

Motivic homotopy theory

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1 Classification of Vector Bundles

Let X be a reasonable space (manifold, CW complex, etc). Then

$$\mathrm{Vect}_r(X) \rightarrow \mathrm{Vect}_r(X \times I)$$

of rank r topological vector bundles on X to those on $X \times I$ is an isomorphism. As a consequence, pullbacks of a vector bundle by homotopic maps are isomorphic.

Let Gr_n be the Grassmannian of n -planes in $\mathbb{C}^\infty$. There is a tautological vector bundle E_n over Gr_n and the map $[X, \mathrm{Gr}_n] \rightarrow \mathrm{Vect}_n(X)$ defined by $f \mapsto f^*E_n$ is a bijection.

We know the low homotopy groups of Gr_n

Theorem 1.1. (*Bott*)

$$\pi_i(\mathrm{Gr}_n) \cong \begin{cases} \mathbb{Z} & i \leq 2n, \text{ even} \\ 0 & i \leq 2n, \text{ odd} \\ \mathbb{Z}/n! & i = 2n + 1 \end{cases}$$

There is the Postnikov tower

$$\begin{array}{ccc} K(k, \pi_k \mathrm{Gr}_n) & \longrightarrow & \tau_{\leq k} \mathrm{Gr}_n \\ & & \downarrow \\ & & \tau_{\leq k-1} \mathrm{Gr}_n \xrightarrow{\kappa_{k-1}} K(k, \pi_{k+1} \mathrm{Gr}_n) \\ & & \downarrow \\ & & \dots \end{array}$$

and obstruction theory says that a map lifts from $\tau_{\leq k-1} \mathrm{Gr}_n$ to $\tau_{\leq k} \mathrm{Gr}_n$ iff $f^*(\kappa_{k-1}) \in H^{k+1}(X; \pi_k \mathrm{Gr}_n)$ vanishes.

Example 1.1. $\tau_{\leq 4} \mathrm{Gr}_2 \cong K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4)$ If X is a 4 dimensional CW complex, then $[X, \mathrm{Gr}_2] \cong [X, \tau_{\leq 4} \mathrm{Gr}_2] \cong H^2(X, \mathbb{Z}) \times H^4(X, \mathbb{Z})$ via $E \mapsto (c_1(E), c_2(E))$

2 Algebraic Vector Bundles

If R is a commutative ring, then algebraic vector bundles on $\text{Spec } R$ correspond to finitely generated projective R modules.

Theorem 2.1. (*Quillen, Suslin 1976*) *If k is a field, every finitely generated projective module on $k[x_1, \dots, x_n]$ is free.*

$$\text{I.e. } \text{Vect}_r(\text{Spec } k) \cong \text{Vect}_r(\mathbb{A}_k^n)$$

Conjecture 2.1. (*Bass-Quillen*) *If X is a regular Noetherian affine scheme of finite Krull dimension then $\text{Vect}_r(X) \rightarrow \text{Vect}_r(X \times \mathbb{A}^n)$ is a bijection*

This has been proven in many cases. For instance, if X is of finite type over a field (Lindel)

3 $\mathbb{A}^1$ -invariant motivic spaces

Aim: construct a homotopy theory from affine varieties such that $\mathbb{A}^1 \simeq *$. Further evidence why this is reasonable is because there are many cohomology theories that satisfy $\mathbb{A}^1$ -invariance:

1. Algebraic K theory: $K_i(X \times \mathbb{A}^1) \rightarrow K_i(X)$ is an equivalence if X is smooth over k
2. Etale cohomology
3. Chow groups.

Definition 3.1. Let k be a field and Sm_k be the category of (finitely presented) smooth k schemes.

We modify this into the category we want.

Step 1: Add homotopy colimits (via $\mathcal{P}(\text{Sm}_k)$, the presheaves on Sm_k . This comes with a Yoneda embedding $\text{Sm}_k \rightarrow \mathcal{P}(\text{Sm}_k)$, and is freely generated by Sm_k under colimits);

Step 2: Remember some covers: we want

$$\begin{array}{ccc} U \times_X V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \hookrightarrow & X \end{array} \quad U \cup V = X$$

to become a pushout square again.

Definition 3.2. A cartesian square

$$\begin{array}{ccc} V & \hookrightarrow & Y \\ \downarrow & & \downarrow f \\ U & \xrightarrow{i} & X \end{array}$$

is an *elementary (Nisnevich) square* if

- i is an open immersion
- f is etale
- $f: (Y \setminus V)_{red} \rightarrow (X \setminus U)_{red}$ is an isomorphism

Example 3.1.

$$\begin{array}{ccc}
 P & \hookrightarrow & \mathbb{A}^1 \setminus \{0, 1\} \\
 \downarrow & & \downarrow x \mapsto x^2 \\
 \mathbb{A}^1 \setminus \{1\} & \xhookrightarrow{i} & \mathbb{A}^1
 \end{array}$$

is Nisnevich

Definition 3.3. If $U \in \mathbf{Sm}_k$, then a collection of maps $\{V_i \rightarrow U\}$ is a *Nisnevich cover* if each of the maps $V_i \rightarrow U$ is etale and for each $x \in U$, there is an i and a point $y \in V_i$ such that $f_i(y) = x$ and $f^*: k(x) \rightarrow k(y)$ is an isomorphism.

$p: X \rightarrow Y$ is *etale* if it locally-on-source looks like $\mathrm{Spec}(A[t]/p(t))_{(g)} \rightarrow \mathrm{Spec} A$ where if $g \neq 0$ then $p'(t) \neq 0$. An *etale cover* is a jointly surjective collection of etale maps

If $k \subseteq L$ is a seperable field extension, $\mathrm{Spec} L \rightarrow \mathrm{Spec} k$ is not a Nisnevich cover.

Definition 3.4. A *Nisnevich sheaf* is a functor $F: \mathbf{Sm}_k^{\mathrm{op}} \rightarrow \mathbf{Spc}$ such that F sends elementary squares to pullback squares and $F(\emptyset) = *$

We denote $\mathcal{P}_{\mathrm{Nis}}(\mathbf{Sm}_k)$ for the subcategory of $\mathcal{P}(\mathbf{Sm}_k)$ on Nisnevich sheaves.

Remark 3.1. The Yoneda embedding embeds in $\mathcal{P}_{\mathrm{Nis}}(\mathbf{Sm}_k)$, and elementary squares become pushouts in $\mathcal{P}_{\mathrm{Nis}}(\mathbf{Sm}_k)$

Step 3: Enforce $\mathbb{A}^1$ invariance.

Definition 3.5. A functor $F: \mathbf{Sm}_k^{\mathrm{op}} \rightarrow \mathbf{Spc}$ is $\mathbb{A}^1$ -invariant if for all $X \in \mathbf{Sm}_k$, the map $F(X) \rightarrow F(X \times \mathbb{A}^1)$ is an equivalence.

We write $\mathcal{P}_{\mathbb{A}^1}(\mathbf{Sm}_k) \subseteq \mathcal{P}(\mathbf{Sm}_k)$ for the full subset on $\mathbb{A}^1$ -invariant presheaves

Definition 3.6. (Movel-Voevodsky) The ∞ -category of $(\mathbb{A}^1)$ -motivic spaces over k is $\mathbf{Spc}_k = \mathcal{P}_{\mathbb{A}^1}(\mathbf{Sm}_k) \cap \mathcal{P}_{\mathrm{Nis}}(\mathbf{Sm}_k)$

The inclusion $\mathbf{Spc}_k \rightarrow \mathcal{P}(\mathbf{Sm}_k)$ has a left adjoint $L_{\mathrm{mot}}: \mathcal{P}(\mathbf{Sm}_k) \rightarrow \mathbf{Spc}_k$. In particular every $X \in \mathbf{Sm}_k$ defines an object in $\mathbf{Spc}_k$ by applying $\mathbf{Sm}_k \rightarrow \mathcal{P}(\mathbf{Sm}_k) \rightarrow \mathbf{Spc}_k$.

We can also consider $\mathbf{Spc}_{k,*} = (\mathbf{Spc}_k)_{\mathrm{Spec} k/}$. As usual we have a pointed-unpointed adjunction.

4 Features of Spc_k

As desired, $\mathbb{A}^1 \cong *$ in Spc_k by Yoneda.

There are two families of spheres in our new category.

Definition 4.1. Let $S_t^1 = \Sigma(S^0) = \Sigma(\mathrm{Spec} k \sqcup \mathrm{Spec} k)$, where Σ is formed as usual for an ∞ -category. This is the *topological circle*.

Let $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$. This is the *Tate circle*.

In general we write $S^{p,q} = S_t^{p-q} \wedge \mathbb{G}_m^q$. q is called the *motivic weight*.

Example 4.1.

$$\begin{array}{ccc} \mathbb{A}^1 \setminus \{0\} & \hookrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow x \mapsto x^2 \\ \mathbb{A}^1 & \xhookrightarrow{i} & \mathbb{P}^1 \end{array}$$

is a pushout in Spc_k so $\mathbb{P}^1$ is the suspension of $\mathbb{G}_m$, so is equal to $S^{2,1}$.

We also have $\mathbb{A}^n \setminus \{0\} \cong S^{2n-1,n}$

Theorem 4.1. (*Asok-Hoyois-Wendt*) Let X be a smooth affine k -scheme and $r \geq 0$. Then

$$\mathrm{Vect}_r(X) \simeq [X, \mathrm{Gr}_r]_{\mathbb{A}^1}$$

Where we will define this all later.