

Lie Algebras

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October 2025

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1 Introduction (Groj style)

We can define matrix groups, such as SL_n, SO_n, Sp_{2n} , which are simple in some sense. If we take these with complex entries, these are in fact differential manifolds as well, and are furthermore Lie groups (ie multiplication and inversion are smooth maps). These have maximal compact subgroups (such as $SU_n \subseteq SL_n$), and it turns out that over the complex numbers, these have the same representations (and SU_n is homotopic to SL_n). This is all complicated. We wish to see how much of these we can study by considering the tangent space at the origin, $\mathfrak{g} = T_e G$, as this reduces to linear algebra.

Definition 1.1. *Linear algebraic groups* are subgroups of GL_n defined by polynomial equations in the matrix coefficients

All the groups we mentioned above are linear algebraic.

We want to compute the tangent space at 1 of $SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1 \right\}$.

If we write $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \dots$. We have $1 = \det g = (1 + \varepsilon a)(1 + \varepsilon d) - \varepsilon^2 bc$, hence $g \in SL_2$ is equivalent to $a + d = 0$, ie $\text{Tr}(g) = 0$. To do this formally, we consider a linear algebraic group G over $E = \mathbb{C}[\varepsilon]/(\varepsilon^2)$. If we consider the map $E \rightarrow \mathbb{C}$ by $\varepsilon \mapsto 0$ gives a projection $\pi: G(E) \rightarrow G(\mathbb{C})$, we can define $\mathfrak{g} = \pi^{-1}(I)$, and this formalises the calculation done above.

Exercise 1.1. Prove that this is equivalent to the tangent space in algebraic geometry/differential geometry, and $G(E)$ is the tangent bundle.

Example 1.1.

1. $G = GL_n$. $G(E) = \{\tilde{A} \in \text{Mat}_n(E) \mid \tilde{A}^{-1} \text{ exists}\} = \{A + B\varepsilon \mid A^{-1} \text{ exists}\}$, as $(A + B\varepsilon)(A^{-1} - A^{-1}BA^{-1}\varepsilon) = I$. Therefore $\mathfrak{g} = \mathfrak{gl}_n = \text{Mat}_n$
2. $G = SL_n$.

Exercise 1.2. Show that $\det(I + \varepsilon X) = 1 + \varepsilon \text{Tr}(X)$

Hence $\mathfrak{g} = \mathfrak{sl}_n = \{X \in \text{Mat}_n \mid \text{Tr}(X) = 0\}$

3. $G = O_n\mathbb{C} = \{AA^T = I\}$. One can compute $\mathfrak{g} = \{X \mid X + X^T = 0\}$. Note $X + X^T = 0$ implies $2\text{Tr}(X) = 0$. As $2 \neq 0$, $\text{Tr}(X) = 0$. So the Lie algebra of the orthogonal group is the same as that of the special orthogonal group, hence $T_1O_n(\mathbb{C}) = T_1SO_n(\mathbb{C})$. This is a demonstration that Lie algebras only care about the connected component of I

What does $\mathfrak{g}$ remember about the multiplication in G ? It does not get a multiplication (as $(I + A\varepsilon)(I + B\varepsilon) = I + (A + B)\varepsilon$ which does not see the multiplication of the group)

Consider $G \times G \rightarrow G$ via $(P, Q) \mapsto PQP^{-1}Q^{-1}$. This sends $(1, 1)$ to 1, so we get a map $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, but what we want is a bilinear map, not a linear one. So we do this in two stages. Fixing P , we get a map $f_P: Q \mapsto PQP^{-1}Q^{-1}$, we differentiate this to get $df_P: \mathfrak{g} \rightarrow \mathfrak{g}$ which is a map $G \rightarrow \text{End}(\mathfrak{g})$, and we differentiate this.

If $G \subseteq \text{Mat}_n(\mathbb{C})$, writing $P = I + A\varepsilon$ and $Q = I + B\delta$, where $\varepsilon^2 = \delta^2 = 0$, and $\varepsilon\delta = \delta\varepsilon$ and expanding out, $PQP^{-1}Q^{-1} = I + (AB - BA)\varepsilon\delta$. Hence we can define $[A, B] = AB - BA$ for $A, B \in \mathfrak{g} \subseteq \text{Mat}_n$

Exercise 1.3. Show directly from the definition (as the derivative of the commutator) that $[X, Y] = -[Y, X]$ (as $PQP^{-1}Q^{-1} = (QPQ^{-1}P^{-1})^{-1}$)

Further show that the Jacobi Identity, $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ by (1) explicit computation for $\mathfrak{gl}_n$ or (2) define $(x, y, z) = y^{-1}((x, y^{-1}), z)y$ and see it by differentiating the identity $(x, y, z)(y, z, x)(z, x, y) = 1$ or (3) deduce it from associativity of multiplication of G

Definition 1.2. A *Lie algebra* is a vector space $\mathfrak{g}$ with a bracket operation $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which is bilinear, satisfies $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$, and satisfies the Jacobi identity $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

Example 1.2.

1. For any vector space V , if the bracket is always 0, we get a Lie algebra (called an *abelian* Lie algebra)
2. $\mathfrak{gl}_n = \text{Mat}_n$ with the bracket $[A, B] = AB - BA$
3. $\mathfrak{so}_n = \{A \in \mathfrak{gl}_n \mid A + A^T = 0\}$
4. $\mathfrak{sl}_n = \{A \in \mathfrak{gl}_n \mid \text{tr } A = 0\}$

$$5. \mathfrak{sp}_{2n} = \{A \in \mathfrak{gl}_{2n} \mid JA^T J^{-1} + A = 0\} \text{ where } J \text{ is } \begin{pmatrix} & & & & -1 \\ & & & & \\ & & & -1 & \\ & & 1 & & \\ & \cdots & & & \\ 1 & & & & \end{pmatrix}$$

6. $\mathfrak{b} = \{\text{upper triangular matrices in } \mathfrak{sl}_n\}$
7. $\mathfrak{n} = \{\text{strictly upper triangular matrices in } \mathfrak{sl}_n\}$

Exercise 1.4.

1. Check $\mathfrak{gl}_n$ is a Lie algebra
2. Check that all the non-abelian Lie algebras above are Lie subalgebras of $\mathfrak{gl}_n$, ie they are closed under $[A, B] = AB - BA$
3. Show that $\left\{ \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \right\}$ is not a Lie subalgebra of $\mathfrak{gl}_2$
4. Find algebraic groups whose Lie algebras are those above
5. Classify all 3 dimensional Lie algebras over $\mathbb{C}$

Definition 1.3. If $\mathfrak{g}, \mathfrak{h}$ are Lie algebras, a linear map $f: \mathfrak{g} \rightarrow \mathfrak{h}$ is a *Lie algebra homomorphism* if for all $x, y \in \mathfrak{g}$, we have $f([x, y]) = [f(x), f(y)]$.

Definition 1.4. A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$. We say $\mathfrak{g}$ acts on V . It is *faithful* if $f(x) = 0$ implies $x = 0$.

The Lie algebra $\mathfrak{g}$ acts on itself via $\text{ad } x: \mathfrak{g} \rightarrow \mathfrak{g}$ by $y \mapsto [x, y]$

Lemma 1.1. $\text{ad}: \mathfrak{g} \rightarrow \text{End } \mathfrak{g}$ is a Lie algebra homomorphism

Proof. We must show $\text{ad}[x, y] = \text{ad } x \text{ ad } y - \text{ad } y \text{ ad } x$
Now $\text{ad}[x, y](z) = [[x, y], z]$, and the right hand side is $[x, [y, z]] - [y, [x, z]]$,
and the equality of these is the Jacobi identity after a couple applications of
antisymmetry $\square$

Definition 1.5. The *centre* of a Lie algebra $\mathfrak{g}$ is $\{x \in \mathfrak{g} \mid [x, y] = 0, \forall y \in \mathfrak{g}\}$.

Remark 1.1. This is the kernel of ad , and so if $\mathfrak{g}$ has no centre, then ad is
an injection, hence $\mathfrak{g}$ admits a faithful finite dimensional representation in this
case.

Example 1.3. $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \subseteq \mathfrak{gl}_2$ has a faithful representation (in fact, we just gave
it), but is abelian, so the above result is not an equivalence

Theorem 1.2. (*Ado's theorem*) Every finite dimensional Lie algebra has a
faithful finite dimensional representation

Proof. No Proof $\square$

We use $\mathfrak{sl}_2$ as an extended (very important) example.
This has basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Exercise 1.5. $[ef] = h, [he] = 2e, [hf] = -2f$

So a representation of $\mathfrak{sl}_2$ on an n -dimensional vector space is a triple $E, F, H \in$
 $\text{Mat}_n(k)$ satisfying $EF - FE = H, [H, E] = 2E, [H, F] = -2F$. Where can we
find these? We will get them as tangent spaces to representations of SL_2

Definition 1.6. If G is an algebraic group, an *algebraic representation* of G on
a vector space V is a group homomorphism $G \rightarrow GL_V$ defined by polynomial
equations in the matrix coefficients of G

So we get maps

$$\begin{array}{ccc} G(E) & \longrightarrow & GL_V(E) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\rho} & GL_V \end{array}$$

where $E = k[\varepsilon]/(\varepsilon^2)$. So $\rho(I) = I$ and $\rho(I + A\varepsilon) = I + \varepsilon M(A)$ where $M(A)$ is
a matrix which is a function of A . We call the function M by $d\rho: \mathfrak{g} \rightarrow \mathfrak{gl}_V$

Exercise 1.6. Check ρ is a group homomorphism implies $d\rho$ is a Lie algebra
homomorphism

Example 1.4. Let $G = SL_n$ and $L(n)$ the space of homogeneous polynomials
in x, y of 2 variables. $GL_2 \curvearrowright L(n)$ by change of basis, ie

$$\rho_n \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) (f)(x, y) = f(ax + cy, bx + dy)$$

So ρ_0 is the trivial representation, ρ_1 is the standard representation, and

$$\rho_2 \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

We restrict this to an action of SL_2 . Now we differentiate.

For e , $\rho(I + e\varepsilon) \cdot x^i y^j = x^i (\varepsilon x + y)^j = x^i y^j + \varepsilon j x^{i+1} y^{j-1}$.

Hence $d\rho(e) \cdot x^i y^j = j x^{i+1} y^{j-1}$.

Exercise 1.7. Show the action of $\mathfrak{sl}_2$ is given by $e: x^i y^j \mapsto j x^{i+1} y^{j-1}$, $f: x^i y^j \mapsto i x^{i-1} y^{j+1}$, $h: x^i y^j \mapsto (i-j)x^i y^j$, and check directly that this is a representation. Further show that $L(2)$ and the adjoint representation are isomorphic. Also show that $e = x \frac{\partial}{\partial y}$, $f = y \frac{\partial}{\partial x}$, $h = e - f = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$ defines a representation of $\mathfrak{sl}_2$ on $k[x, y]$.

Suppose $\text{char } k = 0$. Show $L(n)$ is irreducible

We have a map from representations of an algebraic group G to representations of its Lie algebra $\mathfrak{g}$ by differentiation. For “most” Lie groups, this map is complicated

Example 1.5. Let $\mathfrak{g} = \mathbb{C}$ with the zero bracket. A representation of $\mathfrak{g}$ is uniquely determined by a linear map into V , as $\rho(1)$ determines the representation. $W \leq V$ is a submodule if it is preserved by $\rho(1)$. The representations given by A and A' are isomorphic if and only if they are conjugate as matrices, so isomorphism classes of representations are given by Jordan normal forms. For example if we take $A = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$ (or indeed, for n di-

mensions) this is indecomposable but not irreducible, it has only submodules $\langle e_1 \rangle, \langle e_1, e_2 \rangle, \langle e_1, e_2, e_3 \rangle, \langle e_1, e_2, e_3, e_4 \rangle$

Contrast representations of $G = GL_1 = \mathbb{C}^*$ as an algebraic group. It is a theorem that

Theorem 1.3. Every representation of GL_1 is a direct sum of irreducible representations, and irreducibles are given by $z \mapsto z^n, n \in \mathbb{Z}$

Proof. No Proof □

We get $d\rho_n = n \in \mathbb{C}$, so we don’t even get all the irreducible representations of the Lie algebra

$(\mathbb{C}, +)$ is an algebraic group with Lie algebra $\mathbb{C}$, so the map from algebraic groups to Lie algebras is not injective

Theorem 1.4. (Lie) $\text{Rep } G \rightarrow \text{Rep } \mathfrak{g}$ by $\rho \mapsto d\rho$ is an equivalence if G is a simply connected connected simple (algebraic, Lie) group

Proof. No Proof □

Definition 1.7. A *simple* (algebraic/Lie) group is one whose Lie algebra is simple.

Remark 1.2. This does not line up with the group theoretic definition, eg SL_n is simple, but $\mu_n = \{\lambda I \mid \lambda^n = 1\}$ is normal in SL_n , however $PSL_n = SL_n/\mu_n$ has the same Lie algebra. This holds in general for finite central subgroups $Z \subseteq G$, ie $\text{Lie}(G/Z) = \text{Lie}(G)$

Remark 1.3. The map from algebraic groups to Lie algebras is not injective as we've seen, and is also not surjective. In $\mathfrak{gl}_n$, any element decomposes as $S + N$ where S is diagonalisable, N is nilpotent, and S, N commute (Jordan-Chevalley decomposition), and any Lie algebra of an algebraic group also has such a decomposition, but you can find Lie algebras that don't.

2 Representations of $\mathfrak{sl}_2$

From now on, $k = \mathbb{C}$. We know $\mathfrak{sl}_2$ has basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $[ef] = h, [he] = 2e, [hf] = -2f$.

Theorem 2.1. 1. For every integer $n \geq 0$, there is a unique irreducible representation of $\mathfrak{sl}_2$ of dimension $n + 1$

2. Every finite dimensional representation is a direct sum of such representations.

Proof. deferred □

Definition 2.1. Let V be a representation of $\mathfrak{sl}_2$. We define the λ -weight space of V to be $V_\lambda = \{v \in V \mid hv = \lambda v\}$

Example 2.1. $L(n)_\lambda = \mathbb{C}x^i y^j$ if $i - j = \lambda$

Let $v \in V_\lambda$. Consider ev . Then $hev = (he - eh + eh)v = 2ev + \lambda ev = (\lambda + 2)v$ so $v \in V_\lambda$ implies $ev \in V_{\lambda+2}$, and the converse holds if $ev \neq 0$. Similarly $fv \in V_{\lambda-2}$. Therefore we have

$$\begin{array}{c} V_{\lambda+2} \\ \begin{array}{c} f \updownarrow e \\ \updownarrow \end{array} \\ V_\lambda \\ \begin{array}{c} f \updownarrow e \\ \updownarrow \end{array} \\ V_{\lambda-2} \end{array}$$

Definition 2.2. If $0 \neq v \in V_\lambda \cap \ker e$, we say v is a *highest weight vector* with *weight* λ

Lemma 2.2. Let V be a representation of $\mathfrak{sl}_2$ and $v \in V$ a highest weight vector of weight λ , then $W = \langle v, fv, f^2v, \dots \rangle$ is a $\mathfrak{sl}_2$ -invariant subspace.

Proof. We need to show $fW \leq W, hW \leq W, eW \leq W$. The first holds by construction. As $v \in V_\lambda$, $f^i v \in V_{\lambda-2i}$, so $hW \leq W$. Finally, $ev = 0$, and $ef^i v = (ef^i + fef^{i-1} - fef^{i-1})v = hf^{i-1}v - f(ef^{i-1}v) \in W$ by induction. A more explicit induction shows (exercise) $ef^n v = n(\lambda - n + 1)f^{n-1}v$. $\square$

Lemma 2.3. *Let V be a finite dimensional representation of $\mathfrak{sl}_2$ and v a highest weight vector of weight λ . Then $\lambda \in \mathbb{Z}_{\geq 0}$.*

Proof. $\{f^k v \mid k \in \mathbb{Z}_{\geq 0}\}$ lie in different eigenspaces of h if they are non-zero, hence are linearly independent. But V is finite dimensional, so only finitely many of the $f^k v$ are non-zero. Hence there exists some k such that $f^k v \neq 0, f^{k+1} v = 0$, so $(k+1)(\lambda - k)f^k v = ef^{k+1}v = 0$, hence $\lambda = k$ as $k+1 \neq 0$. $\square$

Proposition 2.1. *If V is a finite dimensional representation of $\mathfrak{sl}_2$ it has a highest weight vector*

Proof. As V is finite dimensional, it has some eigenvector v of h . Consider $v, ev, e^2v, \dots$. These are all eigenvectors of h of different eigenvalues, so are linearly independent, so some $e^{k+1}v = 0$ with $e^k v \neq 0$, so $e^k v$ is a highest weight vector $\square$

Corollary 2.3.1. *If V is an irreducible representation of $\mathfrak{sl}_2$ then $\dim V = n+1$ and V has a basis $v_0, \dots, v_{n+1}$ where $hv_i = (n-2i)v_i, f v_i = v_{i+1}, ev_i = i(n-i+1)v_{i-1}$. In particular there is a unique irreducible n dimensional representation which is isomorphic to $L(n)$.*

Proof. Trivial $\square$

Exercise 2.1. Check that this defines a representation of $\mathfrak{sl}_2$, so existence is part of what we've proven. Write down the isomorphism with $L(n)$ explicitly

Let V be a representation of $\mathfrak{sl}_2$.

Definition 2.3. Define $\Omega = ef + fe + \frac{1}{2}h^2 \in \text{End } V$. This is called the *Casimir element* of $\mathfrak{sl}_2$

Proposition 2.2. Ω commutes with $\mathfrak{sl}_2$ as endomorphisms of V

Exercise 2.2. Provide the proof.

Exercise 2.3. Recall $\mathbb{C}[x, y] = \bigoplus_{n \geq 0} L(n)$ as a representation of $\mathfrak{sl}_2$ where $\mathfrak{sl}_2$ acts as differential operators as given before defines a representation of $\mathfrak{sl}_2$ on $x^\lambda y^\mu \mathbb{C}[x, y]$ for any $\lambda, \mu \in \mathbb{C}$ and describe the submodule structure. Ω acts on this representation as a second order differential operator. Write it out explicitly.

$$\Omega = ef - fe + 2fe + \frac{1}{2}h^2 = 2fe + (\frac{1}{2}h^2 + h)$$

Corollary 2.3.2. *If V is a finite dimensional irreducible representation of $\mathfrak{sl}_2$ then ω acts as a scalar*

Proof. Schur's lemma. $\square$

Lemma 2.4. Ω acts on $L(n)$ by $\frac{1}{2}n^2 + n$

Proof. It suffices to find Ωv for v a highest weight vector, but in this case

$$(2fe + \frac{1}{2}h^2 + h)v = (\frac{1}{2}h^2 + h)v = (\frac{1}{2}n^2 + n)v$$

(note we don't even need Schur's lemma as $f^i v$ span $L(n)$, and $\Omega f^i v = f^i \Omega v$) $\square$

Note $n \mapsto \frac{1}{2}n^2 + n$ is strictly increasing if $n \geq 0$ so $n \neq m$ if and only if $L(n) \not\cong L(m)$ if and only if $\frac{1}{2}n^2 + n \neq \frac{1}{2}m^2 + m$

Definition 2.4. If V is a representation of $\mathfrak{sl}_2$, write $V^\lambda = \{v \in V \mid (\Omega - \lambda)^{\dim V} v = 0\}$, ie the generalised eigenspace for Ω with eigenvalue λ .

Remark 2.1. Observe V^λ is a representation of $\mathfrak{sl}_2$ as if $x \in \mathfrak{sl}_2, v \in V^\lambda$ then $(\Omega - \lambda)^{\dim V} xv = x(\Omega - \lambda)^{\dim V} v = 0$, so $xv \in V^\lambda$.

Corollary 2.4.1. $V = \bigoplus_\lambda V^\lambda$ as representations of $\mathfrak{sl}_2$

Proof. Trivial $\square$

Vague claim: If $V^\lambda \neq 0$ then $\lambda = \frac{1}{2}n^2 + n$ and V^λ is 'glued together' from copies of $L(n)$

Definition 2.5. Let W be a representation of a Lie algebra $\mathfrak{g}$. A *composition series* for W is a sequence $0 = W_0 < W_1 < \dots < W_r = W$ such that W_i/W_{i-1} is irreducible.

Example 2.2.

1. If $\mathfrak{g} = \mathbb{C}$ and $W = \mathbb{C}^r$ with $\mathfrak{g}$ acting by the Jordan block of eigenvalue 0 and size n , then there is a unique composition series for W , $\langle e_1 \rangle < \langle e_1, e_2 \rangle < \dots < W$
2. If $\mathfrak{g} = \mathbb{C}, W = \mathbb{C}^r$, $\mathfrak{g}$ acts by 0, then any chain $W_0 < W_1 < \dots < W_n$ such that $\dim W_i = i$ is a composition series

Lemma 2.5. If W is finite dimensional, then composition series always exist.

Proof. Induct on $\dim W$. Take an irreducible submodule $W_1 \leq W$. Then W/W_1 has lower dimension than W , so induction gives a series $0 \leq W_1 \leq [\text{preimage of composition series of } W/W_1]$. $\square$

Proposition 2.3. V^λ has a composition series with every subquotient $W_i/W_{i-1} \cong L(n)$, where $\lambda = \frac{1}{2}n^2 + n$.

Proof. Take an irreducible submodule of $V(\lambda)$. This is $L(n)$ for some n . So $\lambda = \frac{1}{2}n^2 + n$, and we can quotient by the irreducible submodule, and $\Omega \in V^\lambda/L(n)$ with the same generalised eigenvalue λ , so we are done by induction. $\square$

We now wish to show every V^λ is a direct sum of $L(n)$. This is harder.

Proposition 2.4. h acts on V^λ with generalised eigenvalues in the set $n, n-2, \dots, 2-n, -n$ where $\lambda = \frac{1}{2}n^2 + n$

Proof. This is just a general fact about composition series. If $h \subset W$, and $W' \leq W$ with $hW' \leq W'$, then generalised eigenvalues of h on W are the union of generalised eigenvalues of $h \subset W'$ and $h \subset W/W'$ (make h an upper triangular matrix). So as V^λ has a composition series with subquotients only $L(n)$ and h has eigenvalues $n, \dots, -n$ on $L(n)$, the proposition follows $\square$

Lemma 2.6. h acts on $\ker(e: V^\lambda \rightarrow V^\lambda)$ and the only generalised eigenvalue of h is n

Proof. Same proof, but for composition series of V^λ , intersected with $\ker e$ $\square$

Lemma 2.7. 1. $hf^k = f^k(h-2k)$

$$2. e f^{n+1} = f^{n+1}e + (n+1)f^n(h-n)$$

Proof. (1) was an exercise last time, (2) is an exercise this time. Consistency! $\square$

Proposition 2.5. h acts diagonally on $\ker(e: V^\lambda \rightarrow V^\lambda)$ by multiplication by n , and all vectors on which h acts by multiplication by n are in $\ker e$

Proof. If $hx = nx$ then $ex \in V_{n+2}^\lambda = 0$. Conversely if $x \in \ker e$ then $(h-n)^{\dim V^\lambda} x = 0$. Now $(h-n+2k)^{\dim V^\lambda} f^k x = f^k (h-n)^{\dim V^\lambda} x = 0$, so $f^k x$ is in the $n-2k$ generalised eigenspace for x . We claim for any $0 \neq y \in \ker e$, $f^n y \neq 0$. It holds for $L(n)$. Let $0 = W_0 < W_1 < \dots < W_r = V^\lambda$ be a composition series, which therefore has quotients $L(n)$. Let i be minimal such that $y \in W_i$, then $\bar{y} \neq 0$ in W_i/W_{i-1} . So $\bar{y}$ is a highest weight vector for W_i/W_{i-1} , so $f^n \bar{y} \neq 0$, so $f^n y \neq 0$. We know $f^{n+1}y$ has generalised h -eigenvalue $-n-2$, so is 0. So $0 = e f^{n+1}x = (n+1)f^n(h-n)x + f^{n+1}ex$ and hence $(n+1)f^n(h-n)x = 0$. But $y = (h-n)x \in \ker e$, so if $y = (h-n)x \neq 0$ then $f^n y \neq 0$ by the claim, contradiction as $(n+1)f^n y = 0$. So $hx = nx$, and h is diagonalisable. $\square$

Proof. (Of complete reducibility of finite dimensional representations of $\mathfrak{sl}_2$). Choose a basis $w_1, \dots, w_r$ of $\ker(e: V^\lambda \rightarrow V^\lambda)$. Consider $\langle w_i, f w_i, \dots, f^n w_i \rangle$.

Exercise 2.4. Prove this is a direct sum decomposition of V^λ

In particular, h acts diagonally on all of V . $\square$

Exercise 2.5. Everything here is false in characteristic p . Let $k = \bar{\mathbb{F}}_p$.

1. Show $L(n)$ need not be irreducible (take $n \geq p$).
2. Show not every representation is a direct sum of representations.
3. (Hard exercise) Show there are irreducible representations of $\mathfrak{sl}_2$ which are not highest weight. (hint: show that $e^p, f^p, h^p - h$ all commute with e, f, h)

Let $\mathfrak{g}$ be a Lie algebra and V, W representations of $\mathfrak{g}$

Proposition 2.6. *The map $\mathfrak{g} \rightarrow \text{End}(V \otimes W)$ by $x \mapsto x \otimes 1 + 1 \otimes x$ is a representation*

Exercise 2.6. Prove this

Recall that if V, W are representations of a group G , then so is $V \otimes W$ with g acting by $g \mapsto g \otimes g$, and the above map is its derivative.

In general if A is an associative algebra and V, W are representations of A , then $V \otimes W$ isn't a representation of A , it is a representation of $A \otimes A$. So to make it a representation of A , we need an algebra homomorphism $A \rightarrow A \otimes A$. Such an algebra A is called a bialgebra.

Example 2.3. If $A = kG$, then there is an algebra homomorphism $g \mapsto g \otimes g$ from A to $A \otimes A$

Now let $\mathfrak{g} = \mathfrak{sl}_2$ and consider $L(n) \otimes L(m)$. This can be written as $\bigoplus_{k \geq 0} a_k L(k)$ for some $a_k \in \mathbb{Z}_{\geq 0}$. What are the multiplicities a_k ?

One way to decompose $L(n) \otimes L(m)$ is to find all the highest weight vectors.

Exercise 2.7. Do the above for $L(1) \otimes L(n)$. Then do it for $L(2) \otimes L(n)$.

Remark 2.2. Let v_n be the highest weight vector of $L(n)$. Then $v_n \otimes v_m$ is a highest weight vector of $L(n) \otimes L(m)$. $h(v_n \otimes v_m) = hv_n \otimes v_m + v_n \otimes hv_m = (n+m)v_n \otimes v_m$ and $e(v_n \otimes v_m) = ev_n \otimes v_m + v_n \otimes ev_m = 0$. However there is a lot of other stuff, as $\dim L(n) \otimes L(m) = (n+1)(m+1) \gg n+m+1$

You can write down explicit formulae for the highest weight vectors, but we don't need to.

Definition 2.6. The *character* of V is $\text{ch}(V) = \sum (\dim V_a) z^a \in \mathbb{N}[z, z^{-1}]$ where V_a is the a -eigenspace of h

Remark 2.3. 1. $\text{ch}(V)(1) = \dim V$ as h is diagonalisable

$$2. \text{ch}(L(n)) = z^n + z^{n-2} \dots + z^{-n} = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} = [n+1]_z$$

3. $\text{ch } V = \text{ch } W$ if and only if $V \cong W$, as $\text{ch}(L(n))$ form a basis for $\mathbb{N}[z, z^{-1}]_{ev}$, ie those polynomials with $f(z) = f(z^{-1})$. On the other hand complete reducibility implies V and L are direct sums of copies of the $L(n)$ representations, and they are isomorphic if and only if each factor has the same multiplicity, and the character of $V' \oplus V''$ is $\text{ch } V' + \text{ch } V''$.

4. $\text{ch}(V \otimes W) = \text{ch}(V) \cdot \text{ch}(W)$ (Proof: exercise. Show $V_n \otimes W_m \subseteq (V \otimes W)_{n+m}$. Hence $(V \otimes W)_a = \sum_{n+m=a} V_n \otimes W_m$, but this is exactly how you multiply polynomials.)

Example 2.4. We can now compute $L(1) \otimes L(3)$. $\text{ch } L(1) \cdot \text{ch } L(3) = (z + z^{-1})(z^3 + z + z^{-1} + z^{-3}) = \text{ch } L(4) + \text{ch } L(2)$

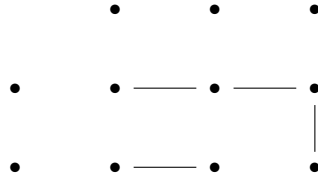
So $L(1) \otimes L(3) = L(2) \oplus L(4)$.

Proposition 2.7. *This implies the Clebsch-Gordon rule,*

$$L(n) \otimes L(m) = \bigoplus_{k=|n-m|, 2|n-m-k}^{n+m} L_k$$

Proof. Basically identical to the example □

Remark 2.4. We get a diagram that looks like this (for $n = 2, m = 3$)



3 Structure and Classification of Simple Lie Algebras

Via 3 theorems of Linear Algebra

Definition 3.1. An *ideal* of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ such that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$

Definition 3.2. A Lie algebra $\mathfrak{g}$ is *simple* if the only ideals of $\mathfrak{g}$ are $\mathfrak{g}$ and 0 and $\dim \mathfrak{g} > 1$

It is *semisimple* if it is a direct sum of simple Lie algebras

Definition 3.3. The derived subalgebra of $\mathfrak{g}$ denoted by $[\mathfrak{g}, \mathfrak{g}]$ is the k -span of $[x, y]$, $x, y \in \mathfrak{g}$

Exercise 3.1. $[\mathfrak{g}, \mathfrak{g}]$ is an ideal, and $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian

Definition 3.4. The *central series* of $\mathfrak{g}$ is given by $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}]$

Definition 3.5. The *derived series* of $\mathfrak{g}$ is given by $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$

Remark 3.1. These series are decreasing, and $\mathfrak{g}^{(n)} \leq \mathfrak{g}^n$

Definition 3.6. $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}^n = 0$ for some n and *solvable* if $\mathfrak{g}^{(n)} = 0$ for some n

Remark 3.2. Nilpotent Lie algebras are solvable

Exercise 3.2. Show that the two dimensional Lie algebra with basis x, y and $[x, y] = x$ is solvable but not nilpotent. Show that the set of strictly upper triangular matrices in $\mathfrak{gl}_n$ is nilpotent, that the set of upper triangular matrices is solvable, and compute the central and derived series of each. Compute their centres.

Definition 3.7. A vector space V with a non-degenerate alternating bilinear form $\langle \cdot, \cdot \rangle: V \times V \rightarrow k$ is called *symplectic*

Example 3.1. Let L be a finite dimensional vector space, $V = L^* \oplus L$, and define $\langle L, L \rangle = 0 = \langle L^*, L^* \rangle$ and $\langle v, v^* \rangle = v^*(v)$ for $v^* \in L^*, v \in L$. This is symplectic.

In fact if V is any finite dimensional vector space, $\{L \leq V \mid (L, L) = 0, \dim L \text{ maximal}\}$ is non-empty and $\dim L = \dim V/2$ and $Sp(V)$ (the automorphisms of V preserving the form) acts transitively on the set of such spaces. Spaces with $(L, L) = 0$ are called *isotropic*, and isotropic spaces of maximal dimension are called *Lagrangian*. Any complement L' to L identifies V with $L \oplus L^*$.

By induction, we can also do this by taking v, w with $\langle v, w \rangle = 1$, then $V = k\{v, w\} \oplus V'$ where V' is the orthogonal complement of $k\{v, w\}$

We write $V = L + L^*$ a symplectic vector space. The Heisenberg Lie algebra of V , $\mathfrak{h}_V = V \oplus k$ has $[v, v'] = \langle v, v' \rangle c$ if $v, v' \in V$, $[c, \mathfrak{h}_V] = 0$

Exercise 3.3. Show:

1. $\mathfrak{h}_V$ is a Lie algebra
2. Show it is nilpotent
3. Show $Sp(V)$ acts on $\mathfrak{h}_V$ via Lie algebra homomorphisms

Example 3.2. If $L = \mathbb{C}$ then $\mathfrak{h}_V$ is a 3-dimensional Lie algebra with basis p, q, c such that $[pq] = c$ and every commutator with c is 0. Show that this has a representation on $\mathbb{C}[x]$ by $p \mapsto \frac{\partial}{\partial x}, q \mapsto x, c \mapsto 1$

Remark 3.3. The following are true (and easy to prove):

1. Quotients and subalgebras of solvable (resp nilpotent) Lie algebras are solvable (resp nilpotent)
2. If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h} \leq \mathfrak{g}$ is an ideal, $\mathfrak{g}$ is solvable if and only if both $\mathfrak{h}, \mathfrak{g}/\mathfrak{h}$ are. Therefore solvable Lie algebras are “built out of” abelian Lie algebras (as the final term in the derived series is always abelian, quotient out and induct)
3. $\mathfrak{g}$ is nilpotent if and only if the centre of $\mathfrak{g}$ is non-zero (look at the final term of the central series) and $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent
4. $\mathfrak{g}$ is nilpotent if and only if $\text{ad } \mathfrak{g}$ is nilpotent. (Note that $\text{ad } \mathfrak{g} \cong \mathfrak{g}/Z(\mathfrak{g})$)

Theorem 3.1. (*Lie’s theorem*) Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a solvable Lie algebra, V a vector space over $k = \bar{k}$, $\text{char } k = 0$. Then there exists a basis $v_1, \dots, v_n$ of V such that with respect to the basis, $\mathfrak{g}$ consists of upper triangular matrices. Equivalently, there is a linear function $\lambda: \mathfrak{g} \rightarrow k$ and an element $v \in V$ such that for all $x \in \mathfrak{g}$, $xv = \lambda(x)v$, ie V has a 1d subrepresentation defined by λ

Proof. No Proof

□

Exercise 3.4. Prove the conclusions of the above theorem are equivalent

Exercise 3.5. Show this is false if $k \neq \bar{k}$. Show it is false for non-zero characteristic (hint: consider $L = k$ in Example 3.2 for the later)

Corollary 3.1.1. *If k is characteristic 0 and $\mathfrak{g}$ is a solvable finite dimensional Lie algebra, then $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Proof. Apply Lie's theorem to the adjoint representation. With respect to some basis of $\mathfrak{g}$, $\text{ad } \mathfrak{g}$ is upper triangular, but the derived subalgebra of the Lie algebra of upper triangular matrices is contained in the strictly upper triangular matrices, and is therefore nilpotent. So $[\text{ad } \mathfrak{g}, \text{ad } \mathfrak{g}]$ is nilpotent, so $\text{ad}[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, and hence $[\mathfrak{g}, \mathfrak{g}]$ is by Remark 3.3(4) $\square$

Exercise 3.6. Complete the above proof by showing we can assume $\mathfrak{g}$ is over an algebraically closed field (by showing $\mathfrak{g}$ is solvable if and only if $\mathfrak{g} \otimes_k \bar{k}$ is)

Theorem 3.2. (Engel's Theorem) *If $\mathfrak{g}$ is a Lie algebra and k is a field, then $\mathfrak{g}$ is nilpotent if and only if $\text{ad } \mathfrak{g}$ consists of nilpotent endomorphisms of $\mathfrak{g}$. Equivalently, if V is a finite dimensional representation, and all elements of $\mathfrak{g}$ act by nilpotent linear maps, then there is $v \neq 0, v \in V$ with $xv = 0$ for all $x \in \mathfrak{g}$ (ie V contains the trivial representation). Equivalently, if $\mathfrak{g} \subseteq \text{End}(V)$ there is a basis of V where $\mathfrak{g}$ consists of upper triangular matrices*

Proof. No Proof $\square$

Exercise 3.7. Prove the conclusions of the above theorem are equivalent

Remark 3.4. This states that if $\mathfrak{g}$ acts by nilpotent matrices, then $\mathfrak{g}$ is nilpotent. The converse is false, take diagonal matrices. This is abelian, hence nilpotent, but has non-trivial irreducible representations. The correct converse is that $\mathfrak{g}$ being nilpotent implies $\mathfrak{g}/Z(\mathfrak{g})$ acts by nilpotent matrices

Definition 3.8. A symmetric bilinear form $(\cdot, \cdot): \mathfrak{g} \times \mathfrak{g} \rightarrow k$ is *invariant* if $([xy], z) = (x, [yz])$

Exercise 3.8. If $\mathfrak{a} \trianglelefteq \mathfrak{g}$ is an ideal and $(\cdot, \cdot)$ is an invariant form, show $\mathfrak{a}^\perp = \{x \in \mathfrak{g} \mid (x, \mathfrak{a}) = 0\}$ is an ideal

Definition 3.9. Let V be a finite dimensional representation of $\mathfrak{g}$. The *trace form* of V is $(x, y)_V = \text{tr}(\rho(x)\rho(y))$.

Exercise 3.9. Show the trace form is symmetric bilinear and invariant

Remark 3.5. For the adjoint representation, this is called the *killing form*.

Theorem 3.3. (Cartan's Criteria) *Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ with V finite dimensional, $\text{char}(k) = 0$. Then $\mathfrak{g}$ is solvable if and only if for all $x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}]$, $(x, y)_V = 0$*

Proof. No Proof $\square$

Remark 3.6. We know one direction from Lie's theorem as if $\mathfrak{g}$ is solvable, we can take x to be triangular and y to be upper triangular (as we can upper triangularise $\mathfrak{g}$), so xy has zeros on the diagonal, so its trace is 0. Therefore the main content is the converse

Corollary 3.3.1. *Let $\text{char } k = 0$. Then $\mathfrak{g}$ is solvable if and only if $(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}])_{\text{ad}} = 0$*

Proof. The forward direction is by Lie's theorem as above. Conversely, apply Cartan's theorem to $\text{ad } \mathfrak{g}$ to get $\text{ad } \mathfrak{g}$ is solvable, so then $\mathfrak{g}/Z(\mathfrak{g}), Z(\mathfrak{g})$ are solvable, so $\mathfrak{g}$ is $\square$

Exercise 3.10. Not every invariant form is a trace form. Let $\tilde{\mathfrak{h}} = \mathbb{C}\{p, q, c, d\}$ with $[p, q] = c, [c, \tilde{\mathfrak{h}}] = 0, [d, p] = p, [d, q] = -q$. Show $\tilde{\mathfrak{h}}$ is solvable and construct a non-degenerate invariant bilinear form with $(\tilde{\mathfrak{h}}, [\tilde{\mathfrak{h}}, \tilde{\mathfrak{h}}]) \neq 0$. Extend the representation of $\mathfrak{h}$ on $\mathbb{C}[x]$ to a representation of $\tilde{\mathfrak{h}}$.

Definition 3.10. If $\mathfrak{g}$ is a Lie algebra, the *radical* of $\mathfrak{g}$, $R(\mathfrak{g})$, is the maximal solvable ideal in $\mathfrak{g}$

Exercise 3.11. Show that a sum of solvable ideals is solvable, and hence $R(\mathfrak{g})$ is well defined. Also show $R(\mathfrak{g}/R(\mathfrak{g})) = 0$.

Theorem 3.4. *In characteristic 0 the following are equivalent*

1. $\mathfrak{g}$ is semisimple
2. $R(\mathfrak{g}) = 0$
3. (Killing criterion) The Killing form is non-degenerate

Proof. Claim: $R(\mathfrak{g}) = 0$ if and only if $\mathfrak{g}$ has no non-zero abelian ideals. Proof: An abelian ideal is solvable, so if there is one, $R(\mathfrak{g}) \neq 0$. If $R(\mathfrak{g}) \neq 0$, the last term in its derived series is abelian.

For the rest of the proof, $(\cdot, \cdot) = (\cdot, \cdot)_{\text{ad}}$.

3 implies 2: We want to show that if $\mathfrak{a}$ is an abelian ideal, then $\mathfrak{a} \subseteq \mathfrak{g}^\perp$, then we are done by the above claim. Choose a vector space complement to $\mathfrak{a}$, so $\mathfrak{g} = \mathfrak{a} + \mathfrak{h}$. Then if $x \in \mathfrak{a}$, $\text{ad } x$ has the matrix $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. On the other hand for

any $x \in \mathfrak{g}$, $\text{ad } x$ has matrix $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ but then $\text{tr}(\text{ad } x \text{ad } x) = 0$ so $(\mathfrak{a}, \mathfrak{g}) = 0$

For 2 implies 3, let $\mathfrak{r} \subseteq \mathfrak{g}^\perp$ be an ideal and suppose $\mathfrak{r} \neq 0$. Then $R(\mathfrak{g}) = 0$ implies the centre of $\mathfrak{g}$ is 0 implies $\mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$ by ad . But $(x, y) = 0$ for all $x, y \in \mathfrak{r}$ as $\mathfrak{r} \subseteq \mathfrak{g}^\perp$ and then Cartan's criteria implies $\mathfrak{r}$ is solvable.

We prove 2, 3 imply 1. Suppose $(\cdot, \cdot)_{\text{ad}}$ is non-degenerate and let $\mathfrak{s} \trianglelefteq \mathfrak{g}$ be minimal non-zero ideal. We claim $(\cdot, \cdot)_{\text{ad}|_{\mathfrak{s}}}$ is either non-degenerate or zero, as the kernel on fixing one argument is an ideal. But then Cartan's criteria implies that if it is 0, the $\mathfrak{s}$ is solvable, but $R(\mathfrak{g}) = 0$ contradiction. So $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}^\perp$. Note $\mathfrak{s}$ is not abelian as $R(\mathfrak{g}) = 0$, so $\mathfrak{s}$ is a simple Lie algebra. Ideals of $\mathfrak{s}^\perp$ are ideals of $\mathfrak{g}$, so $R(\mathfrak{g}) = 0$ implies $R(\mathfrak{s}^\perp) = 0$, so we are done by induction on dimension. $\square$

Exercise 3.12. Show that if $\mathfrak{g}$ is semisimple, it is uniquely a sum of minimal non-zero ideals. Conclude (1) implies (2) and we are done.

Exercise 3.13. Show $R(\mathfrak{g}) \supseteq \mathfrak{g}^\perp \supseteq [R(\mathfrak{g}), R(\mathfrak{g})]$ in general.

Remark 3.7. If $\mathfrak{g}$ is Lie algebra, $0 \rightarrow R(\mathfrak{g}) \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/R(\mathfrak{g})$

Theorem 3.5. (*Levi's Theorem*) The above short exact sequence splits

Exercise 3.14. If $\mathfrak{g}$ is the Lie algebra of (n, m) two block upper triangular matrices. Find $R(\mathfrak{g})$ and a subalgebra isomorphic to $\mathfrak{g}/R(\mathfrak{g})$

Exercise 3.15. Show Levi's theorem fails in characteristic p as follows. Let $\mathfrak{g} = \mathfrak{sl}_p, k = \mathbb{F}_p$. Show $R\mathfrak{g} = kI$, but there is no subalgebra complement

Definition 3.11. A *derivation* $D: \mathfrak{g} \rightarrow \mathfrak{g}$ is a linear map with $D[x, y] = [Dx, y] + [x, Dy]$

Example 3.3. $\text{ad } x$ is a derivation for $x \in \mathfrak{g}$, and such derivations are called *inner*. $\text{Der } \mathfrak{g} / \text{inner derivations} = H^1(\mathfrak{g}, \mathfrak{g})$
(Even in lectures that don't teach [group/Lie algebra] cohomology, you can't avoid this being taught)

Proposition 3.1. If $\mathfrak{g}$ is semisimple then every derivation $D: \mathfrak{g} \rightarrow \mathfrak{g}$ is inner

Proof. Let $\mathfrak{g}$ be semisimple and $D: \mathfrak{g} \rightarrow \mathfrak{g}$ a derivation. Consider a linear function $l: \mathfrak{g} \rightarrow k$ by $x \mapsto \text{tr}(D \cdot \text{ad } x)$. As $\mathfrak{g}$ is semisimple, $\mathfrak{g}^* \cong \mathfrak{g}$ by the killing form, so there is $y \in \mathfrak{g}$ with $l(x) = (y, x)_{\text{ad}}$. Let $E = D - \text{ad } y$, so $\text{tr}(E \text{ad } x) = 0$. We want $E = 0$. This is equivalent to $(Ex, z)_{\text{ad}} = 0$ for all $x, z \in \mathfrak{g}$, but $\text{ad}(Ex)(z) = [Ex, z] = E[x, z] - [x, Ez]$ as E is a derivation, so $\text{ad}(Ex) = E(\text{ad } x) - (\text{ad } x)E = [E, \text{ad } x]$, so $(Ex, z)_{\text{ad}} = \text{tr}(\text{ad}(Ex) \text{ad } z) = \text{tr}([E, \text{ad } x] \text{ad } z) = \text{tr}(E, \text{ad}[x, z]) = 0$ $\square$

Example 3.4. 1. Show that nilpotent Lie algebras have non-inner derivations

2. Show $\mathfrak{g} = k\langle a, b \rangle, [ab] = b$ has only inner derivations. So this condition does not characterise semisimple Lie algebras
3. Let $\mathfrak{g}$ be a simple Lie algebra and suppose $(\cdot, \cdot)_1, (\cdot, \cdot)_2$ are non-degenerate invariant bilinear forms, Show they are multiples of each other.
4. Let $\mathfrak{g} = \mathfrak{sl}_n \mathbb{C}$. Assume it is simple. We have the killing form, and $(A, B) = \text{tr}(AB)$. Compute the multiple.

4 Structure of Simple Lie Algebras

Definition 4.1. A *torus* $\mathfrak{t} \subseteq \mathfrak{g}$ is an abelian Lie algebra such that for all $t \in \mathfrak{t}$, $\text{ad } t$ is a diagonalisable linear map. It is maximal if it is not contained inside a bigger torus.

Exercise 4.1. If $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{g} = \mathfrak{gl}_n$, then the diagonal matrices form a maximal torus, and $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ is not a torus in $\mathfrak{sl}_2$

Let V be a vector space and $t_1, \dots, t_n: V \rightarrow V$ be pairwise commuting diagonalisable maps. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and let $V_\lambda = \{v \in V \mid t_i v = \lambda_i v\}$ be the simultaneous eigenspace. Then $V = \bigoplus_{\lambda \in \mathbb{C}^n} V_\lambda$, by induction on n and the fact that commuting maps preserve each others eigenspaces.

Let $\mathfrak{t}$ be the n dimensional abelian Lie algebra with basis $t_1, \dots, t_n$. If V is a representation of $\mathfrak{t}$ in which each $t \in \mathfrak{t}$ acts diagonalisably, then the lemma says V is a direct sum of 1d representations, $\lambda: \mathfrak{t} \rightarrow \mathbb{C}$ is a linear map sending $t_i \mapsto \lambda_i$ and 1d representations of $\mathfrak{t}$ are the same as $\lambda \in \mathfrak{t}^*$, and $V_\lambda = \{v \in V \mid tv = \lambda(t)v, \forall t \in \mathfrak{t}\}$. We call this the λ -weight space of $\mathfrak{t}$ on V . $V = \bigoplus V_\lambda$ is called the weight space decomposition of V

Exercise 4.2. Show that every irreducible representation of an abelian Lie algebra $\mathfrak{t}$ is some $\mathbb{C}_\lambda$ where $\mathfrak{t}$ acts via $\lambda \in \mathfrak{t}^*$

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{t}$ be a maximal torus. We get a weight space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{0 \neq \lambda \in \mathfrak{t}^*} \mathfrak{g}_\lambda$ for the adjoint representation.

Definition 4.2. The roots of $\mathfrak{g}$ are the non-zero weights in this decomposition.

Example 4.1. $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{t}$ is diagonal matrices. If E_{ij} is the matrix with a 1 in the ij position and 0 elsewhere, and let $t = \text{diag}(t_1, \dots, t_n)$. Then $[t, E_{ij}] = (t_i - t_j)E_{ij}$. If we let $\varepsilon(t) = t_i$, then $E_{ij} \in \mathfrak{g}_{\varepsilon_i - \varepsilon_j}$, so $R = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, $\mathfrak{g}_0 = \mathfrak{t}$ and $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}$

Exercise 4.3. Do the same for $\mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}, \mathfrak{sp}_n$. Take $\mathfrak{t}$ to be diagonal matrices and $\mathfrak{so}_n = \{A \in \text{Mat}_n \mid JA + A^T J = 0\}$ for J the anti-diagonal matrix

Proposition 4.1. $\mathfrak{sl}_n$ is a simple Lie algebra

Proof. We know $\mathfrak{sl}_n = \mathfrak{t} + \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ where $R = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ and $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}$. Now suppose $\mathfrak{r}$ is a non-zero ideal and let $r \in \mathfrak{r}$ such that $r = t + \sum_{\alpha \in R} e_\alpha, e_\alpha \in \mathfrak{g}_\alpha$ with a minimal number of non-zero terms. Suppose $t \neq 0$. Then there is $t_0 \in \mathfrak{t}$ with distinct eigenvalues and $\alpha(t_0) \neq 0$ for all $\alpha \in R$. So $[t_0, r] = \sum_{\alpha \in R} \alpha(t_0)e_\alpha \in \mathfrak{r}$ as $\mathfrak{r}$ is an ideal. But this has fewer non-zero terms, so must be 0, so every $e_\alpha = 0$, so $r = t \in \mathfrak{t}$. As $t \neq 0, \alpha(t) \neq 0$ for some $\alpha \in R$, so $\alpha = \varepsilon_i - \varepsilon_j$, so $[t, E_{ij}] = \alpha(t)E_{ij} \neq 0$ so $E_{ij} \in \mathfrak{r}$. $[E_{ij}, E_{jk}] = E_{ik} \in \mathfrak{r}$ and similarly we get $E_{ab} \in \mathfrak{r}$ and $[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1}$ so $\mathfrak{sl}_n = \mathfrak{r}$. If $t = 0$, then if $r = cE_{ij}$ we are done as above, otherwise $r = e_\alpha + e_\beta + \sum_{\gamma \in R \setminus \{\alpha, \beta\}} e_\gamma$. Take $t \in \mathfrak{t}$ with $\alpha(t) \neq \beta(t)$, so some linear combination of $[t_0, r]$ and r has fewer terms contradiction. $\square$

Recall that if $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{t}$ a maximal torus, $\mathfrak{g} = \mathfrak{g}_0 + \bigoplus_{\lambda \in R} \mathfrak{g}_\lambda, R \subseteq \mathfrak{t}^*$

Proposition 4.2. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Then maximal tori exist, and if $\mathfrak{t}$ is a maximal torus, $\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid [\mathfrak{t}, x] = 0\} = \mathfrak{t}$

Proof. No Proof □

So we know $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$

Theorem 4.1. (*Structure theorem for semisimple Lie algebras, Part I*) Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{t}$ a maximal torus and $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. Then

1. The roots span $\mathfrak{t}^*$
2. $\dim \mathfrak{g}_\alpha = 1$ for $\alpha \in R$
3. If $\alpha, \beta \in R$ and $\alpha + \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. If $\alpha + \beta \notin R$ and $\alpha \neq -\beta$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$
4. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional, and $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is isomorphic to $\mathfrak{sl}_2$. In particular, $\alpha \in R$ implies $-\alpha \in R$

Proof. Suppose the roots don't span $\mathfrak{t}^*$, and there is $t \in \mathfrak{t}$ with $\alpha(t) = 0$ for all $\alpha \in R$. Then $[t, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in R$, $[t, \mathfrak{t}] = 0$ and so $[t, \mathfrak{g}] = 0$ so $\mathbb{C}t$ is an abelian ideal in $\mathfrak{g}$, but $\mathfrak{g}$ is semisimple, so has no abelian ideals.

Now we show some nice statements building towards the other 3 statements.

(a) We claim $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$ if $\lambda, \mu \in \mathfrak{t}^*$. This is because if $x \in \mathfrak{g}_\lambda, y \in \mathfrak{g}_\mu$ and $t \in \mathfrak{t}$, then $[t, [xy]] = [[tx]y] + [x[ty]] = (\lambda(t) + \mu(t))[xy]$ so one direction of (3) is done, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ if $\alpha + \beta \notin R \cup \{0\}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{t}$.

(b) $(\mathfrak{g}_\lambda, \mathfrak{g}_\mu)_{\text{ad}} = 0$ if $\lambda \neq -\mu$ and moreover $\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}$ has non-degenerate Killing form. Proof: If $x \in \mathfrak{g}_\lambda, y \in \mathfrak{g}_\mu$ then we know $(x, y)_{\text{ad}} = \text{tr}(\text{ad } x \text{ ad } y)$. We will show this is 0 for $\lambda + \mu \neq 0$ by showing $\text{ad } x \text{ ad } y$ is nilpotent. But for all $\alpha \in \mathfrak{t}^*$, $(\text{ad } x \text{ ad } y)^N \mathfrak{g}_\gamma \subseteq \mathfrak{g}_{\gamma+N(\lambda+\mu)}$. By finite dimensionality, this is 0 for sufficiently large N if $\lambda + \mu \neq 0$, so we have proven nilpotence. On the other hand, the Killing form is non-degenerate on $\mathfrak{g} = \bigoplus (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda})$ which we have just shown is an orthogonal direct sum, so the whole claim follows.

(c) In particular, the Killing form is non-degenerate on $\mathfrak{g}_0 = \mathfrak{t}$ (Warning: $(\cdot, \cdot)_{\text{ad}}|_{\mathfrak{t}}$ is not $(\cdot, \cdot)_{\text{ad } \mathfrak{t}}$), and hence $(\cdot, \cdot)_{\text{ad}}$ gives an isomorphism $\nu: \mathfrak{t} \rightarrow \mathfrak{t}^*$ by $\nu(t)(t') = (t, t')_{\text{ad}}$, and defines a bilinear form $(\cdot, \cdot)$ on $\mathfrak{t}^*$ by $(\gamma, \gamma') = (\nu^{-1}(\gamma), \nu^{-1}(\gamma'))_{\text{ad}}$. All is immediate.

(d) If $\alpha \in R$, then $(\cdot, \cdot)_{\text{ad}}$ is non-degenerate on $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$, but degenerate on $\mathfrak{g}_\alpha$, so $\mathfrak{g}_{-\alpha} \neq 0$, so $-\alpha \in R$ and $\mathfrak{g}_\alpha \cong \mathfrak{g}_{-\alpha}^*$ by the Killing form.

(e) If $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ then $[x, y] = (x, y)_{\text{ad}} \nu^{-1}(\alpha)$. Proof: $(t, [x, y])_{\text{ad}} = ([t, x], y)_{\text{ad}} = \alpha(t)(x, y)_{\text{ad}}$ so it follows.

(f) Let $e_\alpha \in \mathfrak{g}_\alpha$ be non-zero and pick $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ with $(e_\alpha, e_{-\alpha})_{\text{ad}} \neq 0$. We have $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})_{\text{ad}} \nu^{-1}(\alpha)$ and we want $e_\alpha, [e_\alpha, e_{-\alpha}], e_{-\alpha}$ to form an $\mathfrak{sl}_2$. We compute $[\nu^{-1}(\alpha), e_\alpha] = \alpha(\nu^{-1}(\alpha))e_\alpha = (\alpha, \alpha)e_\alpha$. If $(\alpha, \alpha) \neq 0$ define $h_\alpha = \frac{2}{(\alpha, \alpha)} \nu^{-1}(\alpha)$ and rescale $e_{-\alpha}$ so $(e_\alpha, e_{-\alpha})_{\text{ad}} = \frac{2}{(\alpha, \alpha)}$. Exercise: $(e, h, f) \mapsto (e_\alpha, h_\alpha, e_{-\alpha})$ defines an isomorphism $\mathfrak{sl}_2 \cong m_\alpha = \langle e_\alpha, h_\alpha, e_{-\alpha} \rangle$

Proposition 4.3. $(\alpha, \alpha) \neq 0$ for all $\alpha \in R$

Proof. Suppose otherwise and let $m_\alpha = \langle e_\alpha, e_{-\alpha}, \nu^{-1}(\alpha) \rangle$. Then $[\nu^{-1}(\alpha), e_{\pm\alpha}] = 0$ so $[m_\alpha, m_\alpha] = \mathbb{C}\nu^{-1}(\alpha)$ and m_α is solvable. But then by Lie's theorem,

$\text{ad}[m_\alpha, m_\alpha]$ acts by nilpotents on $\mathfrak{g}$, ie $\nu^{-1}(\alpha)$ acts by nilpotent elements on $\mathfrak{g}$, but $\nu^{-1}(\alpha) \in \mathfrak{t}$, so is semisimple and hence is 0, so $\alpha = 0$ contradiction. $\square$

We claim $\dim \mathfrak{g}_{-\alpha} = 1$ for all $\alpha \in R$. This proves (2) as $-\alpha \in R$ if and only if $\alpha \in R$. Suppose otherwise. Then $\text{ad } e_\alpha: \mathfrak{g}_{-\alpha} \rightarrow \mathbb{C}\nu^{-1}(\alpha) = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ must have a non-trivial kernel. If $x \in \ker \text{ad } e_\alpha$ then as $x \in \mathfrak{g}_{-\alpha}$, $\text{ad } h_\alpha \cdot x = [h_\alpha, x] = -\alpha(h_\alpha)x = -2x$. Also $\text{ad } e_\alpha \cdot x = 0$, so x is a highest weight vector for m_α with highest weight -2 , inside the finite dimensional representation $\mathfrak{g}$, contradiction as $-2 < 0$, so $x = 0$ and $\dim \mathfrak{g}_\alpha = 1$.

Theorem 4.2. (*Structure theorem part II*)

1. For all $\alpha, \beta \in R$, $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$
2. If $\alpha \in R, k\alpha \in R$ then $k = \pm 1$
3. The module $\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$ is irreducible for m_α . In particular the set of all $\beta + k\alpha$ which are roots is of the form $\beta - p\alpha, \beta - (p-1)\alpha, \dots, \beta + q\alpha$ where $p - q = \frac{2(\alpha, \beta)}{(\beta, \beta)}$ and this is called the α string through β

Proof. Let $q = \max\{k \mid \beta + k\alpha \in R\}$ and let $v \in \mathfrak{g}_{\beta + q\alpha}$. As this is maximal, $\text{ad } e_\alpha \cdot v \in \mathfrak{g}_{\beta + (q+1)\alpha} = 0$ and $\text{ad } h_\alpha \cdot v = (\beta + q\alpha)(h_\alpha)v = \left(\frac{2(\beta, \alpha)}{(\alpha, \alpha)} + 2q\right)v$. As highest weights are in $\mathbb{Z}$, we have done (1).

For (3), $\mathfrak{sl}_2$ module structure tells us that $(\text{ad } e_\alpha)^r v \neq 0$ for $0 \leq r \leq \frac{2(\alpha, \beta)}{(\alpha, \alpha)} + 2q = N$, so the corresponding weight spaces are non-zero, and hence are in $R \cup \{0\}$. We need to show these are all. We do the same thing, starting at the bottom. Let $p = \max\{k \mid \beta - k\alpha \in R \cup \{0\}\}$ and let $w \in \mathfrak{g}_{\beta - p\alpha}$ with $w \neq 0$. $\text{ad } h_\alpha \cdot w = \left(\frac{2(\alpha, \beta)}{(\alpha, \alpha)} - 2p\right)w$, so w is a least weight vector, and applying e_α repeatedly we get an irreducible $\mathfrak{sl}_2$ module with weights $\beta - p\alpha, \beta - (p-1)\alpha, \dots, \beta + \left(p - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\right)\alpha$. By construction, $p - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \leq q$ and $q + \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \leq p$, so $p - q = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$ and the two submodules coincide.

For (2), observe $m_\alpha \subseteq \mathfrak{g}$ is an irreducible representation of $\mathfrak{sl}_2$ with weights $-\alpha, 0, \alpha$ and (3) shows this is the only α string through α . $\square$

Remark 4.1. (1) implies $k = \pm 1, \pm 2$ as $\frac{2(\alpha, k\alpha)}{(k\alpha, k\alpha)} = \frac{2}{k} \in \mathbb{Z}$ and similarly if we flip the roles of $\alpha, k\alpha$

Finally $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in R$ as $\bigoplus \mathfrak{g}_{\beta + k\alpha}$ is an irreducible representation of $\mathfrak{sl}_2$ so $\text{ad } e_\alpha$ is an isomorphism from $\mathfrak{g}_{\beta + k\alpha}$ to $\mathfrak{g}_{\beta + (k+1)\alpha}$ is an isomorphism if $k < q$ and as $q \geq 1$ here, we're done. $\square$

If $\alpha \in R$ define $s_\alpha: \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ by $v \mapsto v - \frac{2(\alpha, v)}{(\alpha, \alpha)}v$, the *reflection along α* .

Remark 4.2. (3) implies $s_\alpha\beta$ is a root whenever α, β are roots, as if $r = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$ and $r > 0$, then $p = q + r \geq r$ otherwise if $r \leq 0$, $q = p - r \geq -r$ and in either case $\beta - r\alpha$ is in the α string through β

Proposition 4.4. 1. If $\alpha, \beta \in R$, then $(\alpha, \beta) \in \mathbb{Q}$

2. If we take a basis $\beta_1, \dots, \beta_r$ of $\mathfrak{t}^*$ with each β_i a root, then $\beta \in R$ expands as $\sum q_i \beta_i$ with $q_i \in \mathbb{Q}$, equivalently $\dim \mathbb{Q}R = \dim_{\mathbb{C}} \mathfrak{t}^*$

3. $(\cdot, \cdot)$ is positive definite on $\mathbb{Q}R$

Proof. (1) by structure theorem II(1), it suffices to show every $(\beta, \beta) \in \mathbb{Q}$ for all $\beta \in R$. Now $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus \mathfrak{g}_{\alpha}$. If $h, h' \in \mathfrak{t}$ then $(h, h')_{\text{ad}} = \text{tr}(\text{ad } h \text{ ad } h') = \sum_{\alpha \in R} \alpha(h)\alpha(h')$ and so for $\lambda, \mu \in \mathfrak{t}^*$,

$$\begin{aligned} (\lambda, \mu) &= (\nu^{-1}(\lambda), \nu^{-1}(\mu))_{\text{ad}} \\ &= \sum_{\alpha \in R} \alpha(\nu^{-1}\lambda)\alpha(\nu^{-1}\mu) \\ &= \sum_{\alpha \in R} (\lambda, \alpha)(\mu, \alpha) \end{aligned}$$

In particular, if $\lambda = \mu = \beta$, $(\beta, \beta) = \sum_{\alpha \in R} (\alpha, \beta)^2$, so multiplying by $\frac{4}{(\beta, \beta)^2}$ we

get $\frac{4}{(\beta, \beta)} = \sum_{\alpha \in R} \left(\frac{2(\alpha, \beta)}{(\beta, \beta)} \right)^2 \in \mathbb{Z}$, so $(\beta, \beta) \in \mathbb{Z}$

(2) Let $B = ((\beta_i, \beta_j))_{ij}$ be the Gram matrix of the form. As $(\cdot, \cdot)$ is non-degenerate, B is invertible. Let $\beta = \sum c_i \beta_i$ for $\beta \in R$. Then $(\beta, \beta_i) = \sum c_j (\beta_j, \beta_i)$, so inverting the matrix B , we get the c_i in terms of the (β, β_i) as $B_{ij} \in \mathbb{Q}$ by (1), $c_i \in \mathbb{Q}$ also.

(3) Let $\lambda \in \mathbb{Q}R$, so $\lambda = \sum c_i \beta_i$ with $c_i \in \mathbb{Q}$ and $(\lambda, \alpha) \in \mathbb{Q}$ for all $\alpha \in R$, so $(\lambda, \lambda) = \sum_{\alpha \in R} (\lambda, \alpha)^2 \geq 0$ with equality if and only if every $(\lambda, \alpha) = 0$, so $\lambda = 0$ as R spans $\mathfrak{t}^*$ and $(\cdot, \cdot)$ is non-degenerate. $\square$

5 Root Systems

Let V be a vector space over $\mathbb{R}$ and let $(\cdot, \cdot)$ be a positive definite symmetric bilinear form. If $0 \neq \alpha \in V$, we write $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$, so $(\alpha, \alpha^\vee) = 2$. We define $s_\alpha: V \rightarrow V$ to be $v \mapsto v - \langle v, \alpha^\vee \rangle \alpha$.

Lemma 5.1. s_α is the reflection in the hyperplane orthogonal to α . s_α has one eigenvalue -1 with eigenvector α and all other eigenvalues are 1 and $s_\alpha \in O(V)$. Hence $s_\alpha^2 = 1$.

Proof. Easy as $V = \mathbb{R}\alpha \oplus \alpha^\perp$, $s_\alpha = -\alpha$ and $s_\alpha v = v$ if $v \in \alpha^\perp$ $\square$

Definition 5.1. A root system R in V is a finite set $R \subseteq V$ such that

1. $0 \notin R$ and $\mathbb{R}R = V$
2. For all $\alpha, \beta \in R$, $(\alpha, \beta^\vee) \in \mathbb{Z}$
3. For all $\alpha \in R$, $s_\alpha R \subseteq R$. In particular $s_\alpha \alpha = -\alpha \in R$ if $\alpha \in R$

We say R is *reduced* if $\alpha, k\alpha \in R$ implies $k = \pm 1$

Example 5.1. For semisimple Lie algebra $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, $R \subseteq \mathbb{R}R$ is a reduced root system.

Definition 5.2. Let $W \subseteq GL_V$ be the group generated by the reflections s_α for $\alpha \in R$. This is called the *Weyl group* of R

Proposition 5.1. *The Weyl group W of a root system is finite*

Proof. W acts on R faithfully by permutations as R spans V , so $W \leq S_{|R|}$. $\square$

Definition 5.3. If $(R, V), (R', V')$ are root systems, they are *isomorphic* if there is a linear bijection $\phi: V \rightarrow V'$ such that $\phi(R) = R'$. Note that ϕ need not preserve the inner product

Definition 5.4. If (R, V) and (R', V') are root systems, their *direct sum* is $(R \sqcup R', V \oplus V')$. A root system which is not isomorphic to a non-trivial direct sum is *irreducible*. The *rank* of (R, V) is $\dim V$

Example 5.2.

1. In rank 1, $V = \mathbb{R}$, $(x, y) = xy$, $R = \{\alpha, -\alpha\}$ where $\alpha \neq 0$ and $W = \mathbb{Z}/2\mathbb{Z}$. We call this A_1
2. In rank 2, we take $V = \mathbb{R}^2$ with the usual inner product. We have the following examples
 - (a) $R = \{(0, 1), (0, -1), (1, 0), (-1, 0)\}$, which is $A_1 \times A_1$ is not irreducible has $W = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
 - (b) R is the vertices of a regular hexagon chosen so $\alpha = \alpha^\vee, \beta = \beta^\vee$. Then $(\alpha, \beta^\vee) = -1$ and $W = S_3$. This is the root system of $\mathfrak{sl}_3$
 - (c) R is the vertices and midpoints of a square, spanned by $\alpha = (1, 0), \beta = (-1, 1)$. We have short roots $\alpha, \alpha + \beta$ and long roots $\beta, 2\alpha + \beta$. $W = D_8$. This is the root system of $\mathfrak{sp}_4$ and $\mathfrak{so}_5$
 - (d) Let R be the vertices and intersections of two regular triangles, one facing up and one facing down. Then $W = D_{12}$, and this is the root system of G_2

Exercise 5.1. Verify that these are all reduced root systems, and we have listed all the rank 2 root systems. Find the non-reduced rank 2 root system.

Lemma 5.2. *If R is a root system, so is $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$*

Proof. Trivial $\square$

Definition 5.5. R is *simply laced* if all the roots have the same length

Exercise 5.2. If R is simply laced, $(R, V) \cong (R', V')$ where $(\alpha, \alpha) = 2$ for all $\alpha \in R'$, ie $\alpha = \alpha^\vee$. (Hint: If R is irreducible, rescale the inner product, otherwise break up as a direct sum and do the same)

Definition 5.6. A *lattice* L is a finitely generated free abelian group with a bilinear form $L \times L \rightarrow \mathbb{Z}$ such that $(L \otimes \mathbb{R}, (\cdot, \cdot))$ is an inner product. A *root* of L is any $\alpha \in L$ with $(\alpha, \alpha) = 2$, and $R_L = \{\alpha \in L \mid (\alpha, \alpha) = 2\}$. Observe that for $\alpha \in R_L$, $s_\alpha L \subseteq L$

Lemma 5.3. R_L is a root system in $\mathbb{R}R_L$

Proof. Finiteness is the only part that isn't clear. But R_L is the intersection of the compact set $\{v \in \mathbb{R}R_L \mid (v, v) = 2\}$ with the discrete set L , so is finite. $\square$

Definition 5.7. L is *generated by roots* if $\mathbb{Z}R_L = L$. This implies L is an *even lattice*, ie $(l, l) \in 2\mathbb{Z}$ for all $l \in L$

Example 5.3. Let $L = \mathbb{Z}\alpha$ and $(\alpha, \alpha) = \lambda$. If $\lambda = 2$, $R_L = \{\pm\alpha\}$ and $L = \mathbb{Z}R_L$ and if $k^2\lambda/2 \neq 1$ for all $k \in \mathbb{Z}$, then $R_L = \emptyset$

We now list all the simply laced root systems by giving lattices for which they are R_L . We consider $\mathbb{Z}^n = \bigoplus \mathbb{Z}e_i$ with $(e_i, e_j) = \delta_{ij}$, the square lattice.

1. A_n : $L = \{l \in \mathbb{Z}^{n+1} \mid (l, e_1 + \dots + e_{n+1}) = 0\} \cong \mathbb{Z}^n$, with roots $R_L = \{e_i - e_j \mid i \neq j\}$. $|R_L| = n(n+1)$ and $\mathbb{Z}R_L = L$. If $\alpha = e_i - e_j$ then s_α swaps the i, j coordinates, so $W = S_{n+1}$. This is the root system of $\mathfrak{sl}_{n+1}$.
2. D_n : Now $R_{\mathbb{Z}^n} = \{\pm e_i \pm e_j \mid i \neq j\}$. Let $L = \mathbb{Z}R_{\mathbb{Z}^n} \subsetneq \mathbb{Z}^n$. Then $s_{e_i - e_j}$ swaps the i, j coordinates, $s_{e_i + e_j}$ flips the sign of the i, j coordinate. $|R_L| = 2n(n-1)$ and $W = (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$. This is the root system of $\mathfrak{so}_{2n}$
3. E_8 : Let $\Gamma_n = \{(k_1, \dots, k_n) \mid k \in \mathbb{Z} \cup (\mathbb{Z} + \frac{1}{2}), \sum k_i \in 2\mathbb{Z}\}$. Consider $\alpha = (\frac{1}{2}, \dots, \frac{1}{2}) \in \Gamma_n$. Then $(\alpha, \alpha) = \frac{n}{4}$ so if Γ_n is an even lattice, $8 \mid n$. For Γ_8 , the root system is E_8
4. If we take $\alpha = \frac{1}{2}(1, \dots, 1)$ and $\beta = e_7 + e_8 \in E_8$, then $\alpha^\perp$ has type E_7 , and $\beta^\perp$ in E_7 has type E_6

Exercise 5.3. Check $R_{D_3} = R_{A_3}$, $R_{D_2} = R_{A_1 \times A_1}$

Exercise 5.4.

1. Γ_{8n} is a lattice
2. For $n > 1$, Γ_{8n} has type D_n
3. The roots of Γ_8 are those of the form $\pm e_i \pm e_j$ for $i \neq j$ and $\frac{1}{2}(\pm e_1 \pm \dots \pm e_8)$ with an even number of $-$ signs. The number of such roots is 240 (and the dimension of the Lie algebra is 248, with 8 extra dimensions from the 8d space for the torus)

Exercise 5.5. If R is a root system and $\alpha \in R$, then $\alpha^\perp \cap R$ is also a root system.

Theorem 5.4.

1. (ADE classification) The complete list of simply laced irreducible root systems is $A_n = \mathfrak{sl}_{n+1}, n \geq 1$, $D_n = \mathfrak{so}_{2n}, n \geq 4$ and E_6, E_7, E_8 and no two of those root systems are isomorphic
2. The remaining reduced root systems are $B_2 = C_2$, $B_n = \mathfrak{so}_{2n+1}, C_n = \mathfrak{sp}_{2n}, n \geq 3$, F_4 and G_2 where $R_{B_n} = \{\pm e_i, \pm e_i \pm e_j, i \neq j\} \subseteq \mathbb{Z}^n$, $R_{C_n} = \{\pm 2e_i, \pm e_i \pm e_j, i \neq j\} \subseteq \mathbb{Z}^n$ (so $R_{B_n}^\vee = R_{C_n}$, and $W_{B_n} = W_{C_n} = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$) and we define F_4 by setting $Q = \{(k_1, \dots, k_4) \mid k_i \in \mathbb{Z} \cup (\mathbb{Z} + \frac{1}{2})\}$ and $R_{F_4} = \{\alpha \in Q \mid (\alpha, \alpha) \in \{1, 2\}\}$, which is $\{\pm e_i, \pm e_i \pm e_j, i \neq j, \frac{1}{2}(\pm e_1 \pm \dots \pm e_4)\}$

Proof. Deferred □

Given a root system (R, V) , choose $f: V \rightarrow \mathbb{R}$ such that $f(\alpha) \neq 0$ for all $\alpha \in R$

Definition 5.8. A root $\alpha \in R$ is *positive* if $f(\alpha) > 0$ and *negative* if $f(\alpha) < 0$. It is *simple* if it is positive and not the sum of 2 positive roots. We write Π for the simple roots.

Example 5.4.

1. If we take A_n , with $R = \{\pm(e_i - e_j), i \neq j\}$. Choose $f(e_1) = n + 1, \dots, f(e_{n+1}) = 1$. Then $R^+ = \{e_i - e_j, i < j\}$. As $f(R) \subseteq \mathbb{Z}$, if $f(\alpha) = 1$ then α is simple, so $\Pi \supseteq \{e_1 - e_2, \dots, e_n - e_{n+1}\}$, and these span $\mathbb{Z}R$ so we have equality. We draw a graph with nodes the simple roots, and $(\alpha_i, \alpha_j^\vee)(\alpha_j, \alpha_i^\vee)$ -many edges between α_i, α_j with an arrow towards the short root if we have more than one length (this is the *Dynkin diagram*) For A_n this is



2. For B_n our roots are $\{\pm e_i, \pm e_i \pm e_j, i \neq j\} \subseteq \mathbb{Z}^n$. We take $f(e_j) = n + 1 - j$. Then $R^+ = \{e_i \pm e_j, i < j, e_i\}$ and $\Pi = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n\}$. The Dynkin diagram is



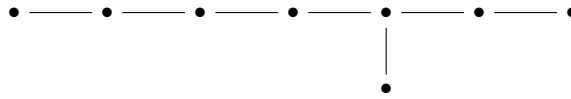
3. For C_n our roots are $\{\pm 2e_i, \pm e_i \pm e_j, i \neq j\} \subseteq \mathbb{Z}^n$. We take $f(e_j) = n + 1 - j$. Then $R^+ = \{e_i \pm e_j, i < j, 2e_i\}$ and $\Pi = \{e_1 - e_2, \dots, e_{n-1} - e_n, 2e_n\}$. The Dynkin diagram is



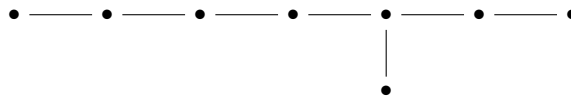
4. For D_n , $\Pi = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}$, and the diagram is



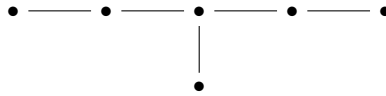
5. For E_8 set $f(e_1) = 28 = 1 + \cdots + 7$ and $f(e_i) = 9 - i$ for $2 \leq i \leq 8$. Then $R^+ = \{e_i \pm e_j, i < j, \frac{1}{2}(e_1 \pm \cdots \pm e_8), \text{ even number of signs}\}$. Then $\Pi = \{e_2 - e_3, \dots, e_7 - e_8, \frac{1}{2}(e_1 + e_8 - e_2 - \cdots - e_7), e_7 + e_8\}$. The Dynkin diagram is



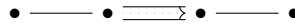
6. For E_7 the Dynkin diagram is



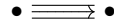
7. For E_6 the Dynkin diagram is



8. For F_4 the Dynkin diagram is



9. For G_2 the Dynkin diagram is



Exercise 5.6. Check all the above and do it for the ones not listed

Proposition 5.2.

1. If α, β are simple roots, then $\alpha - \beta$ is not a root
2. If α, β are simple roots, $\alpha \neq \beta$ then $(\alpha, \beta^\vee) \leq 0$
3. Every $\alpha \in R^+$ can be written as $\sum k_i \alpha_i$ with $\alpha_i \in \Pi$ and $k_i \in \mathbb{Z}_{\geq 0}$, so simple roots integrally span V
4. Simple roots are linearly independent
5. If $\alpha \in R^+ \setminus \Pi$ then there is $\alpha_i \in \Pi$ such that $\alpha - \alpha_i \in R^+$
6. R is irreducible if and only if Π is indecomposable if and only if the Dynkin diagram is connected (can't be written as a union of two sets with 0 inner product)

Exercise 5.7. Prove these.

Definition 5.9. Let $I_{1,N} = \mathbb{Z}^{N+1}$ with basis $e_0, \dots, e_N$ and $(e_i, e_j) = 0$ if $i \neq j$, $(e_0, e_0) = -1$ and $(e_i, e_i) = 1$ for $i \geq 1$. This is the *unimodular lattice of signature $(N, 1)$* . Let $k = (-3, 1, \dots, 1)$. $k \cdot k = N - 9$. The E_N -lattice is $L = K^\perp$

Exercise 5.8. Let $N \geq 3$. Show $\alpha_1 = e_0 - e_1 - e_2 - e_3, \alpha_i = e_{i-1} - e_i, i \geq 2$ form a basis of L and are simple roots, with Dynkin diagram

$$\begin{array}{ccccccc} 2 & \text{---} & 3 & \text{---} & 4 & \text{---} & 5 & \text{---} & \dots & \text{---} & N \\ & & & & | & & & & & & \\ & & & & 1 & & & & & & \end{array}$$

Describe the positive roots for $N \leq 8$, and show that for $N \geq 8$, there are infinitely many positive roots. Conclude that for $6 \leq N \leq 8$ this is the E_n lattice, and for $N \leq 5$ we get $D_5, A_4, A_2 \times A_1, A_1 \times A_1, A_1$

Definition 5.10. Given a set of simple roots $\{\alpha_1, \dots, \alpha_l\}$ for a root system R , we define $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$, set $A = (a_{ij})$, the *Cartan matrix*. In the Dynkin diagram, there are $a_{ij}a_{ji}$ lines between i, j towards the shorter of j, i

Proposition 5.3. For the Cartan matrix, $a_{ii} = 2, a_{ij} \leq 0$ otherwise, $a_{ij} \in \mathbb{Z}$. $a_{ij} = 0$ if and only if $a_{ji} = 0$, $\det A > 0$ and all principal subminors have positive determinant

Proof. Everything except the determinant theorems have already been proven. Now

$$A = \text{diag} \left(\frac{2}{(\alpha_i, \alpha_i)} \right) ((\alpha_i, \alpha_j))_{ij}$$

where the first is a diagonal matrix of positive coefficients, and the latter is the Gram matrix of an inner product, hence has positive determinant. The last comment holds as the subminors come from a span of the subset of the roots. $\square$

If (R, V) is a root system and R^+ is a choice of positive roots

Exercise 5.9. If (R, V) is irreducible, there is a unique root $\theta \in R^+$ such that for all $\alpha \in \Pi$, $\theta + \alpha \notin R$. We call this the *highest root*. (For the moment, see this case by case)

Example 5.5. For A_n , this is $e_1 - e_{n+1}$

Definition 5.11. We define an *extended Cartan matrix* $\tilde{A}$ by setting $\alpha_0 = -\theta$ and $A = ((\alpha_i^\vee, \alpha_j))$ as before. The related Dynkin diagram is the *affine Dynkin diagram*

Example 5.6. $A_1 = (2)$ and $\tilde{A}_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. $\tilde{A}$ satisfies the non-determinant properties of Proposition 5.3, but has $\det A = 0$, and every principal subminor has positive determinant.

Exercise 5.10. Draw the associated affine Dynkin diagram for the reduced ones.

Exercise 5.11. Show that the Dynkin diagram of A^T is the Dynkin diagram of A with arrows reversed.

Remark 5.1. There is an extra Dynkin diagram not already described (including exercises):



called $\tilde{A}_n^{(2)}$

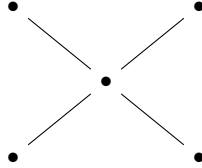
Theorem 5.5. An irreducible Dynkin diagram is one of the $A_n, B_n, \dots, G_2$

Proof. We first classify the rank 2 diagrams. The Cartan matrices are of the form

$$\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$$

As the determinant is positive, $(a, b) = (0, 0), (1, 1), (2, 1), (1, 2), (3, 1), (1, 3)$ which are $A_1 \times A_1, A_2, B_2 = C_2, G_2$

Any subdiagram of a Dynkin diagram is a Dynkin diagram (trivially) so the number of arrows between two vertices is 1, 2, 3 by above. As the affine diagrams have determinant 0, they are not subdiagrams of Dynkin diagrams. A Dynkin diagram has no cycles. Let $\alpha_1, \dots, \alpha_n$ be distinct simple roots and consider $x = \sum \frac{\alpha_i}{\sqrt{(\alpha_i, \alpha_i)}}$. Then $0 < (x, x) = n + \sum_{i < j} \frac{2(\alpha_i, \alpha_j)}{\sqrt{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}} = n - \sum_{i < j} \sqrt{a_{ij}a_{ji}}$ and so $\sum_{i < j} \sqrt{a_{ij}a_{ji}} < n$ but a cycle on n roots needs $n + 1$ edges, contradiction. If the diagram is simply laced, suppose it is not of type A . Since $\tilde{D}_4$, which is



is not a subdiagram, branching is at most 3-way. Since $\tilde{D}_n$ is not a subdiagram, there is at most one branching, and hence exactly 1 as we are not type A , so the diagram is $T_{p,q,r}$ with a branch point, and p, q, r nodes on each branch respectively. To fix notation, $E_8 = P_{5,3,2}$. Now $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ are not subdiagrams, so we are done in the simply laced case. (Alternative: show $\det T_{p,q,r} = pq + pr + qr - pqr$ and conclude similarly). For the non-simply laced case:

Exercise 5.12. If there is a triple bond, it must be G_2 (need to show 2 triple bonds, and double and triple can't happen, as G_2 deals with the single bond case)

If there is a double bond, then by $\tilde{C}_n, \tilde{A}_n^{(2)}$ and their transpose there can only be one. By $\tilde{B}_n$ the diagram does not branch, and by $\tilde{F}_4$, we can have at most one vertex on each side. So we are done. $\square$

Exercise 5.13. Compute the determinant of all the Cartan matrices eg. $\det A_n = n + 1$

Remark 5.2. In general for a simply connected Lie group with Lie algebra $\mathfrak{g}$, the size of the centre has order $\det A$

5.1 Independence of Choice Theorems

Theorem 5.6. *All maximal tori of a semisimple Lie algebra over $k, k = \bar{k}$ and $\text{char } k = 0$ are conjugate*

Proof. No Proof □

Theorem 5.7. *All choices of positive roots are conjugate by the Weyl group*

Proof. No Proof □

These theorems imply that the Cartan matrix depends only on the Lie algebra.

5.2 Existence and Uniqueness Theorems

Theorem 5.8. *Let $\mathfrak{g}_i$ be semisimple Lie algebras with maximal tori $\mathcal{A}_i$, roots R_i , positive roots R_i^+ , simple roots Π_i and Cartan matrices A_i , and suppose A_i are equal after reordering indices. Then there is an isomorphism of Lie algebras sending all the data to each other*

Proof. No Proof □

Theorem 5.9. *Let A be a Cartan matrix. Then there is a semisimple Lie algebra with A as its Cartan matrix*

Proof. No Proof □

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, and choose R^+, Π . We choose $E_i \in \mathfrak{g}_\alpha$, and $F_i \in \mathfrak{g}_\alpha$ forming an $\mathfrak{sl}_2$, and set $H_i = \nu^{-1}(\alpha_i^\vee)$ as usual. We observe $[H_i, H_j] = 0, [H_i, E_j] = \alpha_j(H_i)E_j = a_{ij}E_j, [H_i, F_j] = -a_{ij}F_j$, and

$$[E_i, F_j] = \begin{cases} 0 & i \neq j \\ H_i & i = j \end{cases}$$

. We define $\eta^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ and $\eta^- = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}$, so $\mathfrak{g} = \eta^+ \oplus \mathfrak{t} \oplus \eta^-$. We claim the E_i generate η^+, F_i generate η^- and hence E_i, F_i generate $\mathfrak{g}$ as Lie algebras. Proof: for $\alpha \in R^+, \alpha = \sum k_i \alpha_i$ with $k_i \geq 0, k \in \mathbb{Z}_{\geq 0}$, we prove E_α is generated by $E_1, \dots, E_l$ by induction on the height, $\sum k_i$. If this is 1, we are done by assumption. If this is larger than 1, there is a simple root α_i with $\alpha - \alpha_i = \beta \in R^+$, so β is generated by the E_i . But $E_\alpha \in [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_\beta]$ so E_α is also. We flip our choice of positive roots to get the statement for F_i . As $[E_i, F_i] = H_i$, and the H_i are a basis of $\mathfrak{t}$ we're done.

Corollary 5.9.1. *The Cartan matrix determines $\mathfrak{g}$*

Proof. Let A be a generalised Cartan matrix (ie $a_{ii} = 2, a_{ij} \leq 0, \alpha_{ij} = 0$ implies $a_{ji} = 0$). Define a Lie algebra $\tilde{\mathfrak{g}}$ with the relations described above. If A is the Cartan matrix of $\mathfrak{g}$, then by above $\tilde{\mathfrak{g}}$ surjects onto $\mathfrak{g}$. Let J be the ideal generated by the relations $(\text{ad } E_i)^{1-a_{ij}} E_j = 0$ (and the same for F). It is a theorem that if A is indecomposable, then $\tilde{\mathfrak{g}}$ has a unique maximal ideal, generated by the relations above, and the quotient $\bar{\mathfrak{g}}$ is a simple Lie algebra. Hence if $\mathfrak{g}$ is simple, the map $\bar{\mathfrak{g}}$ is surjective with no kernel, so is an isomorphism. (The theorem is easy) $\square$

Remark 5.3. This implies uniqueness, and implies existence if and only if the following holds: $\bar{\mathfrak{g}}$ is finite dimensional if and only if A is a Cartan matrix. For arbitrary A , $\bar{\mathfrak{g}}$ is called a Kac-Moody algebra. If $\det A = 0$ and every subdiagram is a Cartan matrix, then $\mathfrak{g}_A$ is called an affine Kac-Moody algebra

Theorem 5.10. *Write $s_i = s_{\alpha_i}$ for $\alpha_i \in \Pi$. Then $W = \langle s_1, \dots, s_n \mid s_i^2, (s_i s_j)^{m_{ij}} \rangle$ where*

$$m_{ij} = \begin{cases} 2 & \text{if } a_{ij}a_{ji} = 0 \\ 3 & \text{if } a_{ij}a_{ji} = 1 \\ 4 & \text{if } a_{ij}a_{ji} = 2 \\ 6 & \text{if } a_{ij}a_{ji} = 3 \end{cases}$$

Proof. No Proof $\square$

6 Representations

We write $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$ as before for a finite dimensional semisimple Lie algebra over $\mathbb{C}$ called $\mathfrak{g}$.

Proposition 6.1. *Let V be a finite dimensional representation of $\mathfrak{g}$*

1. $V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$ where $V_\lambda = \{v \in V \mid tv = \lambda(t)v\}$
2. If $V_\lambda \neq 0$ then $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in R$

Proof. Since $\mathfrak{g}$ is a finite dimensional $\mathfrak{g}$ module, it is a finite dimensional $(\mathfrak{sl}_2)_\alpha$ module, hence h_α acts diagonalisably with $\lambda(h_\alpha) \in \mathbb{Z}$ and the result follows as the $\alpha^\vee$ span $\mathfrak{t}$. $\square$

Definition 6.1. $Q = \mathbb{Z}R \subseteq \mathfrak{t}^*$ is the lattice of roots, and

$$\begin{aligned} P &= \{\gamma \in \mathbb{Q}R \mid (\gamma, \alpha^\vee) \in \mathbb{Z}, \forall \alpha \in R\} \\ &= \{\gamma \in \mathbb{Q}R \mid (\gamma, \alpha^\vee) \in \mathbb{Z}, \forall \alpha \in \Pi\} \end{aligned}$$

is the lattice of weights

Remark 6.1. V being finite dimensional implies $V = \bigoplus_{\lambda \in P} V_\lambda$, also $Q \subseteq P$, and $|P/Q| = \det A$ for A the Cartan matrix, and $WP \subseteq P$ (exercise)

Example 6.1. For $\mathfrak{sl}_2$, $R = \{\pm\alpha\}$, $Q = \mathbb{Z}\alpha$, $(\alpha, \alpha) = 2$, $P = \frac{\alpha}{2}\mathbb{Z}$ and $|P/Q| = 2 = \det 2$

Definition 6.2. If V is a finite dimensional representation with a weight space decomposition, we define

$$\text{ch} V = \sum \dim V_\lambda \cdot e^\lambda \in \mathbb{Z}[P]$$

the formal character of V

Example 6.2. For $\mathfrak{sl}_2$, let $z = e^{\alpha/2}$, then $\text{ch} L(n) = z^n + \cdots + z^{-n}$ as before

Example 6.3. If $\mathfrak{g} = \mathfrak{sl}_3$ and $V = \mathfrak{g}$ is the adjoint rep, then $\text{ch} V = 2e^0 + e^\alpha + e^\beta + e^{\alpha+\beta} + e^{-\alpha} + e^{-\beta} + e^{-\alpha-\beta}$. We can decorate the weight lattice with the multiplicities, and it is rotationally symmetric, with the outer most shell being 0s, the next shell being 1s, and the 0 shell being 2

Proposition 6.2. If V is a finite dimensional representation of $\mathfrak{g}$ then $\dim V_\lambda = \dim V_{w\lambda}$ for $\lambda \in P$, $w \in W$, so $\text{ch} V \in \mathbb{Z}[P]^W$

Proof. (Proof sketch 1) If G is an algebraic group with Lie algebra $\mathfrak{g}$, then it is a theorem that $W = N_G(T)/T$. Suppose also $G \curvearrowright V$ (if G is simply connected this is automatic). Then if $\dot{w} \in wT$, then $t\dot{w}v = \dot{w}\dot{w}^{-1}t\dot{w}v = \dot{w}\lambda(\dot{w}^{-1}t\dot{w})v$, so $\dot{w}V_\lambda \subseteq V_{w\lambda}$.

(Proof sketch 2) Mimic the above argument in $\mathfrak{g}$. We have

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Each of these is unipotent, so we can rewrite this as $\exp f \exp(-e) \exp f$ and define $\dot{s}_\alpha = \exp F_\alpha \exp -E_\alpha \exp F_\alpha$ and (exercise) if V is a finite dimensional representation of $\mathfrak{g}$, each E_α, F_α act nilpotently on V , so $\dot{s}_\alpha$ is a well defined finite sum, and $\dot{s}_\alpha = \varepsilon_\alpha: V_\lambda \rightarrow V_\lambda$ with $\varepsilon_\alpha^2 = 1, \varepsilon_\alpha$ a constant (to be determined in terms of λ). Further $\dot{s}_\alpha^2(V_\lambda) = V_{\dot{s}_\alpha \lambda}$.

(Proof 3) Consider V as an $(\mathfrak{sl}_2)_\alpha \times \mathfrak{t}$ representation. It splits as a direct sum of irreducible representations $\mu, \mu - \alpha, \dots, \mu - n\alpha$ where $m = \langle \mu, h_\alpha \rangle$. Such a string is $(\mathfrak{sl}_2)_\alpha$ -invariant $\square$

Definition 6.3. $x: V \rightarrow V$ is locally nilpotent if for all $v \in V$, there is an n such that $x^n v = 0$

Remark 6.2. This is what made the above work, and e_α, f_α are locally nilpotent on V if V is a sum of finite dimensional $(\mathfrak{sl}_2)_\alpha$ -representations. Such a V is called *integrable*

Definition 6.4. If $\lambda, \mu \in \mathfrak{t}^*$, we write $\mu \leq \lambda$ to mean $\lambda - \mu = \sum_{\alpha_i \in \Pi} k_i \alpha_i$ with $k_i \geq 0$. The set of all μ such that $\mu \leq \lambda$ are lattice points in an obtuse cone

Definition 6.5. If V is a rep of $\mathfrak{g}$, μ is a *highest weight* of V if $V_\mu \neq 0$ and if $V_\lambda \neq 0$ then $\lambda \leq \mu$. A non-zero element of such a weight space is a *highest weight vector*. $v \in V_\lambda$ is a *singular vector* if $v \neq 0$ and $c_\alpha v = 0$ for all $\alpha \in R^+$. μ is an *extremal weight* if $w\mu$ is a highest weight for some $w \in W$. Let $P^+ = \{\lambda \in P \mid \langle \lambda, \alpha^\vee \rangle \geq 0\}$ for all $\alpha \in R^+$ (where it suffices to check the inner product only for simple roots) is the cone of *dominant weights*

Remark 6.3. If λ is a highest weight, all $0 \neq v \in V_\lambda$ are singular

Theorem 6.1. (*The Theorem of the Highest Weight*)

1. (*Complete reducibility*) If V is a finite dimensional representation of $\mathfrak{g}$, it is a direct sub of irreducibles
2. Let V be a finite dimensional representation of $\mathfrak{g}$, V irreducible and $v \in V_\lambda$ a singular vector. Then $\dim V_\lambda = 1$, if $V_\mu \neq 0$ then $\mu \leq \lambda$, and $\lambda(h_i) = \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all simple roots α_i , so $\lambda \in P^+$. Moreover if V' is also finite dimensional irreducible with highest weight λ then there is a unique isomorphism $V' \rightarrow V$ mapping given highest weight vectors to each other
3. Given $\lambda \in P^+$, there is a finite dimensional irreducible representation with highest weight λ , $L(\lambda)$
4. There is a nice formula for the character of $L(\lambda)$ called the Weyl character formula

Proof. Proof deferred □

Corollary 6.1.1. $\text{ch } L(\lambda) = e^\lambda + \sum_{\mu < \lambda} a_{\mu\lambda} \in \mathbb{Z}[P]$, so the characters of different $L(\lambda)$ are linearly independent. The character is Weyl group invariant, and $WP^+ = P$, and $P_{\mathbb{Q}}^+$ is a fundamental domain for the action of W . If $m_\lambda = \sum_{\mu \in W\lambda} e^\mu$, then m_λ for $\lambda \in P^+$ is a basis for $\mathbb{Z}[P]^W$, so the characters are also so.

Proof. Trivial □

Corollary 6.1.2. If V, W are finite dimensional representations, $V = W$ if and only if their characters are equal

Proof. Trivial □

Definition 6.6. Let ω_i be the dual basis to the simple coroots $\alpha_j^\vee$.

Remark 6.4. These are a basis of P^+

Exercise 6.1. Compute everything for everything you care to.

Example 6.4. $L(0) = \text{trivial}$ and if $\mathfrak{g}$ is simple, the the adjoint representation is, with highest weight $\theta \in R$ with $\theta + \alpha$ not a root for all positive roots α^+ , so θ is the highest root, ie $\mathfrak{g} = L(\theta)$

Exercise 6.2. Compute the expansion of θ in terms of the ω_i for each Lie algebra. For $\mathfrak{sl}_n$ you have $\theta = e_1 - e_n$, and $\theta = \omega_1 + \omega_{n-1}$

Example 6.5. If $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathbb{C}^n$ is the standard representation with basis $v_1, \dots, v_n$ and weights $e_1, \dots, e_n$. The sum of the e_n is 0, and P is the direct sum of n copies of $\mathbb{Z}$ modulo their sum, so e_1 is the highest weight and $\mathbb{Z}[P] = \mathbb{Z}[z_1, \dots, z_n]/z_1 \dots z_n$ and the character is their sum.

Example 6.6. If V, W are representations, so is the tensor product, exterior power and symmetric power. $\bigwedge^s \mathbb{C}^n$ is a representation of $\mathfrak{sl}_n$ with the usual basis, with weights given by the sum of the usual weights of $\mathbb{C}^n$ with no duplicates. $E_j(v) = 0$ if and only if $v = v_1 \wedge \dots \wedge v_s$, so $\bigwedge^s \mathbb{C}^n$ is irreducible with highest weight ω_s , so for $n-1$, $L(\omega_{n-1}) = L(\omega_1)^*$.

For the symmetric power $S^m \mathbb{C}^n$, with the usual basis, with weights given by the sum of the usual weights of $\mathbb{C}^n$ with duplicates. $E_j(v)$ is 0 for all $1 \leq j \leq n$ if and only if $v = v_1 \dots v_1$. $S^m L(\omega_1) = L(m\omega_1)$

Exercise 6.3. If V, W are $\mathfrak{g}$ -modules and V is finite dimensional, then $V^* \otimes W \cong \text{Hom}(V, W)$. If $\mathfrak{g} = \mathfrak{sl}_n$ and $V = \mathbb{C}^n$, then $V \otimes V^* = \mathbb{C} \oplus \mathfrak{sl}_n$

Exercise 6.4. Let $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2l}$, $n = 2l$ and $V = \mathbb{C}^n$, show V is irreducible and compute its highest weight. Show $V \cong V^*$ as representations of $\mathfrak{g}$ via the form defining $\mathfrak{g}$. Hence $V \otimes V$ has at least 3 components $n \geq 3$ (as the trivial representation is a factor, and $V \otimes V = S^2 V \oplus \bigwedge^2 V$). Show it has exactly 3 and find the highest weights

We now start to prove the theorem of the highest weight.

Definition 6.7. Let $\mathfrak{g}$ be any Lie algebra with a non-degenerate invariant bilinear form $(\cdot, \cdot)$. Let $x_1, \dots, x_n$ be a basis of $\mathfrak{g}$, and $x^1, \dots, x^n$ be the dual basis. The Casimir of $\mathfrak{g}$ is $\Omega = \sum x_i x^i$ (this is a well defined endomorphism of any finite dimensional representation of $\mathfrak{g}$)

Lemma 6.2. For all $x \in \mathfrak{g}$, $[\Omega, x] = 0$

Proof. (v1): In coordinates, $[\Omega, x] = [\sum x_i x^i, x] = \sum [x_i, x] x^i + \sum x_i [x^i, x]$. Writing $[x^i, x] = \sum a_{ij} x^j$ and $[x_i, x] = \sum b_{ij} x_j$. Then $a_{ij} = ([x^i, x], x_j) = ([x_j, x^i], x)$ by invariance of the form, and $b_{ij} = ([x^j, x_i], x) = -a_{ji}$. Then expand everything and you get 0.

(v2) The map $1 \rightarrow \text{End}(\mathfrak{g})$ via $\lambda \cdot 1 \mapsto \lambda 1_{\mathfrak{g}} \cong \mathfrak{g} \otimes \mathfrak{g}^* \cong \mathfrak{g} \otimes \mathfrak{g}$ via the invariant form. The overall map is just $1 \mapsto \sum x^i \otimes x_i$. $\mathfrak{g}$ acts on V and so $\mathfrak{g} \otimes \mathfrak{g}$ acts on $V \otimes V$, so we get $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \text{End}(V) \otimes \text{End}(V) \rightarrow \text{End}(V)$. This map in total sends $1 \rightarrow \Omega$, and is a map of $\mathfrak{g}$ -modules, so $[\Omega, x] = 0$ for all $x \in \mathfrak{g}$. $\square$

Where does the Casimir live? Let $\mathfrak{g}$ be a Lie algebra over k for k a field

Definition 6.8. The universal enveloping algebra of $\mathfrak{g}$, $\mathcal{U}\mathfrak{g}$ is the associative algebra with 1 over k generated by $\mathfrak{g}$ with the relations $xy - yx = [xy]$ for all $x, y \in \mathfrak{g}$. (More formally, the tensor algebra on $\mathfrak{g}$ modulo those relations)

Remark 6.5. So $\Omega \in \mathcal{U}\mathfrak{g}$

Exercise 6.5. An enveloping algebra for $\mathfrak{g}$ is an associative algebra A and a linear map $\mathfrak{g} \rightarrow A$ such that $xy - yx = [x, y]$. Show that any such algebra factors through $\mathcal{U}\mathfrak{g}$

Example 6.7. In particular, any representation factors as $\mathcal{U}\mathfrak{g} \rightarrow \text{End } \mathfrak{g}$

$\mathcal{U}\mathfrak{g}$ is filtered as the relation is in degree below 2. We define $(\mathcal{U}\mathfrak{g})_{\leq n}$ to be the products of $\leq n$ elements of $\mathfrak{g}$ in $\mathcal{U}\mathfrak{g} = \mathcal{U}$ and write $\mathcal{U}_n = (\mathcal{U}\mathfrak{g})_{\leq n}$. Now $\mathcal{U}_n \subseteq \mathcal{U}_m$ for $n \leq m$, and $\mathcal{U}_n \mathcal{U}_m \subseteq \mathcal{U}_{n+m}$ so we have a filtration of algebras.

Exercise 6.6. If $x \in \mathcal{U}_n, y \in \mathcal{U}_m$, then $[x, y] \in \mathcal{U}_{n+m-1}$

We can form the associated graded algebra of the filtration $\bigoplus_{n \geq 0} F_n / F_{n-1} = \text{gr } V$.

Theorem 6.3. (Poincare-Birkhoff-Witt) $\text{gr } \mathcal{U}\mathfrak{g} \cong S\mathfrak{g}$. Equivalently, if $x_1, \dots, x_n$ is a basis of $\mathfrak{g}$, then $\{x_1^{a_1} \dots x_n^{a_n} \mid a_i \in \mathbb{Z}_{\geq 0}\}$ is a basis for $\mathcal{U}\mathfrak{g}$, and in particular $\mathfrak{g} \rightarrow \mathcal{U}\mathfrak{g}$ is an injection

Proof. No Proof □

Exercise 6.7. Show the previous exercise implies there is a well defined surjective map $S\mathfrak{g} \rightarrow \text{gr } \mathcal{U}\mathfrak{g}$

Exercise 6.8. For $v \in V$, $(\mathcal{U}\mathfrak{g})v$ is the subrepresentation spanned by v .

Remark 6.6. If $\mathfrak{g}$ is semisimple, and $(\cdot, \cdot)_{\text{ad}}$ is our invariant form, we write Ω as

If V is a highest weight module and v is a highest weight vector, by the easy direction of PBW, $V = \mathcal{U}\eta^- v$, as the basis of η^+ map v to 0, and monomials in $\mathfrak{t}$ give scalar multiples of v .

Remark 6.7. If V is irreducible and finite dimensional, it is a highest weight module (as it has a singular vector, and the submodule it spans must be all of V)

Proposition 6.3. Let V be a highest weight module for $\mathfrak{g}$ with V_Λ a highest weight vector of weight $\Lambda \in \mathfrak{t}^*$, V not necessarily finite dimensional. Then

1. $\mathfrak{t}$ acts diagonalisably on V , so $V = \bigoplus_{\lambda \in D(\Lambda)} V_\lambda$ where $D(\Lambda) = \{\Lambda - \sum k_i \alpha_i \mid k_i \in \mathbb{Z}_{\geq 0}\}$ (ie the weights less than Λ)
2. $V_\Lambda = \mathbb{C}v_\Lambda$, and all other weight spaces are finite dimensional
3. V is irreducible if and only if the only singular vectors are in V_Λ
4. V contains a unique maximal proper submodule I , which admits a weight space decomposition $I = \bigoplus_{\lambda \in D(\Lambda)} I \cap V_\lambda$, and I is the sum of all proper submodules of V

5. Ω acts on V_Λ as multiplication by $|\Lambda + \rho|^2 - |\rho|^2$ where $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$
6. If v_λ is a singular vector then $|\lambda + \rho|^2 = |\Lambda + \rho|^2$
7. If $\Lambda \in \mathbb{R}R$, then there are only finitely many λ such that V_λ contains a singular vector

Proof. (1) + (2): As $V = \mathcal{U}\mathfrak{g}v_\Lambda$, expressions $(\prod_i e_{-\beta_i})v_\Lambda$ span V with $e_{-\beta} \in \mathfrak{g}_{-\beta}$ for $\beta \in R^+$, and the weight of this is $\Lambda - \sum_i \beta_i$ which implies 1 and 2, as there are only finitely many ways to arrange the e_{β_i} to get into V_{β_i} .

(3): If $v_\lambda \in V_\lambda$ is singular, then $\mathcal{U}\mathfrak{g}v_\lambda$ is a submodule of V , so is equal to V with weights all at most λ , so we have $\lambda \leq \Lambda \leq \lambda$, so they are equal. Conversely, if $N < V$ is a proper submodule, $\mathfrak{t}N \subseteq N$, so $\mathfrak{t}$ grades N , and the weights lie in $D(\Lambda)$. Let $\lambda = \Lambda - \sum k_i \alpha_i$ be a weight in N with $\sum k_i$ minimal. As $N \neq V$, $\lambda \neq \Lambda$ so the sum is larger than 0, and any $v \in N_\lambda \setminus 0$ is a singular vector, as $e_{\alpha_i}v$ would otherwise have smaller $\sum k_i$ for some $\alpha_i \in R^+$.

(4) is immediate, as any proper submodule of V is $\mathfrak{t}$ -graded, and does not contain V_Λ , hence the sum of all such submodules is still proper.

(5) + (6): Choose a basis $u_1, \dots, u_l$ of $\mathfrak{t}$ and $x_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in R$. We get dual bases $u^1, \dots, u^l \in \mathfrak{t}$ and $x^\alpha \in \mathfrak{g}_\alpha$ with respect to the adjoint Killing form. Normalise x_α so $(x_\alpha, x_{-\alpha})_{\text{ad}} = 1$. Then $x^{-\alpha} = x_{-\alpha}$, and $[x_\alpha, x_{-\alpha}] = \nu^{-1}(\alpha)$. Now

$$\begin{aligned} \Omega &= \sum u_i u^i + \sum_{\alpha \in R^+} (x_\alpha x_{-\alpha} + x_{-\alpha} x_\alpha) \\ &= \sum u_i u^i + 2 \sum_{\alpha \in R^+} (x_{-\alpha} x_\alpha) + \sum_{\alpha \in R^+} \nu^{-1}(\alpha) \\ &= \sum u_i u^i + 2 \sum_{\alpha \in R^+} (x_{-\alpha} x_\alpha) + \nu^{-1}(\rho) \end{aligned}$$

So $\Omega v = (\lambda(2\nu^{-1}(\rho)) + \sum \lambda(u_i)\lambda(u^i))v_\lambda = ((\lambda, \lambda) + 2(\rho, \lambda))v_\lambda$ as desired for V_λ , but then it acts on all V by this scalar, so it holds for all of V , and 6 holds as we only needed λ to be singular.

(7): If V_λ contains a singular vector, $|\lambda + \rho|^2 = |\Lambda + \rho|^2$, but then λ is in a sphere in $\mathbb{R}R$ and in $D(\Lambda)$, which is therefore discrete and bounded, hence finite. $\square$

Remark 6.8. (7) is true for all $\Lambda \in \mathfrak{t}^*$, but the proof is harder.

Definition 6.9. Let $\Lambda \in \mathfrak{t}^*$, then a *Verma module* $M(\Lambda)$ is the universal highest weight module with highest weight Λ and highest weight vector v_Λ : If (W, w) is another such pair with highest weight Λ , there is a unique map $(V, v_\Lambda) \rightarrow (W, w)$

Proposition 6.4. *Verma modules exist and are unique. There is therefore a unique irreducible highest weight module of weight Λ for $\Lambda \in \mathfrak{t}^*$ which we denote $L(\Lambda)$*

Proof. Uniqueness is clear by abstract nonsense. For existence, we define $M(\Lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}(\eta^+ + \mathfrak{t})} \mathbb{C}_\Lambda$. We define $\mathfrak{b} = \eta^+ + \mathfrak{t}$, and $\mathbb{C}_\Lambda$ the module where η^+ acts by 0,

and $\mathfrak{t}$ acts by Λ . $M(\Lambda)$ is the free $\mathfrak{g}$ module where the action is what we desire. Let W be any other highest weight module with highest weight Λ . Then usual universal property nonsense does what we desire, so we are done with (1). An irreducible highest weight module must be a quotient of $M(\Lambda)$ by a maximal proper submodule, but there is only one such, so (2) is done. $\square$

Proposition 6.5. *Let $R^+ = \{\beta_1, \dots, \beta_r\}$. Then $(\prod_i e_{-\beta_i})v_\Lambda$ is a basis of M_Λ*

Proof. PBW (the hard part) $\square$

Example 6.8. We've seen this for $\mathfrak{g} = \mathfrak{sl}_2$, it just has basis $v_\Lambda, Fv_\Lambda, \dots$

Exercise 6.9. For $\mathfrak{sl}_2$, show $M(\Lambda) = L(\Lambda)$ unless $\Lambda \in \mathbb{Z}_{\geq 0}$. If $\Lambda \in \mathbb{Z}_{\geq 0}$, show $M(\Lambda)$ has a unique proper submodule $F^{\Lambda+1}v_\Lambda, \dots$ which is itself a Verma module.

Exercise 6.10. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\Lambda \in \mathfrak{t}^*$, $\Lambda(h_i) \in \mathbb{Z}_{\geq 0}$ and $\Lambda \in P^+$. Show $F_i^{\Lambda(h_i)+1}v_\Lambda$ is a singular vector for $M(\Lambda)$

Exercise 6.11. Compute the character of $M(\Lambda)$

We now know that if V is a finite dimensional irreducible representation of $\mathfrak{g}$ then $V = L(\Lambda)$ for a unique $\Lambda \in P^+$. We want to show the converse, that if $\Lambda \in P^+$, then $L(\Lambda)$ is finite dimensional.

Exercise 6.12.

1. $a^k b = \sum \binom{k}{i} ((\text{ad } a)^i b) a^{k-i}$
2. Using (1) and the Serre relations, show that $F_i^N (\prod e_{-\beta_i})v_\Lambda = 0$ if N is sufficiently large

Proposition 6.6. *If $\Lambda \in P^+$, $L(\Lambda)$ is integrable (ie E_i, F_i act locally nilpotently)*

Proof. This is trivial for any highest weight module V in the E_i case. We must show the result for F_i on $L(\Lambda)$ (this is not true for $M(\Lambda)$). We know $F_i^{\Lambda(h_i)+1}v_\Lambda$ is a singular vector of $L(\Lambda)$ by the above, but $L(\Lambda)$ is irreducible, so the only singular vectors have weight Λ , so it is 0, and the above exercise finishes the proof $\square$

Corollary 6.3.1. *$\text{Ch } L(\Lambda)$ is W -invariant.*

Theorem 6.4. *(Cartan's theorem) If $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra over $\mathbb{C}$ and $\Lambda \in P^+$, then $L(\Lambda)$ is finite dimensional.*

Proof. Let $\alpha \in R^+$. We want to show $e_{-\alpha}$ acts locally nilpotently on $L(\Lambda)$. Let $n = (\Lambda, \alpha^\vee)$. $e_{-\alpha}^n v_\Lambda \in L(\Lambda)_{\lambda - n\alpha}$, which is 0 as if it is not, then $s_\alpha(\Lambda - n\alpha) = s_\alpha(\Lambda) + \alpha = \Lambda + \alpha$ is non-zero, contradiction, so by the same argument as above, $e_{-\alpha}$ acts locally nilpotently on $L(\Lambda)$, so $L(\Lambda) = \mathcal{U}\eta^- v_\Lambda$ which is finite dimensional. $\square$

So we have given the complete list of irreducible finite dimensional representations. We aim for complete reducibility next.

Lemma 6.5. $s_i(R^+ \setminus \{\alpha_i\}) = R^+ \setminus \{\alpha_i\}$ for α_i simple.

Proof. Let $\alpha = \sum k_i \alpha_i$. If $\alpha \neq \alpha_i$ then some $j \neq i$ has $k_j \neq 0$. But $s_i(\alpha_j) = \alpha_j + m\alpha_i$, so we have the same $k_j > 0$ coefficient for α_j . As all the k_n have the same sign, we are done. $\square$

Recall that $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.

Lemma 6.6. $\rho = \omega_1 + \dots + \omega_l$, ie $\rho(h_i) = 1$ for all i

Proof. Now $s_i(\rho) = s_i\left(\frac{1}{2}\alpha_i + \frac{1}{2} \sum_{\alpha \in R^+ \setminus \{\alpha_i\}} \alpha\right) = -\frac{1}{2}\alpha_i + \frac{1}{2} \sum_{\alpha \in R^+ \setminus \{\alpha_i\}} \alpha = \rho - \alpha_i$, but using the usual formula for s_i , we see $(\rho, \alpha_i^\vee) = 1$ $\square$

Exercise 6.13. Compute ρ for every simple Lie algebra

Lemma 6.7. Let $\Lambda \in P^+$ and $\mu + \rho \in P^+$ for $\mu \leq \Lambda$. If $|\Lambda + \rho| = |\mu + \rho|$ then $\mu = \Lambda$

Proof.

$$\begin{aligned} 0 &= (\Lambda + \rho, \Lambda + \rho) - (\mu + \rho, \mu + \rho) \\ &= ((\Lambda + \rho) - (\mu + \rho), (\Lambda + \rho) + (\mu + \rho)) \\ &= (\Lambda - \mu, \Lambda + (\mu + \rho) + \rho) \end{aligned}$$

which is larger than 0 if $\Lambda \neq \mu$ $\square$

Theorem 6.8. (Weyl's complete reducibility) If $\mathfrak{g}$ is semisimple over $\mathbb{C}$ then every finite dimensional $\mathfrak{g}$ -module V is a direct sum of irreducibles.

Proof. As V is completely irreducible as an $(\mathfrak{sl}_2)_\alpha$, $V = \bigoplus V_\lambda$ is a weight space decomposition. Consider $V^{\eta^+} = \{v \in V \mid \eta^+ v = 0\}$. This is non-zero by Engel. $[\mathfrak{t}, \eta^+] \subseteq \eta^+$, so $\mathfrak{t}$ acts on V^{η^+} , so $V^{\eta^+} = \bigoplus_{\mu \in P} V_\mu^{\eta^+}$. We claim that for any $v_\mu \in V_\mu^{\eta^+}$, the module $L = \mathcal{U}\mathfrak{g}v_\mu$ is irreducible. We know L is a highest weight module with highest weight μ . We must show there are no other singular vectors. But if λ is the weight of a singular vector $w \in L$, then $\lambda \leq \mu$ and $|\lambda + \rho| = |\mu + \rho|$ as the Casimir acts identically, but V is finite dimensional so $\lambda, \mu \in P^+$. So by the lemma, $\lambda = \mu$ and the claim is proven. Hence $V' = \mathcal{U}\mathfrak{g}V^{\eta^+}$ is completely reducible, and if $v_1 \dots v_r$ is a basis of highest weight vectors of V^{η^+} in weight spaces corresponding to $\lambda_1, \dots, \lambda_r$, then $V' = \bigoplus L(\lambda_i)$. We show $V/V' = 0$, at which point we are finished. If $N \neq 0$, then it has highest weight vectors. Let v_λ be one of weight $\lambda \in P^+$ and let $\tilde{v}_\lambda \in V_\lambda$ project to v_λ . As $v_\lambda \neq 0$ in N , we must have that some $E_i \tilde{v}_\lambda$ is non-zero, but $E_i v_\lambda = 0$ so $E_i \tilde{v}_\lambda \in V'$, so Ω acts on $E_i \tilde{v}_\lambda$ by $|\tilde{\lambda} + \rho|^2 - |\rho|^2$ for some $\tilde{\lambda} \in \{\lambda_1, \dots, \lambda_r\}$. On the other hand, the Casimir acts by $|\lambda + \rho|^2 - |\rho|^2$, so $|\lambda + \rho| = |\tilde{\lambda} + \rho|$, but $\tilde{\lambda} + \alpha_i$ is a weight of $L(\lambda)$, so $\tilde{\lambda} \leq \lambda + \alpha_i < \lambda$, but then the previous lemma gives a contradiction. $\square$

Lemma 6.9. *If $\Lambda \in \mathfrak{t}^*$, $M(\Lambda)$ the Verma module, then $\text{ch } M(\Lambda) = e^\Lambda / (\prod_{\alpha \in R^+} (1 - e^{-\alpha}))$*

Proof. If $R^+ = \{\beta_1, \dots, \beta_N\}$ then PBW gives a basis of $M(\Lambda)$ of the form $(\prod e_{-\beta_i}^{k_i})v_\Lambda$, and the weight of this is $\Lambda - \sum k_i \beta_i$, so $\dim M(\Lambda)_{\Lambda - \beta}$ is the number of ways of writing $\beta = \sum k_i \beta_i$, and this is exactly the claimed formula $\square$

Remark 6.9. We are just saying $S^*[x] = \frac{1}{1-e^\beta}$ if x has weight β , that $S^*(V + W) = S^*V \otimes S^*W$, that the character of the tensor product is the sum of the characters, and $\text{Ch } \mathcal{U}\mathfrak{g} = \text{Ch } S^*\mathfrak{g}$ for any $\mathfrak{g}$.

We write $\Delta = \prod_{\alpha \in R^+} (1 - e^{-\alpha})$, so $\text{Ch } M(\Lambda) = e^\Lambda / \Delta$. For $\mathfrak{sl}_2$ we know $\text{Ch } L(n) = \text{Ch } M(n) - \text{Ch } M(-n-2)$. We want to do this in general

Lemma 6.10. *Let $\Lambda \in P^+$. If $w \in W$, then $w(e^\rho \Delta) = \det(w)e^\rho \Delta$, where $\det w$ is the determinant as a linear map from $\mathfrak{t}^*$ to itself*

Proof. As W is generated by simple reflections, it suffices to check $s_i(e^\rho \Delta) = -e^\rho \Delta$ for s_i simple.

But

$$\begin{aligned} s_i(e^\rho \Delta) &= s_i(e^\rho (1 - e^{-\alpha_i}) \prod_{\alpha_i \neq \alpha \in R^+} (1 - e^{-\alpha})) \\ &= e^{\rho - \alpha_i} (1 - e^{\alpha_i}) \prod_{\alpha_i \neq \alpha \in R^+} (1 - e^{-\alpha}) \\ &= -e^\rho \Delta \end{aligned}$$

$\square$

Lemma 6.11.

1. *For any highest weight module $V(\Lambda)$ with highest weight Λ , there are coefficients $a_\lambda \in \mathbb{Z}_{\geq 0}$ for $\lambda \leq \Lambda$ such that $\text{Ch}(V(\Lambda)) = \sum_{\lambda \leq \Lambda, |\lambda + \rho| = |\Lambda + \rho|} a_\lambda \text{Ch } L(\lambda)$ and moreover $a_\Lambda = 1$*
2. *There are coefficients $b_\lambda \in \mathbb{Z}$ with $b_\Lambda = 1$ such that $\text{Ch } L(\Lambda) = \sum_{\lambda \leq \Lambda, |\lambda + \rho| = |\Lambda + \rho|} b_\lambda \text{Ch } V(\lambda)$.*

Proof. We write $B(\Lambda)$ for the set we are summing over. Recall that we showed this is finite if $\Lambda \in \mathbb{R}R$.

(1) implies (2): Extend the order of $B(\Lambda)$ to a total order. Apply (1) with $V(\Lambda) = M(\Lambda)$ to get an upper triangular system of equations relating $\text{Ch } M(\Lambda)$ and $\text{Ch } L(\lambda)$ with 1s on the diagonal, so we can invert it and get (2) (warning: $b_\Lambda \in \mathbb{Z}$ not necessarily non-negative)

For (1) we induct on $\sum_{\mu \in B(\Lambda)} \dim V(\Lambda)_\mu$, which is finite as weight spaces are finite dimensional and $B(\Lambda)$ is finite. If $V(\Lambda)$ is irreducible, the result is immediate. Otherwise there is $\mu \in B(\Lambda)$ with a singular vector $v_\mu \in V(\Lambda)_\mu$. Choose μ to be such that $\text{ht}(\Lambda - \mu)$ is maximal. Then $\mathcal{U}\mathfrak{g}v_\mu$ is a submodule with no singular vectors other than v_μ , so it is $L(\mu)$. Set $\overline{V(\Lambda)} = V(\Lambda)/L(\mu)$. This is a highest weight module with highest weight Λ with smaller sum of dimensions, so the result holds there, and $\text{Ch } V(\Lambda) = \text{Ch } \overline{V(\Lambda)} + \text{Ch } L(\mu)$ so we are done by induction. $\square$

We know $\text{Ch } L(\Lambda) = \sum_{\lambda \in B(\Lambda)} b_\lambda \text{Ch } M(\lambda) = \sum_{\lambda \in B(\Lambda)} b_\lambda e^\lambda / \Delta$. We also know that $w \text{Ch}(L(\Lambda)) = \text{Ch } L(\Lambda)$ and $w(e^\rho \Delta) = \det(w) e^\rho \Delta$, so $e^\rho \Delta \text{Ch}(L(\Lambda))$ is W anti-invariant. But $\sum_{\lambda \in B(\Lambda)} b_\lambda e^{\lambda+\rho} = \sum_{W\text{-orbits on } B(\Lambda)+\rho} b_\lambda \sum_{w \in W} \det(w) e^{w(\lambda+\rho)}$. Now fix $\Lambda \in P^+$, so $B(\Lambda) \subseteq \mathbb{R}R$, then $W(\lambda + \rho)$ intersects $P_{\mathbb{R}}^+ = \{x \in \mathbb{R}R \mid (x, \alpha_i^\vee) \geq 0\}$ in exactly one point, as the latter is a fundamental domain for the action of W . So we can take dominant weights as orbit representatives, but we know there is only one dominant weight in $B(\Lambda)$ by Lemma 6.7, which is Λ , so there is only one term in the sum, $b_\Lambda = 1$.

Theorem 6.12. (*Weyl Character Formula*) If $\Lambda \in P^+$,

$$\text{Ch } L(\Lambda) = \frac{\sum_{w \in W} \det(w) e^{w(\Lambda+\rho)}}{e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha})} = \sum \det w \text{Ch } M(w(\lambda + \rho) - \rho)$$

Proof. We just proved it $\square$

Example 6.9. If $\mathfrak{g} = \mathfrak{sl}_2$, $z = e^{\alpha/2}$, then $\mathbb{Z}[P] = \mathbb{Z}[z, z^{-1}]$ and $\rho = z$, so $\text{Ch } L(m \frac{\alpha}{2}) = \frac{z^{m+1} - z^{-m-1}}{z - z^{-1}}$ as before

Corollary 6.12.1. (*Weyl denominator formula*) $e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha}) = \sum \det(w) e^{w\rho}$

Proof. As $L(0) = \mathbb{C}$, $\text{Ch } L(0) = 1$ $\square$

Exercise 6.14. For $\mathfrak{g} = \mathfrak{sl}_n$ show the Weyl denominator is equivalent to the Vandemonde determinant

Remark 6.10. If $w = s_{i_1} \dots s_{i_r}$ with s_{i_k} simple reflections and r minimal, we call this a *reduced decomposition* of w , and call r its *length*, denoted $l(w)$. Note $\det w = (-1)^{l(w)}$

Exercise 6.15. $R^+ \cap (-wR^+) = \{\alpha_1, s_{i_1}\alpha_1, \dots, s_{i_1} \dots s_{i_{r-1}}\alpha_{i_r}\}$, so $l(w) = |R^+ \cap (-wR^+)| = l(w^{-1})$

Lemma 6.13. $P_{\mathbb{R}}^{0+} = \{\mu \in \mathbb{R}R \mid \langle \mu, \alpha_i^\vee \rangle > 0, \forall \alpha_i \in \Pi\}$ has closure a fundamental domain for W on $\mathbb{R}R$

Proof. If $\mu \in P_{\mathbb{R}}^{0+}$, we must show $w\mu \notin \overline{P_{\mathbb{R}}^{0+}}$. But $w\mu \in P_{\mathbb{R}}^{0+}$ if and only if $\langle w\mu, \alpha \rangle > 0$ for all $w \in R^+$, which is true if and only if $w^{-1}\alpha \in R^+$ for all $\alpha \in R^+$, which holds if and only if $l(w) = 0$.

If $\lambda \in \mathbb{R}R$, pick $\mu = w\lambda \in W\lambda$ maximal within this set with respect to the ordering. Then $s_i\mu = \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i$ by choice of μ , so $\langle \mu, \alpha_i^\vee \rangle \geq 0$ and $\mu \in \overline{P_{\mathbb{R}}^{0+}}$ $\square$

Corollary 6.13.1. (*Weyl Dimension Formula*) $\dim L(\Lambda) = \prod_{\alpha \in R^+} \frac{(\alpha, \Lambda + \rho)}{(\alpha, \rho)}$

Proof. We know $\text{Ch } L(\Lambda) = \sum \dim L(\Lambda)_\lambda e^\lambda$, so we want to set $e^\lambda \mapsto 1$ and all others to 0, but this gives 0/0 in the character formula. We get around this in the following way. We define a homomorphism $F_\mu: \mathbb{C}[P] \rightarrow \mathbb{C}(q)$ by $e^\lambda \mapsto q^{-(\lambda, \mu)}$ for each μ . We want $F_0(\text{Ch}(L(\Lambda)))$. Applying F_μ to the Weyl

denominator formula, we get $q^{(\rho, \mu)} \prod_{\alpha \in R^+} (1 - q^{(\alpha, \mu)}) = \sum (\det w) q^{-(w\rho, \mu)} = \sum (\det w) q^{-(\rho, w^{-1}\mu)}$ as the inner product is W -invariant. Apply F_μ to the Weyl character formula. We get

$$F_\mu(\text{Ch}(\Lambda)) = \frac{\sum_{w \in W} (\det w) q^{-(w(\Lambda + \rho), \mu)}}{q^{-(\rho, \mu)} \prod_{\alpha \in R^+} (1 - q^{(\alpha, \mu)})}$$

which is defined if $(\alpha, \mu) \neq 0$ for all $\alpha \in R^+$. We take $\mu = \rho$, as $(\rho, \alpha_i^\vee) = 1$ for all $\alpha_i = 1$, hence $(\rho, \alpha) > 0$ for all $\alpha \in R^+$. Then by the Weyl denominator identity,

$$\begin{aligned} F_\rho(\text{Ch } L(\Lambda)) &= \sum (\dim L(\Lambda)_\lambda) q^{-(\lambda, \rho)} \\ &= \sum q^{-(\rho, \Lambda + \rho)} \frac{\prod_{\alpha \in R^+} (1 - q^{(\alpha, \Lambda + \rho)})}{q^{-(\rho, \rho)} \prod_{\alpha \in R^+} (1 - q^{\alpha, \rho})} \end{aligned}$$

Now set $q = 1$ and use l'Hopital, and the result follows. $\square$

Example 6.10. $\mathfrak{g} = \mathfrak{sl}_3$, $R = \{\alpha, \beta, \alpha + \beta\}$ then $\rho = \omega_1 + \omega_2$. If $\Lambda = m_1\omega_1 + m_2\omega_2$, then we get

	α	β	$\alpha + \beta$	
$(\cdot, \Lambda + \rho)$	$m_1 + 1$	$m_2 + 1$	$m_1 + m_2 + 2$	$\text{so } \dim L(\Lambda) = \frac{1}{2}(m_1 + 1)(m_2 +$
$(\cdot, \rho)$	1	1	2	$1)(m_1 + m_2 + 2)$

Exercise 6.16. Compute the dimension of the irreducible representations for B_2, G_2

We can now use characters to understand representations as we did for $\mathfrak{sl}_2$. We define $\text{ct}: \mathbb{Z}[P] \rightarrow \mathbb{Z}$ to be the coefficient of 1, and we define to send $e^\lambda \mapsto e^{-\lambda}$. Define $\Delta = \prod_{\alpha \in R^+} (1 - e^{-\alpha})$, and define an inner product $(\cdot, \cdot): \mathbb{Z}[P] \times \mathbb{Z}[P] \rightarrow \mathbb{Z}$ by $(f, g) = \frac{1}{|W|} \text{ct}(f \bar{g} \Delta \bar{\Delta})$. Let $\chi_\lambda = \text{Ch } L(\Lambda)$. We claim $(\chi_\lambda, \chi_\mu) = \begin{cases} 1 & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases}$. To prove this,

$$(\chi_\lambda, \chi_\mu) = \frac{1}{|W|} \text{ct} \left(\sum_{x, w \in W} e^{w(\lambda + \rho) - \rho} \overline{e^{w(\mu + \rho) - \rho}} \det w \det x \right)$$

by Weyl character formula. But the constant term of this exists exactly when $x = w, \lambda = \mu$ as $\lambda + \rho, \mu + \rho \in P^+$ which is the interior of a fundamental domain. We can now decompose $L(\lambda) \otimes L(\mu) = \sum m_{\lambda\mu}^\nu L(\nu)$ with $m_{\lambda\mu}^\nu = \langle \chi_\mu \chi_\lambda, \chi_\nu \rangle$

Exercise 6.17. Compute $\text{Ch } L(n\rho)$ for $n \in \mathbb{Z}_{\geq 0}$

7 Principal $\mathfrak{sl}_2$

We define a new character $\rho^L \in \mathfrak{t}^*$ by $(\rho^L, \alpha_i) = 1$ for all $\alpha_i \in \Pi$. (contrast with $(\rho, \alpha_i^\vee) = 1$ for ρ)

Exercise 7.1. Show that this is the ρ of the dual system, so if R is simply laced, then $\rho = \rho^L$.

Now $(\rho^L, \alpha) = \text{ht } \alpha$.

Exercise 7.2. Show that $F_{\rho^L} = q^{-(\Lambda, \rho^L)} \prod_{\alpha \in R^+} \frac{(1 - q^{(\Lambda + \rho, \alpha^\vee)})}{1 - q^{(\rho, \alpha^\vee)}}$ by applying the Weyl denominator formula to the Lie algebra with root system $R^\vee$. Note $\frac{(\lambda + \rho, \alpha^\vee)}{(\rho, \alpha^\vee)} = \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}$. This is called the q -dimension of $L(\Lambda)$, $\dim_q L(\Lambda) \in \mathbb{N}[q, q^{-1}]$

Proposition 7.1. $\dim_q L(\Lambda)$ is a unimodal polynomial (increasing coefficients, then decreasing coefficients). In fact it is the character of an $\mathfrak{sl}_2$ module where all the strings have the same parity

Proof. Set $H = 2\nu^{-1}(\rho^L)$. Now $\dim_q L(\Lambda)$ is the weight space decomposition of H . Set $E = \sum_{\alpha_i \in \Pi} E_i$. Now write $H = \sum c_i h_i$ as a sum where $h_i = \alpha_i^\vee$, and define $F = \sum c_i F_i$. The proposition follows from the following exercise. $\square$

Exercise 7.3. H, F, E above are an $\mathfrak{sl}_2$, called the *principal* $\mathfrak{sl}_2$, and if $\Lambda - \gamma$ is a weight of $L(\Lambda)$ then $(\Lambda - \gamma, 2\rho^L) = (\Lambda, 2\rho^L) \bmod 2$

Exercise 7.4. Let $[n] = \frac{q^n - 1}{q - 1}$ and $[n]! = [n][n-1] \dots [1]$. Show that $\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}$ is unimodal, and show the same of $(1+q)(1+q^2) \dots (1+q^n)$ by applying the above to $\mathfrak{g} = \mathfrak{sl}_n$, $V = S^k \mathbb{C}^n, \bigwedge^k \mathbb{C}^n$ and the spin representation of $\mathfrak{so}_{2n+1}$.

Remark 7.1. When is $L(\Lambda) \cong L(\Lambda)^*$? The lowest weight of $L(\Lambda)$ is $w_0 \Lambda$ where w_0 is the unique element of maximal length in W , and the lowest weight of $L(\Lambda)^*$ is $-\Lambda$, so it happens when $-\Lambda = w_0 \Lambda$.

8 Crystals

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with simple roots $\alpha_1, \dots, \alpha_l$ and P the weight lattice

Definition 8.1. A *crystal* is a set B not including 0 and functions $\text{wt}: B \rightarrow P$, $\tilde{e}_i: B \rightarrow B \cup \{0\}$, $\tilde{f}_i: B \rightarrow B \cup \{0\}$ such that

1. If $\tilde{e}_i b \neq 0$ then $\text{wt}(\tilde{e}_i b) = \text{wt } b + \alpha_i$ and if $\tilde{f}_i b \neq 0$ then $\text{wt}(\tilde{f}_i b) = \text{wt } b - \alpha_i$.
2. If $b, b' \in B$, $\tilde{e}_i b = b'$ is equivalent to $b = \tilde{f}_i b'$

3. We draw a graph with vertices B and edges labelled i from b to $\tilde{e}_i b$ for all i , called the *crystal graph*. We define $\varepsilon_i(b) = \max\{n \mid \tilde{e}_i^n b \neq 0\}$ and $\phi_i(b) = \max\{n \mid \tilde{f}_i^n b \neq 0\}$ and require

$$\phi_i(b) - \varepsilon_i(b) = \langle \text{wt } b, \alpha_i^\vee \rangle$$

Define $B_\mu = \{b \in B \mid \text{wt}(b) = \mu\}$.

Example 8.1. For $\mathfrak{sl}_2$ we get $n \leftarrow n-2 \leftarrow n-4 \leftarrow \dots \leftarrow -n$ is a crystal with $\text{wt } a = a \frac{\alpha}{2}$. In this case $\phi_i(b) + \varepsilon_i(b) = n$ if the string is that of highest weight representation $L(n)$, and $\text{wt}(b) = n - 2\varepsilon(b) = 2\phi(b) - n$, so $\phi(b) - \varepsilon(b) = \langle \text{wt } b, \alpha^\vee \rangle$

Definition 8.2. If B_1, B_2 are crystals, we define $B_1 \otimes B_2$ to be $B_1 \times B_2$, $\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2)$. We define $\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \phi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i(b_2) & \phi_i(b_1) < \varepsilon_i(b_2) \end{cases}$, $\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \phi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i(b_2) & \phi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$

Exercise 8.1. Check this is a crystal, and tensor products are associative (it is not commutative)

Definition 8.3. $B^\vee$ (the *dual* of B) is B with arrows reversed.

Exercise 8.2. Tensor product and duals commute

Theorem 8.1. (Kashiwara) Let $L(\lambda)$ be an irreducible highest weight representation with highest weight $\lambda \in P^+$. Then

1. There is a crystal $B(\lambda)$ whose elements are in bijection with a basis of $L(\lambda)$, with $B(\lambda)_\mu$ parameterising a basis of $L(\lambda)_\mu$, so $\text{ch } L(\lambda) = \sum_{b \in B(\lambda)} e^{\text{wt } b}$
2. For any simple $\alpha_i \in \Pi$, the decomposition of $L(\lambda)$ as an $(\mathfrak{sl}_2)_i$ -module is given by the i -coloured strings
3. $B(\lambda) \otimes B(\mu)$ is the crystal for $L(\lambda) \otimes L(\mu)$

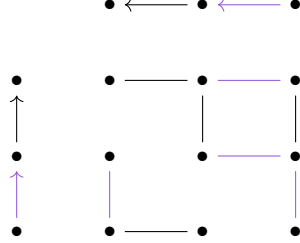
In particular, as every element of $L(\lambda)$ is obtained by acting on $V(\lambda)$ by various f_i , the graph $B(\lambda)$ is connected ignoring the colours.

Proof. No Proof □

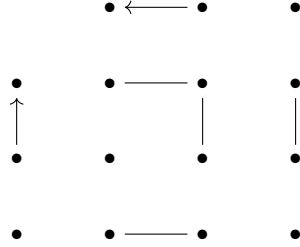
Example 8.2. If V is a representation of $\mathfrak{g}$ with 1d weight spaces, it is obvious what the crystal graph looks like. Take $\mathfrak{g} = \mathfrak{sl}_3$ and $V = L(\omega_1)$. This has crystal graph

$$\omega_1 \xleftarrow{e_1} \omega_1 - \alpha_1 \xleftarrow{e_2} \omega_1 - \alpha_1 - \alpha_2$$

We take the tensor product of this with itself. We get



where colours indicate labels. The colours make more sense if we consider them one at a time, for the black arrows:



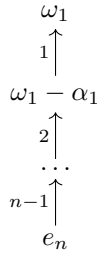
the standard product for $\mathfrak{sl}_2$.

Warning: there is no basis of $L(\lambda)$ with this properties for e_i, f_i instead of $\tilde{e}_i, \tilde{f}_i$. Instead we replace $\mathcal{U}\mathfrak{g}$ with a quantum group over $\mathbb{C}[q, q^{-1}]$.

Definition 8.4. A crystal is *integrable* if it is the crystal of a highest weight module.

Remark 8.1. Tensor products of integrable crystals are commutative, and there is an extra axiom which characterises integrability (due to Stembridge)

We work in $\mathfrak{g} = \mathfrak{sl}_n$, $L(\omega_1) = \mathbb{C}^n$ with crystal



We will use this to construct the crystal of every representation. Let $\lambda \in P^+$, $\lambda = \sum k_i \omega_i$ for $k_i \geq 0$. This means $L(\lambda)$ is a summand of $L(\omega_1)^{\otimes k_1} \otimes \cdots \otimes L(\omega_n)^{\otimes k_n}$ as it contains a highest weight vector of highest weight λ . But $L(\omega_i) = \bigwedge^i \mathbb{C}^n$, and this is a summand of $(\mathbb{C}^n)^{\otimes i}$, so every $L(\lambda)$ for $\mathfrak{sl}_n$ occurs as a summand of some $(\mathbb{C}^n)^{\otimes N}$. So the crystal of $L(\omega_1)$ determines all the crystals of $\mathfrak{sl}_n$. We write $1 \xleftarrow{1} 2 \xleftarrow{2} \cdots \xleftarrow{n-1} n$ for $B(\omega_1)$, and for $i < n$, let $b_i = 1 \otimes 2 \otimes \cdots \otimes i \in B(\omega_i)^{\otimes i}$. This will correspond to $e_1 \wedge \cdots \wedge e_i \in \bigwedge^i \mathbb{C}^n$.

Exercise 8.3. Show that b_i is a highest weight vector in $B(\omega_1)^{\otimes i}$ with weight $\omega_i = \sum_{j=1}^i \varepsilon_j$, so the connected component of the crystal $B(\omega_1)^{\otimes i}$ containing b_i is $B(\omega_i)$

The connected component of $B(\omega_i)$ consists precisely of $a_1 \otimes a_2 \otimes \cdots \otimes a_i$ with

the a_i strictly increasing. We write this element as $\begin{bmatrix} a_1 \\ \vdots \\ a_i \end{bmatrix}$ Now let $\lambda = \sum k_i \omega_i$ and

embed $B(\lambda) \hookrightarrow B(\omega_1)^{\otimes k_1} \otimes \cdots \otimes B(\omega_{n-1})^{\otimes k_{n-1}}$ by $b \mapsto b_1^{\otimes k_1} \otimes \cdots \otimes b_n^{\otimes k_n}$. We can represent elements of the right hand side as a sequence of column vectors, where entries strictly increase down the columns

Definition 8.5. A *semi-standard Young tableau* is such a sequence of column vectors with non-strictly decreasing rows

Exercise 8.4. The semi-standard Young tableaux are precisely the connected component of $B(\lambda)$

Describe the action of $\tilde{e}_i, \tilde{f}_i$ explicitly in terms of Young tableaux

We do similar for the other classical Lie algebras. We start with the standard representations

For B_n , the Lie algebra is $\mathfrak{so}_{2n+1}$ and representation $\mathbb{C}^{2n+1}$ we have crystal

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} 0 \xrightarrow{n} \bar{n} \xrightarrow{n-1} \cdots \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$$

For $D_n = \mathfrak{so}_{2n}$, we have

$$\begin{array}{ccccccc} & & & n & & & \\ & & n-1 & \swarrow & n & \searrow & \\ 1 & \xrightarrow{1} & 2 & \xrightarrow{2} & \cdots & \xrightarrow{n-1} & n \\ & & n & \searrow & \bar{n} & \swarrow & n-1 \\ & & & \bar{n} & & & \end{array}$$

and for $C_n = \mathfrak{sp}_{2n}$ we have

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} \bar{n} \xrightarrow{n-1} \cdots \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$$

Exercise 8.5. Check that these are the crystals of the standard representation. What subcategory of the (monoidal) category of representations do these generate?

We should not expect everything to be generated: If a group G is simply connected, the only representations we can get from V this way are those whose centre acts by powers of the character of V on the centre. In terms of Lie algebras, if G is simply connected, then $Z(G) = P/Q$ and $|P/Q| = \det C$ where C is the Cartan matrix. If λ is the highest weight of the standard representation, λ defines some $\bar{\lambda} \in P/Q$. We claim any irreducible summand of

$V_\lambda^{\otimes n}$ has highest weight in $\langle \bar{\lambda} \rangle$. So if P/Q is not cyclic, we don't get everything. $\frac{P/Q}{\left| \begin{array}{cccc} B_n & C_n & D_{2n} & D_{2n+1} \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} \end{array} \right|}$ Show this, and in each case compute the image of the standard representation. Write down a combinatorial set that gives the crystals generated by the standard representations. For type B_n , we have fundamental weights $\omega_1, \dots, \omega_n$. The representation $L(\omega_n)$ is called the *spin representation*. Using the Weyl dimension formula, $\dim L(\omega_n) = 2^n$. If we let $B = \{(i_1, \dots, i_n) \mid i_j = \pm 1\}$ and set $\text{wt}(i_1, \dots, i_n) = \frac{1}{2} \sum i_j e_j \in P$, $\tilde{e}_j(i_1, \dots, i_n) = \begin{cases} (i_1, \dots, i_{j-1}, 1, -1, \dots, i_n) & (i_j, i_{j+1}) = (-1, 1) \\ 0 & \text{otherwise} \end{cases}$ for $j < n$, and $\tilde{e}_n(i_1, \dots, i_n) = \begin{cases} (i_1, \dots, i_{n-1}, 1) & i_n = -1 \\ 0 & \text{otherwise} \end{cases}$

Exercise 8.6. Check this is a crystal

We claim this is the crystal of $L(\omega_n)$. We know $\mathfrak{gl}_n \hookrightarrow \mathfrak{so}_{2n+1}$ by $\text{diag}(A, 0, -JA^T J^{-1})$. Now $L(\omega_n)|_{\mathfrak{gl}_n} = \bigwedge^* \mathbb{C}^n$. Check that $B(\omega_n)|_{\mathfrak{gl}_n}$ is the right crystal. For D_n , the representation decomposes into half-spin representations instead of being irreducible.

9 Littlemann Paths

Set $P_{\mathbb{R}} = P \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 9.1. A *path* in $P_{\mathbb{R}}$ is a piecewise linear map $\pi: [0, 1] \rightarrow P_{\mathbb{R}}$ up to piecewise linear reparameterisation. We denote $\mathcal{P} = \{\text{paths } \pi: [0, 1] \rightarrow P_{\mathbb{R}} \mid \pi(0) = 0, \pi(1) \in P\}$ and define a crystal structure on $\mathcal{P}$. We set $\text{wt } \pi = \pi(1)$, we define $\tilde{e}_i(\pi)$ as follows: $h_i = \min \mathbb{Z} \cap \{\langle \pi(t), \alpha_i^\vee \rangle \mid 0 \leq t \leq 1\} \leq 0$. If $h_i < 0$, let t_1 be the first time h_i is achieved, and t_0 the last time before that that it hits $h_i + 1$. We reflect this part of the path through the hyperplane $\langle x, \alpha_i^\vee \rangle = h_i + 1$, and continuing via the translation of the rest of the path. We define $\tilde{e}_i(\pi)$ to be the resulting path. If $h_i = 0$, then $\tilde{e}_i(\pi) = 0$. If π is a path, let $\pi^\vee$ be the reverse path. Then $\tilde{f}_i(\pi) = (\tilde{e}_i(\pi^\vee))^\vee$.

Exercise 9.1. Show that this is a crystal, and $\varepsilon_i(\pi) = -h_i$

Example 9.1. If $\mathfrak{g} = \mathfrak{sl}_2$, and π is the linear path $0 \rightarrow -\frac{\alpha_i}{2}$, then $\tilde{e}_i(\pi) = 0 \rightarrow \frac{\alpha_i}{2}$. If we take the path $0 \rightarrow -\alpha_i$, the resulting path is the one from $0 \rightarrow -\frac{\alpha_i}{2} \rightarrow 0$

Define $\mathcal{P}^+$ to be those paths staying wholly within P^+ . It is clear that such paths have $\tilde{e}_i(\pi) = 0$. If $\pi \in \mathcal{P}^+$, we let B_π be the subcrystal of $\mathcal{P}$ generated by π

Theorem 9.1. (*Littlemann*)

1. If $\pi, \pi' \in \mathcal{P}^+$, then $B_\pi = B_{\pi'}$ if and only if $\pi(1) = \pi'(1)$

2. Let $\lambda = \pi(1)$. Then $B\pi = B(\lambda)$ up to unique isomorphism sending $\pi \rightarrow b_{\pi(1)}$
3. For the path $\pi(t) = \lambda t$, there is a combinatorial description of the paths in $B(\pi)$

Proof. No Proof □

Example 9.2. Compute the adjoint representation crystal for $\mathfrak{sl}_3$

Definition 9.2. If $\pi_1, \pi_2 \in \mathcal{P}$, denote by $\pi_1 * \pi_2$ be their concatenation.

Exercise 9.2. Concatenation of paths induces a morphism of crystals

Implicit in Littlewood's theorem is that we can apply the Weyl character formula to B_π . This is a purely combinatorial statement, and has a purely combinatorial proof: The idea is to build up $B(\lambda)$ and the character of $B(\lambda)$ inductively. (Demazure character formula) For $\lambda \in P^+$, define $L_w(\lambda)$ for $w \in W$ to be the η^+ submodule of $L(\lambda)$ generated by ${}^w v_\lambda$.

Theorem 9.2. $\text{Ch } L_w(\lambda) = D_w(e^\lambda)$ where $D_w = D_{s_{i_1}} \dots D_{s_{i_n}}$ (where $w = s_{i_1} \dots s_{i_n}$ where $D_{s_i}(f) = \frac{f - e^{-\alpha_i} s_i(f)}{1 - e^{-\alpha_i}}$)

Proof. No Proof □

This is easy, and implies what we want.