

Hochschild Cohomology and Deformation Theory

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1 Introduction

Hochschild cohomology was developed by Hochschild (1945) to develop a cohomology theory for associative algebras, in particular finite dimensional A over a field k in order to generalise/interpret some theorems of Wedderburn.

Gerstenhaber (1962/3) showed the relevance of Hochschild cohomology to deformation theory. In a series of papers he developed additional algebraic structure on the cohomology ring. We have an associative product and a lie bracket on such rings. This was used in the deformation theory. Let t be an indeterminate, k a field and A a k -algebra. We can define the usual product on $A[[t]]$ of formal power series with coefficients in A .

$$\left(\sum_{n \geq 0} a_n t^n \right) \left(\sum_{n \geq 0} b_n t^n \right) = \left(\sum_{n \geq 0} \left(\sum_{i+j=n} a_i b_j \right) t^n \right)$$

A is a k -algebra, so we can regard k as a subalgebra of A and $A[[t]]$ is a $k[[t]]$ -module. Denote the underlying vector space of A as V . We're interested in associative products on $V \otimes_k k[[t]]$ which reduce to the usual multiplication on $A \bmod t$

Definition 1.1. A Formal Deformation $(A_t, *)$ of A or equivalently, a deformation of A over $k[[t]]$, is an associative $k[[t]]$ bilinear multiplication $*$ on the $k[[t]]$ -module $V \otimes_k k[[t]]$ of the form

$$\left(\sum_{n \geq 0} a_n t^n \right) * \left(\sum_{n \geq 0} b_n t^n \right) = \left(\sum_{n \geq 0} \left(\sum_{i+j=n} a_i * b_j \right) t^n \right)$$

where $a * b = ab + F_1(a, b)t + F_2(a, b)t^2 + \dots$

Set $F_0(a, b) = ab$. Each F_i is k -bilinear and so corresponds to a k -linear function $A \otimes_k A \rightarrow A$

Remark. 1. There is an analogous definition for deformations over $k[t]$ (polynomials) or $k[t]/(t^n)$ (truncated polynomials) or $k[[t_1, \dots, t_n]]$

2. Physicists often refer to star products

Example 1.1. We have examples of deformations

1. $A = k[x, y]$. We can define a formal (or a polynomial) deformation via $x^i * x^j = x^{i+j}$, $y^i * y^j = y^{i+j}$, $x^i * y^j = x^i y^j$, $y * x = xy + t$ and use associativity. This has an ideal generated by $t = 1$ which gives you the Weyl Algebra on quotienting ($A_1 = k\langle x, y \rangle / (yx - xy - 1)$) called the differential algebras on $k[x]$, with y corresponding to $\frac{d}{dx}$)
2. Set $k = \mathbb{C}$, $A = \mathbb{C}[x, y]$. Define a formal deformation via $x^i * x^j = x^{i+j}$, $y^i * y^j = y^{i+j}$, $x^i * y^j = x^i y^j$, $y * x = xy \exp(t)$. Taking $t_0 \in \mathbb{C}$ and set $t = t_0$ in the subalgebra generated by x, y we get the quantum plane by letting $q = \exp(t_0)$, denoted $\mathbb{C}_q[x, y] = \mathbb{C}\langle x, y \rangle / (yx - qxy)$
3. Let $\text{char}(k) \neq 2$ and $A = k[e, f, t]$. Define the deformation over $k[t]$ by $f * e = ef - ht$, $h * e = eh + 2et$, $h * f = fh - 2ft$ and all other products of monomials as usual. Specialising to $t = 1$, we get the universal enveloping algebra of sl_2 , $\mathcal{U}(sl_2) = k\langle e, f, h \rangle / (fe - ef + h, he - eh - 2e, hf - fh + 2f)$

For a formal deformation of A we consider the effect of associativity on $*$,

$$(a * b) * c = a * (b * c)$$

We get

$$\begin{aligned} (a * b) * c &= abc + (F_1(ab, c) + F_1(a, b)c)t \\ &\quad + (F_2(ab, c) + F_1(F_1(a, b), c) + F_2(a, b)c)t^2 \\ a * (b * c) &= abc + (F_1(a, bc) + aF_1(b, c))t \\ &\quad + (F_2(a, bc) + F_1(a, F_1(b, c)) + aF_2(b, c))t^2 \end{aligned}$$

(ignoring higher order terms). Equating coefficients

$$F_1(ab, c) + F_1(a, b)c - F_1(a, bc) - aF_1(b, c) = 0$$

In cohomology, this says F_1 is a 2-cocycle in Hochschild cohomology

Definition 1.2. This 2-cocycle, F_1 , is the **Infinitesimal Deformation**

It allows us to define a deformation modulo t^2 .

Definition 1.3. Given a 2-cocycle F_1 , it is **Integrable** if it arises from a formal deformation

Question: When is an infinitesimal deformation integrable?
Equating t^2 terms now

$$F_1(F_1(a, b), c) - F_1(a, F_1(b, c)) = aF_2(b, c) - F_2(ab, c) + F_2(a, bc) - F_2(a, b)c$$

Both sides of this correspond to k -linear functions $A \otimes_k A \otimes_k A \rightarrow A$. We will see that the left hand side is a 3-cocycle and the right hand side is the coboundary of F_2 . So this can be interpreted as a cohomological statement. The left hand side is regarded as an obstruction to deforming mod t^3 , as it needs to be a coboundary, so that F_1 isn't integrable if it isn't

Theorem 1.1. (Gerstenhaber) *If $HH^3(A, A) = 0$ then all infinitesimal deformations are integrable*

Proof. Proof deferred. □

Definition 1.4. Two star products, f and g on $V \otimes_k k[[t]]$, where V is the underlying vector space of A are **equivalent** if there is a $k[[t]]$ -linear automorphism, φ of $V \otimes_k k[[t]]$ such that $\varphi(f(a, b)) = g(\varphi(a), \varphi(b))$.
If f is equivalent to the usual star product on $A[[t]]$ we say f is **trivial**.
If all formal deformations of A are trivial, A is **rigid**

Remark. This raises the question of which k -algebras are rigid

Theorem 1.2. (Gerstenhaber) *If $HH^2(A, A) = 0$ then A is rigid (the converse is not true)*

Proof. No Proof □

Remark. The universal enveloping algebra of a finite dimensional semi-simple complex lie algebra is rigid but if we also consider the tensor product and so think about bialgebras there are non-trivial deformations

2 Hochschild Cohomology: Definitions and Low Degrees

Let k be a field and A an associative k -algebra.

We consider $A - A$ bimodules M . That is, A acts on the left and right and the actions commute. For example, A is an $A - A$ bimodule.

Definition 2.1. The **opposite algebra** of A , denoted A^{op} has the same underlying vector space as A , with multiplication defined by $a \cdot_{\text{op}} b = ba$

Example 2.1. $kG \cong kG^{\text{op}}$ by $g \mapsto g^{-1}$ for $g \in G$ and extending linearly

An $A - A$ bimodule M can be viewed as a left $A \otimes_k A^{\text{op}}$ by $(x \otimes y)(m) = xmy$ for $x \otimes y \in A \otimes_k A^{\text{op}}$ and as a right $A \otimes_k A^{\text{op}}$ module via $m(x \otimes y) = ymx$

Remark. We notate $A \otimes_k A^{\text{op}} = A^e$ called the **enveloping algebra** of A

Example 2.2. $A \otimes_k A$ is an $A - A$ bimodule where $x(a \otimes b)y = xa \otimes by$ or equivalently a left $A \otimes_k A^{\text{op}}$ module, freely generated by $1 \otimes 1$. This is a "free bimodule of rank 1"

Similarly we can take $A \otimes_k A \otimes_k A$ as an $A - A$ bimodule via $x(a \otimes b \otimes c)y = ax \otimes b \otimes cy$. It is freely generated by $1 \otimes a_1 \otimes 1, \dots, 1 \otimes a_n \otimes 1$ for $a_1, \dots, a_n$ a free k basis of A .

This generalises to $A^{\otimes(n+2)}$ for $n \geq 0$ in the obvious manner

Definition 2.2. The **acyclic Hochschild chain complex** is

$$\dots \xrightarrow{d_3} A \otimes A \otimes A \otimes A \xrightarrow{d_2} A \otimes A \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{\mu} A \longrightarrow 0$$

where $d_n(a_0 \otimes \dots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i a_0 \otimes \dots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \dots \otimes a_n$ and $\mu: A \otimes A \rightarrow A$ is the multiplication.

We say $A \otimes A$ is the degree 0 term, $A \otimes A \otimes A$ the degree 1 term and so on.

Remark. $\text{Im}(d_{n+1}) = \ker(d_n)$ and $\text{Im}(d_1) = \ker(\mu)$ with the usual proof from algebraic topology.

We have a contracting homotopy $s_n(a_0 \otimes \dots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \dots \otimes a_{n+1}$.

Every module except A is free as an $A - A$ bimodule, and hence we have a free resolution of A as an $A - A$ bimodule. This is also known as the **Bar/Standard resolution**

Remark. There is a truncated version of the chain complex

$$\dots \xrightarrow{d_3} A \otimes A \otimes A \otimes A \xrightarrow{d_2} A \otimes A \otimes A \xrightarrow{d_1} A \otimes A \longrightarrow 0$$

which is no longer exact, with homology A at $A \otimes A$, soetimes called the **Bar complex**

Definition 2.3. The **Hochschild Homology** for $A - A$ bimodules M is obtained by applying $M \otimes_{A \otimes_k A^{\text{op}}} \cdot$ to the bar complex and the **Hochschild Cohomology** for $A - A$ bimodules M is obtained by applying $\text{Hom}_{A \otimes_k A^{\text{op}}}(\cdot, M)$ to the bar complex

We will mostly stick to cohomology in this course.
Applying $\text{Hom}_{A \otimes_k A^{\text{op}}}(\cdot, M)$ to the Hochschild chain complex gives a chain complex. It's easier to describe $A - A$ bimodule maps $A \otimes \cdots \otimes A \rightarrow M$ by say where the free generators are mapped. So we identify

$$\text{Hom}_{A \otimes_k A^{\text{op}}}(A \otimes \cdots \otimes A, M) = \text{Hom}_k(A \otimes \cdots \otimes A, M) = C^n(A, M)$$

where the right hand side has two less A terms than the left.

In particular, $\text{Hom}_{A \otimes_k A^{\text{op}}}(A \otimes A, M) = \text{Hom}_k(k, M) \cong M = C^0(A, M)$.

Definition 2.4. The **Hochschild cochain complex** is

$$M \cong \text{Hom}_k(k, M) \xrightarrow{\delta_1} \text{Hom}_k(A, M) \xrightarrow{\delta_2} \text{Hom}_k(A \otimes A, M) \xrightarrow{\delta_3} \dots$$

Now $\delta_n(f) = fd_n$ so

$$\begin{aligned} \delta_1(f)(a) &= af(1) - f(1)a \\ \delta_2(f)(a_1 \otimes a_2) &= a_1f(a_2) - f(a_1a_2) + f(a_1)a_2 \\ \delta_3(f)(a_1 \otimes a_2 \otimes a_3) &= a_1f(a_2 \otimes a_3) - f(a_1a_2 \otimes a_3) + f(a_1 \otimes a_2a_3) - f(a_1 \otimes a_2)a_3 \\ \delta_n(f)(a_1 \otimes \cdots \otimes a_n) &= a_1f(a_2 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n-1} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &\quad + (-1)^n f(a_1 \otimes \cdots \otimes a_{n-1})a_n \end{aligned}$$

Note that $\text{Im}(\delta_n) \subset \ker(\delta_{n+1})$. The n -cocycles are $\ker(\delta_{n+1}) = Z_n(A, M)$ and the n -coboundaries are $\text{Im}(\delta_n) = B_n(A, M)$.

We have $HH^0(A, M) = \ker(\delta_1)$ and $HH^n(A, M) = \ker(\delta_{n+1})/\text{Im}(\delta_n)$ are k -vector spaces. We will think about $HH^*(A, M) = \bigoplus_{i \geq 0} HH^i(A, M)$

Remark. As usual for (co)homology, by the usual proof, (co)homology is independent of the choice of resolution. Usually for calculation is is better to use other free resolutions

Example 2.3. If $A = k$, then all tensor powers are isomorphic to k , so the standard complex is

$$\dots \longrightarrow k \longrightarrow k \longrightarrow k \longrightarrow 0$$

but we need to consider the maps. Alternatively, take a better resolution

$$0 \longrightarrow k \otimes k \xrightarrow{\mu} k \longrightarrow 0$$

then truncate it to get

$$0 \longrightarrow k \longrightarrow 0$$

, apply $\text{Hom}_{A \otimes_k A^{\text{op}}}(\cdot, M)$ to get

$$0 \longrightarrow \text{Hom}_{A \otimes_k A^{\text{op}}} k \longrightarrow 0$$

and then $HH^0(A, M) = \{m \in M \mid am = ma\}$ and all other cohomologies are 0

Definition 2.5. The **Hochschild cohomological dimension**, $\text{Dim } A$ of A is $\sup\{n \mid HH^i(A, M) \neq 0 \text{ for some bimodule } M\}$ if this exists, and ∞ otherwise

Example 2.4. We just discovered that $\text{Dim}(k) = 0$
It is a hard exercise that $\text{Dim}(M_{m \times m}(k)) = 0$

Example 2.5. Let $A = k[x]$. Then $A \otimes A = k[x, y]$ and so on. So we use the resolution

$$0 \longrightarrow k[x] \otimes k[x] \xrightarrow{x \otimes 1 - 1 \otimes x} k[x] \otimes k[x] \xrightarrow{\mu} k[x] \longrightarrow 0$$

where the first map is multiplication by $x \otimes 1 - 1 \otimes x$ and the latter is multiplication. Now $k[x]^{\text{op}}$ as A is commutative and this is a free resolution. We truncate it and apply $\text{Hom}_{A \otimes_k A^{\text{op}}}(\cdot, k[x])$ to get

$$0 \longleftarrow k[x] \longleftarrow k[x] \longleftarrow 0$$

and let $a \in k[x]$ identified with the function $f_a \in \text{Hom}_{k[x] \otimes_k k[x]^{\text{op}}}(k[x] \otimes k[x], k[x])$ taking $1 \otimes 1$ to a .

Composing, we get $f_a((x \otimes 1 - 1 \otimes x)(1 \otimes 1)) = x f_a(1 \otimes 1) - f_a(1 \otimes 1)x = xa - ax = 0$ and therefore all maps in the sequence below are zero maps.

$$0 \longleftarrow k[x] \longleftarrow k[x] \longleftarrow 0$$

Taking homology, $HH^0(k[x], k[x]) \cong k[x]$ and $HH^1(k[x], k[x]) \cong k[x]$ and all other terms are 0.

Observe that in general, for bimodule M , $HH^i(k[x], M) = 0$ for $i \geq 2$ and this $\text{Dim}(k[x]) = 1$

Example 2.6. Consider $A = k[x]/(x^m)$. We have the sequence

$$\dots \longrightarrow A \otimes_k A^{\text{op}} \xrightarrow{v} A \otimes_k A^{\text{op}} \xrightarrow{u} A \otimes_k A^{\text{op}} \xrightarrow{\mu} A \longrightarrow 0$$

with u, v repeating alternately to the left where u, v are the bimodule maps given by $u = \cdot(x \otimes 1 - 1 \otimes x)$ and $v = \cdot(x^{m-1} \otimes 1 + x^{m-2} \otimes x \cdots + 1 \otimes x^{m-1})$. The sequence is exact. Apply $\text{Hom}_{A \otimes_k A^{\text{op}}}(\cdot, A)$ to the truncated sequence to get

$$\dots \xleftarrow{\cdot \eta x^{m-1}} A \xleftarrow{0} A \xleftarrow{\cdot \eta x^{m-1}} A \xleftarrow{0} A \longleftarrow 0$$

Thus if the characteristic of k divides m , all maps are 0 and $HH^i(A, A) = A$ for all i , and otherwise we get the Hochschild homology alternating on parity given by $HH^0(A, A) = A, HH^{2i+1}(A, A) \cong (x), HH^{2i} \cong A/(x^{m-1})$.

Therefore $\text{Dim}(A)$ is infinite

2.1 Small Degrees

We return to the standard resolution of A , and let M be an $A - A$ bimodule

Definition 2.6. $HH^0(A, M) = \ker(\delta_1) = \{m \in M \mid am - ma = 0, \forall a \in A\}$ with δ_1 given as before. In particular, if $M = A$, then $HH^0(A, A) = Z(A)$

Definition 2.7. The 1-cocycles are

$$\ker(\delta_2) = \{f \in \text{Hom}_k(A, M) \mid f(a_1 a_2) = a_1 f(a_2) + f(a_2) a_2\}$$

. Such f are called **derivations from A to M** denoted $\text{Der}(A, M)$.

The 1-coboundaries,

$$\text{im}(\delta_1) = \{f \in \text{Hom}_k(A, M) \mid \exists m \in M \text{ such that } f = a \mapsto am - ma\}$$

are called **inner derivations from A to M** and we have

$$HH^1(A, M) = \text{Der}(A, M) / \text{Inner}(A, M)$$

Setting $M = A$ we get the **outer derivations** of A . If A is commutative, $\text{Inner}(A) = 0$

Example 2.7. If $A = k[x]$ and $\text{char}(k) = 0$, then we have that the set of derivations is $\text{Der}(A) = \{p(x) \frac{d}{dx} \mid p(x) \in k[x]\}$ and $HH^1(A, A) = \text{Der}(A)$.

In general $\text{Der}(A)$ forms a lie algebra and $\text{Inner}(A)$ is an ideal, where the lie bracket is given by $[D_1, D_2] = D_1 D_2 - D_2 D_1 \in \text{End}_k(A)$. Note that this is a derivation.

Example 2.8. If $k = \mathbb{C}$ and $A = \mathbb{C}[x, x^{-1}]$ and $D_n = -x^{n+1} \frac{d}{dx}$ for $n \in \mathbb{Z}$ then $\{D_n \mid n \in \mathbb{Z}\}$ form a vector space basis of $\text{Der}(A)$. The lie bracket is $[D_m, D_n] = (m - n)D_{m+n}$. $\text{Der}(A)$ is sometimes called the Witt algebra and has a central extension, the Virasoro Lie Algebra, denoted Vir

$$0 \longrightarrow \mathbb{C} \longrightarrow \text{Vir} \longrightarrow A \longrightarrow 0$$

which appears in mathematical physics

Example 2.9. Let $\text{char}(k) = p$, $A = k[x]/(x^p)$. We have another algebra called the Witt Algebra, $\text{Der}(A)$ spanned by the D_m , $-1 \leq m \leq p-2$ as in the notation of Example 2.8. This is a simple lie algebra (we can do the same for several variables)

Definition 2.8. Given a $A - A$ bimodule M we can form the **semidirect product** of A and M , $A \ltimes M$ with elements (a, m) for $a \in A, m \in M$ and multiplication $(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2)$. Thus $A \ltimes M \cong A + M\epsilon$ where ϵ is central with $\epsilon^2 = 0$. If $M = A$ we get back to $A[[t]]/(t^2)$. In general M forms a 2-sided ideal with $(M\epsilon)^2 = 0$

Lemma 2.1. $\text{Der}_k(A, M) \cong \{\text{Algebra complements to } M \text{ in } A \ltimes M\}$, which are definitionally embedded copies of A in $A \ltimes M$. Given $A \rightarrow A \ltimes M$ we write $a \mapsto (a, D(a))$

Proof. The function $D: A \rightarrow M$ is a derivation by definition of the semi-direct product and the converse is similarly true $\square$

Corollary 2.1.1. *We have*

$\text{Der}(A, M) = \{\text{Automorphisms of } A + M\epsilon \text{ of the form } a \mapsto a + f(a)\epsilon, m \mapsto m\epsilon\}$
and

$\text{Inner}(A, M) = \{\text{automorphisms obtained by conjugation by } 1 + m\epsilon, m \in M\}$
(note that $1 + m\epsilon$ is always a unit).

Proof. Trivial $\square$

Remark. For $M = A$ the lie algebra derivations of A may be thought of as corresponding to the tangent space to the identity within the automorphisms of A

Now we consider $HH^2(A, M)$.

Definition 2.9. An **extension** of A by M is a k -algebra B which contains a 2-sided ideal I with $I^2 = 0$ and $B/I \cong A$, $I \cong M$ as an $A - A$ bimodule. (Note that I can be viewed as an $A - A$ bimodule since $I^2 = 0$ induced from left/right multiplication in B)

Denote $\pi: B \rightarrow A$ be the canonical map. Two extensions are **equivalent** if there is a k -algebra isomorphism $\theta: B_1 \rightarrow B_2$ with $\pi_2 \circ \theta = \pi_1$ and θ is the identity on A

Remark. The semi-direct product is such an extension

Proposition 2.1. $HH^2(A, M)$ corresponds bijectively to the equivalence classes of extensions of A by M

Remark. We need to prove extension equivalent to a semi-direct product is a semi-direct product

Proof. Let B be an extension and $\pi: B \rightarrow A$ has kernel I such that $I^2 = 0$ and $I \cong M$. Let ρ be a k -linear map $A \rightarrow B$ such that $\pi(\rho(a)) = a$ for all $a \in A$. Define a map $f_\rho: A \otimes A \rightarrow I$ by $a_1 \otimes a_2 \mapsto \rho(a_1)\rho(a_2) - \rho(a_1a_2)$. The image is easily seen to be in I by applying π . Check that f_ρ is a 2-cocycle.

If we replace ρ with ρ' which is a different k -linear map then

$$\begin{aligned} f_\rho(a_1 \otimes a_2) - f_{\rho'}(a_1 \otimes a_2) &= \rho(a_1)(\rho(a_2) - \rho'(a_2)) \\ &\quad - (\rho(a_1a_2) - \rho'(a_1a_2)) \\ &\quad + (\rho(a_1) - \rho'(a_1))\rho'(a_2) \end{aligned}$$

. This can be seen to be a 2-coboundary, $\delta(\rho - \rho')$, noting that $\rho - \rho': A \rightarrow I$ so we are one direction.

Given a $A - A$ bimodule A and a 2-cocycle $f: A \otimes A \rightarrow M$. Let $B = A \oplus M$ as vector spaces, with product

$$(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2 + f(a_1 \otimes a_2))$$

which is associative since f is a cocycle. The map $(a, m) \mapsto a$ yields a homomorphism $B \rightarrow A$ with kernel being a 2-sided ideal with square 0 which is isomorphic to M as a bimodule. Taking $\rho: A \rightarrow B$ by $a \mapsto (a, 0)$ we get the 2-cocycle back by the previous construction and if f' is another cohomologous cocycle, then there is a linear map $\tau: A \rightarrow M$ with $f - f' = \delta\tau$. Hence $f(a_1 \otimes a_2) = f'(a_1 \otimes a_2) + a_1\tau(a_2) - \tau(a_1a_2) + \tau(a_1)a_2$ so $(a, m) \mapsto (a, m + \tau(a))$ gives an algebra isomorphism between their induced algebras inducing the identity on A $\square$

Corollary 2.1.2. $HH^2(A, M) = 0$ if and only if all extensions of A by M are semidirect products

Theorem 2.2. (Wedderburn, Malcev) Let B be a k -algebra such that

1. $\dim(B/J(B)) = 1$
2. $J(B)^n = 0$ for some $n \geq 1$

where $J(B)$ is the **Jacobson Radical** of B , the intersection of all maximal (right or left) ideals

Then there is a subalgebra $A \cong B/J(B)$ of B such that $B = A \ltimes J(B)$

Furthermore, if $\dim(B/J(B)) = 0$ then any two subalgebras A, A' that are complements of $J(B)$ in B have an $x \in J(B)$ such that $A' = (1 + x)A(1 + x)^{-1}$

Proof. (For $n = 2$ - the general case is a relatively straightforward induction). Since $\dim(B/J(B)) \leq 1$, for $A = B/J(B)$, we know that $HH^2(A, J(B)) = 0$ and so all extensions of A by $J(B)$ are semidirect. Furthermore, when we have $\dim(B/J(B)) = 0$, then $HH^1(A, J(B)) = 0$ and so all the complements are conjugate under conjugation by some $1 + x$ for $x \in J(B)$ $\square$

So a natural question arises: When do k -algebras have dimension 0 over k ?

Definition 2.10. A k -algebra A is **k -separable** if the multiplication map which is $\mu: A \otimes A \rightarrow A$ admits a bimodule map section $\sigma: A \rightarrow A \otimes A$. The **separating idempotent** is $\sigma(1)$, $\sigma(1) = \sum_{i=1}^n a_i \otimes a'_i \in A \otimes A$

Remark. This means that $A \hookrightarrow A \otimes A$ as a direct summand and so is a projective bimodule. The separating idempotent is the idempotent defining A as a summand of $A \otimes A$, and determines the bimodule map σ uniquely. We have that $\sum_{i=1}^n a_i a'_i = 1$. The condition that σ is a bimodule map is equivalent to saying that $\sum_{i=1}^n a a_i \otimes a'_i = \sum_{i=1}^n a_i \otimes a'_i a$

Example 2.10. Let $A = K$ a finite separable field extension of k . We can write $K = k[a]$ by Galois theory. Let $f(x) = (x - a) \sum_{i=0}^{n-1} b_i x^i$ be the minimal polynomial of a . Then a separating idempotent is given by $\sum_{i=0}^{n-1} a^i \otimes \frac{b_i}{f'(a)}$. The tensorands are dual bases with respect to the bilinear form $(x, y) \mapsto \text{Tr}(xy)$. We know from previous courses that this form is non-degenerate as the extension is separable, and hence we can talk of the dual basis. We have therefore shown that K is k -separable

Example 2.11. $M_n(k)$ is a k -separable k -algebra. Let E_{ij} be the matrix with a 1 in the ij th position, and 0 otherwise.

Fix j . Then $\sum_{i=1}^n E_{ij}E_{ji} \in A \otimes A$ is a separating idempotent (they are hence non-unique)

Example 2.12. We consider kG , the group algebra, where G is finite and $|G|$ is invertible in k . Then $\frac{1}{|G|} \sum_{g \in G} g \otimes g^{-1}$ is a separating idempotent

Remark. If A is k -separable then A is a projective bimodule and so we can pick a short projective resolution of A by

$$0 \longrightarrow A \longrightarrow A \longrightarrow 0$$

. If we apply $\text{Hom}_{A \otimes_k A^{\text{op}}}(\cdot, M)$ gives that $HH^i(A, M) = 0$ for all $i \geq 1$ for any bimodule M . Hence $\text{Dim}(A) = 0$.

Example 2.13. If $A = K$ is a finite inseparable field extension of k . There is an intermediate field L such that $K = L(c)$, $K \neq L$ where $c^p \in L$. Define the linear map $D: K \rightarrow K$ that restricts to the zero map on L and $D(\lambda c^q) = \lambda q c^{q-1} d_0$ for $\lambda \in L$ and a fixed $0 \neq d_0 \in K$. D is a non-zero derivation over L and hence over k . Therefore $HH^1(K, K) \neq 0$ because $\text{Inner}(K) = 0$ as K is commutative. We deduce that $\text{Dim}_k(K) \neq 0$ and so K is not k -separable

Lemma 2.3. If $\text{Dim}_k(A) = 0$ then A is k -separable

Proof. If $\text{Dim}_k(A) = 0$ then $HH^1(A, M) = 0$ for any $A - A$ bimodule M . This means that when we apply $\text{Hom}_{A \otimes_k A^{\text{op}}}(\cdot, M)$ to the Hochschild complex, we are saying that any map $\phi: \ker(\mu) \rightarrow M$ is the restriction of a map of the form $\phi': A \otimes A \rightarrow M$. Take $M = \ker(\mu)$ and $\phi = \text{Id}$. Thus we get ϕ' projects onto $\ker(\mu)$ and so

$$0 \longrightarrow \ker \mu \longrightarrow A \otimes A \xrightarrow{\mu} A \longrightarrow 0$$

splits and we are done. $\square$

Going back to (2.2),

Definition 2.11. An algebra A with $\dim_k(A) < \infty$ is **semisimple** if $J(A) = 0$.

Remark. It is easy to see that $J(B/J(B)) = 0$ for a k -algebra B . Thus $B/J(B)$ is semisimple if $\dim_k(B) < \infty$

Theorem 2.4. (Artin-Wedderburn) *A k -algebra A with $\dim_k(A) < \infty$ is semisimple if and only if it is the direct sum of finitely many subalgebras of the form $M_n(D)$ for D a finite dimensional division algebra over k . In particular, if $k = \bar{k}$ then $D = k$ for all D*

Proof. Well known (also in Modular Representation Theory notes) $\square$

Corollary 2.4.1. *A semisimple k -algebra A with $\dim_k A < \infty$ is k -separable if and only if it is a finite direct sum of subalgebras of the form $M_n(D)$ where D is a division algebra where the centre of D is a separable field extension of k . In particular, if $k = \bar{k}$ then A is k -separable and $\text{Dim}(A) = 0$*

Proof. Trivial □

The upshot of this is that if k is algebraically closed, we get the following theorem

Theorem 2.5. (Wedderburn) *For a k -algebra B with $\dim_k B < \infty$, there is a subalgebra A of B with $A \cong B/J(B)$ where $B = A \ltimes J(B)$ and moreover A is unique up to conjugation by units of the form $1 + x, x \in J(B)$ (note that $J(B)^n = 0$ automatically for finite dimensional B)*

Proof. $B/J(B)$ is semi-simple and hence $\text{Dim}(B/J(B)) = 0$. Apply 2.2. □

3 Cup Products, Gerstenhaber Brackets and Gerstenhaber Algebras

Recall that $HH^0(A, A) = Z(A)$. If U, V are $A - A$ bimodules then $Z(A)$ acts on $\text{Hom}_{A \otimes_k A^{\text{op}}}(U, V)$ by $(a \cdot f)(u) = af(u)$ for $a \in Z(A), u \in U$ and for all $f \in \text{Hom}_{A \otimes_k A^{\text{op}}}(U, V)$. Taking $U = A^{\otimes(n+2)}$ and $V = A$, we get an action which commutes with δ in the Hochschild cochain complex and so there is an action of $Z(A)$ on $HH^n(A, A)$ and thus on $HH^*(A, A) = \bigoplus HH^n(A, A)$. The idea behind the cup product is to extend this to make the cohomology into a graded-commutative algebra. On the standard/bar resolution, recall that we have the cochains $C^n(A, A) = \text{Hom}_k(A^{\otimes n}, A) \cong \text{Hom}_{A \otimes_k A^{\text{op}}}(A^{\otimes(n+2)}, A)$

Definition 3.1. On $C^m(A, A) \otimes C^n(A, A) \rightarrow C^{n+m}(A, A)$ we have $f \otimes g \mapsto f \cup g$ by $(f \cup g)(a_1 \otimes \cdots \otimes a_{n+m}) = (-1)^{nm} f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{n+m})$. If $m = 0$, we have $f \cup g(a_1 \otimes \cdots \otimes a_n) = f(1)g(a_1 \otimes \cdots \otimes a_n)$ and similarly for $n = 0$. This is called the **cup product**

Remark. Not everyone uses the same sign conventions. Gerstenhaber didn't have the $(-1)^{mn}$ but Witherspoon does

Under this product, $C^*(A, A) = \bigoplus C^n(A, A)$ becomes an associative graded algebra. As a consequence of the $(-1)^{mn}$ in the definition, we need to consider δ' where $\delta'(f) = (-1)^m \delta_{m+1}(f)$ for an m -cochain f . Then

$$\delta'(f \cup g) = \delta'(f) \cup g + (-1)^m f \cup (\delta'g)$$

. Thus δ' is a (left) graded derivation of the graded algebra $C^*(A, A)$. Consequently $\cup$ induces a graded cup product on Hochschild cohomology which is associative

Remark. 1. The cup product of two elements of degree 0 is their usual product in $Z(A)$

2. The cup product of a term of degree 0 with one of degree m corresponds to the previous action of $Z(A)$ on $HH^m(A, A)$. Thus $HH^m(A, A)$ is a $HH^0(A, A)$ module

Example 3.1. Let $A = k[x]$. Recall that $HH^0(A, A) = k[x]$ and that we know $HH^1(A, A) = k[x] \frac{\partial}{\partial x} \cong k[x]$, and all further cohomology is 0. So as a graded vector space, $HH^*(A, A) = k[x] \oplus k[x]$. In degree 0, the cup product is normal multiplication. The cup product of a term of degree 0 with one of degree 1 corresponds to multiplication by the appropriate polynomial. The cup product of two elements of degree 1 is 0, as degree 2 is 0. Hence

$$HH^*(A, A) = k[x, y]/(y^2)$$

where y corresponds to the $1 \in k[x] = HH^1(A, A)$

We will prove graded commutativity after introducing the Gerstenhaber bracket

Definition 3.2. The **Gerstenhaber bracket** is defined by, for $f \in C^m(A, A)$, $g \in C^n(A, A)$, by $[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f \in \text{Hom}_k(A^{\otimes(m+n-1)}, A)$ where the **circle product** is

$$\begin{aligned} f \circ g(a_1 \otimes \dots \otimes a_{m+n-1}) \\ = \sum_{i=1}^m (-1)^u f(a_1 \otimes \dots \otimes a_i \otimes g(a_i \otimes \dots \otimes a_{i+n-1}) \otimes a_{i+n} \otimes \dots \otimes a_{m+n+1}) \end{aligned}$$

where $u = (n-1)(i-1)$. If $m = 0$ then $f \circ g = 0$ and if $n = 0$, consider the g term to be $g(1)$

Definition 3.3. The **divided square**, $\text{Sq}(f)$ for $f \in \text{Hom}_k(A^{\otimes m}, A)$. Assume $\text{char}(k) = 2$ or m is even. Then $\text{Sq}(f) = f \circ f$

Lemma 3.1. Let $f \in C^m(A, A)$, $g \in C^n(A, A)$, $h \in C^p(A, A)$

1. $[f, g] = -(-1)^{(m-1)(n-1)}[g, f]$

- 2.

$$\begin{aligned} (-1)^{(m-1)(p-1)}[f, [g, h]] + (-1)^{(n-1)(m-1)}[g, [h, f]] \\ + (-1)^{(n-1)(p-1)}[h, [f, g]] = 0 \end{aligned}$$

3. $\delta([f, g]) = (-1)^{m-1}[\delta f, g] + [f, \delta g]$

Proof. Trivial □

Remark. These give us the following facts

1. We have graded commutativity of $[\cdot, \cdot]$ after shifting homological degree by 1

2. We have the graded Jacobi identity
3. Hence $C^*(A, A)$ forms a graded Lie algebra after shifting degree
4. Setting δ' as before, we have that δ' is a (left) graded derivation of degree 1 with square 0
5. Hence we have a differential graded Lie algebra

So it follows that we get a bracket on $HH^*(A, A)$ induced by $[\cdot, \cdot]$ which gives the structure of a graded Lie algebra and each $HH^m(A, A)$ is a $HH^1(A, A)$ module. Thus $HH^*(A, A)$ is a $HH^1(A, A)$ module

Lemma 3.2. *For $f \in C^m(A, A), g \in C^n(A, A)$ then*

1. $(-1)^{mn} f \cup g - g \cup f = \delta(g) \circ f + (-1)^m \delta(g \circ f) + (-1)^{m-1} g \circ \delta(f)$
2. $[f, \mu] = -\delta(f)$ where $\mu: A \otimes A \rightarrow A$ is multiplication

Proof. Trivial □

Theorem 3.3. *The cup product on $HH^*(A, A)$ is graded commutative. If $\alpha \in HH^m(A, A), \beta \in HH^n(A, A)$ then $\alpha \cup \beta = (-1)^{mn}(\beta \cup \alpha)$*

Proof. Take f, g to be cocycles representing α, β respectively. Thus we have that $\delta(f) = \delta(g) = 0$ and so $(-1)^{mn}(f \cup g) = g \cup f + (-1)^m \delta(g \circ f)$ so modulo coboundaries, the result holds, hence we are done. □

Lemma 3.4. *Let $\alpha \in HH^m(A, A), \beta \in HH^n(A, A), \gamma \in HH^p(A, A)$ then we have $[\gamma, \alpha \cup \beta] = [\gamma, \alpha] \cup \beta + (-1)^{m(p-1)} \alpha \cup [\gamma, \beta]$*

Proof. See Gerstenhaber □

Remark. This says that $[\gamma, \cdot]$ is a graded derivation with respect to the cup product

Definition 3.4. A **Gerstenhaber algebra**, $(H, \cup, [\cdot, \cdot])$ is a free $\mathbb{Z}$ graded k -module H for which $(H, \cup)$ is a graded commutative associative algebra and $(H, [\cdot, \cdot])$ is a graded Lie algebra of degree -1 and we have the graded derivation identity $[\gamma, \alpha \cup \beta] = [\gamma, \alpha] \cup \beta + (-1)^{|\alpha|(|\gamma|-1)} \alpha \cup [\gamma, \beta]$ (ie $[\gamma, \cdot]$ acts as a degree $|\gamma| - 1$ graded derivation of $\cup$)

Theorem 3.5. *Hochschild cohomology $HH^*(A, A)$ is a Gerstenhaber algebra under the cup product and Gerstenhaber bracket*

Proof. This is the content of this section up to now (essentially) □

Remark. 1. In a Gerstenhaber algebra, H^0 is a commutative algebra and $\cup: H^0 \times H^i \rightarrow H^i$ is a module action

2. H^1 is a Lie algebra and $[\cdot, \cdot]: H^1 \times H^0 \rightarrow H^0$ is a Lie module action

3. There is a Lie algebra map $H^1 \rightarrow \text{End}_k(H^0)$ by $\gamma \mapsto [\gamma, \cdot]$ giving a map of Lie algebras $H^1 \rightarrow \text{Der}(H^0)$

The prototypical example of a Gerstenhaber algebra is the exterior algebra $\bigwedge_A \text{Der}(A)$ for a commutative algebra A (with A in degree 0)

Definition 3.5. The **Schouten bracket** is

$$\begin{aligned} & [\lambda_1 \wedge \dots \wedge \lambda_m, \lambda'_1 \wedge \dots \wedge \lambda'_{m'}] \\ &= (-1)^{(m-1)(n-1)} \sum_{i,j} (-1)^{i+j} [\lambda_i, \lambda'_j] \wedge \lambda_1 \wedge \dots \wedge \lambda_m \wedge \lambda'_1 \wedge \dots \wedge \lambda'_{m'} \end{aligned}$$

with λ_i, λ'_j missing on $\bigwedge_A \text{Der}(A)$ with the bracket in the sum taken in $\text{Der}(A)$

For any Gerstenhaber algebra $H = \bigoplus H^i$, there is a canonical homomorphism of Gerstenhaber algebras $\bigwedge_{H^0} H^1 \rightarrow H$

Definition 3.6. A is **smooth** if there is a projective $A - A$ bimodule resolution of A which is finite

Theorem 3.6. (*HKR theorem - Hochschild-Kostant-Rosenberg*) If A is a smooth commutative k -algebra and $\text{char}(k) = 0$ then

$$\bigwedge_A \text{Der}_k(A) \rightarrow HH^*(A, A)$$

given above is an isomorphism

Proof. No Proof □

Remark. 1. This applies when $A = k[x_1, \dots, x_n]$

2. It applies if $k = \mathbb{C}, \mathbb{R}$ and A is the appropriate functions on an algebraic variety or smooth manifold
3. The original HKR theorem from the 1960s concentrated on the algebraic structure of $HH^*(A, A)$ as an exterior algebra of derivations

Example 3.2. If $A = k[x, y]$ is the polynomial algebra in 2 variables, we know that $HH^0(A, A) = A$, $HH^1(A, A) = \text{Der}(A)$ and $HH^2(A, A) = \text{Der}(A) \wedge \text{Der}(A)$ generated by $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ as an A module with all higher cohomology 0

4 Deformation Theory

Recall from the introduction that, for A a k -algebra with underlying vector space V , we define a star product or formal deformation as

$$a * b = ab + tF_1(a, b) + t^2F_2(a, b) + \dots$$

and $F_0(a, b) = ab$ so each F_i is a k -linear map $A \otimes_k A \rightarrow A$. We obtained explicit conditions for $*$ to be associative for each t^i . The t condition says that F_1 is a Hochschild cocycle and the left hand side of the coefficient of t^2 is the circle product $F_1 \circ F_1$. If $\text{char}(k) \neq 2$, this is also $\frac{1}{2}[F_1, F_1]$ applied to the triple $a \otimes b \otimes c$ and the right hand side is the coboundary of F_2 applied to $a \otimes b \otimes c$. Therefore the t^2 coefficient is equivalent to $[F_1, F_1] = 2\delta(F_2)$ and in characteristic 2, we have $\text{Sq}(F_1) = \delta(F_2)$. By looking at the coefficients of t^3 , we get $[F_1, F_2] = \delta(F_3)$ and more generally we get

$$\sum_{j=1}^{i-1} F_j(F_{i-j}(a \otimes b) \otimes c) - F_j(a \otimes F_{i-j}(b \otimes c)) = \delta(F_i)(a \otimes b \otimes c) \quad (1)$$

for $i \geq 2$. Applying lemma 3.2(2), we can rephrase δ as $(-1)[\cdot, F_0]$ and changing indexing to include $j = 0, i$ we get

$$\sum_{j=0}^i F_j(F_{i-j}(a \otimes b) \otimes c) - F_j(a \otimes F_{i-j}(b \otimes c)) = 0 \quad (2)$$

Definition 4.1. The left hand side of (1) as a function of $A^{\otimes 3}$ is called the $i - 1$ st **obstruction**. We see that it needs to represent 0 in $HH^3(A, A)$

Thus if the left hand side of (1) is not zero in cohomology then our star product can't be associative.

Assume $\text{char}(k) \neq 2$. Then (2) for $i \geq 2$, together with associativity of F_0 and F_1 being a Hochschild 2-cocycle, can be combined as saying that $[F_t, F_t] = 0$ where $F_t(a, b) = a * b$. Set $F_t = F_0 + F'$. Associativity gives that $[F_0, F_0] = 0$ and F_0, F' being 2-cochains, combined with graded commutativity gives $[F', F_0] = [F_0, F']$. The equation $[F_t, F_t] = 0$ is therefore equivalent to $2[F_0, F'] + [F', F'] = 0$

Definition 4.2. The **Maurer-Cartan** equation is

$$-\delta(F') + \frac{1}{2}[F', F'] = 0$$

Remark. We have shown that the associativity of the formal deformation is equivalent to the Maurer-Cartan equation

Definition 4.3. A **Poisson algebra** is a commutative associative algebra A that is also a Lie algebra under a bracket $[\cdot, \cdot]$ for which $[a, bc] = [a, b]c + b[a, c]$. The bracket is called the **Poisson bracket**

Remark. By definition, a Poisson algebra is a Lie algebra, so the bracket is antisymmetric and satisfies the Jacobi identity

Example 4.1. Let $F_1: A \otimes A \rightarrow A$ be a Hochschild 2-cocycle whose primary obstruction at the chain level, $[F_1, F_1]$, is 0. Let $[a, b] = \frac{1}{2}(F_1(a \otimes b) - F_1(b \otimes a))$ for all $a, b \in A$. Calculating, using commutativity and that F_1 is a 2-cocycle, we get that $[\cdot, \cdot]$ is a Poisson bracket. Conversely calculation shows that $[\cdot, \cdot]$ is a Hochschild 2-cocycle

Definition 4.4. Let A be a Poisson algebra. Then a **deformation quantisation** of A is a formal deformation for which $a * b - b * a = [a, b]t \mod t^2$

Example 4.2. The Weyl algebra is the specialisation (ie where we set $t = 1$) of a deformation quantisation of $k[x, y]$ with Poisson bracket $[x, y] = -1$. Similarly for the quantum plane and other examples

Definition 4.5. For A commutative, $\text{char}(k) \neq 2$ and the graded Lie algebra $\bigwedge^{*+1}(\text{Der}(A))$ with zero differential then the **Maurer Cartan elements** are the **formal Poisson structures**. They are formal power series $\sum t^i F_i$ where $F_1 \in \text{Der}(A) \wedge \text{Der}(A)$ satisfies the Maurer Cartan equation, using the Schouten bracket

Kontsevich, around 2000, considered deformation quantisations of smooth Poisson manifolds (ie where the function algebras are smooth Poisson algebras). A consequence of his work is

Theorem 4.1. (*Kontsevich*) *The equivalence classes of star products biject with the equivalence classes of formal Poisson structures where A is a smooth commutative algebra in characteristic 0 (and hence HKR theorem applies and we know that $HH^*(A, A) = \bigwedge^*(\text{Der } A)$ where two Poisson structures are equivalent if they are conjugate under the group $\exp(t \text{Der } A[t])$. In particular this applies for $k[x_1, \dots, x_n]$*

Proof. No Proof □

From the introduction, recall

Theorem 4.2. (*Gerstenhaber*) *If $HH^3(A, A) = 0$ then all infinitesimal deformations are integrable*

Proof. Last time, we had equations for associativity from each t^i , (1). If we have a 2-cocycle F_1 (an infinitesimal deformation corresponds to such, working mod t^2). We know that the left hand side of each equation is a 3-cocycle. Thus if $HH^3(A, A) = 0$, each is a coboundary so the left hand side of (1) in the case $i = 2$ is $\delta(F_2)$ for some F_2 and so we can continue inductively to define all F_i . This gives us a formal deformation extending the infinitesimal deformation and so it is integrable. □

Also in the introduction we met

Theorem 4.3. (*Gerstenhaber*) *If $HH^2(A, A) = 0$ then A is rigid (ie all formal deformations are trivial)*

We will prove the more general result

Theorem 4.4. (*Gerstenhaber*) *A formal deformation f is equivalent to $g(a, b) = ab + t^n G_n(a, b) + \text{higher order}$ where G_n is a 2-cocycle that is not cohomologous to 0. If $HH^2(A, A) = 0$ then f is trivial and hence 4.3 follows.*

Proof. Suppose that $f(a, b) = ab + tF_1(a, b) + \dots$ and $F_1, \dots, F_{n-1}$ are zero. Then it follows from (1) (at n) that $\delta F_n = 0$ since all terms below F_n are zero. So if F_n is not a coboundary, then we are done. If F_n is a coboundary for some $\phi_n: A \rightarrow A$ we have $F_n = -\delta\phi_n$. We set $\Phi_n(a) = a + t^n\phi_n(a)$. Then $\Phi_n^{-1}f(\Phi_n(a), \Phi_n(b))$ has no terms below $n+1$ and we can repeat inductively. We are done if we reach a non-coboundary at some point. If the process continues without obstruction, we deduce that f is trivial since $\dots \circ \Phi_3 \circ \Phi_2$ converges in $\text{End}_{k[[t]]}(V \otimes k[[t]])$ for V the underlying vector space of A . $\square$

We can similarly consider $k[[t]]$ -linear maps $\Phi(a) = a + t\phi_1(a) + t^2\phi_2(a) + \dots$ and when it defines an automorphism of $A[[t]]$ with the usual product. Working mod t^2 , we are thinking about automorphisms of $A[\epsilon]/(\epsilon^2)$ and we saw that these correspond exactly with derivations of A . Thus ϕ_1 is a derivation. Given ϕ_1 , when is it integrable in this sense? (when does an infinitesimal automorphism rise to an automorphism?)

Theorem 4.5. (*Gerstenhaber*) Suppose $HH^2(A, A) = 0$. Then all derivations are integrable

Proof. Exercise $\square$

5 Hodge Decomposition

In this section we assume A is commutative and k is characteristic 0

Remark. We will look at $HH^n(A, A)$ and see that it decomposes as a direct sum, but there's a similar decomposition for Hochschild homology, and for $HH^n(A, M)$ for M a symmetric bimodule (where both actions are the same) Historically this is due to Gerstenhaber and Schack, and Loday (ca 1988). They were building on a work of Barr (1987) showing that Harrison cohomology is a direct summand of Hochschild cohomology

Recall that the Hochschild cochain complex has terms

$$C^n(A, A) = \text{Hom}_{A \otimes_k A^{\text{op}}}(A^{\otimes(n+2)}, A) \cong \text{Hom}_k(A^{\otimes n}, A)$$

The symmetric group S_n acts on the left of $A^{\otimes(n+2)}$ by permuting the inner terms and hence we get a right action of kS_n on $C^n(A, A)$.

Remark. A crucial point: There is a mutually orthogonal set of idempotents in $\mathbb{Q}S_n \subset kS_n$ called the **Eulerian idempotents** $e_n(1), \dots, e_n(n)$ such that $1 = e_n(1) + \dots + e_n(n)$ for which $d_n e_n(j) = e_{n-1}(j) d_n$ as operating on the chain complex. So the Hochschild chain complex decomposes into a direct sum of subcomplexes

Theorem 5.1. (*Hodge decomposition*) For each n ,

$$HH^n(A, A) = \bigoplus_{i=1}^n HH^n(A, A) e_n(i)$$

The component $HH^n(A, A)e_n(1)$ is the Harrison cohomology of degree n . The component $HH^n(A, A)e_n(n) = \bigwedge^n(\text{Der } A)$

As a consequence, if A is a smooth commutative k algebra, we know that the HKR theorem applies, $HH^*(A, A) = \bigwedge_A(\text{Der } A)$ and so we read off that the other components are 0. In particular, for degree at least 2, the Harrison cohomology is 0. Calculation shows that any non-trivial commutative formal deformation gives rise to a 2-cocycle that is a Harrison 2-cocycle which is not a coboundary. But such a cocycle does not exist, so all commutative formal deformations of a smooth commutative k -algebra are trivial. On the other hand, for example, $k[x_1, \dots, x_n]$ is smooth and has non-commutative deformations as we've seen

Definition 5.1. For non-negative integers p, q a (p, q) -**shuffle** is some $\sigma \in S_{p+q}$ such that $\sigma(i) < \sigma(j)$ whenever $1 \leq i < j \leq p$ or $p+1 \leq i < j \leq p+q$. Let $S_{p,q}$ denote the subset of S_{p+q} of all (p, q) -shuffles

Definition 5.2. If $f \in \text{Hom}_k(A^{\otimes n}, A)$, then f is a **Harrison cochain** if

$$\sum_{\sigma \in S_{p,q}} (\text{sgn}(\sigma)) f(a_{\sigma^{-1}(1)} \otimes \dots \otimes a_{\sigma^{-1}(n)}) = 0$$

for each p, q such that $p+q=n$. The Harrison cochains form a subcomplex of the Hochschild cochain complex with cohomology called the **Harrison cohomology**

Remark. Harrison's original definition was different

We denote $s_{i,n-i} = \sum_{\sigma \in S_{i,n-i}} (\text{sgn } \sigma) \sigma \in kS_n$ and $s_n = \sum_{i=1}^{n-1} s_{i,n-i}$ with special cases $s_0 = s_1 = 0$

Barr points out that it's useful, because A is a symmetric bimodule, to use the complex $B_n = A \otimes A^{\otimes n}$ viewed as a symmetric bimodule by acting on the first term on the left and right. We denote elements of B_n as $a[a_1, \dots, a_n]$. The boundary maps on B_n are defined by

$$\begin{aligned} d(a[a_1, \dots, a_n]) &= aa_1[a_2, \dots, a_n] \\ &\quad + \sum_{i=1}^{n-1} (-1)^i a[a_1, \dots, a_i a_{i+1}, \dots, a_n] \\ &\quad + (-1)^n aa_n[a_1, \dots, a_{n-1}] \end{aligned}$$

Lemma 5.2. (*Barr*)

$$\begin{aligned} ds_{i,n-i}[a_1, \dots, a_n] &= s_{i-1,n-i}d[a_1, \dots, a_i] \otimes [a_{i+1}, \dots, a_n] \\ &\quad + (-1)^i s_{i,n-i-1}[a_1, \dots, a_i] \otimes d[a_{i+1}, \dots, a_n] \end{aligned}$$

Proof. Trivial □

Corollary 5.2.1. $ds_n = s_{n-1}d$

Proof. Trivial □

Proposition 5.1. *For each n , there is $e_n \in \mathbb{Q}S_n \subset kS_n$ such that*

1. e_n is a polynomial in s_n with no constant term
2. $\text{sgn } \sigma = 1$ where $\text{sgn}: \mathbb{Q}S_n \rightarrow \mathbb{Q}$ extends $\text{sgn}: S_n \rightarrow \mathbb{Q}$
3. $de_n = e_{n-1}d$
4. $e_n^2 = e_n$
5. $e_n s_{i,n-i} = s_{i,n-i}$

From this we see that the cochains which vanish for all $s_{i,n-i}$ are those which vanish for s_n which is equivalent to vanishing for e_n

Proof. Deferred □

Let $\epsilon_n = \frac{1}{n!} \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma$, which is a central idempotent in $\mathbb{Q}S_n$ with $\epsilon_n u = \text{sgn } u$ for $u \in \mathbb{Q}S_n$. For any chain $[a_1, \dots, a_n]$ we have $d\epsilon_n[a_1, \dots, a_n] = 0$. If $u \in \mathbb{Q}S_n$ satisfies $du[a_1, \dots, a_n] = 0$ for all chains $[a_1, \dots, a_n]$, then we have $u = (\text{sgn } u)\epsilon_n$

Remark. $\text{sgn } s_{i,n-i}$ is a binomial coefficient $\binom{n}{i}$, and $\text{sgn } s_n = 2^n - 2$

Proposition 5.2. *The minimum polynomial of s_n acting on B_n is a product of distinct linear factors, $\mu_n(x) = \prod_{i=1}^n (x - 2^i + 2)$ which is $(x - 2^n + 2)\mu_{n-1}(x)$*

Proof.

Lemma 5.3. *For any chain $[a_1, \dots, a_n]$ we have $d\epsilon_n[a_1, \dots, a_n] = 0$ where $\epsilon_n = \frac{1}{n!} \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma$. If $u \in \mathbb{Q}S_n$ satisfies $du[a_1, \dots, a_n] = 0$ for all chains then $u = (\text{sgn } u)\epsilon_n$*

Proof. In the computation of $n!d\epsilon_n[a_1, \dots, a_n]$, the term $a_{\sigma(1)}[a_{\sigma(2)}, \dots, a_{\sigma(n)}]$ appears with coefficient $\text{sgn } \sigma$ in the first term of the boundary of $\sigma^{-1}[a_1, \dots, a_n]$ and with coefficient $\text{sgn } \sigma \text{sgn } \tau (-1)^n$ in the last term of the boundary of $(\sigma\tau)^{-1}[a_1, \dots, a_n]$ where $\tau = (12 \dots n)$, but $\text{sgn } (\tau) = (-1)^{n-1}$ and so the terms cancel. Similarly every other term cancels so we are done. Now if $du = 0$ then for each $\sigma \in S_n$ and $1 \leq i \leq n$, the coefficients of $\sigma, \sigma(i, i+1)$ must have opposite sign (by the calculation above), hence u is a multiple of ϵ_n , which must be given by the sign of u □

By induction, $n = 1, n = 2$ are trivial. Assume $n \geq 3$ and use induction. Now as $ds_n = s_{n-1}d$, $d\mu_n(s_n) = \mu_n(s_{n-1})d = \mu_{n-1}(s_{n-1})(s_{n-1} - 2^n + 2)d = 0$ by induction and so if $u = \mu_n(s_n) \in \mathbb{Q}S_n$, by the lemma, $\mu_n(s_n) = \text{sgn}(\mu_n(s_n))\epsilon_n$. But $\text{sgn}(\mu_n(s_n)) = \mu_n \text{sgn}(s_n) = \mu_n(2^n - 2) = 0$. Now if v is a non-zero polynomial with $v(s_n) = 0$, then $dv(s_n) = 0$, so

$$\text{sgn}(v(s_n)) = v(\text{sgn } s_n) = v(2^n - 2) = 0$$

, so $(x - 2^n + 2) \mid v$. Now as $dv(s_n) = v(s_{n-1})d$, $v(s_{n-1})$ annihilates every boundary. This works for the case A is a polynomial algebra. Only the zero element of $\mathbb{Q}S_{n-1}$ annihilates $d[x_1, \dots, x_n]$ and so $v(s_{n-1}) = 0$, thus $\mu_{n-1} \mid v$ and so $\mu_n \mid v$ and we are done $\square$

Thus $\mu_n(x) = (x - \lambda_1) \dots (x - \lambda_n)$ for $\lambda_i = 2^i - 2$. Let

$$e_n(j) = \frac{\prod_{i \neq j} (s_n - \lambda_i)}{\prod_{i \neq j} (\lambda_j - \lambda_i)}$$

and $e_n(j) = 0$ for $j > n$, so that $e_n(j)$ are the projections onto the λ_j eigenspaces

Theorem 5.4. *The $e_n(j)$ are mutually orthogonal idempotents with sum 1 and $\sum_i \lambda_i e_n(i) = s_n$*

Proof. Trivial $\square$

Theorem 5.5.

1. Barr's idempotent e_n from 5.1 is $e_n(2) + \dots + e_n(n)$ (and so 5.1 is proven)
2. $de_n(j) = e_{n-1}(j)d$ so $de_n(n) = 0$
3. $\text{sgn } e_n(j) = 0$ for $j \neq n$, and $\text{sgn } e_n(n) = 1$
4. $e_n(n) = \epsilon_n$

Proof. 1. Since e_n is a polynomial in s_n with no constant term, we can write $e_n = \sum \alpha_i e_n(i)$ with each α_i either 0 or 1. As $\lambda_1 = 0, \alpha_1 = 0$ and as $e_n s_n = s_n$, all others are 1

2. If $j < n$ then the formula for $e_n(j)$ together with $ds_n = s_{n-1}d$ gives $de_n(j) = (\lambda_j - \lambda_n)^{-1}(s_{n-1} - \lambda_n)e_{n-1}(j)d$ so it suffices to show

$$de_n(j) = (\lambda_j - \lambda_i)^{-1}(s_{n-1} - \lambda_n)e_{n-1}(j) = e_{n-1}(j)$$

. If we write $s_{n-1} = \lambda_1 e_{n-1}(1) + \dots + \lambda_{n-1} e_{n-1}(n-1)$, this is clear
For $j = n$ we get $de_n(n) = \prod (\lambda_i - \lambda_n) \prod (s_{n-1} - \lambda_i) d = 0$

3. For $j < n$, $\text{sgn } e_n(j)$ contains $\text{sgn}(s_n - \lambda_n)$ as a factor, but that is 0. Since the $e_n(j)$ sum to give 1, it follows
4. For any $[a_1, \dots, a_n]$, $de_n(n)[a_1, \dots, a_n] = 0$ so $e_n(n) = (\text{sgn } e_n(n))\epsilon_n = \epsilon_n$ $\square$

Remark. 1. The Hodge decomposition works for A commutative in characteristic 0 and for any symmetric $A - A$ bimodule M , so we get a decomposition of $HH^*(A, M)$, eg. $M = A^*$ the dual of A

2. In general, without the commutativity restriction, the Hochschild chain complex with coefficients A^* has a subcomplex whose (co)homology gives rise to cyclic (co)homology, and for A commutative with characteristic k being 0, this subcomplex is invariant under s_n , so we have a Hodge decomposition for cyclic (co)homology
3. Bergeron-Wolfgang have an interpretation of the other components of the decomposition in terms of more complicated shuffles
4. There is a shuffle product on the B_n complex inducing a product of the Hochschild homology, for commutative A

6 Cyclic Cohomology

Cyclic cohomology is related to Hochschild cohomology with coefficients in A^* which is defined as $\text{Hom}_k(A, k)$. This is an A - A bimodule via $(afb)(c) = f(bca)$. The Hochschild cocomplex has terms

$$C^n(A) = C^n(A, A^*) = \text{Hom}_k(A^{\otimes n}, \text{Hom}(A, k))$$

We recall a basic lemma that $\text{Hom}_R(M, \text{Hom}_R(N, P)) \cong \text{Hom}_R(M \otimes_R N, P)$ for R -modules M, N, P and therefore $C^n(A) \cong \text{Hom}_k(A^{\otimes(n+1)}, k)$. More generally since $\text{Hom}(A^{\otimes n}, M^*) \cong \text{Hom}(M \otimes A^{\otimes n}, k)$ is compatible with our boundary/coboundary maps, in characteristic 0 we have a natural isomorphism of the form $HH^n(A, A^*) \cong HH_n(A, A)^*$. Recall, using $C^n(A, A^*) = \text{Hom}(A^{\otimes(n+1)}, k)$,

$$\begin{aligned} (\delta f)(a, \dots, a_{n+1}) &= \sum_{i=0}^n (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots, a_n) \\ &\quad + (-1)^{n+1} f(a_{n+1} a_0, a_1, \dots, a_n) \end{aligned}$$

Remark. $HH^0(A, A^*) = \{f \in \text{Hom}(A, k) \mid f(ab) = f(ba)\}$ is the space of traces. The dual result is that $HH^0(A, A) = A/[A, A]$

The original definition of cyclic (or λ -)cohomology, due to Connes, looking at $k = \mathbb{C}$, and generalising de Rham homology for smooth complex manifolds aimed to generalise to non-commutative differential geometry.

Definition 6.1. An n -cyclic cochain is an $f \in C^n(A)$ such that the identity $f(a_n, a_0, \dots, a_{n-1}) = (-1)^n f(a_0, \dots, a_n)$ holds for all $a_0, \dots, a_n \in A$. The space of cyclic cochains is called $C_\lambda^n(A)$

Lemma 6.1. $\delta C_\lambda^n(A) \subset C_\lambda^{n+1}(A)$

Proof. Define $\lambda: C^n(A) \rightarrow C^n(A)$ by $\lambda f(a_0, \dots, a_n) = (-1)^n f(a_n, a_0, \dots, a_{n-1})$ and define $\delta': C^n(A) \rightarrow C^{n+1}(A)$ by

$$(\delta' f)(a_0, \dots, a_{n+1}) = \sum_{i=1}^n (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots, a_n)$$

as the Hochschild cohomology with the last term missing. We can check that $(1 - \lambda)\delta = \delta'(1 - \lambda)$. Since $C_\lambda^n(A) = \ker(1 - \lambda)$, we're done $\square$

We therefore have a subcomplex of the Hochschild cochain complex

Definition 6.2. The **cyclic complex** is

$$C_\lambda^0(A) \xrightarrow{\delta} C_\lambda^1(A) \xrightarrow{\delta} \dots$$

. The cohomology of this subcomplex is the **cyclic (or λ -cohomology, $HC_\lambda^*(A)$**

The inclusion of complexes from the cyclic complex to the Hochschild complex induces a map $I: HC_\lambda^n(A) \rightarrow HH^n(A, A^*)$ for $n \geq 0$

Example 6.1. Let $A = \mathbb{C}$. We have $C_\lambda^{2n}(\mathbb{C}) = \mathbb{C}$ and $C_\lambda^{2n+1}(\mathbb{C}) = 0$ and so the cyclic complex looks like

$$0 \longrightarrow \mathbb{C} \longrightarrow 0 \longrightarrow \mathbb{C} \longrightarrow \dots$$

but $HH^n(\mathbb{C}, \mathbb{C}) = 0$ for $n \geq 1$ while $HC_\lambda^{2n}(\mathbb{C}) = \mathbb{C}$ and $HC_\lambda^{2n+1}(\mathbb{C}) = 0$ so the map I on cohomology need not be injective

Remark. $HH_\lambda^0(A) = HH^0(A, A^*)$ is the space of traces for all A

Example 6.2. For M a closed smooth orientable manifold of dimension n , and $A = C^\infty(M)$ the smooth complex valued functions on M , define

$$\phi(f^0, \dots, f^n) = \int_M f^0 df^1 \dots df^n$$

. This is a cyclic cochain - we can check it is a Hochschild cochain, and the cyclic is

$$\int_M f^n df^0 \dots df^{n-1} - (-1)^n \int_M f^0 df^1 \dots df^n = \int_M d(f^n f^0 df^1 \dots df^{n-1})$$

which is zero by Stokes theorem

We have another definition of Cyclic cohomology for general k . We introduced $\lambda: C^n(A) \rightarrow C^n(A)$ so we can define $N: C^n(A) \rightarrow C^n(A)$ where $N = \sum_{i=0}^n \lambda^i$. We also met δ' . These give rise to a first quadrant bicomplex

Definition 6.3. The **cyclic bicomplex**, $CC(A)$ is given by

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \delta \uparrow & & -\delta' \uparrow & & \delta \uparrow \\ C^2(A) & \xrightarrow{1-\lambda} & C^2(A) & \xrightarrow{N} & C^2(A) \xrightarrow{1-\lambda} \dots \\ \delta \uparrow & & -\delta' \uparrow & & \delta \uparrow \\ C^1(A) & \xrightarrow{1-\lambda} & C^1(A) & \xrightarrow{N} & C^1(A) \xrightarrow{1-\lambda} \dots \\ \delta \uparrow & & -\delta' \uparrow & & \delta \uparrow \\ C^0(A) & \xrightarrow{1-\lambda} & C^0(A) & \xrightarrow{N} & C^0(A) \xrightarrow{1-\lambda} \dots \end{array}$$

Definition 6.4. For general k , we have $HC^*(A)$, the cohomology of the total complex of the cyclic bicomplex $CC(A)$

Remark. When the characteristic is 0, it can be shown that the inclusion of complexes $C_\lambda^*(A), \delta \rightarrow C^*(A), \delta$ induces an isomorphism $HC_\lambda^n(A) \rightarrow HC^n(A)$ for $n \geq 0$.

The more general definition is more difficult to compute, so we use the original in characteristic 0

The short exact sequence of complexes

$$0 \rightarrow C_\lambda \rightarrow C_{\text{Hochschild}} = C \rightarrow C/C_\lambda \rightarrow 0$$

yields a long exact sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & HC_\lambda^n(A) & \longrightarrow & HH^n(A, A^*) & & \\ & & & \searrow & & & \\ H^n(C/C_\lambda) & \longrightarrow & HC_\lambda^{n+1}(A) & \longrightarrow & \dots & & \end{array}$$

so we need to consider $H^*(C/C_\lambda)$. Recall operators λ, N . We have the identities $N(1 - \lambda) = (1 - \lambda)N = 0$ and $\delta N = N\delta'$. Consider the short exact sequence $0 \rightarrow C/C_\lambda \xrightarrow{1-\lambda} (C, \delta') \xrightarrow{N} C_\lambda \rightarrow 0$. The middle term is exact as we can write down the contracting homotopy $s: C^n \rightarrow C^{n-1}$ defined by the expression $(s\phi)(a_0, \dots, a_{n-1}) = (-1)^{n-1}\phi(a_0, \dots, a_{n-1}, 1)$, and hence we get isomorphisms from the long exact sequence $HC^{n-1}(A) \rightarrow H^n(C/C_\lambda)$ and so

Definition 6.5. (Connes' long exact sequence) We have a long exact sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & HC_\lambda^n(A) & \xrightarrow{I} & HH^n(A, A^*) & & \\ & & & \searrow B & & & \\ H^{n-1}(C/C_\lambda) & \xrightarrow{s} & HC_\lambda^{n+1}(A) & \longrightarrow & \dots & & \end{array}$$

where B is the composition $HH^n(A, A^*) \rightarrow H^n(C/C_\lambda) \rightarrow HC_\lambda^{n+1}(A)$ and I arises from the inclusion of cyclic cochains into Hochschild cochains

At the level of cochains we have the **Connes operator**, $B = Ns(1 - \lambda)$ where s is the contracting homotopy $s: C^n \rightarrow C^{n-1}$. We have

$$\begin{aligned} Ns(1 - \lambda)(1 - \lambda)^{-1}\delta'N^{-1}[\phi] &= Ns\delta'N^{-1}[\phi] \\ &= N(1 - \delta's)N^{-1}[\phi] \\ &= (1 - \delta NsN^{-1})[\phi] \\ &= [\phi] \end{aligned}$$

so at the level of cohomology, B is the inverse of $(1 - \lambda)^{-1}\delta'N^{-1}$. We can also write $B = NB_0$ where

$$B_0\phi(a_0, \dots, a_{n-1}) = \phi(1, a_0, \dots, a_{n-1}) - (-1)^n\phi(a_0, \dots, a_{n-1}, 1)$$

. Using the relations $(1 - \lambda)\delta = \delta'(1 - \lambda)$, $(1 - \lambda)N = N(1 - \lambda) = 0$, $\delta N = N\delta'$ and $s\delta' + \delta's = 1$, it is easy to see that $\delta B + B\delta = 0$ and $B^2 = 0$. Let s' be the composition $s': HC^{n-1}(A) \rightarrow H^n(C/C_\lambda) \rightarrow HC^{n+1}(A)$. Therefore $s'[\phi] = [\delta(1 - \lambda)^{-1}\delta'N^{-1}\phi]$. If we take a cochain $\psi \in [(1 - \lambda)^{-1}\delta'N^{-1}\phi]$ then $\delta\psi$ is cyclic and $B[\psi] = \phi$. We compute for later that

$$\begin{aligned} B(1 - \lambda)^{-1}\delta'N^{-1}\phi &= Ns(1 - \lambda)(1 - \lambda)^{-1}\delta'N^{-1}\phi \\ &= N(1 - \delta's)N^{-1}\phi \\ &= \phi - \delta Ns\phi \end{aligned}$$

giving $s'[\phi] = [\delta\psi] = [\delta B^{-1}\phi]$. We hence get a long exact sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & HC_\lambda^n(A) & \xrightarrow{I} & HH^n(A, A^*) & & \\ & & & \searrow B & & & \\ HC_\lambda^{n-1}(A) & \xrightarrow{s'} & HC_\lambda^{n+1}(A) & \longrightarrow & \dots & & \end{array}$$

Setting $s[\phi] = n(n + 1)s'[\phi]$ for $n - 1$ cocycles, we get the Connes long exact sequence

Definition 6.6. We define **Periodic cyclic cohomology** as

$$HP^i(A) = \varinjlim_S HC_\lambda^{2n+i}(A)$$

for $i = 0, 1$

Example 6.3. If $A = \mathbb{C}$ then $HP^0(\mathbb{C}) = \mathbb{C}$, $HP^1(\mathbb{C}) = 0$

Definition 6.7. We define a new bicomplex, $B(A)$, by

$$\begin{array}{ccccccc} \vdots & \xrightarrow{B} & \vdots & \xrightarrow{B} & \vdots & \xrightarrow{B} & \dots \\ \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\ C^2(A, A^*) & \xrightarrow{B} & C^1(A, A^*) & \xrightarrow{B} & C^0(A, A^*) & & \\ \delta \uparrow & & \delta \uparrow & & & & \\ C^1(A, A^*) & \xrightarrow{B} & C^0(A, A^*) & & & & \\ \delta \uparrow & & & & & & \\ C^0(A, A^*) & & & & & & \end{array}$$

. We have $\delta^2 = B^2 = \delta B + B\delta = 0$

Theorem 6.2. The map $\phi \mapsto (0, \dots, 0, \phi)$ is a quasi-isomorphism of complexes

$$(C_\lambda^*, \delta) \rightarrow (\text{Tot}(B(A)), \delta + B)$$

and so we have an isomorphism of cohomology

Proof. No Proof □

Proposition 6.1. *The total complexes $\text{Tot}(B(A))$ and $\text{Tot}(CC(A))$ are homotopy equivalent*

Proof. We define chain maps and show chain homotopy explicitly. Define $I: \text{Tot}(B(A)) \rightarrow \text{Tot}(CC(A))$ (not the same as the Connes sequence) by $I = \text{Id} + Ns$ and $J: \text{Tot}(CC(A)) \rightarrow \text{Tot}(B(A))$ by $J = \text{Id} + sN$. I, J are chain maps. Define $g: \text{Tot}(B(A)) \rightarrow \text{Tot}(B(A))$ by $g = Ns^2B_0$ and $h: \text{Tot}(CC(A)) \rightarrow \text{Tot}(CC(A))$ by $h = s$, where $B_0 = s(1 - \lambda)$. Direct computation shows that these are the desired chain maps and homotopies so we are done. □

In the cyclic bicomplex, note that the rows are exact except in degree 0 by using a homotopy operator

$$H = \frac{1}{n+1}(1 + 2\lambda + \cdots + (n+1)\lambda^n): C^n(A, A^*) \rightarrow C^n(A, A^*)$$

Then $(1 - \lambda)H = \frac{1}{n+1}N - \text{Id}$ which implies exactness in positive degrees, so working out cohomology of the cyclic bicomplex, we have 0 except in the first column. Therefore in characteristic 0, the cohomology of $\text{Tot } CC(A)$ is the λ -cohomology

7 Bi-algebras and Hopf Algebras

An algebra A over a field k has a multiplication $\mu: A \otimes A \rightarrow A$ and a unit $u: k \rightarrow A$ satisfying certain commutative diagrams. We dualise this to get

Definition 7.1. A k -coalgebra C is a k -vector space with k -linear maps of the form $\Delta: C \rightarrow C \otimes C$, the comultiplication, and the counit $\epsilon: C \rightarrow k$ such that

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \Delta \otimes \text{Id} \\ C \otimes C & \xrightarrow{\text{Id} \otimes \Delta} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccccc} & & C & & \\ & \swarrow & \downarrow \Delta & \searrow & \\ k \otimes C & \xleftarrow{\epsilon \otimes \text{Id}} & C \otimes C & \xrightarrow{\text{Id} \otimes \epsilon} & C \otimes k \end{array}$$

commute. A **morphism of coalgebras** is a k -linear map $f: C \rightarrow D$ such that $\Delta_D f = (f \otimes f) \Delta_C$ and $\epsilon_D f = \epsilon_C$.

A subspace $I \subseteq C$ is a **coideal** if $\Delta(I) \subseteq C \otimes I + I \otimes C$ and $\epsilon(I) = 0$.

In this case, C/I is a coalgebra with the induced coproduct and counit

C is **cocommutative** if $\tau \Delta = \Delta$ where $\tau: V \otimes W \rightarrow W \otimes V$ is the natural flip map

- Example 7.1.**
1. If G is a group then kG is a coalgebra with $\Delta(g) = g \otimes g$ and ϵ the augmentation map, $\sum_{g \in G} \lambda_g g = \sum_{g \in G} \lambda_g$. This is cocommutative
 2. If $\mathfrak{g}$ is a Lie algebra over k then $\mathcal{U}(\mathfrak{g})$, its universal enveloping algebra, is a coalgebra by $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$ for any monomial $x = x_1 \dots x_n, x_i \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g})_i$. This is again cocommutative
 3. If $\mathcal{O}(M_n(k)) = k[x_{ij} \mid 1 \leq i, j \leq n]$, the polynomial functions on the space of $n \times n$, we can define $\Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}$ and $\epsilon(x_{ij}) = \delta_{ij}$, which is the dual of matrix multiplication. It is not cocommutative

Remark. Notation (Sweedler convention): We denote $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$

Example 7.2. The co-unit axiom can be denoted as

$$\sum \epsilon(c_{(1)}) c_{(2)} = \sum c_{(1)} \epsilon(c_{(2)}) = c$$

Definition 7.2. A **bi-algebra** B is a k -vector space with maps μ, u, Δ, ϵ such that (B, μ, u) is an algebra, (B, Δ, ϵ) is a coalgebra, and

1. Δ and ϵ are algebra maps
2. μ and u are coalgebra maps

Definition 7.3. A **Hopf algebra** H is a bi-algebra for which there is an antipode $S: H \rightarrow H$, a k -linear map such that

$$\mu \circ (S \otimes \text{Id}) \circ \Delta = \mu \circ (\text{Id} \otimes S) \circ \Delta = u \otimes \epsilon$$

Remark. The use of S is to enable left H -modules to be regarded as right H -modules and vice versa

Example 7.3. 1. kG is a Hopf algebra with $S(g) = g^{-1}$

2. $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra with $S(x) = -x$

3. We have a bi-algebra $\mathcal{O}(M_n(k))$. Let $D = \det(x_{ij}) \in \mathcal{O}(M_n(k))$. Then $\epsilon(D) = 1, \Delta(D) = D \otimes D$ and so $D - 1$ generates a bi-ideal and we can define $\mathcal{O}(SL_n(k)) = \mathcal{O}(M_n(k))/(D - 1)$. This is a Hopf algebra with antipode $S(x_{ij})$ being the ij th entry of $(x_{ij})^{-1}$ modulo $D - 1$

Remark. Note that these examples are either commutative or cocommutative

Lemma 7.1. *If C is a coalgebra then C^* is an algebra with multiplication Δ^* and unit ϵ^* . If C is cocommutative then C^* is commutative*

Proof. Trivial □

Remark. The same doesn't hold if we flip "algebra" and "coalgebra" in the Lemma, if A is infinite dimensional, as $A^* \otimes A^*$ is a proper subspace of $(A \otimes A)^*$, and the dual of the multiplication may not land within it. However if $A = \bigoplus_i A_i$ is graded with A_i finite dimensional, we can take the graded dual $\bigoplus_i A_i^*$ which is a coalgebra

Example 7.4. $(kG)^*$ is a commutative Hopf algebra for finite G and has comultiplication $\Delta(\phi_g) = \sum_{g_1 g_2 = g} \phi_{g_1} \otimes \phi_{g_2}$ with ϕ_g the dual basis

Example 7.5. Quantum groups are non-commutative, non-cocommutative Hopf algebras. The easiest example is the **Drinfeld (or quantum) double** of kG for finite G (the construction works for any finite dimensional Hopf algebra). As a vector space, $D(G) = (kG)^* \otimes_k kG$. Now G acts by automorphisms on $(kG)^*$ via $f^g(x) = f(gxg^{-1})$ and we use this to define the multiplication structure by $(f_1 \otimes g_1)(f_2 \otimes g_2) = f_1 f_2^{g_1^{-1}} \otimes g_1 g_2$ (so as algebras, $kG, (kG)^*$ embed into $D(G)$). The comultiplication is $\Delta(\phi_g \otimes h) = \sum_{g_1 g_2 = g} \phi_{g_1} \otimes h \otimes \phi_{g_2} \otimes h$. $D(G)$ is **quasi-triangular**, ie there is an involution R of $D(G) \otimes D(G)$ such that $R\Delta(x)R^{-1} = \tau\Delta(x)$. This is given by $R = \sum_{g \in G} \phi_g \otimes g$ with $R^{-1} = \sum \phi_g \otimes g^{-1}$ where we think g, ϕ_g as elements of $D(G)$. This gives an isomorphism between $V \otimes U, U \otimes V$ for $D(G)$ modules U, V . The unit of $D(G)$ is $k \mapsto k \sum_{g \in G} \phi_g \otimes 1_G$ and the counit is $\epsilon: D(G) \rightarrow k$ by $\phi_g \otimes h \mapsto 1$ if $g = 1$ and 0 otherwise. The antipode is $S: D(G) \rightarrow D(G)$ by $\phi_g \otimes h \mapsto h^{-1} \phi_{g^{-1}} = \phi_{h^{-1}g^{-1}h} \otimes h^{-1}$

Example 7.6. (Quantum elementary abelian groups). These are finite dimensional non-cocommutative Hopf algebras. Let m, n be positive integers with $n \geq 2$ and let q be a primitive n th root of unity. We define the k algebra H to be generated by $x_1, \dots, x_m, g_1, \dots, g_m$ with the relations $x_i^n = 0, g_i^n = 1$ and commutativity relations $x_i x_j = x_j x_i, g_i g_j = g_j g_i, g_i x_j = q x_j g_i$. We define the coalgebra structure of H by $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$ and $\Delta(g_i) = g_i \otimes g_i$. We define the counit by $\epsilon(x_i) = 0, \epsilon(g_i) = 1$ and the antipode by $S(x_i) = -g_i^{-1} x_i$ and $S(g_i) = g_i^{-1}$

Example 7.7. (Quantum enveloping algebras). We define $H = \mathcal{U}_q(sl_2)$, where $0 \neq q$ is a scalar such that $q^2 \neq 1$. As a k algebra, H is generated by E, F, K, K^{-1} with relations

$$KK^{-1} = K^{-1}K = 1, KE = q^2EK, KF = q^{-2}FK, EF = FE + \frac{K - K^{-1}}{q - q^{-1}}$$

We define the comultiplication by

$$\begin{aligned}\Delta(E) &= E \otimes 1 + K \otimes E \\ \Delta(F) &= F \otimes K^{-1} + 1 \otimes F \\ \Delta(K) &= K \otimes K\end{aligned}$$

with counit $\epsilon(E) = \epsilon(F) = 0, \epsilon(K) = 1$ and antipode

$$S(E) = -K^{-1}E, S(F) = -FK, S(K) = K^{-1}$$

. If q is a primitive n th root of unity, $n > 2$ then we can consider the quotient by an ideal generated by $E^n, F^n, K^n - 1$. This quotient is finite dimensional and not cocommutative. We denote this quotient by $u_q(sl_2)$, a ‘small quantum group’

7.1 Hochschild Cohomology of Finite Group Algebras

We want to use group cohomology, $H^*(G, M) = \text{Ext}_{kG}^*(k, M)$, to understand Hochschild cohomology, where the group cohomology is defined by taking a projective/free resolution of the trivial module k and applying $\text{Hom}_{kG}(\cdot, M)$ where M is a left kG module and taking homology, which is independent of resolution as usual.

The key point for us is that we have an embedding $\Delta: kG \rightarrow kG \otimes kG$ by $g \mapsto g \otimes g$. We thus have a copy of kG in $kG \otimes kG$. If we take the trivial module for $\Delta(kG)$ and induce to a module for $kG \otimes kG \cong k(G \times G)$, we get kG as a $kG \otimes kG$ module. Recall that $kG \cong kG^{op}$ and so we have kG as a $kG \otimes kG^{op}$ module, or as a $kG - kG$ bimodule. Taking a resolution of the trivial $k\Delta(G)$ module and induce to get a projective/free $kG \otimes kG^{op}$ resolution of kG . Taking $\text{Hom}_{kG - kG}(\cdot, M)$ where M is a bimodule is isomorphic to taking $\text{Hom}_{k\Delta(G)}(\cdot, M)$ where M is regarded as a $k\Delta(G)$ module. We then have that

$$HH^*(kG, kG) \cong H^*(G, kG)$$

as k -vector spaces, where the latter kG is regarded as a $\Delta(kG)$ module.

Remark. For any Hopf algebra H , H itself is a left H module under the left adjoint action given by, for $a, b \in H$, $a \cdot b = \sum a_{(1)} b S(a_{(2)})$. Denote this by H^{ad} . In the group algebra case $H = kG$, $H^{ad} = kG$ where G acts by conjugation. Furthermore, kG^{ad} splits as a direct sum of submodules,

$$kG = \bigoplus_{ccl X_i} kX_i$$

Taking conjugacy class representatives $g_1, \dots, g_r$, we have $kX_i = k \uparrow_{C(g_i)}^G$ for each i . If $C(g_i)$ is a finite index subgroup of G then we have that

$$k \uparrow_{C(g_i)}^G \cong \text{Coind}_{C(g_i)}^G k$$

Recall that $H^*(G, \text{Coind}_L^G(U)) \cong H^*(L, U)$ for $L \leq G$ and U a kL module. Thus $H^*(G, kX_i) = H^*(C(g_i), k)$

Theorem 7.2. $HH^*(kG, kG) = \bigoplus_{i=1}^r H^*(C(g_i), k)$ as k vector spaces

Proof. Trivial □

Corollary 7.2.1. If G is finite abelian then

$$H^*(kG, kG) \cong kG \otimes H^*(G, k)$$

as k vector spaces

Proof. Trivial □

Remark. Holm showed that this is an isomorphism of algebras under the cup product

In general, we have

Theorem 7.3. (Siegel-Witherspoon, 1999) If G is a finite group and $g_1, \dots, g_r$ is a set of conjugacy class representatives let $\alpha \in H^*(C(g_i), k)$ and $\beta \in H^*(C(g_j), k)$. Then let $\gamma_i(\alpha), \gamma_j(\beta)$ be the corresponding elements in $HH^*(kG, kG)$ under the vector space isomorphism of 7.2. Then

$$\gamma_i(\alpha) \cup \gamma_j(\beta) = \sum_{x \in D} \gamma_l(\text{cores}_{W(x)}^{C(g_i)} \text{res}_{W(x)}^{yC(g_i)} y^* \alpha) \cup \text{res}_{W(x)}^{yxC(g_j)} (gx)^\gamma(\beta)$$

where D is a set of double coset representatives for $C(g_i) \backslash G / C(g_j)$ and the integer $l = l(x)$ and group element $y = y(x)$ are chosen so that $g_l = (y g_i)(y^x g_j)$ and $W(x) = yC(g_i) \cap y^x C(g_j)$

We have restriction $\text{res}_H^G: H^*(G, M) \rightarrow H^*(H, M)$ for $H \leq G$ since a free kG resolution is also a free kH resolution

For corestriction, we have $\text{cores}_H^G: H^*(H, M) \rightarrow H^*(G, M)$ defined at the cochain level of a projective resolution $\mathcal{P}$ of the trivial module by

$$\text{Hom}_{kH}(\mathcal{P}_n, M) \ni f \mapsto \left(x \mapsto \sum_{g \in [G/H]} g \cdot f(g^{-1}x) \right)$$

This does not depend on the choice of coset representatives and commutes with the coboundary map, so induces a map on cohomology

We also have conjugation, where for $H \leq G$ and $g \in G$ we write ${}^g H$ for gHg^{-1} , and $g^*: H^*(H, k) \rightarrow H^*({}^g H, k)$ given on cochain level by $g^* f(x) = gf(g^{-1}x)$ for all $x \in \mathcal{P}_n$

Definition 7.4. A finite dimensional k algebra A is **symmetric** if A is isomorphic to A^* as an $A - A$ bimodule, or equivalently if there is a symmetric non-degenerate bilinear form

$$\langle \cdot, \cdot \rangle: A \times A \rightarrow k$$

such that $\langle ab, c \rangle = \langle a, bc \rangle$

Remark. $HH^*(A, A) \cong HH^*(A, A^*)$ for symmetric A , but the isomorphisms depend on the choice of bilinear form

Example 7.8. For a finite group G , we have a symmetric bilinear form of the form $kG \times kG \rightarrow k$ by $(g, g') \mapsto 1$ if $gg' = 1$ and 0 otherwise, so kG is symmetric

Remark. Some finite dimensional Hopf algebras are symmetric but some aren't

Definition 7.5. A **Batalin-Vilkovisky (BV) algebra** is a Gerstenhaber algebra with an operator V (although usually called Δ) of degree -1 satisfying $V^2 = 0$ and $[x, y] = (-1)^{|x|}V(xy) - (-1)^{|x|-1}(Vx)y - x(Vy)$

Tradler (2008) showed that for symmetric algebras the operator V can be defined at the cochain level. For $f \in C^n(A, A)$, we define $C^{n-1}(A, A) \ni V(f)$ by

$$\langle Vf(a_1, \dots, a_{n-1}), a_n \rangle = \sum_{i=1}^n (-1)^{i(n-1)} \langle f(a_i, \dots, a_n, a_1, \dots, a_{i-1}), 1 \rangle$$

This depends on the choice of the bilinear form

Theorem 7.4. (Tradler) $HH^*(kG, kG)$ is a BV-algebra

Proof. Trivial □

The map V behaves well on the centraliser decomposition and is related to the Connes operator. The Connes exact sequence therefore also behaves well under the centraliser decomposition.

Recall that

$$\dots \rightarrow HH^{n+1}(kG, kG) \xrightarrow{B} HC_\lambda^n(kG) \xrightarrow{S} HC_\lambda^{n+2}(kG) \xrightarrow{I} HH^n(kG, kG) \rightarrow \dots$$

is dual to an analogous long exact sequence in homology. Burghelea (1985) showed that this sequence respects the centraliser decomposition, and in fact we get a direct product of sequences corresponding to each component.

Benson, Kesser and Linckelmann (2021) “On the BV-Structure of the Hochschild Cohomology of Finite Group Algebras” gives a relatively straightforward way of describing the map V for kG with G finite - it breaks up as a product of operators V_g on the components of the centraliser decomposition. On the components, by renaming we may assume $g \in Z(G) \subseteq G$

Theorem 7.5. (BKL) Let G be a finite group and $g \in Z(G)$. Consider

$$z_g: \mathbb{Z} \times G \rightarrow G: (m, h) \mapsto g^m h$$

We get an induced map

$$z_g: H^n(G, k) \rightarrow H^n(\mathbb{Z} \times G, k) = H^n(G, k) \oplus H^{n-1}(G, k)$$

This has the form $(1, V_g)$ where $V_g = I \circ B$. The map V_g is a derivation with respect to the cohomology cup product

Proof. No Proof □

Example 7.9. If G is a finite cyclic p group the $H^*(G, k)$ contains a polynomial subalgebra $k[x]$ where x is of degree 2 and some y of degree 1 such that $H^*(G, k)$ is a $k[x]$ module generated by $1, y$. We know that

$$HH^*(kG, kG) \cong kG \otimes_k H^*(G, k)$$

and V sends $g \otimes x^n \mapsto 0$ and $g \otimes x^n y \mapsto y(g)g \otimes x^n$

Example 7.10. Let G be an extraspecial p group (ie nilpotent of class 2). Then $HH^1(kG, kG)$ is soluble as a Lie algebra of derived length 2 or 3

Theorem 7.6. (BKL) If G is a finite p group and $|Z(G): Z(G) \cap \Phi(G)| \geq 3$ then $HH^1(kG, kG)$ is not soluble