

Non-Commutative Noetherian Rings

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1 Generalities

In this course, all rings are unital and associative, but not necessarily commutative.

Example 1.1.

1. All commutative rings are rings.
2. The zero ring is a ring.
3. If k is a commutative ring, then so is $\text{Mat}_n(k)$ under the usual matrix operations, and is only commutative for $n < 2$.

Given a ring R we can define R^{op} with the same underlying set and addition, and multiplication $a * b = ba$. R is commutative if and only if $R = R^{op}$. The *units* of R , denoted $R^\times$ are those elements with two sided multiplicative inverses. This forms a group under multiplication

Example 1.2. $\text{Mat}_n(k)^\times = GL_n(k)$ and $k[x]^\times = k^\times$ if k is an integral domain

The *centre* of R is $Z(R) = \{z \in R \mid rz = zr, \forall r \in R\}$. This is a subring of R .

Example 1.3. $Z(\text{Mat}_n(k)) = kI_n$

A (*two-sided*) *ideal* I of a ring R is an additive subgroup of R such that $axb \in I$ for all $x \in I$ and $a, b \in R$. We write $I \trianglelefteq R$. Given an ideal I , R/I is a ring in the usual way.

A *ring homomorphism* $\alpha: R \rightarrow S$ is a function which is an abelian group homomorphism, $\alpha(ab) = \alpha(a)\alpha(b)$ and $\alpha(1) = 1$

Exercise 1.1. Show that $\alpha(R^\times) = S^\times$

The *kernel* of α is $\alpha^{-1}(\{0\})$ and the *image* of α is $\alpha(R)$. If k is a commutative ring, a *k-algebra* R is a ring equipped with a ring homomorphism $k \rightarrow Z(R)$

Example 1.4. $\text{Mat}_n(k)$ is a k -algebra via $a \mapsto aI_n$

A *k-algebra homomorphism* $R \rightarrow S$ is a ring homomorphism such that $\alpha \circ \iota_R = \iota_S$.

Remark 1.1. Every ring is uniquely a $\mathbb{Z}$ algebra and every ring homomorphism is a $\mathbb{Z}$ -algebra homomorphism

Proposition 1.1. *If $\alpha: R \rightarrow S$ is a homomorphism, then $\ker \alpha \trianglelefteq R$, $\text{im } \alpha$ is a subring of S , and $\bar{\alpha}: R/\ker \alpha \rightarrow S$ is an isomorphism*

Proof. Trivial □

Given a ring R , a (left) R -module M is an abelian group with a $\mathbb{Z}$ -bilinear operation $R \times M \rightarrow M$ denoted as am such that $(ab)m = a(bm)$ for all $a, b \in R$ and $m \in M$, and $1m = m$ for all $m \in M$. A right R -module can be defined similarly for $M \times R \rightarrow M$. We will assume R -module without qualification is a left R -module.

If M, N are R -modules, an R -module homomorphism is a group homomorphism $\phi: M \rightarrow N$ such that $\phi(am) = a\phi(m)$ for all $m \in M, a \in R$. We write $\text{Hom}_R(M, N)$ for the abelian group of all such homomorphisms under pointwise addition.

Remark 1.2. Warning: if R is not commutative, then $\text{Hom}_R(M, N)$ is not necessarily naturally an R -module, it is only a $Z(R)$ -module in general via $(a \cdot \phi)(m) = a \cdot \phi(m)$ (as outside the centre, the right hand side need not be an R -module homomorphism). It follows that if R is a k -algebra, $\text{Hom}_R(M, N)$ is always a k -module

For every ring R and set X , the free R module on X , denoted RX , is defined as the set of finitely supported functions $X \rightarrow R$ (ie, all but finitely many elements of X map to 0) under pointwise operations. When R is a field, this is the vector space with basis corresponding to X , $\delta_x = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$

Lemma 1.1. *For any R -module N , restriction along $X \rightarrow RX$ (by $x \mapsto \delta_x$) gives a bijection $\text{Hom}_R(RX, N) \rightarrow N^X$*

Proof. Trivial □

Therefore RX together with $\iota: X \rightarrow RX$ is determined up to unique isomorphism by this lemma (by usual abstract nonsense). For fields this states that if two vector spaces have bases indexed by the same set, there is a canonical bijection between them induced by the bijection of bases.

2 Some examples of non-commutative rings

Example 2.1.

1. If M is an abelian group, $\text{End}(M)$ (the set of endomorphisms of M) is a ring under pointwise addition and composition.

2. More generally, M is an R -module for a k -algebra R , then $\text{End}_R(M)$ is a k -algebra with the same operations, with units $\text{Aut}_R(M)$
3. If $M = \mathbb{Z}^n$, then $\text{End}(M) = \text{Mat}_n(\mathbb{Z})$. If k is a field, $V = k^n$, then $\text{End}_k(V) = \text{Mat}_n(k)$ and the automorphism group is $\text{GL}_n(k)$

Remark 2.1.

1. Suppose M is an abelian group. Then there is a bijection between R -module structures on M and ring homomorphisms $\theta: R \rightarrow \text{End}(M)$ given by $\theta(r)(m) = rm$.
2. Similarly, a right R -module structure on M corresponds to a ring homomorphism $\theta: R \rightarrow \text{End}(M)^{op}$
3. If R is a k -algebra, we can define $\theta: R \rightarrow \text{End}_k(R)$ by $\theta(a)(r) = ar$ and this is an injective k -algebra homomorphism. In fact $\text{im } \theta$ is the subring of $\text{End}_k(R)$ consisting of right R -endomorphisms of R (as a right module)

Definition 2.1. Let G be a group and k a commutative ring. The *group algebra* kG is the k -algebra structure on the free k -module kG with multiplication given by convolution $(f_1 \cdot f_2)(g) = \sum_{h_1, h_2=g} f_1(h_1)f_2(h_2)$

Remark 2.2. This is the only multiplication which makes $g \rightarrow \delta_g$ a group homomorphism and such that $k \rightarrow kG$ by $\lambda \rightarrow \lambda\delta_e$ makes kG into a k -algebra

Lemma 2.1. Suppose R is a k -algebra. Restriction along $G \rightarrow kG^\times$ by $g \rightarrow \delta_g$ defines a natural bijection

$$\Theta: \text{Hom}_{k\text{-alg}}(kG, R) \rightarrow \text{Hom}_{Grp}(G, R^\times)$$

Proof. If $\alpha: kG \rightarrow R$ is a k -algebra homomorphism, then $\Theta(\alpha)(g) = \alpha(\delta_g)$ defines Θ , and if $\beta: G \rightarrow R^\times$ is a group homomorphism, define

$$\Phi(\beta)(f) = \sum_{g \in G} f(g)\beta(g)$$

to give the inverse to Θ . The remainder of the lemma is routine checks of everything we have claimed. $\square$

Definition 2.2. Suppose k is a field. A *k -linear representation* of G is a group homomorphism $\phi: G \rightarrow \text{Aut}_k(V)$ where V is a k -vector space

Corollary 2.1.1. There is a natural bijection between *k -linear representations of G* , and *left kG modules*

Proof. This follows from our remarks and the above lemma. $\square$

Definition 2.3. A *Lie algebra over k* is a k -module $\mathfrak{g}$ equipped with a multiplication $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$ (and hence $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$)

2. $[x[yz]] + [y[zx]] + [z[xy]] = 0$ for all $x, y, z \in \mathfrak{g}$

Example 2.2.

1. Any associative k -algebra R becomes a k -Lie algebra under the commutator $[x, y] = xy - yx$
2. $\mathfrak{gl}(V) = \text{End}_k(V)$ with the commutator bracket is a k -Lie algebra

Definition 2.4. A k -Lie algebra homomorphism is a k -linear map $\theta: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that $[x, y] = [\theta(x), \theta(y)]$ for all $x, y \in \mathfrak{g}$. A representation of $\mathfrak{g}$ is a k -linear homomorphism $\mathfrak{g} \rightarrow \text{End}_k(V)$ for some k -module V

Definition 2.5. Suppose V is a k -module. Define $T^0V = k, T^1V = V$ and $T^nV = V \otimes_k T^{n-1}V$ for $n \geq 2$. The tensor algebra is $TV = \bigoplus_{n=0}^{\infty} T^nV$ with multiplication given by concatenation. There is a natural map $V \rightarrow TV$ via $v \mapsto v \in T^1V$. If V is the free k -module on $\{x_1, \dots, x_d\}$ we write $k\langle x_1, \dots, x_d \rangle$ and call it the free associative algebra on $x_1, \dots, x_d$

Remark 2.3. Note that $k\langle x_1, \dots, x_d \rangle$ is not finitely generated as a k -module for $d > 0$, and is not commutative or Noetherian for $d > 1$

Lemma 2.2. Suppose V is a k -module and R is a k -algebra. Then there is a natural bijection $\text{Hom}_{k\text{-alg}}(TV, R) \rightarrow \text{Hom}_k(V, R)$ given by restriction

Proof. The other direction is given by $f \mapsto (v_1 \otimes \dots \otimes v_d \mapsto \prod_{i=1}^d f(v_i))$. The verification is left to the reader. $\square$

Definition 2.6. Suppose $\mathfrak{g}$ is a k Lie algebra. The universal enveloping algebra of $\mathfrak{g}$ denoted $\mathcal{U}\mathfrak{g}$ is $T\mathfrak{g}/I$ where I is the two sided ideal of $T\mathfrak{g}$ generated by $\{xy - yx - [xy] \mid x, y \in \mathfrak{g} \subseteq T^1\mathfrak{g}\}$. There is a natural map of k -Lie algebras $\iota_{\mathfrak{g}}: \mathfrak{g} \rightarrow \mathcal{U}\mathfrak{g}$ by $x \mapsto x + I$.

Remark 2.4. If $\mathfrak{g}$ is an abelian Lie algebra free on $x_1, \dots, x_d$ then $\mathcal{U}\mathfrak{g} = k[x_1, \dots, x_d]$. More generally if $\mathfrak{g} = V$ is abelian, $\mathcal{U}V$ is called the symmetric algebra on V written $S(V)$

Lemma 2.3. Suppose $\mathfrak{g}$ is a k Lie algebra and R is an associative k algebra. Then there is a natural bijection $\Phi: \text{Hom}_{k\text{-alg}}(\mathcal{U}\mathfrak{g}, R) \rightarrow \text{Hom}_{k\text{-lie}}(\mathfrak{g}, R)$ given by restriction, where R has the commutator bracket.

[LECTURE MISSING]

Theorem 2.4. If $S = \langle R, x \rangle$, $R + Rx = R + xR$ and R is left Noetherian, then S is left Noetherian

Proof. The hypotheses imply $R + Rx + \dots + Rx^n = R + xR + \dots + x^nR$ for every n by induction as $Rx^n = Rx^{n-1}x \subseteq (R + xR + \dots + x^{n-1}R)x$. It follows that

1. $S_n = R + xR + \dots + x^nR$ is a left and right R -module

2. $S = \bigcup S_n$ as $\bigcup S_n$ is a subring of S containing R and x , as $S_n S_m \subseteq S_{n+m}$ (we don't necessarily have $S = \bigoplus x^n R$)
3. For $r \in R$ and $n \geq 0$, there is some $r' \in R$ such that $r'x^n - x^n r \in S_{n-1}$

Now suppose $I \trianglelefteq S$. We will show I is finitely generated. Let $I_n = \{r_n \in R \mid \exists s \in I, s - x^n r_n \in S_{n-1}\}$. We claim this is a left ideal. It is easy to see that I_n is a subgroup as I is. Suppose $r \in R, r_n \in I_n$. Then by (3), there is r' such that $r'x^n - x^n r \in S_{n-1}$, so $r'x^n r_n - x^n r r_n \in S_{n-1}$, and so $r's - x^n r r_n \in S_{n-1}$ and so I_n is a left ideal. Moreover if $s = \sum_{i=0}^n x^i r_i$ then $xs = s = \sum_{i=0}^n x^{i+1} r_i$, so $I_n \subseteq I_{n+1}$, so by the ascending chain condition in R , there is some m such that $I_m = I_{m+r}$ for all $r \geq 0$. For $i = 0, \dots, m$ we choose a finite generating set $\{r_{ij}\}$ for I_i as a left ideal, and choose $s_{ij} \in I$ such that $s_{ij} - x^i r_{ij} \in S_{i-1}$. We claim $X = \{s_{ij} \mid i = 0, \dots, m\}$ generates I as a left ideal. Pick $s = \sum_{i=0}^n x^i r_i \in I_n S_n$. We show by induction on n that $s \in SX$, then we are done by (2). If $n = 0$, then $I_n S_0 = I_n R = I_0$ and $s_{0j} = r_{0j}$ for all j , so $s \in RX \subseteq SX$. If $0 < n \leq m$ then $r_n = \sum a_j r_{nj}$ for some $a_j \in R$ and we can choose $a'_j \in R$ such that $a'_j x^n - x^n a_j \in S_{n-1}$. Then $s - \sum a'_j s_{nj} \in I_n S_{n-1}$, so by induction, we are done. If $n > m$, the proof is basically the same. $\square$

Remark 2.5. Passing to S^{op} gives a right Noetherian version of this. Note that if k is commutative, then $kG \cong (kG)^{op}$, so kG is left Noetherian if and only if it is right Noetherian

Definition 2.7. A group G is *polycyclic* if there is a chain of subgroups $1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$ such that G_i/G_{i-1} is cyclic.

Example 2.3.

1. Every cyclic group is polycyclic.
2. Any finitely generated abelian group is polycyclic.
3. $G = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$ is polycyclic via $1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G$ where $G_1 = \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $G_2 = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
4. Any subgroup of unitriangular matrices in $M_n(\mathbb{Z})$ is polycyclic.

Theorem 2.5. Let G be a polycyclic group. Then G is isomorphic to a subgroup of $GL_n(\mathbb{Z})$, and has a finite index subgroup H isomorphic to a subgroup of upper triangular matrices with coefficients in a ring of integers of an algebraic number field

Proof. No Proof $\square$

Theorem 2.6. Let k be a commutative Noetherian ring and let G be polycyclic. Then kG is a Noetherian ring

Theorem 2.7. Choose a chain of subgroups $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ such that G_i/G_{i-1} is cyclic. An induction argument reduces us to proving kG_{i-1} is left Noetherian implies kG_i is left Noetherian. Pick $x \in G_i$ such that xG_{i-1} generates G_i/G_{i-1} . Let $R = kG_{i-1}$ and $S = \langle R, \delta_x \rangle \trianglelefteq kG_i$ and suppose R is left Noetherian. Since $x^{-1}G_{i-1}x = G_{i-1}$ we see that $\delta_x R = R\delta_x$ so $R + R\delta_x = R + \delta_x R$ so S is left Noetherian. Now if $I \trianglelefteq_l kG_i$ then $I \cap S \trianglelefteq_l S$ so $I \cap S$ has a finite generating set. Now if $t \in I$, then there is some $m \geq 0$ such that $\delta_x^m t \in I \cap S$, so that finite generating set generates $I \trianglelefteq_l kG_i$ and we are done.

Remark 2.6. It remains an open problem whether the only Noetherian group algebras $\mathbb{Z}G$ are those where G has a finite index polycyclic subgroup. It is not hard to see that $\mathbb{Z}G$ Noetherian implies subgroups of G has the maximal condition, but there are groups with the maximal condition that are not virtually polycyclic, eg Tarski monsters. In 2019, Kropholler and Lorreson published a proof that $\mathbb{Z}G$ Noetherian implies G is amenable and every subgroup of G is finitely generated. The only groups known to satisfy both are virtually polycyclic ones. We will prove later that $\mathcal{U}(\mathfrak{g})$ is Noetherian whenever k is and $\mathfrak{g}$ is finitely generated as a k module. The converse is open. The Weyl algebras are Noetherian (ES1)

3 Simple Modules and Artinian Rings

For now R is an arbitrary ring. As for commutative rings, there is a first isomorphism theorem for R -modules

Proposition 3.1. *The first isomorphism theorem holds for R -modules*

Proof. Trivial □

Definition 3.1. An R -module is *simple/irreducible* if it has no proper non-zero submodules

Remark 3.1. Suppose M is a simple R -module and $x \in M \setminus \{0\}$. Then $Rx = M$, so $M = R/\text{ann}(x)$ where $\text{ann}(x) = \{r \in R \mid rx = 0\} = \ker(R \rightarrow M, r \mapsto rx)$ is the *point annihilator* of x . Note that if $y \in M \setminus \{0\}$, it is not necessarily true that $\text{ann}(x) = \text{ann}(y)$ unless R is commutative

It is easy to see that a cyclic R -module $M = Rx$ is simple if the annihilator of x is a maximal left ideal.

Lemma 3.1. *Suppose $L \trianglelefteq R$ is a proper left ideal. Then L is contained in a maximal left ideal of R*

Proof. As in the commutative case. □

Corollary 3.1.1. *Every non-zero finitely generated R -module has a simple quotient*

Proof. Suppose $M = \sum_{i=1}^n Rx_i$ but is not generated by $x_1, \dots, x_{n-1}$. Then $N = \sum_{i=1}^{n-1} Rx_i \leq M$ is proper, and M/N is cyclic, generated by x_i , so we appeal to the lemma and are done. $\square$

Remark 3.2. The finitely generated condition is necessary, as the rationals have no simple quotient as a $\mathbb{Z}$ -module.

Definition 3.2. The *annihilator*, $\text{Ann}_R(M) = \bigcap_{m \in M} \text{ann}_R(m)$ is a two sided ideal of R and if $I \leq R$ then $I = \text{Ann}_R(R/I)$. We say M is *faithful* if $\text{Ann}_R(M) = 0$, ie $R \rightarrow \text{End}(M)$ is injective

Definition 3.3. Let $I \leq R$. Then I is *left primitive* if $I = \text{Ann}_R(M)$ for some simple module M . R is called *left primitive* if the zero ideal is left primitive (equivalently, if R has a faithful simple module)

George Bergan found examples of rings that are left, but not right, primitive.

Lemma 3.2. Let $M = Rx$ be a cyclic left R -module. Then $\text{Ann}_R(M)$ is the largest ideal contained in $L = \text{ann}_R(x)$

Proof. Now $I = \sum_{J \leq R, J \subseteq L} J \leq R$ is the largest two sided ideal of R contained in L . Therefore $\text{Ann}_R(M) \subseteq L$, and hence $\text{Ann}_R(M) \subseteq I$. Conversely, $IM = IRx = Ix \subseteq Lx = 0$ so $I \subseteq \text{Ann}_R(M)$ $\square$

Corollary 3.2.1. Every maximal ideal of R is left and right primitive. If R is commutative, every primitive ideal is maximal.

Proof. Choose a maximal left/right ideal containing our maximal ideal, and apply the former. $\square$

Definition 3.4. The *Jacobson radical* $J(R)$ is the intersection of all left primitive ideals. Thus $J(R) = \{r \in R \mid rM = 0, M \text{ is simple}\}$

Lemma 3.3. $J(R)$ is the intersection of all maximal left ideals

Proof. Denote the intersection by K , suppose $x \in J(R)$. Then if $L \leq_l R$, then $xR/L = 0$ so $x \in L$ and so $J(R) \subseteq K$. Conversely if $P \leq R$ is left primitive, then $P = \text{Ann}_R(M)$ for some simple module M . Then $P = \bigcap_{m \in M} \text{ann}_R(m)$ is an intersection of maximal left ideals, so $K \subseteq P$ and hence as P is arbitrary, $K \subseteq J(R)$. $\square$

Lemma 3.4. (Nakayama) Let M be a non-zero finitely generated R -module. Then $J(R)M \subsetneq M$

Proof. Since M is finitely generated, it has a simple quotient M/K . Then $J(R)M/K = 0$, so $J(R)M \leq K \subsetneq M$ $\square$

Example 3.1. If $R = \mathbb{Z}_{(p)}$ for some prime p and $M = \mathbb{Q}$ then $J(R) = p\mathbb{Z}_{(p)}$ but $J(R)\mathbb{Q} = \mathbb{Q}$ so finite generation is needed.

Corollary 3.4.1. *Let M be a finitely generated R -module and $N \subseteq M$ be a submodule such that $M = N + J(R)M$. Then $M = N$*

Proof. Apply the lemma to M/N □

Proposition 3.2. $J(R) = \{x \mid 1 - axb \in R^\times, a, b \in R\}$

Proof. We denote the right hand side by K . Since $J(R)$ is an ideal in R , to prove $J(R) \subseteq K$, it suffices to prove $x \in J(R)$ implies $1 - x \in R^\times$. Now $R = R(1 - x) + Rx$, so as $Rx \subseteq J(R)$, Nakayama implies $R(1 - x) = R$, so there is $y \in R$ such that $y(1 - x) = 1$. Now $1 - y = -yx \in J(R)$, so there is $z \in R$ such that $zy = 1$. Then $z = 1 - x$ by usual argument, so $J(R) \subseteq K$. Now let $x \in K$ and suppose I is a maximal left ideal. Suppose $x \notin I$. Then $\langle x, I \rangle = R$, so there is $a \in R$ such that $1 - ax \in I$. Since $x \in K$, $(1 - ax)$ is a unit, so $I = (1)$ contradiction. □

Thus $J(R)$ is the largest two sided ideal $A \trianglelefteq R$ such that $1 - A \subseteq R^\times$, so $J(R) = J(R^{op})$. In other words, the intersection of all maximal left ideals is the intersection of all maximal right ideals.

Suppose that V is a left R -module and let $D = \text{End}_R(V)^{op}$. Then V is a right D -module. Indeed, we have an R - D bimodule (ie M has a left R module structure, a right D module structure, and these commute)

Theorem 3.5. (*Schur's Lemma*) *Let V be a simple left R -module. Then its endomorphism ring is a division ring.*

Proof. Why are you reading these notes if you don't know this? □

Thus we can view any simple R -module as a right vector space over the division ring D . We prove a technical lemma.

Lemma 3.6. *Let V be a simple R module, $D = \text{End}_R(V)^{op}$. Suppose $X \subseteq V$ is finite and D -linearly independent. Let $I = \text{ann}_R(X) = \bigcap_{x \in X} \text{ann}_R(x)$. Suppose $Iy = 0$ for some $y \in V$. Then $y \in XD$*

Proof. We induct on the size of X . If $n = 0$ then $I = R$, so $y = 0 \in \emptyset D$ as $1 \in I$. Suppose $n \geq 1$. Let $x \in X$ and let $J = \text{ann}_R(X \setminus \{x\})$. We know $I = J \cap \text{ann}_R(x)$. If $J \subseteq \text{ann}_R(x)$ then $J = I$ so $Jy = 0$ and we are done by induction. If not, then $0 \neq Jx \leq V$ and hence $Jx = V$ by simplicity. Let $d \in D$ be given by $(rx)d = ry$ for $r \in J$. Note that this is well defined as if $rx = r'x$ for $r, r' \in J$ then $r - r' \in J \cap \text{ann}_R(x) = I$ so $(r - r')y = 0$ and $ry = r'y$. Now $J(y - xd) = 0$, so by induction hypothesis, $y - xd \in (X \setminus \{x\})D$, so $y \in XD$. □

Theorem 3.7. (*Jacobson Density Theorem*) *Let V be a simple R module, $X \subseteq V$ be a finite D -linearly independent subset. Then for all $\alpha \in D$, there is some $r \in R$ such that $\alpha(x) = rx$ for all $x \in X$*

Proof. Let $X = \{x_1, \dots, x_n\}$ and $X_i = X \setminus \{x_i\}$. As $x_i \notin X_i D$, $\text{ann}_R(X_i)x_i \neq 0$ so there is some $r_i \in \text{ann}_R(X_i)$ such that $r_i x_i \neq 0$. Since V is simple, $Rr_i x_i = V$, so there is $s_i \in R$ with $s_i r_i x_i = \alpha(x_i)$. Then $r = \sum s_i r_i$ works. □

We also have another consequence of the lemma.

Lemma 3.8. *Let R be a left Artinian ring and a simple left R module. Let $D = \text{End}_R(V)^{op}$. Then V is a finite dimensional right D module.*

Proof. Consider $\{\text{ann}_R(X) \mid |X| < \infty, X \subseteq V\}$. This is a collection of left ideals, so by left Artinianness, this has a minimal element $I = \text{ann}_R(X)$. Let $y \in V$ and consider $\text{ann}_R(X \cup \{y\})$. This is an ideal and a subset of I , and hence is equal to I by minimality. Thus $Iy = 0$, and so $y \in XD$, so X is a finite spanning set. $\square$

Lemma 3.9. *Suppose S is a ring, N is a right S module and $m \geq 1$. Then $\text{End}_S(N^m) = M_m(\text{End}_S(N))$*

Proof. Trivial $\square$

Theorem 3.10. (Artin-Wedderburn) *Let R be a primitive left Artinian ring. Then $R \cong M_n(D)$ where D is some division ring.*

Proof. Since R is left primitive, it has a simple left R -module V . Let $D = \text{End}_R(V)^{op}$, which is a division ring by Schur. Since R is left Artinian, $V \cong D^n$ as a right D module by Lemma 3.8, so $\text{End}_D(V) \cong M_n(\text{End}_D(D))$. Moreover, $D \cong \text{End}_D(D)$ via $d \mapsto (d' \mapsto dd')$, so $\text{End}_D(V) \cong M_n(D)$, so it remains to show that $R \cong \text{End}_D(V)$. We have a map $\phi: R \rightarrow \text{End}_D(V)$ via $r \mapsto (v \mapsto rv)$, a ring homomorphism. It is injective as V is faithful. It is surjective by the Jacobson Density Theorem. So we are done. $\square$

Theorem 3.11. (Chinese Remainder Theorem) *Let R be a ring and $P_1, \dots, P_n \leq R$ such that $P_i + P_j = R$ for all $i \neq j$. Then $R/(P_1 \cap \dots \cap P_n) \rightarrow R/P_1 \times \dots \times R/P_n$*

Proof. We define $\phi: R \rightarrow \prod R/P_i$ by $r \mapsto (r + P_i)_{i=1, \dots, n}$. This is a ring homomorphism with kernel $(P_1 \cap \dots \cap P_n)$. We prove surjectivity by induction on n . $n = 0, 1$ are clear. Since $P_i + P_n = R$ for all $i < n$, there are $a_i \in P_i, b_i \in P_n$ such that $a_i + b_i = 1$. Let $a = a_1 a_2 \dots a_{n-1} \in P_1 \cap \dots \cap P_{n-1}$ and $b = 1 - a$. $b = (a_1 + b_1) \dots (a_{n-1} + b_{n-1}) - a_1 \dots a_{n-1} \in P_n$. Given $(r_i + P_i) \in \prod R/P_i$, we can find $s \in R$ such that $s - r_i \in P_i$ for $1 \leq i \leq n-1$. Let $r = sb + r_n a$. Then $r + P_n = r_n a = r_n(1 - b) = r_n$ and identical for P_i , so done. $\square$

Definition 3.5. A ring R is called *semi-primitive* if $J(R) = 0$. A non-zero ring R is called *simple* if it has no non-zero proper ideals

Exercise 3.1. If D is a division ring and $d \geq 1$, then $M_d(D)$ is simple

Corollary 3.11.1. *If R is a semi-primitive left Artinian ring, there are division rings $D_1, \dots, D_n$ and positive integers $d_1, \dots, d_n \geq 1$ such that*

$$R \cong \prod_{i=1}^n M_{d_i}(D_i)$$

Proof. Let $S = \{P_1 \cap \cdots \cap P_n \mid P_i \trianglelefteq R \text{ left primitive}\}$. This is non-empty since R has a simple left module, and since R is left Artinian, S has a minimal element $I = P_1 \cap \cdots \cap P_n$. If Q is any left primitive ideal, then $Q \cap I \subseteq I$ is an element of S , and so by minimality, $Q \cap I = I$, so $I \subseteq Q$, so $I \subseteq J(R) = 0$. Now each R/P_i is left primitive left Artinian, so by Artin-Wedderburn it is $M_{d_i}(D_i)$, which is simple and hence P_i is maximal. Therefore P_i, P_j are coprime and so by the Chinese remainder theorem, we have

$$R \cong \prod_{i=1}^n R/P_i \cong \prod_{i=1}^n M_{d_i}(D_i)$$

□

Exercise 3.2. By directly computing the left ideals of R , show that $M_{d_i}(D_i)$ is a semiprimitive left Artinian ring for every division ring D_i and positive integer $d_i \geq 1$. Deduce that a left Artinian semiprimitive ring is also right Artinian.

Definition 3.6. A left R -module M is *semisimple* if it can be written as a direct sum of simple modules.

Remark 3.3. R is semiprimitive if and only if R has a faithful semisimple R module (this is easy to prove from the definitions)

Lemma 3.12. *Any subquotient of a semisimple R module is semisimple*

Proof. It suffices to prove that every submodule of every quotient module is semisimple. Suppose $M = \bigoplus_{i \in I} M_i$ and $N \leq M$. We will show N has a complement of the form $\bigoplus_{j \in J} M_j$ for some $J \subseteq I$. To this end, we consider the set

$$S = \left\{ J \subseteq I \mid N \cap \bigoplus_{j \in J} M_j = 0 \right\}$$

By Zorn, we have a maximal element J of S . If $N + \bigoplus_{j \in J} M_j \neq M$, there is some M_i not contained in it. But then $J \cup \{i\} \in S$, contradiction so we have a complement. $N \cong M / \bigoplus_{j \in J} M_j = \bigoplus_{j \notin J} M_j$ so is semisimple, and $M/N \cong \bigoplus_{j \in J} M_j$, so we are done. □

Lemma 3.13. *Let M be a semisimple R module. The following are equivalent*

1. M is Noetherian as an R module
2. M is Artinian as an R module
3. M is a finite direct sum of simples

We call such a module semisimple Artinian

Proof. Since a simple module is both Noetherian and Artinian, (3) implies (1) and (2). If (3) does not hold, then M has a submodule $N = \bigoplus_{n \geq 1} N_n$ with each N_n simple, and so $N_1 \leq N_1 \oplus N_2 \leq \dots$ and $N \geq \bigoplus_{n \geq 2} N_n \geq \dots$, so neither (1) nor (2) holds □

We note that any subquotient of a semisimple Artinian module is semisimple Artinian

Exercise 3.3. Show that the following are equivalent

1. All left R modules are semisimple
2. All right R modules are semisimple
3. R is a semisimple left R module
4. R is a semisimple right R module
5. R is semiprimitive and left Artinian

Proposition 3.3. *The Jacobson radical of a left Artinian ring R is nilpotent*

Proof. Let $J = J(R)$. Then we have a descending chain $J \supseteq J^2 \supseteq \dots$ which must terminate, so $J^n = J^{n+1}$ for some n . Consider $I = \{x \in R \mid J^n x = 0\} \trianglelefteq R$. It suffices to show $I = R$. Suppose that we instead have strict inclusion $I \subsetneq R$. Then R/I has a minimal non-zero left ideal K/I as it is left Artinian. Then K is a simple R/I (and hence R) module. Now $JK/I = 0$, so $JK \subseteq I$, and so $J^{n+1}K = 0$, so $J^n K = 0$ and $K \subseteq I$ contradiction. $\square$

Theorem 3.14. *(Hopkins-Levitski, 1939) Let R be a left Artinian ring. Then R is also left Noetherian*

Proof. Let J be the Jacobson radical of R . For any $i \geq 0$, J^i/J^{i+1} is a left Artinian R module and so an Artinian left R/J module. Since R/J is semiprimitive and left Artinian, Exercise 3.3 gives that J^i/J^{i+1} is semisimple left Artinian, and hence left Noetherian. But eventually $J^n = 0$, so R is a finite extension of left Noetherian modules by left Noetherian modules, and hence is left Noetherian $\square$