Logarithmic Algebraic Geometry

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1 Motivation

1.1 Euclidean Topology

If we have $\mathbb{A}^n = \operatorname{Spec}\mathbb{C}[x_1 \ldots, x_n]$ over $\mathbb{C}$ and X is a closed subscheme of $\mathbb{A}^n$ we have the closed and generic points. Looking at only the closed points we get X^{top} in $\mathbb{C}^n$, which we can now consider with the Euclidean topology.
This construction extends to functor on schemes locally of finite type over $\mathbb{C}$, which we denote $\mathrm{Sch}^{lft}_{\mathbb{C}}$.

1.2 Calabi-Yau manifolds

Let $X/\operatorname{Spec}\mathbb{C}$ be smooth proper and irreducible, of dimension d. We have the sheaf of Kähler differential forms, denoted Ω^1_X, a vector bundle of rank d. We define $\Omega^d_X = \bigwedge^d_{\mathcal{O}_X} \Omega^1_X$ called the canonical sheaf, which is a line bundle.

Definition 1.1. An *algebraic Calabi-Yau manifold* (ACYM) is a smooth proper irreducible scheme X over $\mathbb{C}$ with $\Omega^{\dim X}_X \cong \mathcal{O}_X$
X is *strict* if $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < d$.

Remark. We know $H^0(X, \mathcal{O}_X) = \mathbb{C}$ as X is irreducible and proper and $H^d(X, \mathcal{O}_X) = H^0(X, \Omega^d_X)^\vee = H^0(X, \mathcal{O}_X)^\vee \cong \mathbb{C}$ by Serre duality.

We now wish to construct ACYMs.
We begin with two facts:

- If $X \subseteq \mathbb{P}^{d+1}$ is a smooth hypersurface of degree δ then $\Omega^d_X = \mathcal{O}_X(-d - 2 + \delta)$.
- If X, Y are smooth proper irreducible over $\mathbb{C}$ with dimensions d_1, d_2 respectively, then so is $X \times Y$, and $\Omega^{d_1+d_2}_{X\times Y} = p^*_X\Omega^{d_1}_X \otimes p^*_Y\Omega^{d_2}_Y$. Therefore $X \times Y$ is a ACYM.

Example 1.1.

1. If $E = \{y^2z - x^3 - x^2z - z^3 = 0\}$, a smooth elliptic curve, then $\Omega^1_E \cong \mathcal{O}_E(-1 - 2 + 3) = \mathcal{O}_E$, so it is a strict ACYM. E^{top} is a torus.
2. $E \times E$ is an ACYM, and in fact is an abelian surface. It is not strict, as H^1 is $\mathbb{C}^2$
3. If $C = \{x^4 + y^2 + 10xyz^2 - z^4 = 0\}$, then $\Omega^1_C \cong \mathcal{O}_C(1)$ is not ACYM. The topological picture is a genus 2 surface.
4. If $Q = \{x^4 + y^4 + z^4 + w^4 = 0\}$, then $\Omega^2_Q \cong \mathcal{O}_Q$ so it is a ACYM, and is in fact strict.
5. If $F = \{x^5 + y^5 + z^5 + w^5 + v^5 = 0\}$, then $\Omega^3_F \cong \mathcal{O}_F$ so it is a ACYM, and is in fact strict. We have $h^1(F, \Omega^1_F) = 1$, $h^2(F, \Omega^1_F) = 101$, and using this we can compute the Betti numbers of F^{top} to be $b_2 = 1, b_3 = 204$.

Remark. Products of ACYMs can't be strict ACYMs by the Künneth decomposition.

There is a family of strict ACYMs in every dimension above 1 given by smooth hypersurfaces. In fact it turns out that if $X, Y \subseteq \mathbb{P}^{d+1}$ of the same degree, then $X^{top} \cong Y^{top}$. This raises the question of how many strict ACYMs of a fixed dimension not pairwise homeomorphic exist.
If $d = 1$ then all smooth elliptic curves are real tori, and hence are homeomorphic so it is 1.
If $d = 2$ then it turns out that all strict ACYMs of dimension 2 are homeomorphic.

1.3 Complete Intersections

Definition 1.2. $X = \{f_1 = \cdots = f_r = 0\}$ is a *complete intersection* if $\dim X = n - r$.

It is true that if $X \subseteq \mathbb{P}^n$ is a smooth complete intersection then $\Omega_X^d = \mathcal{O}_X(-n - 1 + \delta)$ where $\delta_{j=1}^r = \sum \deg f_j$. We can play the same example game as before. I'm not typing it out. We find another few non-homeomorphic (by computing Betti numbers, and possibly by checking cohomology rings) ACYMs of dimension 3 as complete intersections.

Lemma 1.1. *Up to homeomorphism, all CI ACYMs in $\mathbb{P}^n$ of dimension 3 comes from CI of degree*

- $3, 3$
- $4, 2$
- $3, 2, 2$
- $2, 2, 2, 2$

Proof. WLOG $\deg f_i \geq 2$. Then $\delta \geq 2r$, $n - r = 3$ and $n + 1 = \delta \geq 2r$ and then check the list. □

We can obtain more by replacing the ambiant space with something other than $\mathbb{P}^n$, eg products of projective spaces, weighted projective space, Gorenstein toric Fano variety and resolving singularities where we can. There are hundreds of millions fourfolds in Gorenstein toric Fano varieties. How can we tell if they are homeomorphic? We can compute the Hodge numbers (and therefore the Betti numbers), and this gives us 30000 distinct pairs (note this is sufficient but not necessary to distinguish them).

1.4 Degenerating Families

Can we construct infinitely many ACYMs of dimension 3 by another method?

Example 1.2. Let $E = \{y^2z - x^3 - x^2z - tz^3 = 0\} \subseteq \mathbb{P}^2 \times \mathbb{A}^1_t \to \mathbb{A}^1_t$. We call the composite map ϕ. This is flat and proper, and for general t, $\phi^{-1}(t)$ is a smooth elliptic curve, but at 0 we have a cusp so we get a pinched torus. If we normalise this we get a $\mathbb{P}^1$.

So we have a new strategy to construct ACYMs.

1. Construct a scheme V by methods like taking a $\mathbb{P}^1$ and identifying two points
2. Show that there is some flat, proper $\phi\colon X \to \mathbb{A}^1$ with $\phi^{-1}(0) \cong V$ and $\phi^{-1}(t)$ a strict ACYM for general t.

Construction: (Hashimoto-Sano, 2023) If $a \in \mathbb{Z}_{\geq 1}$, let $Y(a) = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $S(a) \subseteq Y(a)$ a general hypersurface of type (2, 2, 2). We have some smooth projective invertible threefold $W(a)$ with an embedding $S(a) \to W(a)$ depending on a. We then take the pushout of $W(a) \leftarrow S(a) \to Y(a)$, call it $X_0(a)$. This has two irreducible components $Y(a)$ and $W(a)$ with intersection $S(a)$. If $p \in S(a)$, then around p, $X_0(a)$ is etale locally of the form $\operatorname{Spec}(\mathbb{C}[x,y]/(xy)) \times \mathbb{A}^2$. $X_0(a) \to \operatorname{Spec}\mathbb{C}$ is proper but not projective.

1.5 Formal Deformations

How can we show the existence of $\phi\colon X \to \mathbb{A}^1_t$? For now, assume we have ϕ. Let $S_k = \operatorname{Spec}\mathbb{C}[t]/(t^{k+1})$. We have

$$\begin{array}{ccccccc} X_0(a) = X_0 & \longrightarrow & X_1 & \longrightarrow & \dots & \longrightarrow & X \\ \downarrow & & \downarrow & & & & \downarrow \\ S_0 & \longrightarrow & S_1 & \longrightarrow & \dots & \longrightarrow & \mathbb{A}^1_t \end{array}$$

where every map is a pullback of ϕ, and S_k maps to the point 0. This is called a *formal flat deformation*. Now assume we have a formal flat deformation but not ϕ. It is a fact that there is a flat and proper map (as analytic spaces) that works up to order c, at least on a small disk near 0.

Remark. This is false algebraically. Note that X_t may not be smooth.

Definition 1.3. $(\phi_k\colon X_k \to S_k)_{k\geq 0}$ is *formal smoothing* if there is an m such that $t^m \in \operatorname{Fitt}_d(\Omega^1_{X_m/S_m})$

It is a fact that for formal smoothings and the previous construction, X_t is smooth for small t.

1.6 Unobstructedness

How do we obtain our formal smoothing?
Inspiration:

Theorem 1.2. *(Special case of Bogomolov-Tian-Todorov) Let $\phi_0\colon X_0 \to \operatorname{Spec}\mathbb{C}$ be an ACYM. Fix $k \geq 0$ and suppose we have the deformation diagram for k. Then we can extend it to l.*

Proof. No Proof □

We use a similar theorem for X_0. The problems are as follows

1. Assume that the theorem holds for $X_0(a)$. Then we obtain a formal flat deformation, but it might not be a formal smoothing.
2. The theorem is presumably false for $X_0(a)$.

The solution is logarithmic deformation theory.
We have a category of log schemes Sch^{log} and a forgetful functor to Sch. For $S_0 = \operatorname{Spec}\mathbb{C}$ there is a canonical log scheme $S_0^\dagger$ with underlying scheme S_0, called the standard log point, and similarly for $S_k^\dagger$..
It is a fact that if $V = X_0(a)$ then we have a morphism of log schemes $X_0^\dagger \to S_0^\dagger$ which is logarithmically smooth. We have log differential forms forming a vector bundle of rank 3, and $X_0(a)$ is log CY, and we have a formal log smooth deformation, which is a formal smoothing. It is therefore a theorem that what we want to do works and gives us strict 3-ACYMs, with $b_2(X_t) = a + 3$, but is neither algebraic nor admits a Kähler metric.

2 Toric Geometry and Log Structures

2.1 Rational Polyhedral Cones and Toric Monoids

Let $N = \mathbb{Z}^{\oplus r}$ and set $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$

Definition 2.1. A *rational polyhedral cone (rpc)* is a subset $\sigma = \{\lambda_1 n_1 + \dots \lambda_k n_k \mid \lambda_j \geq 0\} \subseteq N_\mathbb{R}$ where $n_j \in N$. This is called *strongly convex* if $\sigma \cap -\sigma = 0$.

Example 2.1.
1. $N_\mathbb{R} = \mathbb{R}$, $\sigma_a = [0, \infty)$ is a strongly convex rpc.
2. $N_\mathbb{R} = \mathbb{R}$, $\sigma_b = \mathbb{R}$ is a not strongly convex rpc.
3. $\sigma_c = \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(2,1) \subseteq N_\mathbb{R} = \mathbb{R}^2$ is a strongly convex rpc.
4. $\sigma_d = \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(0,1) \subseteq N_\mathbb{R} = \mathbb{R}^2$ is a strongly convex rpc.
5. $\sigma_e = \mathbb{R}_{\geq 0}(1,1) \subseteq N_\mathbb{R} = \mathbb{R}^2$ is a strongly convex rpc.
6. $\sigma_f = \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}(0,1) \subseteq N_\mathbb{R} = \mathbb{R}^2$ is a not strongly convex rpc.

Definition 2.2. We define the *dual spaces* $M_\mathbb{Z} = \operatorname{Hom}_\mathbb{Z}(N, \mathbb{Z})$, $M_\mathbb{R} = \operatorname{Hom}_\mathbb{R}(N_\mathbb{R}, \mathbb{R})$ and the *dual cone* $\sigma^\vee = \{m \in M_\mathbb{R} \mid \forall n \in \sigma, \langle m, n\rangle \geq 0\} \subseteq M_\mathbb{R}$.

Remark. We have $(\sigma^\vee)^\vee = \sigma$ and $\sigma^\vee$ is a rpc.

Corollary 2.0.1. *If* $\sigma^\vee = \mathbb{R}_{\geq 0}\{m_1, \dots, m_l\}$ $\sigma = \bigcap_{j=1}^{l} \{n \in N_\mathbb{R} \mid \langle m_j, n\rangle \geq 0\}$

Proof. Trivial □

Example 2.2.
1. $\sigma_a^\vee = [0, \infty) \subseteq M_\mathbb{R} \cong \mathbb{R}$
2. $\sigma_b^\vee = 0$
3. $\sigma_c^\vee = \mathbb{R}_{\geq 0}(-1, 2) + \mathbb{R}_{\geq 0}(1, 0)$
4. $\sigma_d^\vee = \sigma_d$
5. $\sigma_e^\vee$ is the half plane in the direction of σ_e

6. $\sigma_f^\vee = [0,\infty) \times \{0\}$

Definition 2.3. For $\sigma \subseteq N_{\mathbb{R}}$ a rpc, we define the *dual monoid* to be $P_\sigma = \sigma^\vee \cap M_{\mathbb{Z}} \subseteq M_{\mathbb{Z}}$, the lattice points in the dual cone.

Remark. This is a easily seen to be a commutative monoid.

Lemma 2.1. *If σ is a rpc, then P_σ is finitely generated.*

Proof. We have $m_1, \ldots, m_l \in P_\sigma$ which span $\sigma^\vee$. Then any element not hit by the $\mathbb{N}$-span of these is in a cocompact set, so there are finitely many. □

Example 2.3. If you've got this far you're capable of finding generating sets for $\sigma_a, \ldots, \sigma_f$

Remark. The group of units functor is right adjoint to the inclusion of abelian groups in commutative monoids. The left adjoint to the inclusion is the Grothendiek group functor. Again, you can come up with examples.

Lemma 2.2. *If σ is a strongly convex rpc then $(P_\sigma)^{grp} \cong M$.*

Proof. We define $(P_\sigma)^{grp} \to M$ via $(p,d) \to p - d$. It is trivially injective. Let $m_1, \ldots, m_l \subseteq P_\sigma$ span $\sigma^\vee$ and define $u = \sum_{j=1}^{l} m_j$. If $n_1, \ldots n_k$ span σ then $\langle u, n_j \rangle \geq 0$ for each i. If we have equality for some i, then $\langle m_j, n_i \rangle = 0$ so $n_i \in \sigma \cap -\sigma = 0$, contradiction so $\langle u, n_i \rangle > 0$ for all i. Now let $m \in M$ and set $\mu_i = \langle m, n_i \rangle \in \mathbb{Z}$. Write $m' = m - \min\{\mu_i, 0\}u$. Then $(m', -\min\{\mu_i, 0\}u) \in P_\sigma$ maps to m (where $-\min\{\mu_i, 0\}u \in \sigma$ as shown before). □

Definition 2.4. $P \in \mathrm{CMon}$ is *integral* if $P \to P^{grp}$ is injective.

Remark. P_σ is integral.

Definition 2.5. $P \in \mathrm{CMon}$ is *fine* if it is finitely generated integral.
P is saturated if it is integral and for $m \in P^{grp}$, $km \in P$ implies $m \in P$. It is *sharp* if $P^\times = 0$.

Example 2.4. Examples are an exercise for the reader.

Lemma 2.3. *If σ is a rpc, then P_σ is saturated.*

Proof. We know P_σ is integral. Suppose $m \in P_\sigma^{grp}$ and $km \in P_\sigma$, so $km \in \sigma^\vee$, so $m \in \sigma^\vee$, so it is in P_σ. □

Definition 2.6. $P \in \mathrm{CMon}$ is *toric* if it is fine and saturated with $P_\sigma^{grp} = \mathbb{Z}^r$.

Remark. We know P_σ is toric. It is a fact that the converse holds.

2.2 Affine Toric Varieties

Let σ be a rpc with toric monoid P_σ.

Definition 2.7. We define the *monoid algebra* $\mathbb{C}[P_\sigma]$ where $p \cdot q = p + q$. The associated *affine toric variety* is $A_{P_\sigma} = \operatorname{Spec} \mathbb{C}[P_\sigma]$

Example 2.5.
1. For σ_a, $A_{P_{\sigma_a}} = \mathbb{A}^1_{\mathbb{C}}$.
2. For σ_d, $A_{P_{\sigma_d}} = \mathbb{A}^2_{\mathbb{C}}$.
3. For σ_b, $A_{P_{\sigma_b}} = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^2)$.

Remark. Let σ be a rpc, then $\mathbb{C}[P_\sigma]$ is normal, integral and of finite type of dimension $\operatorname{rk} P^{grp})\sigma$. It is a fact that the ring is Cohen-Macauley.

Definition 2.8. $F \subseteq P$ is a *face* if F is not empty and for $x, y \in P$, we have $x + y \in F \implies x, y \in F$

Example 2.6. The faces of $\mathbb{N}$ are 0 and $\mathbb{N}$.

If $F \subseteq P_\sigma$ is a face, we have a map $\iota^\sharp_{P,F} \colon \mathbb{C}[P] \to \mathbb{C}[F]$ which sends everything outside F to 0, and is the identity on F. This is a closed immersion with domain $V_{F,P}$.
We can also localise at the face, where we define $P_F = (P \times F)/\sim$ where $(p_1, f_1) \sim (p_2, f_2)$ if there is $f \in F$ with $p_1 + f_2 + f = p_2 + f_1 + f$, and we have a map of commutative monoids $P \to P_F$. This defines the ring theoretic localisation at the set of F. We can realise this in the toric case by sending $(p, f) \to p - f \in M$.
$\varepsilon_{P,F} \colon U_{P,F} = A_{P,F} \to A_P$ is an open immersion, and if F is a face of P, then $V^\circ_{P,F} = V_{P,F} \cap U_{P,F} \cong (\mathbb{A}^1 \setminus \{0\})^{\operatorname{rk} F^{grp}}$

Lemma 2.4. *If $P = P_\sigma$ then $A_P = \bigsqcup_{F \subseteq P} V^\circ_{P,F}$ is a disjoint decomposition.*

Proof. If $\mathfrak{p} \subseteq \mathbb{C}[P]$ is a prime ideal, set $F_\mathfrak{p} = \{f \mid f \notin \mathfrak{p}\}$ is a face, and $\mathfrak{p} \in V^\circ_{P,F_\mathfrak{p}}$. The converse is an exercise. □

2.3 Log Structures

If X is a topological space and $\mathcal{P}$ a sheaf of commutative monoids, we get a sheaf $W \to \Gamma(W, \mathcal{P})^\times$ a sheaf of abelian groups which we denote $\mathcal{P}^\times$.

Definition 2.9. If X is a scheme, we make the following definitions

1. A *pre-log structure* on X is a sheaf of commutative monoids $\mathcal{M}_X$ with a monoid homomorphism $\alpha_X \colon \mathcal{M}_X \to (\mathcal{O}_X, \cdot)$.
2. A *log structure* is a pre-log structure with $\alpha_X^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ an isomorphism.
3. A *pre-log scheme* $X^\dagger = (X, \mathcal{M}_X)$ is a scheme with a pre-log structure.

4. A *log scheme* $X^\dagger = (X, \mathcal{M}_X)$ is a scheme with a log structure.

5. A *morphism of (pre-)log schemes* $X^\dagger, Y^\dagger$ is a morphism of schemes $f\colon X \to Y$ with a monoid homomorphism $f^\flat\colon \mathcal{M}_Y \to f_*\mathcal{M}_X$ such that

$$\begin{array}{ccc} \mathcal{M}_Y & \xrightarrow{\alpha_Y} & \mathcal{O}_Y \\ \downarrow{\scriptstyle f^\flat} & & \downarrow{\scriptstyle f^\sharp} \\ f_*\mathcal{M}_X & \xrightarrow{f_*\alpha_X} & f_*\mathcal{O}_X \end{array}$$

   commutes.

6. We denote the category of (pre-)log schemes by $\mathrm{Sch}^{\mathrm{plog}}$ and $\mathrm{Sch}^{\log}$, and projection to the underlying scheme is a forgetful functor to Sch.

Example 2.7.

1. If X is a scheme and we set $\mathcal{M}_X = \mathcal{O}_X^\times$, we get the trivial log structure.

2. If X is a scheme and we set $\mathcal{M}_X = \mathcal{O}_X$, we get the empty log structure.

3. If $D \subseteq X$ is a closed subscheme, and we set $\Gamma(W, \mathcal{M}_{(X|D)}) = \{r \in \Gamma(W, \mathcal{O}_X) \mid r|_{W\setminus D} \text{ is invertible}\}$ is the compactifying or divisional log structure on X. We denote this $(X \mid D)$.

4. If $P = P_\sigma$ is toric, then we have $P \to \mathbb{C}[P]$ and we have the constant sheaf on A_P, denoted $\underline{P}_{A_P}$ and a map $\beta\colon \underline{P}_{A_P} \to \mathcal{O}_{A_P}$, which is a pre-log structure but never a log structure by a countability argument.

5. $D_P = A_P \setminus A_P^\circ$, the *toric boundary*. We set $A_p^\dagger = (A_P \mid D_P)$. There is a morphism of pre-log schemes $A_P^\dagger \to (A_P, \underline{P}_{A_P})$. We will see later this is a "logification"

6. If $X = \mathbb{A}^1_{\mathbb{C}}$ and $D = \{0\}$, then $\Gamma(X, \mathcal{M}_{(X|D)} = \mathbb{C}^\times \times \{t^m \mid m \geq 0\}$

Lemma 2.5. *If $X^\dagger \in \mathrm{Sch}^{log}$ then $\alpha_X^{-1}(\mathcal{O}_X^\times) = \mathcal{M}_X^\times$.*

Proof. Definition spam. □

2.4 Integrality and Saturatedness

Definition 2.10. $(X, \mathcal{M}_X) \in \mathrm{Sch}^{\log}$ is *m-integral (resp. saturated)* if for all $W \subseteq X$, $\Gamma(W, \mathcal{M}_X)$ is integral (resp. saturated).

Example 2.8. The trivial log structure is m-integral and saturated. If X is integral with $D \subseteq X$ then $(X \mid D)$ is integral. $A_P^\dagger$ is m-integral and saturated. For $X = \operatorname{Spec} \mathbb{C}[x, y]/(x^2 = y^3)$ and $D = 0$, $(X \mid D)$ is not saturated.

Lemma 2.6. *Suppose we have a span $P_2 \leftarrow Q \to P_1$. Then we have an equivalence relation on $P_1 \times P_2$ where $(p_1, p_2) \sim (r_1, r_2)$ if there are q, q' such that $p_1 + q = r_1 + q'$ and $p_2 + q' = r_2 + q$, and the monoid structure descends to the quotient. If one of the three monoids is a group, this is the pushout of the span.*

Proof. No Proof □

Remark. CMon has all colimits, but this is not trivial to prove. It has all limits as well and this is trivial to prove.

Proposition 2.1. *Suppose X is a topological space. Then* $\mathrm{Shv}(X, \mathrm{CMon})$ *is complete and cocomplete.*

Proof. Limits are constructed pointwise. For colimits, we construct pointwise and sheafify. □

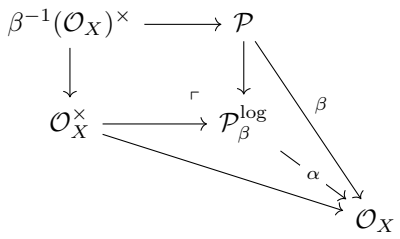
2.5 Logification

We wish to construct (non-trivial, non-empty) log structures. We suppose we have $X \in \mathrm{Sch}$ and a pre-log structure $\beta\colon \mathcal{P} \to \mathcal{O}_X$.

Example 2.9. Suppose $P \in \mathrm{CMon}$ and we have $\beta\colon P \to \Gamma(X, \mathcal{O}_X)$. Then we get such a $\beta\colon \underline{P}_X \to \mathcal{O}_X$.

Proposition 2.2. *Suppose we have $X \in \mathrm{Sch}$ and a pre-log structure $\beta\colon \mathcal{P} \to \mathcal{O}_X$. Then there is a factorisation $\mathcal{P} \to \mathcal{P}^{\log}_\beta \xrightarrow{\alpha} \mathcal{O}_X$ such that α is a log structure, and if $\mathcal{P}$ factors through a log structure $\mathcal{M}$, then $\mathcal{P}^{\log}_\beta$ does uniquely.*

Proof. We have a diagram



and this is how we construct the log structure. You can confirm this is the desired structure yourself. □

Definition 2.11. The log structure constructed above is the *logification* of β.

Example 2.10.

1. Let $S_0 = \operatorname{Spec} \mathbb{C}$ and $\beta\colon \underline{\mathbb{N}}_{S_0} \to \mathcal{O}_{S_0}$ given by $\beta(1) = 0$. Then the logification is $\Gamma(S_0, (\underline{\mathbb{N}}_{S_0})^{\log}) = \mathbb{N} \oplus \mathbb{C}^\times$. This is not a divisorial log structure; there are only two of those so you can check yourself!

2. Let $\mathcal{P} = \mathcal{P}_\sigma$ be a toric monoid, ten we have $\underline{P}_{A_\mathcal{P}} \to (\underline{P}_{A_\mathcal{P}})^{\log} \to \mathcal{O}_{A_\mathcal{P}}$, and in fact this is isomorphic to $\mathcal{M}_{(A_\mathcal{P} | D_\mathcal{P})}$.

3. Let $(X, \mathcal{M}_X) \in \mathrm{Sch}^{\log}$. Sometimes this is locally of the form $(\underline{P}_U)^{\log}$. Eg for $(\mathbb{P}^1 \mid \{0, \infty\})$ and the open cover given by removing 0 (resp ∞) $\mathcal{M}_{U_i}$ is locally $\underline{\mathbb{N}}^{\log}_{U_i}$ by $1 \mapsto x$.

4. If $X = \operatorname{Spec} \mathbb{C}[t,x,y,v]/(xy - tv)$ and $D = \{t = 0\}$ and we consider the divisorial log structure $(X, \mathcal{M}_X) = (X \mid D)$, this is not locally the logification of a constant sheaf (no proof).

2.6 Spectra of pre-log rings

Definition 2.12. A *pre-log ring* is a monoid homomorphism $\beta\colon P \to (R, \cdot)$. A *log ring* is a pre-log ring such that $\beta^{-1}(R^\times) \to R^\times$ is an isomorphism. Homomorphisms of pre-log rings are morphisms of arrows. The map $P \to Q$ is $\phi_\flat$ and the map $A \to R$ is $\phi_\sharp$.

Example 2.11. $(A_P \mid D_P) \cong (\operatorname{Spec} P \to \mathbb{C}[P])$ where D_P is the toric boundary.

Remark. The global sections of a log scheme form a log ring, and hence a pre-log ring. We wish to form an adjoint to this global sections functor.

For a pre-log ring $P \to R$, we have a map $\underline{P}_{\operatorname{Spec} R} \to \mathcal{O}_{\operatorname{Spec} R}$, and we take the logification $(\underline{P}_{\operatorname{Spec} R})^{\log} \to \mathcal{O}_{\operatorname{Spec} R}$.

Example 2.12. If Q is a sharp toric monoid, then $S_0^\dagger = \operatorname{Spec}(Q \to \mathbb{C})$ is a log point. For $\mathbb{N}$ this is the *standard log point* and for 0 it is the *trivial log point*. There is a map from the standard log point to the trivial one, but they have different log structures.

Remark. The following are facts:

1. Γ, the global sections map, is a functor $\operatorname{Sch}^{\log}$ to pre-log rings.
2. Spec from pre-log rings to $\operatorname{Sch}^{\log}$ is also a functor.
3. These functors are adjoint, $\Gamma \dashv \operatorname{Spec}$.

Example 2.13. If we consider $\mathbb{N} \to \mathbb{C}$ as a pre-log ring, then $\operatorname{Aut}_{0 \to \mathbb{C}}(\mathbb{N} \to \mathbb{C}) = 1$. However $\operatorname{Aut}_{(\mathbb{C}^\times \to \mathbb{C})}(\mathbb{N} \oplus \mathbb{C}^\times \to \mathbb{C}) \cong \mathbb{C}^\times$ (by $\phi_\flat(n, \lambda) = (n, \lambda\mu^n)$), so Spec is not fully faithful.

Remark. Warning: In scheme theory, Spec is fully faithful, but the same does not hold for log-scheme theory (being a log ring doesn't save you from the above example!), as $(P \to R) \to \Gamma \operatorname{Spec}(P \to R)$ is not fully faithful (see example below).

Example 2.14. If $R = \mathbb{C}[t]$ and $P = \mathbb{N} \oplus \mathbb{C}^\times$ with $\beta\colon P \to R$ given by $(n, \lambda) \to \lambda t^n (t-1)^n$, then $\operatorname{Spec}(P \to R) \cong (\mathbb{A}^1 \mid \{0, 1\})$. We have $\mathbb{N} \oplus \mathbb{C}^\times \to t \in \Gamma(X, \mathcal{M}_X) \to \Gamma(X, \mathcal{O}_X)$, but the first map is not an isomorphism (t is not in the image).

2.7 The Ghost Sheaf

Definition 2.13. Let $P \in \mathrm{CMon}$ and F a face of P. We define the quotient P/F by $p_1 \sim p_2$ if there are $f_1, f_2 \in F$ such that $p_1 + f_1 = p_2 + f_2$. If $F = P^\times$, then $\bar{P} = P/P^\times$ is the *sharpening*.

Definition 2.14. If $(X, \mathcal{M}_X) \in \mathrm{Sch}^{\log}$, then we define $\bar{\mathcal{M}}_X = \mathcal{M}_X/\mathcal{M}_X^\times$, the sheafification of the pointwise sharpening, as the *ghost sheaf*.

Example 2.15. If $\mathcal{M}_X = \mathcal{O}_X^\times$, the ghost sheaf is 0, and the converse holds.

Proposition 2.3. *Suppose $P = P_\sigma$ is a toric monoid, and $(X, \mathcal{M}_X) = A_P^\dagger = \mathrm{Spec}(P \to \mathbb{C}[P])$. Suppose $x \in V_{P,F}^\circ$ and we have a map $P \to \bar{\mathcal{M}}_{X,x}$. Then $P/F \cong \mathcal{M}_{X,x}$.*

Proof. $x \in V_{P,F}^\circ$ is equivalent to $F = \{p \in P \mid p \notin \mathfrak{p}\}$ where $\mathfrak{p}$ is the prime ideal of x. We can use the formula for the pushout and sharpening to conclude. □

Lemma 2.7. *If $f\colon (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a morphism of log schemes, then we have an induced map $f^+\colon \bar{\mathcal{M}}_Y \to f_*\bar{\mathcal{M}}_X$.*

Proof. Trivial □

2.8 Strict Morphisms

Suppose $f\colon X \to Y$ is a morphism of schemes and $\mathcal{M}_Y \to \mathcal{O}_Y$ a log structure on Y. Then $f^{-1}\mathcal{M}_Y \to \mathcal{O}_X$ is a pre-log structure on X and we can logify this to get $f_{\log}^*\mathcal{M}_Y = (f^{-1}\mathcal{M}_Y)^{\log}$.

Definition 2.15. $f\colon (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is *strict* if $f_{\log}^*\mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism.

Remark.

1. Composition of strict morphisms is strict.
2. If f is strict, then f^+ is an isomorphism.
3. If $\mathcal{M}_X$ is integral and f^+ is an isomorphism, then f is strict.
4. If ϕ is a map of log rings, the associated map of spectra is strict.

2.9 Fibre Products

Proposition 2.4. *The pullback*

$$\begin{array}{ccc} (W, \mathcal{M}_W) & \longrightarrow & (X, \mathcal{M}_X) \\ \downarrow & \lrcorner & \downarrow \\ (Y, \mathcal{M}_Y) & \longrightarrow & (Z, \mathcal{M}_Z) \end{array}$$

exists.

Proof. Let $W = X \times_Z Y$ as underlying schemes. We get a diagram

$$\begin{array}{ccc} r^*_{\log}\mathcal{M}_Z & \longrightarrow & g^*_{\log}\mathcal{M}_Z \\ \downarrow & & \downarrow \\ f^*_{\log}\mathcal{M}_Z & \longrightarrow & P_W \\ & & \searrow \\ & & P_W^{\log} = \mathcal{M}_W \end{array}$$

where $r\colon W \to Z$. You may verify this works. □

Lemma 2.8. *We have a diagram.*

$$\begin{array}{ccc} (X, \mathcal{M}_X) & \xrightarrow{f} & (Y, \mathcal{M}_Y) \\ \downarrow & & \downarrow \\ (X, \mathcal{O}_X^\times) & \longrightarrow & (Y, \mathcal{O}_Y^\times) \end{array}$$

f is strict if and only if this is a pullback.

Proof. Trivial □

Corollary 2.8.1. *If we have $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y) \leftarrow (Z, \mathcal{M}_Z)$ and a map of schemes (not log schemes) $X \to Z$ such that the underlying triangle of schemes commutes, then $X \to Y$ is strict if and only if there is a unique morphism of log schemes $X \to Z$ such that the diagram commutes.*

Proof. Trivial □

Remark. Spec sends pushouts of pre-log rings to fibre products of log schemes, and we can use this to compute the latter as with schemes.