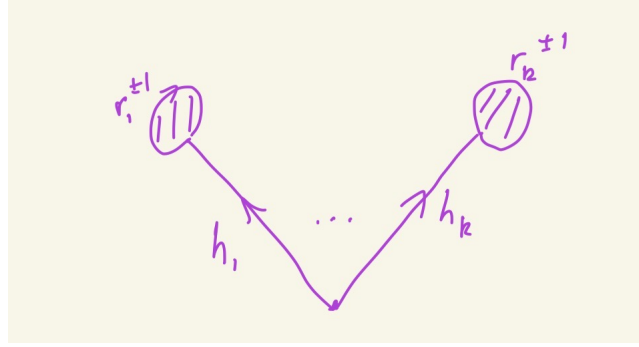


Figure 3: D_0

If w_0 is reduced, then $w_0 = w$ and we are done. Otherwise w_0 contains a cancelling pair of consecutive edges e_1, e_2 labelled a, a^{-1} . We now simplify D_0 to produce a new disc diagram D_1 with shorter ∂D_1 . If the origin of e_1 is the terminus of e_2 . Call the interior of the disc enclosed by e_1, e_2, D' . Then $D_0 = D_1 \vee D'$ for some other disc diagram D_1 with $\partial D_1 = w_1$, the word obtained by deleting a, a^{-1} and we are done. Otherwise the origin of e_1 is distinct from the terminus of e_2 . Then we identify these vertices (folding the edges) and we are done. This terminates as the length of the boundary cycle decreases. $\square$

Remark 3.2. So equations in groups correspond to pictures in the plane. The smallest k that may be chosen is equal to the smallest number of 2-cells in a van Kampen diagram. We call this the *area* of w , $\text{Area}(w)$

Example 3.1. For $G = \mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$, $w_n = a^n b^n a^{-n} b^{-n}$ we have

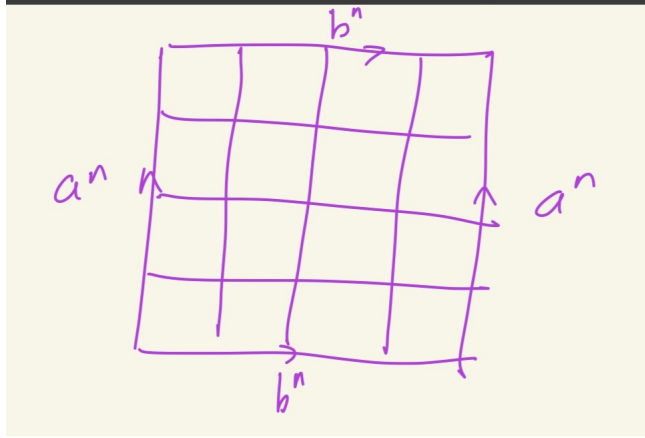


Figure 4: van Kampen Diagram

We will see that D_n has minimal area (ES1), so $\text{Area}(w_n) = n^2$

Definition 3.3. Let $P = \langle A \mid R \rangle$ be a finite presentation. Define $\delta_P: \mathbb{N} \rightarrow \mathbb{N}$ by $l \mapsto \max\{\text{Area}(w) \mid w =_G 1, \text{length}(w) \leq l\}$. This is called the *Dehn function* of P

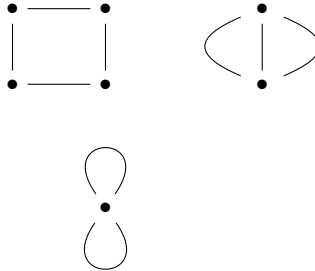
Remark 3.3. In fact, the word problem is solvable if and only if δ_P is computable

4 Basics of Geometric Group Theory

4.1 Cayley Graphs

A graph is a 1-dimensional cell complex.

Example 4.1. These are all graphs



Let G be a group and $S \subseteq G$ be a finite generating set

Definition 4.1. The *Cayley graph*, $\text{Cay}_S(G)$ is defined as follows.

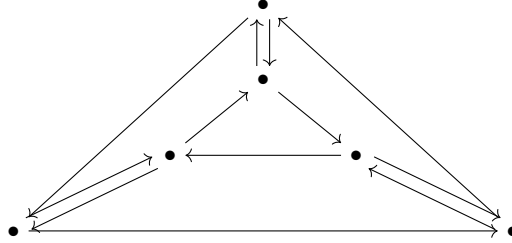
- The vertex set is G
- For each $(g, s) \in G \times S$ there is an edge (g, gs)

Example 4.2.

1. $1 \cong \langle a, b \mid a, b \rangle$ has this Cayley graph



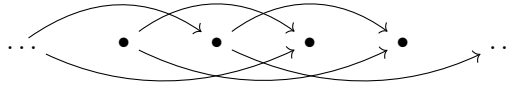
2. $S_3 = \langle r, s \mid r^3, s^2, sr sr \rangle$ has Cayley graph



3. $\mathbb{Z} = \langle 1 \rangle$ has Cayley graph



4. $\mathbb{Z} = \langle 2, 3 \rangle$ has Cayley graph



Remark 4.1. All groups are symmetry groups in some sense (of their Cayley graph) as G naturally acts on $\text{Cay}_S(G)$ on the left as $G \curvearrowright G$ via $\gamma \mapsto g\gamma$ extends to edges: $(\gamma, \gamma s) \mapsto (g\gamma, g\gamma s)$. Note that this action is free, ie point stabilisers are trivial. Some authors collapse multiple edges. This loses freeness.

We also lose the following

Proposition 4.1. (*Cayley Graphs and Presentation Complexes*) Let $G = \langle S \mid R \rangle$ and let X be the corresponding presentation complex. There is a G -equivariant isomorphism of graphs between $\text{Cay}_S(G)$ and $\tilde{X}_{(1)}$

Proof. Choose a base vertex $\tilde{x}_0 \in \tilde{X}_{(0)}$. There is only one G -orbit in $\tilde{X}_{(0)}$ as X has a single 0-cell, and so 0-cells of $\tilde{X}$ are in bijection with G via $g \mapsto gx_0$. We extend Φ over edges as follows: By path lifting, the edge, say corresponding to s , in X lifts to an edge at $\tilde{x}_0$ from $\tilde{x}_0$ to $s\tilde{x}_0$, hence there is an edge $g\tilde{x}_0 \rightarrow gs\tilde{x}_0$ for each $g \in G$, so we can extend Φ over the edges of the Cayley graph. This is trivially a bijection on vertices, and is a bijection on edges as an edge in $\tilde{X}$ is exactly a lift of an edge in X at a given vertex, which corresponds to g (the origin vertex) and s (the edge of X we lift). $\square$

Remark 4.2. Since S generates G , $\text{Cay}_S(G)$ is path connected.

Proposition 4.2. *Let $\tilde{X}$ be a path connected topological space. Suppose that $G \curvearrowright \tilde{X}$ be homeomorphisms. If $U \subseteq \tilde{X}$ is open and $GU = \tilde{X}$, then $S = \{g \in G \mid gU \cap U \neq \emptyset\}$ generates G .*

Proof. Fix $x_0 \in U$. For $g \in G$, take a path $\gamma: x_0 \rightarrow gx_0$. This is covered by the open cover $\{hU \mid h \in G\}$, so we take a finite subcover $g_1U, g_2U, \dots, g_nU$, so that $0 = t_0 < t_1 < \dots < t_n = 1$, $g_1 = e$, $g_n = g$ and $\gamma([t_i, t_{i+1}]) \subseteq g_{i+1}U$. Then $g_iU \cap g_{i+1}U \neq \emptyset$, so $g_i^{-1}g_{i+1} \in S$. But we can then write $g_n = e(g_1^{-1}g_2) \dots (g_{n-1}^{-1}g_n)$, so $g \in \langle S \rangle$. $\square$

Example 4.3. If we consider the tiling of the plane by equilateral triangles, the isometry group of this is finitely generated, by the stabiliser of a fixed triangle T , and symmetries mapping it onto any of those triangles with non-empty intersection with it.

As long as the action is ‘nice enough’, we can conclude that G is finitely generated. Henceforth we assume $\tilde{X}$ is Hausdorff

Definition 4.2. An action $G \curvearrowright \tilde{X}$ is called *properly discontinuous* if for any compact $K \subseteq \tilde{X}$ the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite. It is called *cocompact* if there is some compact K such that $GK = \tilde{X}$. Finally $\tilde{X}$ is *locally compact* if for every open $V \subseteq \tilde{X}$ and every $x \in V$ there is an open neighbourhood W and compact K such that $x \in W \subseteq K \subseteq V$

Corollary 4.0.1. *Let $\tilde{X}$ be path connected and locally compact. If $G \curvearrowright \tilde{X}$ properly discontinuously and cocompactly then G is finitely generated.*

Proof. Let K be a compact set such that $GK = \tilde{X}$. By local compactness, we can find open $U \supseteq K$ such that $\bar{U}$ is compact. Now $GU = \tilde{X}$ and $\{g \mid gU \cap U \neq \emptyset\}$ is finite (by proper discontinuity, applied to $\bar{U}$), so we are done by Proposition 4.2. $\square$

Corollary 4.0.2. *If X is compact, locally compact and X has a universal cover, then $\pi_1(X, x_0)$ is finitely generated*

Proof. Apply the above corollary to $\pi_1(X) \curvearrowright \tilde{X}$. $\square$

5 Schwarz-Milnor Lemma

Definition 5.1. Let S generate G . Define $l_S(g) = \min\{n \mid g = \prod_{i=1}^n s_i^{\pm 1}, s_i \in S\}$. We define, for $g, h \in G$, $d_S(g, h) = l_S(g^{-1}h)$, and call this the *word metric*

Remark 5.1. It is an easy exercise that the word metric is a metric and $d_S(\gamma g, \gamma h) = d_S(g, h)$. So the word metric is left-invariant

Example 5.1. Let $G = \mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$ and let $S = \{a, b\}$. The word metric is d_1 on $\mathbb{Z}^2$. This is equal to the length metric on the Cayley graph.

Lemma 5.1. Suppose S, T are finite generating sets for a group G . Then there are constants $C, C' \geq 1$ such that

$$\frac{1}{C}d_T \leq d_S \leq C'd_T$$

Proof. Let $C = \max\{l_T(s) \mid s \in S\}$. Then for any $g \in G$, if $g = \prod_{i=1}^n s_i^{\pm 1}$ where $n = l_S(g)$, then $l_T(g) \leq Cn$ by induction on n . The other direction follows by symmetry. $\square$

Definition 5.2. A map of metric spaces $f: X \rightarrow Y$ is a (C, D) -quasi-isometric embedding (where $C \geq 1, D \geq 0$) if

$$\frac{1}{C}d_X(x, x') - D \leq d_Y(f(x), f(x')) \leq Cd_X(x, x') + D$$

for all $x, x' \in X$. If in addition there is $E \geq 0$ such that for every $y \in Y$ there is an $x \in X$ such that $d_Y(f(x), y) \leq E$, then we say f is a *quasi-isometry* and we write $X \sim_{qi} Y$

Remark 5.2. Quasi-isometry is an equivalence relation (exercise)

Example 5.2. All bounded metric spaces are quasi-isometric to a point.

Definition 5.3. A metric space X is called *proper* if closed balls are compact. A *geodesic* in a metric space X is an isometrically embedded interval. X is called *geodesic* if every pair of points can be joined by a geodesic.

Theorem 5.2. (*Schwarz-Milnor lemma*) Suppose X is a proper geodesic metric space. Let $G \curvearrowright X$ properly discontinuously cocompactly by isometries. Then G is finitely generated, and $(G, d_S) \sim_{qi} X$ for any finite generating set S .

Proof. Fix a basepoint $x_0 \in X$ and let $B = \bar{B}(x_0, K)$ be a closed ball such that $GB = X$. By proper discontinuity and properness,

$$\{g \in G \mid d(x_0, gx_0) \leq 3K\} = \{g \mid g\bar{B}(x_0, \frac{3}{2}K) \cap \bar{B}(x_0, \frac{3}{2}K) \neq \emptyset\}$$

is finite. Hence there is $\varepsilon > 0$ such that if $d(x_0, gx_0) < 2K + \varepsilon$ then $d(x_0, gx_0) \leq 2K$, which is equivalent to $g\bar{B} \cap \bar{B} \neq \emptyset$. In particular if $U = B(x_0, K + \frac{1}{2}\varepsilon)$, we have

$$S = \{gU \cap U \neq \emptyset\} = \{gB \cap B \neq \emptyset\}$$

Applying the earlier proposition to U , $G = \langle S \rangle$. Since all word metrics are quasi-isometric, we may work with this generating set. Consider the map $G \rightarrow X$ by $g \mapsto gx_0$. This is clearly quasi-surjective, so it remains to show there are $C \geq 1, D \geq 0$ such that

$$\frac{1}{C}l_S(g) - D \leq d(x_0, gx_0) \leq Cl_S(g) + D$$

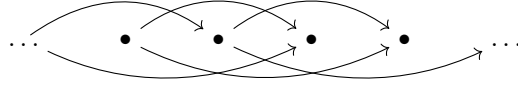
For the upper bound, take $C = \max_{s \in S} d(x_0, sx_0)$. By triangle inequality, the upper bound holds.

For the lower bound consider a geodesic γ from x_0 to gx_0 . Let $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = d(x_0, gx_0)$ such that $\frac{\varepsilon}{2} \leq t_i - t_{i-1} < \varepsilon$ for all $1 \leq i \leq n+1$. Since $GB = X$, there is, for each i , some $g_i \in G$ such that $d(g_i x_0, \gamma(t_i)) \leq K$, and WLOG $g_{n+1} = g, g_0 = e$. For each i , $d(g_i x_0, g_{i+1} x_0) < 2K + \varepsilon$, so $g_{i+1}^{-1} g_i \in S$. Hence $l_S(g) \leq n+1$. Furthermore, since $t_i - t_{i-1} \geq \varepsilon/2$, $d(x_0, gx_0) \geq \frac{n\varepsilon}{2}$, and so we get an upper bound on $l_S(g)$ in terms of $d(x_0, gx_0)$, and so we are done. $\square$

Example 5.3. Recall $\text{Cay}_{\{1\}}(\mathbb{Z})$



and $\text{Cay}_{\{2,3\}}(\mathbb{Z})$



These are quasi-isometric to each other by the above

Corollary 5.2.1. *If G is finitely generated and H is a finite index subgroup of G , then $G \sim_{qi} H$*

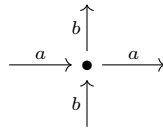
Proof. H acts on $\text{Cay}_S(G)$ in a way that satisfies the hypotheses of the Schwarz-Milnor lemma. $\square$

Example 5.4. $\mathbb{Z}^2 \sim_{qi} \mathbb{R}^2$ via the obvious action.

Let $G = \pi_1(\Sigma_g)$. Choose a Riemannian metric of constant curvature -1 on Σ_g . This pulls back to a Riemannian metric of constant curvature -1 on $\tilde{\Sigma}_g$. By a classical theorem of differential geometry, this means $\tilde{\Sigma}_g = \mathbb{H}^2$. Hence $G \curvearrowright \mathbb{H}^2$ is properly discontinuous and cocompactly by isometries, so $G \sim_{qi} \mathbb{H}^2$

6 Free Groups

Let $A = \{a_1, \dots, a_n\}$ and $F_n = F(A)$. The *Cayley tree* is the infinite $2n$ -valent tree T_{2n} . Note that all vertices look the same, like



for $n = 2$. The group $F_n \wr T_{2n}$ freely and transitively, and the quotient is the $2n$ -valent graph with 1 vertex n edges.